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Almost Split Sequences in Dimension Two

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INTRODUCTION

Let **C** be a full subcategory of an abelian category **A** which is closed under extensions, i.e., if $0 \to C_1 \to C_2 \to C_3 \to 0$ is an exact sequence in **A** with C_1 and C_3 in **C**, then C_2 is in **C**. An exact sequence $(*)$ $0 \to C_1 \to^f C_2 \to^g C_3 \to 0$ in **C** is said to be an almost split sequence in **C** if it satisfies the following conditions:

(a) $(*)$ does not split.

(b) If $h: X \to C_3$ is a morphism in **C** which is not a splittable epimorphism in **C**, then there is a $t: X \to C_2$ such that $gt = h$.

(c) If $j: C_1 \to Y$ is a morphism in **C** which is not a splittable monomorphism in **C**, then there is an $s: C_2 \to Y$ in **C** such that $j = sf$.

Almost split sequences were first introduced by us in the case **A** is the category of finitely generated modules over an artin algebra Λ and **C** = **A** (see [6, 7]). The reader is referred to the expository articles [11, 12] for an account of the role almost split sequences have played in the theory of the representation theory of artin algebras. That almost split sequences exist for various subcategories of module categories over a much wider class of rings than artin algebras was first shown in [2, 3], where various existence theorems for almost split sequences were given. These results seemed to indicate that there might be connections between the structure and existence of almost split sequences and algebraic geometry. Recent developments show that this is indeed the case.

In [4] it is shown that the structure of the almost split sequences in the category of reflexive modules of a complete rational double point over the complex numbers determines the desingularization graph of the singularity. Also it has been shown in [5] that a complete Cohen–Macaulay local ring S is an isolated singularity if and only if the category of Cohen–Macaulay

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S-modules has almost split sequences. A consequence of this result is that if S is a complete Cohen–Macaulay local ring of finite Cohen–Macaulay type (i.e., S has only a finite number of isomorphism classes of indecomposable Cohen–Macaulay modules), then S is an isolated singularity [5]. In addition, almost split sequences have also been used to show that certain complete Cohen–Macaulay singularities are of finite Cohen–Macaulay type [8].

In view of these developments, it is natural to ask to what extent these results about almost split sequences can be carried over to graded modules over graded rings and consequently to coherent sheaves over projective varieties via the standard sheafification of graded modules. In this paper we begin to investigate this question by considering two-dimensional Cohen–Macaulay rings and projective curves. In other papers we will consider higher-dimensional rings and projective varieties. It is to be hoped that almost split sequences will prove to be as interesting in this context as they have turned out to be in other situations.

One final general comment. From the point of view of almost split sequences, complete two-dimensional Cohen–Macaulay rings are rather special, primarily because of the existence of the fundamental exact sequence of the ring. This notion was first introduced in [4] in the more restrictive setting of complete two-dimensional integrally closed local domains. Here we show that fundamental exact sequences exist for certain types of algebras which need not be commutative, and use this fact to derive the existence of almost split sequences in the commutative case. As in [4] we also show how almost split sequences can be constructed from the fundamental exact sequence of the ring.

This paper is divided into two parts. The first part studies (maximal) Cohen–Macaulay modules over commutative noetherian equidimensional Cohen–Macaulay rings of dimension 2. The second part is devoted to establishing the analogues of the results in Part I for $\mathbb{Z}$-graded Cohen–Macaulay modules over two-dimensional, $\mathbb{Z}$-graded Cohen–Macaulay rings. These results are then applied to studying Cohen–Macaulay coherent sheaves over connected, Cohen–Macaulay projective curves. We now give a brief section by section description of the contents of this paper.

Part I

1. This section deals with some generalities concerning S_2-modules over commutative noetherian S_2-rings which are needed throughout the rest of the paper.

2. In this section a basic existence theorem for morphisms determined by modules is established, a notion first introduced in [3]. This gives the existence of various types of exact sequences from which all our existence theorems for almost split sequences are derived.

3. This section is devoted to studying Cohen–Macaulay modules over a complete, noetherian two-dimensional Cohen–Macaulay ring S. The first part of the section is devoted to showing that an indecomposable nonprojective Cohen–Macaulay module C has the property that $C_\mathfrak{p}$ is $S_\mathfrak{p}$-free for all prime ideals $\mathfrak{p}$ in S of height at most one if and only if there is an almost split sequence $0 \to A \to B \to C \to 0$ in the category of Cohen–Macaulay S-modules. The second part of the section deals with explaining how almost split sequences can be constructed from the fundamental exact sequence for S.

Part II

1. Let k be a field, $k[X_1,..., X_n]$ the polynomial ring with the usual $\mathbb{Z}$-grading and let S be the graded ring which is a factor ring of $k[X_1,..., X_n]$ by a homogeneous ideal. We assume that S is a graded Cohen–Macaulay ring of dimension 2, and prove results analogous to those of Part I. As a special case we get that if C is a nonfree indecomposable, Cohen–Macaulay graded module in the category of graded S-modules with degree zero morphisms, then C has the property that $C_\mathfrak{p}$ is $S_\mathfrak{p}$-free for all prime ideals $\mathfrak{p}$ of S of height at most one if and only if there is an almost split sequence $0 \to A \to B \to C \to 0$ in the category of Cohen–Macaulay graded modules with degree zero morphisms.

2. In this section it is shown how to compute almost split sequences in the category of Cohen–Macaulay modules with degree zero morphisms using the fundamental exact sequence for S.

3. In this section X is a connected, Cohen–Macaulay projective curve over an infinite field k. It is shown that an indecomposable Cohen–Macaulay coherent sheaf $\mathscr{H}$ on X is locally free if and only if there is an almost split sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ in the category of coherent Cohen–Macaulay sheaves on X. It is also shown how almost split sequences in the category of Cohen–Macaulay coherent sheaves on X can be computed using the almost split sequences $0 \to \omega_X \to \mathscr{E}_X \to \mathscr{O}_X \to 0$, where $\mathscr{O}_X$ is the structure sheaf and ω_X is the dualizing sheaf. A. Schofield has given an independent proof of these results when X is a nonsingular curve and k is algebraically closed using sheaf theoretic methods.

I

1. Preliminaries

In this section we give some basic facts about S_2-rings and modules. Since the theory of such modules and rings is similar to that of reflexive modules over integrally closed rings, proofs are only sketched.

Throughout this section S is a commutative noetherian ring. We denote by mod S the category of finitely generated S-modules. An M in mod S is said to be an S_2-module if for each prime ideal $\mathfrak{p}$ of S we have $\operatorname{depth}_{S_\mathfrak{p}} M_\mathfrak{p} \geqslant \min(2, \dim S_\mathfrak{p})$. The full subcategory of mod S consisting of the S_2-modules will be denoted by S_2 (mod S). The ring S is said to be an S_2-ring if S is an S_2-module.

By an S-algebra Γ we always mean an S-algebra Γ which is a finitely generated S-module. We say that an M in mod Γ is an S_2-module, if when viewed as an S-module it is an S_2-module. The S-algebra Γ is said to be an S_2-algebra or an S_2-ring if Γ is in S_2 (mod Γ).

Suppose now that A and B are S-modules. Then it is easily checked that $\operatorname{Hom}_S(A, B)$ is an S_2-module whenever B is an S_2-module. Therefore if A and B are S_2-modules, then $\Gamma = \operatorname{End}(A)^{\mathrm{op}}$ is an S-algebra which is S_2 and $\operatorname{Hom}_S(A, B)$ is a Γ-module which is S_2. Also if Λ is an S-algebra and U, V are Λ-modules with V an S_2-module, then the S-module $\operatorname{Hom}_\Lambda(U, V)$ is an S_2 module. Finally, we say for an integer $d \geqslant 0$ that an S-module M is free at height d primes if for each prime ideal $\mathfrak{p}$ with $\dim S_\mathfrak{p} \leqslant d$, we have that $M_\mathfrak{p}$ is $S_\mathfrak{p}$-free. We will be particularly interested in S-modules which are free at height one primes. The following result is a basic tool in this paper.

THEOREM 1.1. *Let S be an S_2-ring and M an S-module which is S_2 and free at height one primes and let $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$. Then the functors $\operatorname{Hom}_S(M, \ )$: $\operatorname{mod} S \to \operatorname{mod} \Gamma$ and $\operatorname{Hom}_\Gamma(M^*, \ )$: $\operatorname{mod} \Gamma \to \operatorname{mod} S$ induce inverse equivalences S_2 (mod S) $\to S_2$ (mod Γ), where $M^* = \operatorname{Hom}_S(M, S)$.*

Before giving the proof of Theorem 1.1, we point out the following easily verified and basically well-known fact concerning S_2-modules.

LEMMA 1.2. *A morphism $f: A \to B$ of S_2-modules is an isomorphism if and only if $f_\mathfrak{p}: A_\mathfrak{p} \to B_\mathfrak{p}$ is an isomorphism for all primes p with $\dim S_\mathfrak{p} \leqslant 1$.*

Proof. Same as for reflexive modules over integrally closed noetherian domains.

We now return to the proof of Theorem 1.1. For each X in mod S we have the following composition of S-morphisms which is obviously functorial in X,

$$X \xrightarrow{\sim} \operatorname{Hom}_S(S, X) \xrightarrow{\alpha} \operatorname{Hom}_S\left(M \underset{\Gamma}{\otimes} M^*, X\right)$$

$$\xrightarrow{\sim} \operatorname{Hom}_\Gamma(M^*, \operatorname{Hom}_S(M, X)),$$

where α is induced by the morphism of S-modules $\beta: M \otimes_\Gamma M^* \to S$ given by $\beta(m \otimes f) = f(m)$. Since M is free at height one primes, it follows that

$\beta_p: M_p \otimes_{\Gamma_p} M_p^* \to S_p$ is an isomorphism if dim $S_p \leqslant 1$. Therefore α_p is also an isomorphism if dim $S_p \leqslant 1$. Hence by Lemma 1.2, we have that α is an isomorphism whenever X is in S_2 (mod S). Therefore we have the canonical isomorphism

$$1_{S_2(\text{mod } S)} \longrightarrow \text{Hom}_\Gamma(M^*, \text{Hom}_S(M, \ )),$$

which we will usually consider an identification.

Also for each Y in mod Γ we have the following composition of Γ-morphisms which is obviously functorial in Y,

$$Y \simeq \text{Hom}_\Gamma(\Gamma, Y) \xrightarrow{\gamma} \text{Hom}_\Gamma\left(M^* \underset{S}{\otimes} M, Y\right)$$
$$\simeq \text{Hom}_S(M, \text{Hom}_\Gamma(M^*, Y)),$$

where γ is induced by $\delta: M^* \otimes_S M \to \text{End}(M)$ given by $\delta(f \otimes m)(x) = f(x)(m)$. Since M is free at height one primes, we know that δ_p and hence γ_p is an isomorphism if dim $S_p \leqslant 1$. Thus γ is an isomorphism for all Y in S_2 (mod Γ). Therefore we have the canonical isomorphism

$$1_{S_2(\text{mod } \Gamma)} \to \text{Hom}_S(M, \text{Hom}_\Gamma(M^*, \ )),$$

which we will usually view as an identification. This completes the proof of Theorem 1.1.

In addition to Theorem 1.1, we will have need of several other facts concerning S_2-modules which are free at height one primes later on in this paper. For convenience, we give their statements and sketch some proofs.

PROPOSITION 1.3. *Let M be a module in S_2(mod S) which is free at height one primes. Then we have the following.*

(a) *M is a reflexive S-module, i.e., the natural S-morphism $M \to M^{**}$ is an isomorphism which we often view as an identification.*

(b) *Let $\Gamma = \text{End}_s(M)$. For each X in S_2 (mod S), the morphism*

$$u_x: \text{Hom}_S(\Gamma, X) \to \text{Hom}_S(M, \text{Hom}_S(M^*, X)),$$

given by $(u_x(f))(m)(g) = f((g \otimes m))$ for all f in $\text{Hom}_S(\Gamma, X)$ and g in M^*, is an isomorphism of two-sided Γ-modules which is functorial in X and where $\delta: M^* \otimes_S M \to \Gamma$ is given by $\delta(g \otimes m)(x) = g(x)\, m$ for all g in M^* and m and x in M.

Proof. (a) Since M is free at height one primes, the map $M \to M^{**}$ is an isomorphism when localized at height one primes. Therefore by Lemma 1.2, $M \to M^{**}$ is an isomorphism since M and M^{**} are in S_2 (mod S).

(b) Essentially the same as the proof for reflexive modules over integrally closed domains given in [4, Lemma 6.3].

Letting $X = S$ in the previous proposition we obtain the two-sided Γ isomorphism $u_S: \mathrm{Hom}_S(\Gamma, S) \to \mathrm{Hom}_S(M, M^{**}) = \mathrm{Hom}_S(M, M) = \Gamma$. We denote by $t: \Gamma \to S$ the S-linear functional $u_S^{-1}(1_M)$, and we call t the trace on Γ. We now point out some other descriptions of the trace.

LEMMA 1.4. *Let M in $S_2\,(\mathrm{mod}\, S)$ be free at height one primes and let $\Gamma = \mathrm{End}\, M$. Then the trace $t: \Gamma \to S$ can be described as follows.*

(a) *$t: \Gamma \to S$ is the unique S-morphism such that for all height one primes $\mathfrak{p}$ in S the map $t_{\mathfrak{p}}: \Gamma_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is the ordinary trace map on the endomorphism ring $\Gamma_{\mathfrak{p}}$ of the free $S_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$.*

(b) *$t: \Gamma \to S$ is the unique S-morphism such that the composition $M^* \otimes_S M \to^{\delta} \Gamma \to S$ is the evaluation map $e: M^* \otimes_S M \to S$ given by $e(f \otimes m) = f(m)$ for all f in M^* and m in M, where δ is defined as in Proposition* 1.3.

Proof. Follows from routine calculations.

In establishing the last property of the trace that is of interest to us, we assume that S is a domain in addition to being S_2. It would be interesting to know if the same result holds without this additional assumption on S.

PROPOSITION 1.5. *Let S be a local domain with maximal ideal $\mathfrak{m}$ which is also S_2. Assume M in $S_2\,(\mathrm{mod}\, S)$ is free at height one primes and that $\Gamma = \mathrm{End}\, M$. Then the trace map $t: \Gamma \to S$ has the property that $t(\mathrm{rad}\, \Gamma) \subset \mathfrak{m}$. Consequently $t(\mathrm{rad}\, \Gamma) \subset \mathfrak{m}$ for all f in $\mathrm{Hom}_S(\Gamma, S)$.*

Proof. The first part of this result was proven in [4, Proposition 5.1] under the additional hypothesis that S is integrally closed. After slight modifications, the same proof works for domains which are not necessarily integrally closed as required here. The second part follows from the fact that t in $\mathrm{Hom}_S(\Gamma, S)$ is a two-sided Γ-generator for $\mathrm{Hom}_S(\Gamma, S)$.

2. SUBFUNCTORS DETERMINED BY MODULES

Throughout this section S is an equidimensional commutative noetherian ring of dimension 2. For such rings the notions of a module being S_2 and being a (maximal) Cohen–Macaulay module coincide. To simplify notation we denote the category $S_2(\mathrm{mod}\, S)$ by **CM***S*. In addition we assume that S is a Cohen–Macaulay ring and an R-algebra which is a finitely generated R-module where is a two-dimensional Gorenstein ring.

Clearly S and R are in **CM**R. Our main aim in this section is to show the following.

Suppose M in **CM**S is free at height one primes and F: **CM**$S \to \text{mod } S$ is a finitely generated (presented) contravariant additive functor. Then every M-determined subfunctor of F is finitely generated (presented). This result is used in the next sections to give information concerning almost split sequences in **CM**S when S is a complete local ring or a certain type of graded ring. These results not only generalize considerably previous work about lattices over two-dimensional orders (see [3]), but also give some new information about the earlier work.

Since S is an S_2-ring and **CM**$S = S_2$ (mod S), the results of Section 1 apply to S and **CM**S. In particular, if M is in **CM**S and is free at height one primes, then we have the equivalence of categories **CM**$S \to$ **CM**Γ described in Theorem 1.1, where $\Gamma = \text{End}(M)^{\text{op}}$. We now show how to use this result to prove our announced result in the special case where F is representable, from which the general result follows easily.

Let Γ be an R-algebra which is a finitely generated Cohen–Macaulay module. We now establish some basic facts concerning the module $\text{Hom}_R(\Gamma, R)$ in **CM**Γ. While we state and prove this result for R being of dimension 2, the result and proof obviously generalize to Gorenstein R of arbitrary dimension.

LEMMA 2.1. (a) *For each A in mod Γ and all $i \geqslant 0$ we have isomorphisms* $\text{Ext}^i_\Gamma(A, \text{Hom}_R(\Gamma, R)) \simeq \text{Ext}^i_R(A, R)$ *which are functorial in A.*

(b) *If A is in* **CM**Γ, *then* $\text{Ext}^i_\Gamma(A, \text{Hom}_R(\Gamma, R)) = 0$ *for all $i \geqslant 1$.*

(c) *If A is nonzero and of finite length, then* $\text{Ext}^i_\Gamma(A, \text{Hom}_R(\Gamma, R)) = 0$ *for $i < 2$ and is not zero for $i = 2$.*

Proof. (a) Let $0 \to R \to I_0 \to I_1 \to I_2 \to 0$ be a minimal injective resolution of R (remember R is Gorenstein of dimension 2). Then

$$0 \to \text{Hom}_R(\Gamma, R) \to \text{Hom}_R(\Gamma, I_0) \to \text{Hom}_R(\Gamma, I_1) \to \text{Hom}_R(\Gamma, I_2) \to 0$$

is an injective Γ resolution of $\text{Hom}_R(\Gamma, R)$. It is exact because Γ is a Cohen–Macaulay R-module and the $\text{Hom}_R(\Gamma, I_j)$ are injective for all j since Γ is Γ-projective and I_j is R-injective for all j. Since we have the isomorphisms $\text{Hom}_\Gamma(A, \text{Hom}_R(\Gamma, B) \simeq \text{Hom}_R(A, B)$ which are functorial in A and B, it follows that $\text{Ext}^i_\Gamma(A, \text{Hom}_R(\Gamma, R)) \simeq \text{Ext}^i_R(A, R)$ for all Γ-modules A and all i.

(b) and (c) These are trivial consequences of (a).

PROPOSITION 2.2. *Suppose Γ is an R-algebra which is in* **CM**R. *Let*

$B \to C \to 0$ be an exact sequence of Γ-modules with B a Cohen–Macaulay module and C a module of finite nonzero length. Moreover assume that Q is a projective Γ^{op}-module such that there is an exact sequence of Γ^{op}-modules $Q \to \mathrm{Ext}^2_R(C, R) \to 0$. Then there is an exact sequence of Γ-modules

$$0 \to \mathrm{Hom}_R(Q, R) \to A \xrightarrow{h} B \to C \to 0$$

with A in $\mathbf{CM}\Gamma$ and any such exact sequence has the following properties.

(a) *$\mathrm{Hom}_R(Q, R) \to A$ is not a splittable monomorphism.*

(b) *A morphism $g\colon X \to B$ in $\mathbf{CM}\Gamma$ can be lifted to A if $\mathrm{Im}\, g \subset \mathrm{Im}\, h$.*

Proof. We first show that our desired type of exact sequence actually exists.

Let $0 \to L \to P \to B \to C \to 0$ be an exact sequence of Γ-modules with P projective. Then $\mathrm{Ext}^i_R(B, R) = \mathrm{Ext}^i_R(P, R) = \mathrm{Ext}^i_R(L, R) = 0$ for all $i > 0$ since the B, P, L are all in $\mathbf{CM}R$. Also $\mathrm{Ext}^i_R(C, R) = 0$ for $i = 0, 1$ since C has finite length. From this it follows that if we apply the functor $\mathrm{Hom}_R(\ , R)$ to the exact sequence $0 \to L \to P \to B \to C \to 0$ we obtain the exact sequence of Γ^{op}-modules

$$0 \to B^* \to P^* \to L^* \to \mathrm{Ext}^2_R(C, R) \to 0,$$

where $X^* = \mathrm{Hom}_R(X, R)$. Since B, P, L are reflexive R-modules, applying $\mathrm{Hom}_R(\ , R)$ to this exact sequence we obtain the commutative exact diagram

$$\begin{array}{ccccccccc} 0 \to & L & \to & P & \to & B & \to & C & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & L^{**} & \to & P^{**} & \to & B^{**} & \to & \mathrm{Ext}^2_R(\mathrm{Ext}^2_R(C, R), R) & \to 0, \end{array}$$

where $C \to \mathrm{Ext}^2_R(\mathrm{Ext}^2_R(C, R), R)$ is the usual isomorphism occurring in the duality between Γ-modules and Γ^{op}-modules of finite length given by $Y \mapsto \mathrm{Ext}^2_R(Y, R)$ for all finite length Γ-modules (Γ^{op}-modules) Y.

It is also not difficult to see that we have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 \to & B^* & \to & U & \to & Q & \to & \mathrm{Ext}^2_R(C, R) & \to 0 \\ & \wr\!\!\downarrow & & \downarrow & & \downarrow & & & \\ 0 \to & B^* & \to & P^* & \to & L^* & \to & \mathrm{Ext}^2_R(C, R) & \to 0. \end{array}$$

Applying $\mathrm{Hom}_R(\ ,R)$ to this diagram we obtain the exact commutative diagram

$$\begin{array}{ccccccccc}
0\to & L & \to & P & \to & B & \to & C & \to 0\\
& \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & \\
0\to & L^{**} & \to & P^{**} & \to & B^{**} & \to & \mathrm{Ext}^2_R(\mathrm{Ext}^2_R(C,R),R) & \to 0\\
& \downarrow & & \downarrow & & \| & & \| & \\
0\to & P^{*} & \to & U^{*} & \to & B^{**} & \to & \mathrm{Ext}^2_R(\mathrm{Ext}^2_R(C,R),R). &
\end{array}$$

Hence we obtain the exact sequence $0\to Q^*\to U^*\to B\to C\to 0$ of Γ-modules, which is our desired sequence since U^* is in $\mathbf{CM}\Gamma$.

We now proceed to the proof of the rest of the proposition.

(a) If $0\to Q^*\to A$ is a splittable monomorphism then $\mathrm{Im}\, h$ is a summand of A and therefore Cohen–Macaulay, which contradicts the fact that $C\neq 0$.

(b) Suppose $f\colon X\to B$ has the property that $\mathrm{Im} f\subset K=\mathrm{Im}\, h$. Then there is the pullback exact commutative diagram

$$\begin{array}{ccccccc}
0\to & \mathrm{Hom}_R(\Gamma,R) & \to & E & \to & X & \to 0\\
& \| & & \downarrow & & \downarrow & \\
0\to & \mathrm{Hom}_R(\Gamma,R) & \to & A & \to & K & \to 0.
\end{array} \qquad (*)$$

But $(*)$ splits since $\mathrm{Ext}^1_\Gamma(X,\mathrm{Hom}_R(\Gamma,R))=0$ because X is Cohen–Macaulay (see Lemma 2.1(b)). This implies trivially that $f\colon X\to B$ can be lifted to A.

We now want to show how Theorem 1.1 and Proposition 2.2 together imply our desired result about additive finitely generated contravariant functors from $\mathbf{CM}S$ to $\mathrm{mod}\, S$. However, before doing this it is convenient to recall some definitions and results about such functors given in [3].

Let $F\colon \mathbf{CM}S\to \mathrm{mod}\, S$ be a contravariant additive functor. A subfunctor F' of F is said to be determined by an X in $\mathbf{CM}S$ if a subfunctor F'' of F is contained in F' whenever $F''(X)\subset F'(X)$. It is not difficult to see that if X is in $\mathbf{CM}S$ and $H\subset F(X)$ is an $(\mathrm{End}\, X)^{\mathrm{op}}$ submodule of the $(\mathrm{End}\, X)^{\mathrm{op}}$-module $F(X)$, then there is a unique X-determined subfunctor $F_H\subset F$ such that $F_H(X)=H$ (see [3]).

Next we recall that a morphism $f\colon B\to C$ in $\mathbf{CM}S$ is said to be right determined by X in $\mathbf{CM}S$ if the induced morphism $\mathrm{Hom}_S(\ ,f)\colon \mathrm{Hom}_S(\ ,B)\to \mathrm{Hom}_S(\ ,C)$ has the property that the subfunctor $\mathrm{Im}\,\mathrm{Hom}_S(\ ,f)$ of $\mathrm{Hom}_S(,C)$ is determined by X. It is easily seen that $f\colon B\to C$ is right determined by X if and only if a morphism

$g: Y \to C$ in **CM**S can be lifted to B provided $\operatorname{Im}(\operatorname{Hom}_S(X, Y) \to^{\operatorname{Hom}_S(X,g)} \operatorname{Hom}_S(X, C))$ is contained in $\operatorname{Im}(\operatorname{Hom}_S(X, B) \to \operatorname{Hom}_S(X, C))$ (see [3]).

Combining Theorem 1.1 and Proposition 2.2, we obtain the following result.

PROPOSITION 2.3. *Let M in* **CM**S *be free at height one primes and let* $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$. *Suppose C is in* **CM**S *and H is a Γ-submodule of* $\operatorname{Hom}_S(M, C)$ *such that* $\operatorname{Hom}_S(M, C)/H$ *is of finite length. Finally, let n be an integer such that there is an epimorphism* $(\Gamma^{\mathrm{op}})^n \to \operatorname{Ext}^2_R(\operatorname{Hom}_S(M, C)/H, R)$.

Then there is a right M-determined morphism $f: B \to C$ in **CM**S *having the property that* $H = \operatorname{Im}(\operatorname{Hom}_S(M, B) \to^{\operatorname{Hom}_S(M,f)} \operatorname{Hom}_S(M, C))$ *and such that:*

(a) $0 \to \operatorname{Hom}_R(\operatorname{Hom}_S(M, S), R)^n \to B \to^f C$ *is exact.*

(b) *f is a splittable epimorphism if and only if* $H = \operatorname{Hom}_S(M, C)$.

(c) *$f: B \to C$ is onto if and only if $H \supset P(M, C)$, where $P(M, C)$ is the* $\operatorname{End}(M)^{\mathrm{op}}$*-submodule of* $\operatorname{Hom}_S(M, C)$ *consisting of all $h: M \to C$ which can be factored through a projective module.*

Proof. (a) Clearly the split exact sequence $0 \to \operatorname{Hom}_R(\operatorname{Hom}_S(M, S), R) \to B \to^f C \to 0$ gives a right M-determined morphism when $H = \operatorname{Hom}_S(M, C)$. So assume now that $H \neq \operatorname{Hom}_S(M, C)$.

Now Γ is a Cohen–Macaulay ring and $\operatorname{Hom}_S(M, C)$ is in **CM**Γ. Since $\operatorname{Hom}_S(M, C)/H$ is a Γ-module of nonzero finite length, there is by Proposition 2.2 an exact sequence of Γ-modules $0 \to \operatorname{Hom}_R(\Gamma, R)^n \to A \to \operatorname{Hom}_S(M, C) \to \operatorname{Hom}_S(M, C)/H \to 0$ where all except the right-hand module is a **CM**S-module. Applying $\operatorname{Hom}_\Gamma(\operatorname{Hom}_S(M, S),)$ to this exact sequence, we obtain an exact sequence of Cohen–Macaulay S-modules $0 \to \operatorname{Hom}_R(\operatorname{Hom}_S(M, S), R) \to B \to^f C$, using the fact that $\operatorname{Hom}_S(M,):$ **CM**$S \to$ **CM**Γ and $\operatorname{Hom}_\Gamma(\operatorname{Hom}_S(M, S),):$ **CM**$\Gamma \to$ **CM**S are inverse equivalences. It is also clear using these equivalences that $\operatorname{Im}(\operatorname{Hom}_S(M, B) \to^{\operatorname{Hom}_S(M,f)} \operatorname{Hom}_S(M, C)) = H$. Therefore to finish the proof of (a), we need only show that $B \to C$ is right X-determined.

Let $h: Y \to C$ be a morphism in **CM**S such that $\operatorname{Im}(\operatorname{Hom}_S(M, Y) \to \operatorname{Hom}_S(M, C)) \subset \operatorname{Im}(\operatorname{Hom}_S(M, B) \to \operatorname{Hom}_S(M, C))$. But then by Proposition 2.2 we know that there is a Γ-morphism $g: \operatorname{Hom}_S(M, Y) \to \operatorname{Hom}_S(M, B)$ such that $\operatorname{Hom}_S(M, f)\, g = \operatorname{Hom}_S(M, h)$. This means that the S-morphism $\operatorname{Hom}_\Gamma(\operatorname{Hom}_S(M, S): Y \to B$ has the property $h = f \operatorname{Hom}_\Gamma(\operatorname{Hom}_S(M, S), g)$. Therefore $f: B \to C$ is right M-determined.

(b) and (c) These are easy to see.

We now point out the following generalization of Proposition 2.2.

THEOREM 2.4. *Let M in* **CM**S *be free at height one primes. Suppose C is in* **CM**S *and H is an arbitrary* $\operatorname{End}(M)^{\mathrm{op}}$*-submodule of* $\operatorname{Hom}_S(M, C)$. *Then there is an exact sequence* $0 \to \operatorname{Hom}_R(\operatorname{Hom}_S(M, S), R)^n \to B \xrightarrow{f} C$ *in* **CM**S *such that f is right X-determined with* $\operatorname{Im}\operatorname{Hom}_S(M, f) = H$. *Moreover f is surjective if and only if $H \supset P(M, C)$.*

Proof. Let $(\ , C)_H$ be the M-determined subfunctor of $(\ , C)$ whose value at M is H. Then $(\ , C)_H(S) \subset \operatorname{Hom}_S(S, C) = C$ is an S-submodule of C which we denote by C'. Then by Proposition 3.18 of [3], we know that $(\ , C)_H \subset \operatorname{Hom}_S(\ , C')$ and H is an $\operatorname{End}(M)^{\mathrm{op}}$-submodule of $\operatorname{Hom}_S(M, C')$ containing $P(M, C')$. Because M is free at height one primes, we know that $\operatorname{Hom}_S(M, C')/(P(M, C')$ and consequently $\operatorname{Hom}_S(M, C')/H$ is of finite length.

Since $C' \subset C$, we have that $C' \subset (C')^{**} \subset C^{**} = C$, where $X^* = \operatorname{Hom}_R(X, R)$ because R is Gorenstein and C is in **CM**R. Letting $C'' = (C')^{**}$ we have that C''/C' is of finite length since $\operatorname{depth}_{R_\mathfrak{p}} C'_\mathfrak{p} \geqslant \dim R_\mathfrak{p}$ for all height at most one primes p in R. From this and the fact that $\operatorname{Hom}_S(M, C')/H$ is of finite length, it follows that $\operatorname{Hom}_S(M, C'')/H$ is of finite length. Since C'' is in **CM**R, it follows from Proposition 2.2 that there is an exact sequence $0 \to \operatorname{Hom}_R(M, S)^* \to B \xrightarrow{f} C''$ such that $\operatorname{Im}\operatorname{Hom}_S(\ , f')$ is the M-determined subfunctor of $(\ , C'')$ whose value at M is H. Therefore $\operatorname{Im}\operatorname{Hom}_S(\ , f) = (\ , C)_H$ since $(\ , C)_H \subset (\ , C'')$ is also M-determined with H as value at M. Consequently, the induced exact sequence $0 \to \operatorname{Hom}_R(\operatorname{Hom}_S(M, S), R)^n \to B \xrightarrow{f} C$ is our desired exact sequence.

The proof of the rest of the theorem is trivial.

We now conclude this section by showing how Theorem 2.4 gives the result announced at the beginning of the section. Before doing this we recall some definitions. A contravariant additive functor F: **CM**$S \to \operatorname{mod} S$ is said to be finitely generated if there is an epimorphism $\operatorname{Hom}_S(\ , C) \to F \to 0$ for some C in **CM**S and F is said to be finitely presented if there is an exact sequence

$$\operatorname{Hom}_S(\ , B) \to \operatorname{Hom}_S(\ , C) \to F \to 0 \qquad \text{with } B \text{ and } C \text{ in } \mathbf{CM}S.$$

THEOREM 2.5. *Let M in* **CM**S *be free at height one primes. Let F' be a subfunctor of F determined by M, where F:* **CM**$S \to \operatorname{mod} S$ *is contravariant. Then*

(a) *F' is finitely generated if F is finitely generated, and*

(b) *F' is finitely presented if F is finitely presented.*

Proof. (a) Let $\alpha: (\ , C) \to F$ be an epimorphism. Then it is easily seen that F' being determined by M in F implies that $\alpha^{-1}(F')$ is determined by M in $(\ , C)$. Then by Theorem 2.4, we know that $\alpha^{-1}(F')$ is finitely presented and therefore finitely generated. Hence F' is finitely generated.

(b) In the case F is finitely presented, we have that the kernel of the epimorphism $\alpha: \alpha^{-1}(F') \to F' \to 0$ is finitely generated. Since $\alpha^{-1}(F')$ is finitely presented, it follows that F' is finitely presented.

The rest of this part is devoted to giving various applications of the results of Sections 1 and 2.

3. Almost Split Sequences

Throughout this section S is a complete local Cohen–Macaulay ring of dimension 2. We can also assume without loss of generality that S is an R-algebra, which is a finitely generated R-module with R a complete local Gorenstein ring of dimension two. Thus the results of Section 2 apply to the R-algebra S. It is our aim in this section to give various existence theorems for almost split sequences in **CM**S. We start with those which are direct consequences of the results in the previous section.

Let C be an indecomposable Cohen–Macaulay S-module which is free at height one primes. Then $\Gamma = \mathrm{End}(C)^{\mathrm{op}}$ is a local ring (remember S is a complete local ring) whose unique maximal ideal is rad Γ, the radical of Γ. Let rad $\mathrm{Hom}_S(\ , C)$ be the C-determined subfunctor of $\mathrm{Hom}_S(\ , C)$ whose value at C is rad Γ. It is easily seen and well known that rad $\mathrm{Hom}_S(X, C)$ consists precisely of the $f: C \to M$ which are not splittable surjections for all X in **CM**S.

We now apply the results of Section 2 to obtain the following existence theorem.

Theorem 3.1. *Let C be an indecomposable module in* **CM**S *which is free at height one primes. Then:*

(a) *There is a unique (up to isomorphism) exact sequence of* $\Gamma = \mathrm{End}\, C^{\mathrm{op}}$*-modules* $(*)$ $0 \to \mathrm{Hom}_R(\Gamma, R) \to A \to \Gamma \to \Gamma/\mathrm{rad}\,\Gamma \to 0$ *with A a Cohen–Macaulay module.*

(b) *Applying the functor* $\mathrm{Hom}_\Gamma(\mathrm{Hom}_S(C, S), \ )$ *to* $(*)$ *we obtain an exact sequence in* **CM**S, $0 \to \mathrm{Hom}_R(\mathrm{Hom}_S(C, S), R) \to B \to^f C$, *which has the property that* $\mathrm{Im}(\ , f) =$ rad $\mathrm{Hom}_S(\ , C)$ *and* $\mathrm{Hom}_R(\mathrm{Hom}_S(C, S), R)$ *is indecomposable.*

(c) *If $C \not\cong S$, then* $0 \to \mathrm{Hom}_R(\mathrm{Hom}_S(C, S), R) \to B \to^f C \to 0$ *is exact and is an almost split sequence in* **CM**S.

(d) *If $C \cong S$, then $\Gamma = S$ and we have the up to isomorphism, uniquely determined, exact sequence*

$$0 \to \operatorname{Hom}_R(S, R) \to B \xrightarrow{f} S \to S/\mathfrak{m} \to 0$$

with B in **CMS**.

Proof. (a) It is easily seen that $\operatorname{Ext}^2_\Gamma(\Gamma/\operatorname{rad}\Gamma, \operatorname{Hom}_R(\Gamma, R)) \cong \operatorname{Ext}^2_R(\Gamma/\operatorname{rad}\Gamma, R)$ is a 1-dimensional vector space over the division ring $\Gamma/\operatorname{rad}\Gamma$. Then it follows that all exact sequences $0 \to \operatorname{Hom}_R(\Gamma, R) \to A \to \Gamma \to \Gamma/\operatorname{rad}\Gamma \to 0$ representing a nonzero element of $\operatorname{Ext}^2_\Gamma(\Gamma/\operatorname{rad}\Gamma, \operatorname{Hom}_R(\Gamma, R)$ are isomorphic. We also know by Proposition 2.2 that the A occurring in such exact sequences are Cohen–Macaulay Γ-modules.

(b) Special case of the proof of Proposition 2.3.

(c) and (d) Trivial consequences of (b).

Before going on with our discussion of the existence of almost split sequences, we recall the notion of right almost split morphisms in **CMS**. A morphism $g: U \to C$ in **CMS** is said to be right almost split in **CMS** if f is not a splittable surjection and any morphism $X \to C$ in **CMS** which is not a splittable surjection can be lifted to U. It is well known and easily shown that if $g: U \to C$ is right almost split, then C is indecomposable and f is surjective if $C \not\cong S$. Next recall that a right almost split morphism $g: U \to C$ in **CMS** is said to be a minimal right almost split morphism in **CMS** if the restriction of g to any summand of U is not a right almost split in **CMS**. It is well known that if $g: U \to C$ is right almost split, then there is a summand U' of U such that g restricted to U' is minimal right almost split. Moreover, any two minimal right almost split morphisms $g: U \to C$ and $g': U' \to C$ are isomorphic. Finally if $0 \to V \to U \to^g C$ is an exact sequence in **CMS**, then g is minimal right almost split if and only if it is isomorphic to the exact sequence $0 \to \operatorname{Hom}_R(C, S), R) \to B \to^f C$ described in Theorem 3.1. Thus by Theorem 3.1 we have that if C in **CMS** is indecomposable and C is free at height one primes, then there is a minimal right almost split morphism $g: U \to C$ in **CMS**. It is our aim now to prove the converse of this statement.

PROPOSITION 3.2. *The following are equivalent for an indecomposable C in* **CMS**.

(a) *C is free at height at one primes.*

(b) *There is a right almost split morphism $f: B \to C$ in* **CMS**.

(c) *There is a minimal right almost split morphism $f: B \to C$ in* **CMS**.

Proof. In view of our previous discussion it is clear that it suffices to show that (c) implies (a). Since (c) implies (a) if $C \cong S$, we can assume that C is not free. Therefore to finish the proof it suffices to prove the following.

LEMMA 3.3. *Let C be an indecomposable nonprojective module in* **CM***S. Suppose there is an almost split sequence* $0 \to A \to B \to C \to 0$ *in* **CM***S. Then C is free at height one primes.*

Proof. Since $\operatorname{Hom}_R(\ , R)\colon \to \mathbf{CM}S$ given by $X \mapsto \operatorname{Hom}_R(X, R)$ for all X in **CM***S* is a duality, it follows that $0 \to \operatorname{Hom}_R(C, R) \to \operatorname{Hom}_R(B, R) \to \operatorname{Hom}_R(A, R) \to 0$ is an almost split sequence in **CM***S*. It follows from Proposition 3.1 and Lemma 3.2 of [5] that $\operatorname{Ext}^1_S(X, \operatorname{Hom}_R(C, R))$ is of finite length for all X in **CM***S*. Now let $0 \to K \to P \to C \to 0$ be exact with P a projective S-module. Then the exact sequence $0 \to \operatorname{Hom}_R(C, R) \to \operatorname{Hom}_R(P, R) \to \operatorname{Hom}_R(K, R) \to 0$ in **CM***S* splits when localized at prime ideals $\mathfrak{p}$ in S of height at most one. Hence the exact sequence $0 \to K \to P \to C \to 0$ splits when localized at primes of height at most one. Therefore C is free at height one primes, which is our desired result.

As a consequence of Proposition 3.1 we have the following.

COROLLARY 3.4. *Let α:* **CM***S* $\to$ **CM***S be an equivalence of categories. If C in* **CM***S is free at height one primes, then $\alpha(C)$ is also free at height one primes.*

Proof. It clearly suffices to prove this in case C is indecomposable. Since C is free at height one primes, we have by Proposition 3.2 that there is a minimal right almost split morphism $f\colon B \to C$ in **CM***S*. But then $\alpha(f)\colon \alpha(B) \to \alpha(C)$ is also a minimal right almost split morphism. Therefore applying Proposition 3.2 again we have that $\alpha(C)$ is free at height one primes.

As a trivial consequence of Theorem 3.1 and Lemma 3.3, we have the following characterization of modules in **CM***S* which are free at height one primes.

PROPOSITION 3.5. *Let C be an indecomposable module in* **CM***S which is not free. Then the following are equivalent.*

(a) *C is free at height one primes.*

(b) *There is an almost split sequence* $0 \to A \to B \to C \to 0$ *in* **CM***S*.

In [4], the uniquely determined, up to isomorphism, exact sequence (*) $0 \to \operatorname{Hom}_R(S, R) \to^g B \to S \to S/\mathfrak{m} \to 0$ *with* B in **CM***S* described in Theorem 3.1 was called the fundamental sequence of S in the case S was an integrally closed domain, and we will continue to use the same terminology in this more general setting. While $(*)$ is not strictly speaking an almost split sequence, it has many similar properties. In particular, the injection $g\colon \operatorname{Hom}_R(S, R) \to B$ which is not a splittable injection has the property that any morphism $h\colon \operatorname{Hom}_R(S, R) \to B$ which is not a splittable injection can be extended to B. As in [4] we denote $\operatorname{Hom}_R(S, R)$ by ω, which is sometimes called the dualizing module since there are canonical isomorphisms $\operatorname{Hom}_S(A, \omega) \xrightarrow{\sim} \operatorname{Hom}_R(A, R)$ for all A in mod S which are functorial in A. We end this section by pointing out how almost split sequences can often be constructed from the fundamental sequence for S. This generalizes results in [4].

Let M be an indecomposable nonprojective module in S_2 (mod S) which is free at height one primes. Then applying the functor $\operatorname{Hom}_S(M^*, \ )$ to the fundamental sequence $0 \to \omega \to B \to^h S \to S/m \to 0$ we obtain the exact sequence

$$0 \to \operatorname{Hom}_S(M^*, \omega) \to \operatorname{Hom}_S(M^*, B) \to \operatorname{Hom}_S(M^*, S) \to 0$$

since M being indecomposable and nonprojective implies the same for M^*. Using our standard identifications, we have the exact sequence

$$0 \to \operatorname{Hom}_R(M^*, R) \to \operatorname{Hom}_S(M^*, B) \to M \to 0, \qquad (*)$$

which looks temptingly like an almost split sequence. To see whether or not it is, we apply the functor $\operatorname{Hom}_S(M, \ )$ to $(*)$ and obtain the exact sequence of Γ-modules ($\Gamma = \operatorname{End} M^{\mathrm{op}}$)

$$\begin{aligned} 0 \to \operatorname{Hom}_S(M, \operatorname{Hom}_R(M^*, R)) &\to \operatorname{Hom}_S(M, \operatorname{Hom}_S(M^*, B)) \\ &\to \operatorname{Hom}_S(M, \operatorname{Hom}_S(M^*, S)). \end{aligned} \qquad (**)$$

Applying Proposition 1.3, we obtain the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_S(\Gamma, B) & \xrightarrow{\operatorname{Hom}_S(\Gamma, h)} & \operatorname{Hom}_S(\Gamma, S) \\ \Big\| \downarrow u_B & & \Big\| \downarrow u_S \\ \operatorname{Hom}_S(M, \operatorname{Hom}_S(M^*, B)) & \xrightarrow[\operatorname{Hom}_S(M, \operatorname{Hom}_S(M^*, h))]{} & \operatorname{Hom}_S(M, \operatorname{Hom}_S(M^*, S)) = \Gamma \end{array}$$

with vertical morphism isomorphisms. Now we know that an $f\colon \Gamma \to S$ can be lifted to $\operatorname{Hom}_S(\Gamma, B)$ if and only if $\operatorname{Im} f \subset \mathfrak{m}$. From the definitions of the isomorphisms u_B and u_S we have that $\operatorname{Im}(\operatorname{Hom}_S(M, \operatorname{Hom}_S(M^*, h))$ consists of those γ in Γ such that $t(\gamma x) \subset \mathfrak{m}$ for all x in Γ. Suppose now we know that $t(\operatorname{rad} \Gamma) \subset \mathfrak{m}$, for instance, when S is a domain as stated in

Proposition 1.5. Then $\operatorname{Im}(\operatorname{Hom}_S(M, \operatorname{Hom}_S(M^*, h))$ contains rad Γ and therefore $\operatorname{Im}(\operatorname{Hom}_S(M, \operatorname{Hom}(M^*, h))$ is either rad Γ or Γ depending on whether $t\colon \Gamma \to S$ is onto or not onto. Therefore making the obvious identification in (**) we obtain the exact sequence of Γ-modules

$$0 \to \operatorname{Hom}_R(\Gamma, R) \to \operatorname{Hom}_S(\Gamma, B) \xrightarrow{v} \Gamma$$

with $\operatorname{Im} v = \operatorname{rad} \Gamma$ or Γ depending on whether or not the trace t is onto.

Suppose $t\colon \Gamma \to S$ is onto. Then we have the exact sequence of Γ-modules

$$0 \to \operatorname{Hom}_R(\Gamma, R) \to \operatorname{Hom}_S(\Gamma, B) \to \Gamma \to \Gamma/\operatorname{rad} \Gamma \to 0$$

with $\operatorname{Hom}_S(\Gamma, B)$ Cohen–Macaulay. Now applying the functor $\operatorname{Hom}_\Gamma(\operatorname{Hom}_S(M, S),)$ to this exact sequence, we obtain by Theorem 3.1 the exact sequence

$$0 \to \operatorname{Hom}_R(M^*, R) \to \operatorname{Hom}_S(M^*, B) \to M \to 0,$$

which is almost split.

Now if $t(\Gamma) \subset \mathfrak{m}$, then $0 \to \operatorname{Hom}_R(\Gamma, R) \to \operatorname{Hom}_S(\Gamma, B) \to \Gamma \to 0$ is our exact sequence of Γ-modules and therefore split. Applying the functor $\operatorname{Hom}_\Gamma(\operatorname{Hom}_S(M, S),)$ we obtain the split exact sequence

$$0 \to \operatorname{Hom}_R(M^*, R) \to \operatorname{Hom}_S(M^*, B) \to M \to 0.$$

We now summarize this discussion in the following.

THEOREM 3.6. *Suppose M is an indecomposable nonprojective module in* S_2 (mod S) *which is free at height one primes and let* $\Gamma = \operatorname{End} M$. *Assume also that* $t(\operatorname{rad} \Gamma) \subset \mathfrak{m}$, *for instance, if S is a domain. Finally, assume that* $0 \to \omega \to B \to S \to S/\mathfrak{m} \to 0$ *is the fundamental sequence. Then the induced sequence* $0 \to \operatorname{Hom}_S(M^*, \omega) \to \operatorname{Hom}_S(M^*, B) \to M \to 0$ *is exact and has the following properties.*

(a) *It splits if and only if* $t(\Gamma) \subset \mathfrak{m}$.

(b) *If it is not split, then it is an almost split sequence in* **CMS**.

In view of Theorem 3.6, it is of interest to know which indecomposable modules M in **CMS** which are free at height one primes have the property that the trace $t\colon \operatorname{End}(M) \to S$ is surjective. Since our comments on this matter are valid if S is a complete local S_2-ring of arbitrary dimension $\geqslant 2$, we assume for the rest of this section that S has these properties rather than the more restrictive property of being Cohen–Macaulay of dimension 2.

We say that an indecomposable module M is a splitting trace module if it is in **CMS**, is free at height one primes and $t(\operatorname{End} M) = S$. Since t

generates $\mathrm{Hom}_S(\Gamma, S)$ as a left Γ-module, we have that M is a splitting trace module if and only if $S \mid \mathrm{End}_S(M)$, i.e., S is an S-summand of End M.

Our description of splitting trace modules involves the rank of a module A which is free at height one primes, a notion we now introduce.

Since S is Cohen–Macaulay, given any two height zero prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$, there is a height one prime $\mathfrak{p}$ containing both $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Therefore if M is free at height one primes then the ranks of the free modules $M_{\mathfrak{p}_1}$, $M_{\mathfrak{p}_2}$, $M_{\mathfrak{p}}$ are all the same. From this it follows that the ranks of the free modules $M_{\mathfrak{p}}$ with height $\mathfrak{p} \leqslant 1$ are all the same. This number is called the rank of M and is denoted by rank M.

Let M be an indecomposable module in **CM**S which is free at height one primes. Then claim $t(\mathrm{Id}_M) = 1 \cdot \mathrm{rank}\, M$ in S. This follows from the fact that two elements x and y in an S_2-ring S' are the same if and only if $x_{\mathfrak{p}} = y_{\mathfrak{p}}$ in $S'_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of S' satisfying height $\mathfrak{p} \leqslant 1$. Since $t_{\mathfrak{p}}(\mathrm{Id}_{M_{\mathfrak{p}}}) = 1 \cdot \mathrm{rank}$ in $S_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ satisfying height $\mathfrak{p} \leqslant 1$, it follows that $t(\mathrm{Id}_M) = 1 \cdot \mathrm{rank}$ M in S.

We now have the following criteria for determining when a module is a splitting trace module.

PROPOSITION 3.7. *Suppose M in* **CM**S *is indecomposable and free at height one primes. Then we have the following.*

(a) *M is a splitting trace module if rank M is a unit in S.*

(b) *If $t(\mathrm{End}_S(M)) \subset \mathfrak{m}$, and $S/\mathfrak{m}$ is algebraically closed, then M is a splitting trace module if and only if rank M and the characteristic of $S/\mathfrak{m}$ are relatively prime.*

Proof. (a) Obvious.

(b) Essentially the same as the proof of [4, Proposition 5.7].

As our final remark about splitting trace modules, we have the following, whose proof is essentially the same as the proof of [4, Proposition 5.10].

PROPOSITION 3.8. *Suppose M and N in* **CM**S *are indecomposable, free at height one primes and have the property that $t(\mathrm{End}\, M)$ and $t(\mathrm{End}\, N)$ are contained in $\mathfrak{m}$.*

(a) *If $S \mid \mathrm{Hom}_S(M, N)$, then $M \cong N$ and M is a splitting trace module.*

(b) *If M is a splitting trace module, then $M \cong N$ if and only if $S \mid \mathrm{Hom}_S(M, N)$.*

II

1. Graded Almost Split Sequences

Throughout this section we assume that $S = \coprod_{i \in \mathbb{Z}} S_i$ is a $\mathbb{Z}$-graded noetherian commutative ring of Krull dimension 2 satisfying the following conditions:

(a) $S_i = 0$ for $i < 0$.

(b) $S_0 = k$ is a field.

(c) Each S_i is a finite dimensional k-vector space.

(d) S_1 generates S over k (as a k-algebra).

(e) S is a graded Cohen–Macaulay ring.

Suppose k is infinite. Then the fact that S is a graded Cohen–Macauley ring implies that we can find a graded regular S-sequence x_1, x_2 in S_1 [10]. Then the x_1, x_2 are algebraically independent over k and the subring $R = k[x_1, x_2]$ of S is a graded subring of S with the x_i of degree 1. Furthermore, the grading in S makes S a graded R-module which is a finitely generated free graded R-module. Since we need that k is infinite only to construct this subring R, rather than making the assumption that k is infinite, we assume the following:

(f) There are x_1, x_2 in S_1 which are algebraically independent over k and such that S is a finitely generated free graded module over the graded subring $R = k[x_1, x_2]$ of S.

From this hypothesis on S and R we see that S is an R-algebra which is an equidimensional Cohen–Macaulay ring of dimension 2 and a finitely generated free R-module with R a Gorenstein ring since it is regular. Thus the R-algebra S satisfies the hypothesis of Section 2 of Part I. Our aim in this section is to show how the results in Part I can be transferred to the category of finitely generated graded S-modules with degree zero maps.

Suppose A and B are graded S-modules. We denote by $\operatorname{Hom}_{\mathscr{g}\imath S}(A, B)$ the S-submodule of $\operatorname{Hom}_S(A, B)$ generated by the homogeneous morphisms from A to B and we denote by $\operatorname{Hom}_{\mathscr{g}\imath S}(A, B)_i$ the k-vector subspace of $\operatorname{Hom}_{\mathscr{g}\imath S}(A, B)$ consisting of the homogeneous morphisms of degree i. Then $\operatorname{Hom}_{\mathscr{g}\imath S}(A, B) = \coprod_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathscr{g}\imath}(A, B)_i$ is a graded S-module. It should be noted that if A is a finitely generated S-module, then $\operatorname{Hom}_{\mathscr{g}\imath S}(A, B) = \operatorname{Hom}_S(A, B)$. From now on, unless stated to the contrary, we assume that all graded S-modules are finitely generated.

Suppose now that A is a graded S-module. Then A is a graded (maximal) Cohen–Macaulay module if and only if it is a graded free R-module, i.e., $A \cong \coprod_{i=1}^{n} R(p_i)$, where $R(p_i)$ means the module R "shifted" by the integer p_i. Thus A is a graded Cohen–Macaulay S-module if and only if it is an ungraded Cohen–Macaulay S-module. We denote by $\mathbf{CM}S(\mathscr{gr}S)$ the category of graded Cohen–Macaulay S-modules. As in the ungraded case, if A and B are graded S-modules, and B is Cohen–Macaulay, then the graded S-module $\operatorname{Hom}_S(A, B)$ is also a finitely generated graded Cohen–Macaulay S-module.

Suppose now that A is a graded S-module. Then End A is a $\mathbb{Z}$-graded ring which is a graded S-algebra, i.e., the natural ring morphism $S \to \operatorname{End}(A)$ is homogeneous of degree 0. Moreover, A is a graded End A-module by means of the given grading on A. Now the grading on any ring is also a grading for the opposite ring, and this is the way we consider the opposite of a graded ring a graded ring. In particular, $\Gamma = \operatorname{End} A^{\mathrm{op}}$ is a graded S-algebra and if B is an arbitrary graded S-module (not necessarily finitely generated) then $\operatorname{Hom}_S(A, B)$ is a graded Γ-module in the usual way. Further if we define a finitely generated graded Γ-module to be a graded Cohen–Macaulay Γ-module if it is a graded Cohen–Macaulay S-module, then $\operatorname{Hom}_S(A, B)$ is a graded Cohen–Macaulay Γ-module whenever B is a finitely generated graded Cohen–Macaulay S-module.

For notational convenience we denote by mod $\mathscr{gr}S$ the category of finitely generated graded S-modules with all morphisms and by $(\text{mod } \mathscr{gr}S)_0$ the abelian category of finitely generated graded S-modules with homogeneous degree zero morphisms. We say that an A in mod $\mathscr{gr}S$ is free at height one primes if it is free at height one primes when viewed as an ungraded module. Then we have the following analogue of Theorem 1.1 in Part I.

PROPOSITION 1.1. *Let M in $\mathbf{CM}(\mathscr{gr}S)$ be free at height one primes and let Γ be the graded S-algebra $\Gamma = \operatorname{End}_S(M)^{\mathrm{op}}$. Then the functors* $\operatorname{Hom}_S(M, \ )$: mod $\mathscr{gr}S \to$ mod $\mathscr{gr}\Gamma$ *and* $\operatorname{Hom}_\Gamma(M^*, \ )$: mod $\mathscr{gr}\Gamma \to$ mod $\mathscr{gr}S$ *induce inverse equivalences* $\mathbf{CM}(\mathscr{gr}S)_0 \to \mathbf{CM}(\mathscr{gr}\Gamma)_0$, *where* $M^* = \operatorname{Hom}_S(M, S)$.

Proof. The proof consists of checking that the isomorphisms $\mathrm{Id}_{\mathbf{CM}(\mathscr{gr}S)_0} \to \operatorname{Hom}_\Gamma(M^*, \operatorname{Hom}_S(M, \ ))$ and $\mathrm{Id}_{\mathbf{CM}_{\mathscr{gr}}(\Gamma)_0} \to \operatorname{Hom}_S(M, \operatorname{Hom}_\Gamma(M^*, \ ))$ described in the proof of Part I, Theorem 1.1, are homogeneous of degree 0.

The proof of Proposition 1.1 is typical of many of the proofs we give in this section; i.e., they consist mainly of showing that the morphisms involved in the proof in the ungraded situation are homogeneous of degree zero. We will usually not carry out this checking.

Let Γ be a graded R-algebra which is a finitely generated free graded R-module. We now point out the following analogue of Part I, Lemma 2.1, whose statement involves the following observation.

Let A be a finitely generated graded Γ-module. Then A has a projective Γ-resolution P in $(\text{mod } \mathscr{gr}\Gamma)_0$ consisting of finitely generated free Γ-modules with degree zero morphisms. Therefore if B is any graded

Γ-module (not necessarily finitely generated), the complex $\operatorname{Hom}_\Gamma(\Gamma, B)$ is a complex of graded R-modules with degree zero morphisms. Hence its homology, $\operatorname{Ext}^i_\Gamma(A, B)$ has the structure of a graded R-module whose k-component we denote by $\operatorname{Ext}^i_\Gamma(A, B)_k$ for each k in $\mathbb{Z}$.

LEMMA 1.2. *The graded Γ-module* $\operatorname{Hom}_R(\Gamma, R(p))$ *in* $\mathbf{CM}(\mathit{gr}\,\Gamma)$ *has the following properties for each p in $\mathbb{Z}$.*

(a) *For each A in* $\operatorname{mod} \mathit{gr}\,\Gamma$ *and all $i \geq 0$ we have homogeneous degree zero isomorphisms of graded R-modules* $\operatorname{Ext}^i_\Gamma(A, \operatorname{Hom}_R(\Gamma, R(p))) \simeq \operatorname{Ext}^i_R(A, R(p))$ *which are functorial in A.*

(b) *If A is in* $\mathbf{CM}(\mathit{gr}\,\Gamma)$, *then* $\operatorname{Ext}^i_\Gamma(A, \operatorname{Hom}_R(\Gamma, R(p)) = 0$ *for all $i \geq 1$.*

(c) *If A is of finite length and $A_k \neq 0$, then* $\operatorname{Ext}^i_\Gamma(A, \operatorname{Hom}_R(\Gamma, R(k-2)) = 0$ *for $i < 2$ and* $\operatorname{Ext}^2_\Gamma(A, \operatorname{Hom}_R(\Gamma, R(k-2))_0 \neq 0$.

Proof. (a) Check that the isomorphisms described in the proof of Part I, Lemma 2.1, are homogeneous of degree 0.

(b) Trivial consequence of (a).

(c) The Koszul complex gives the minimal projective resolution in $(\operatorname{mod} \mathit{gr}\, R)_0$

$$0 \to R(-2) \to 2R(-1) \to R \to R/(x_1, x_2) \to 0.$$

Thus we have that $\operatorname{Ext}^i_R(R/(x_1, x_2), R(-2)) = 0$ for $i = 0, 1$ and $\operatorname{Ext}^2_R(R/(x_1, x_2), R(-2))_0 \cong R/(x_1, x_2)$. Now let $0 \to R(-2) \to I_0 \to I_1 \to I_2 \to 0$ be a minimal injective resolution in the category of graded R-modules with degree zero morphisms. It then follows that $\operatorname{Hom}_R(R/(x_1, x_2), I_j) = 0$ for $j = 1, 2$ and that I_2 is the minimal injective envelope of $R/(x_1, x_2)$, i.e., $(I_2)_k = 0$ for $k < 0$, $(I_2)_0 \cong k$ [9]. From this it follows that if $0 \to R(k-2) \to I_0(R(k-2)) \to I_1(R(k-2)) \to I_2(R(k-2)) \to 0$ is a minimal injective resolution, then $I_j(R(k-2))$ has no nonzero submodules of finite length for $j = 0, 1$ while $I_2(R(k-2))$ is the injective envelope of $R/(x_1, x_2)(k)$.

Suppose now that A is a graded R-module of finite length and suppose $A_k \neq 0$. Let $B \subset A$ be the submodule generated by A_k. Then there is a nonzero degree zero map $f: B \to I_2(R(k-2))$ and hence a nonzero degree zero map $g: A \to I_2(R(k-2))$. Thus we have that $\operatorname{Ext}^2_R(A, R(k-2))_0 \cong \operatorname{Hom}_R(A, I_2(R(k-2))_0 \neq 0$. Since $\operatorname{Ext}^i_\Gamma(A, \operatorname{Hom}_R(\Gamma, R(k-2)) \cong \operatorname{Ext}^i_R(A, R(k-2))$ for all $i \geq 0$, we have our desired result.

Slight modifications of the proof of Part I, Proposition 2.2, gives a proof of the following graded analogue of that proposition.

PROPOSITION 1.3. *Let $B \to C \to 0$ be an exact sequence in* $(\operatorname{mod}_{gr}\Gamma)_0$ *with C of finite nonzero length and B a Cohen–Macaulay module. Moreover assume that Q is a free Γ^{op} graded module such that there is an exact sequence of Γ^{op}-modules $Q \to \operatorname{Ext}^2_R(C, R(p)) \to 0$ for some fixed p in $\mathbb{Z}$. Then there is an exact sequence in* $(\operatorname{mod}_{gr}\Gamma)_0$

$$0 \to \operatorname{Hom}_R(Q, R(p)) \to A \xrightarrow{h} B \to C \to 0$$

with A in $\mathbf{CM}_{gr}\Gamma$ *and any such exact sequence has the following properties.*

(a) $\operatorname{Hom}_R(Q, R(p)) \to A$ *is not a splittable monomorphism.*

(b) *Suppose X is a Cohen–Macaulay Γ-module (not necessarily graded) and $f: X \to B$ a morphism of Γ-modules (not necessarily homogeneous) such that* $\operatorname{Im} f \subset \operatorname{Im} h$. *Then there is a $t: X \to A$ such that $f = ht$. Further, if X is graded and $f: X \to B$ is homogeneous of degree zero, then the $t: X \to A$ such that $f = ht$ can be chosen to be homogeneous of degree zero.*

As a consequence of Proposition 1.3, we have the following graded analogue of Part I, Theorem 2.3.

PROPOSITION 1.4. *Let M in* $\mathbf{CM}_{gr}S$ *be free at height one primes. Suppose B is in* $\mathbf{CM}_{gr}S$ *and H a homogeneous* $\Gamma = \operatorname{End}(M)^{\mathrm{op}}$*-submodule of the graded Γ-module* $\operatorname{Hom}_S(M, B)$ *such that* $\operatorname{Hom}_S(M, B)/H$ *is of finite length. Moreover suppose that* $\coprod_{i=1}^n \Gamma^{\mathrm{op}}(p_i) \to \operatorname{Ext}^2_R(\operatorname{Hom}_S(M, B)/H, R(p)) \to 0$ *is exact with p a fixed integer. Then there is a morphism $h: A \to B$ in* $(\mathbf{CM}_{gr}S)_0$ *having the property* $H = \operatorname{Im}(\operatorname{Hom}_S(M, A) \xrightarrow{\operatorname{Hom}_S(M,h)} \operatorname{Hom}_S(M, B))$ *and such that:*

(a) *If $f: X \to B$ is a morphism in* $\mathbf{CM}S$ *satisfying* $\operatorname{Im}(\operatorname{Hom}_S(M, X) \xrightarrow{\operatorname{Hom}_S(M,f)} \operatorname{Hom}_S(M, B) \subset H$, *then there is a $t: X \to A$ in* $\mathbf{CM}S$ *such that $ht = f$. Moreover, if $f: X \to B$ is in* $(\mathbf{CM}_{gr}S)_0$, *then t can be chosen in* $(\mathbf{CM}_{gr}S)_0$.

(b) $\operatorname{Ker} h \simeq \coprod_{i=1}^n \operatorname{Hom}_R(\operatorname{Hom}_S(M, S)(p_i), R(p))$.

(c) *h is a splittable surjection if and only if* $H = \operatorname{Hom}_S(M, B)$.

(d) *$h: A \to B$ is onto if and only if $H \supset P(M, B)$, where $P(M, B)$ is the graded Γ-submodule of* $\operatorname{Hom}_S(M, B)$ *consisting of all $M \to B$ which can be factored through a projective module.*

Proof. Slight modifications of the proof of Part I, Proposition 2.3, give a proof of this result.

Our aim now is to apply Proposition 1.4 to obtain an existence theorem for almost split sequences in $(\mathbf{CM}_{gr}S)_0$. We begin with some preliminary observations. The first one is a special case of [9, Theorem 3.1].

LEMMA 1.5. *Let $\Lambda = \amalg_{i \in Z} \Lambda_i$ be a finite dimensional algebra over a field such that Λ_0 is a local ring with radical $\mathfrak{r}$. Then the subgroup $V = \amalg_{i \neq 0} \Lambda_i \amalg \mathfrak{r}$ is a two-sided graded ideal in Λ.*

Proof. Since Λ is finite dimensional we know that there is a positive integer n such that $\Lambda_i = 0$ for $i > n$ and $i < -n$. Consequently if x is in Λ_i with $i \neq 0$, then $x^n = 0$. Now suppose x is in Λ_i with $i \neq 0$ and y is in Λ_j. If $j \neq -i$, then xy and yx are in Λ_{i+j} with $i+j \neq 0$. If y is in Λ_{-i}, then yx is in Λ_0. If yx is not in $\mathfrak{r}$, then yx is invertible, which means that x is not nilpotent, a contradication. Therefore yx is in $\mathfrak{r}$. A similar argument shows that xy is in $\mathfrak{r}$. Thus we have that if x is a homogeneous element not in Λ_0, then yx and xy are in V for all y in Λ. Since yx and xy are in V if x is in $\mathfrak{r}$, we have that $V = \amalg_{i \neq 0} \Lambda_i \amalg \mathfrak{r}$ is a two-sided ideal in Λ.

Next we observe that since $\mathrm{Hom}_S(A, B)_0$ is a finite dimensional k-vector space for all A and B in $\mathrm{mod}(\mathscr{gr}S)_0$, the abelian category $(\mathrm{mod}\,\mathscr{gr}S)_0$ is a Krull–Schmidt category. Therefore for each C in $(\mathrm{mod}\,\mathscr{gr}S)_0$, there is a projective cover $P \to^h C \to 0$ in $(\mathrm{mod}\,\mathscr{gr}S)_0$ with P a free graded S-module. Then for each X we have that $\mathrm{Hom}_S(X, P) \to^{(X,h)} \mathrm{Hom}_S(X, C)$ is in $(\mathrm{mod}\,\mathscr{gr}S)_0$ and so $\mathrm{Coker}(X, h)$ which we denote by $\underline{\mathrm{Hom}}_S(X, C)$ is a graded S-module. Then we have the following characterization of C in $(\mathrm{mod}\,\mathscr{gr}S)_0$ being free at height one primes.

LEMMA 1.6. *The following are equivalent for a C in $(\mathrm{mod}\,\mathscr{gr}S)_0$.*

(a) *C is free at height one primes.*

(b) *$\underline{\mathrm{Hom}}_S(X, C)$ is of finite length for all X in $(\mathrm{mod}\,\mathscr{gr}S)_0$.*

(c) *$\underline{\mathrm{Hom}}_S(C, C)$ is of finite length.*

Proof. Left as an exercise.

Combining some of our previous remarks we have the following.

LEMMA 1.7. *Let C in $(\mathrm{mod}\,\mathscr{gr}S)_0$ be a nonfree indecomposable module which is free at height one primes and let $\Gamma = \mathrm{End}(C)^{\mathrm{op}}$. Then we have the following.*

(a) *Γ_0 is a local ring whose radical we denote by $\mathfrak{r}$.*

(b) *$H = \amalg_{i \neq 0} \Gamma_i \amalg \mathfrak{r}$ which we call the graded radical of Γ is a two-sided homogeneous ideal in Γ containing $P(C, C) = \mathrm{Ker}(\mathrm{Hom}_S(C, C) \to \underline{\mathrm{Hom}}_S(C, C))$.*

Proof. It is easily checked that $P(C, C)$, the kernel of the homogeneous degree zero surjection $\mathrm{Hom}_S(C, C) \to \underline{\mathrm{Hom}}_S(C, C)$, is a two-sided homogeneous ideal in $\mathrm{End}_S(C)$. Since C is indecomposable, $\mathrm{End}_S(C)_0$ and therefore $\underline{\mathrm{End}}_S(C)_0$ is a local ring. Since C is free at height one primes, it follows by Lemma 1.6 that $\underline{\mathrm{End}}_S(C)$ is a finite dimensional algebra. Therefore by Lemma 1.5, we know that $V = \amalg_{i \neq 0} \underline{\mathrm{End}}_S(C)_i \amalg \mathrm{rad}\ \mathrm{End}_S(C)_0$ is a two-sided homogeneous ideal in $\underline{\mathrm{End}}_S(C)$. Therefore H, which is the preimage of V in $\mathrm{End}_S(C)$, is a homogeneous two-sided ideal in $\mathrm{End}_S(C)$.

We are now in position to give our existence theorem for almost split sequences in $(\mathbf{CM}(\mathscr{gr}S))_0$.

THEOREM 1.8. *Let C be a nonfree indecomposable module in* $\mathbf{CM}(\mathscr{gr}S)_0$ *which is free at height one primes. Then there is an almost split sequence* $0 \to \mathrm{Hom}_R(\mathrm{Hom}_S(C, S), R(-2)) \to E \to^h C \to 0$ *in* $(\mathbf{CM}(\mathscr{gr}S))_0$. *Further, the morphism $h: E \to C$ has the following more general property. If $f: X \to C$ is a morphism in* $\mathbf{CM}S$ *such that* $\mathrm{Im}(\mathrm{Hom}_S(C, X) \to \mathrm{Hom}_S(C, C))$ *is contained in the graded radical of* $\mathrm{End}_S C$, *then there is $t: X \to E$ in* $\mathbf{CM}S$ *such that $ht = f$.*

Proof. Let H be the graded radical of $\mathrm{End}(C)$. Then $(\mathrm{End}\ C)/H = \mathrm{End}(C)_0/\mathrm{rad}\ \mathrm{End}(C)_0 = (\mathrm{End}\ C/H)_0 \neq 0$. Since $H \supset P(C, C)$, we have by Proposition 1.4 that there is an exact sequence in $(\mathbf{CM}\mathscr{gr}S)_0$

$$0 \to \mathrm{Hom}_R(\mathrm{Hom}_S(C, S), R(-2)) \to E \xrightarrow{h} C \to 0, \qquad (*)$$

with the property that if $f: X \to C$ is in $(\mathbf{CM}\mathscr{gr}S)_0$ such that $\mathrm{Im}(\mathrm{Hom}_S(C, X) \to \mathrm{Hom}_S(C, C)) \subset H$, then there is a $t: X \to E$ in $(\mathbf{CM}(\mathscr{gr}S))_0$ such that $ht = f$. But the condition that H contains $\mathrm{Im}(\mathrm{Hom}_S(X, f)$ is precisely the condition that $f: X \to C$ is not a splittable surjection in $\mathbf{CM}(\mathscr{gr}S)_0$. Thus the exact sequence $(*)$ in $(\mathbf{CM}(\mathscr{gr}S))_0$ is the almost split sequence provided $\mathrm{Hom}_R(\mathrm{Hom}_S(C, S), R(-2))$ is indecomposable. This follows from the fact that $\mathrm{Hom}_R(\ , R(-2))$ is a duality in $(\mathbf{CM}\mathscr{gr}S)_0$ and C being indecomposable implies $\mathrm{Hom}_S(C, S)$ is indecomposable since C being in $(\mathbf{CM}\mathscr{gr}S)_0$ and free at height one primes means that C is reflexive.

The rest of the theorem also follows from Proposition 1.4 in a straightforward way.

Suppose now that C is free, i.e., $C = S(p)$ for some integer p. Since $\mathrm{End}_S(S(p)) = S$, if we let $H = \amalg_{i>0} S_i$ in Proposition 1.4, we obtain the exact sequence in $(\mathrm{mod}\ \mathscr{gr}S)_0$

$$0 \to \mathrm{Hom}_R(S(-p), R(-2)) \to E \xrightarrow{h} S(p) \to k(p) \to 0, \qquad (*)$$

with the first three terms in $\mathbf{CM}S$ having the property that if $f: X \to S(p)$ in $(\mathbf{CM}\mathscr{gr}S)_0$ is not surjective, then there is a homogeneous degree zero map $t: X \to E$ such that $ht = f$. Moreover, this exact sequence has the further property that if $f: X \to S(p)$ is a morphism in $\mathbf{CM}S$ with $\mathrm{Im} f \subset H$, then there is a $t: X \to E$ in $\mathbf{CM}S$ such that $ht = f$. It should be noted that since

$\mathrm{Ext}_S^2(k(p), \mathrm{Hom}_R(S(-p), R(-2))_0 = k$ all exact sequences representing nonzero elements of this group are isomorphic and therefore isomorphic to $(*)$. When $p=0$ we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(S, R(-2) \to E \to S \to k \to 0$$

representing the nonzero element of $\mathrm{Ext}_S^2(k, \mathrm{Hom}_R(S, R(-2))_0$, which is called the fundamental exact sequence.

We end this section by pointing out the graded analogues of Proposition 3.2, Lemma 3.3 and Corollary 3.4 in Part I.

PROPOSITION 1.9. *The following are equivalent for an indecomposable C in $(\mathbf{CM}_{gr}S)_0$.*

(a) *C is free at height one primes.*

(b) *There is a right almost split morphism $f: B \to C$ in $(\mathbf{CM}(_{gr}S))_0$.*

(c) *There is a minimal right almost split morphism $f: B \to C$ in $\mathbf{CM}S$.*

Proof. An argument analogous to the one given in the proof of Part I, Proposition 3.2, shows that to prove this proposition it suffices to show the following.

LEMMA 1.10. *Let C be an indecomposable nonprojective module in $(\mathbf{CM}_{gr}S)_0$. Suppose there is an almost split sequence $0 \to A \to B \to C \to 0$ in $(\mathbf{CM}_{gr}S)_0$. Then C is free at height one primes.*

Proof. The main point to observe is that the obvious graded versions of Proposition 3.1 and Lemma 3.2 of [5] cited in the proof of Part I, Lemma 3.3, are valid. The proof of our desired result then proceeds exactly as the proof of Part I, Lemma 3.3.

As a consequence of Proposition 1.9, we have the following.

COROLLARY 1.11. *Let $\alpha: (\mathbf{CM}_{gr}S)_0 \to (\mathbf{CM}_{gr}S)_0$ be an equivalence of categories. If C in $(\mathbf{CM}_{gr}S)_0$ is free at height one primes, then $\alpha(C)$ is also free at height one primes.*

Proof. Same as that for Part I, Corollary 3.4.

We end this section with the following graded analogue of Part I, Proposition 3.4.

PROPOSITION 1.12. *Let C be an indecomposable module in $(\mathbf{CM}_{gr}S)_0$ which is not free. Then the following are equivalent.*

(a) *C is free at height one primes.*

(b) *There is an almost split sequence $0 \to A \to B \to C \to 0$ in $(\mathbf{CM}_{gr}S)_0$.*

2. Calculating Almost Split Sequence

Throughout this section we assume that the graded R-algebra S satisfies the same hypothesis as in the previous section. Our aim in this section is to discuss how almost split sequences in $(\mathbf{CM}_{gr}S)$ can be computed using the fundamental exact sequence $0 \to \mathrm{Hom}_R(S, R(-2) \to E \to S \to k \to 0$. The results here are entirely analogous to the results already obtained in Section 3 of Part I.

Suppose M is an indecomposable module in $\mathbf{CM}(_{gr}S)$ which is free at height one primes. Let $\Gamma = \mathrm{End}(M)^{\mathrm{op}}$. Then Γ is a graded ring whose graded radical was defined in the previous section to be the two-sided graded ideal $\amalg_{i \neq 0} \Gamma_i \amalg \mathfrak{r}(\Gamma_0)$, where $\mathfrak{r}\Gamma_0$ is the radical of the finite dimensional local ring Γ_0. When $M = S(p)$ then $\mathrm{End}_S(S(p)) = S$ and so the graded radical of S is the unique maximal homogeneous ideal $\mathfrak{m}$ of S. In Section 1 of Part I we defined a two-sided Γ isomorphism $u_S\colon \mathrm{Hom}_S(\Gamma, S) \to \mathrm{Hom}_S(M, N) = \Gamma$, which is not difficult to check, is a homogeneous isomorphism of degree zero. Then the trace $t\colon \Gamma \to S$ was defined to be the S-linear functional $u_S^{-1}(1_M)$. Since u_S is a degree zero isomorphism, the fact that 1_M is homogeneous of degree zero implies $t\colon \Gamma \to S$ is homogeneous of degree zero.

Proposition 2.1. *Let M be an indecomposable module in $(\mathbf{CM}_{gr}S)_0$ which is free at height one primes. Then the trace $t\colon \Gamma \to S$ has the property that $t(_{gr}\,\mathrm{rad}\,\Gamma) \subset \mathfrak{m}$. Consequently, $f(_{gr}\,\mathrm{rad}\,\Gamma) \subset \mathfrak{m}$ for all f in $\mathrm{Hom}_S(\Gamma, S)$.*

Proof. We first show that $t(_{gr}\,\mathrm{rad}\,\Gamma) \subset \mathfrak{m}$. For each $i \neq 0$, we have that $t(\Gamma_i) \subset S_i$ since t is homogeneous of degree zero. Therefore it suffices to show that $t(\mathrm{rad}\,\Gamma_0) = 0$ to finish the proof.

Let x be in $\mathrm{rad}\,\Gamma_0$. Since Γ_0 is a finite dimensional algebra we know that x is nilpotent. Suppose $\mathfrak{p}$ is a prime of height at most 1. Since $M_\mathfrak{p}$ is $S_\mathfrak{p}$-free, $x_\mathfrak{p}$ is a nilpotent element in $\Gamma_\mathfrak{p}$, a full matrix algebra over $S_\mathfrak{p}$. Since $t_\mathfrak{p}\colon \Gamma_\mathfrak{p} \to S_\mathfrak{p}$ is the ordinary trace on a full matrix algebra we have that $t_\mathfrak{p}(x_\mathfrak{p}) = 0$. Thus the homogeneous element $t(x)$ in S has the property that $t(x)_\mathfrak{p} = 0$ for all primes of height at most one. Thus if $t(x) \neq 0$, $\mathrm{Ass}(S \cdot t(x))$ would be $\mathfrak{m}$, which contradicts the fact that S is Cohen–Macaulay. Therefore $t(x) = 0$, which is what we wanted to show. The rest of the proposition follows from the fact that the trace is a free Γ-generator on both sides for $\mathrm{Hom}_S(\Gamma, S)$.

As an easy consequence of Proposition 2.1, we have the folowing.

Corollary 2.2. *Let M be an indecomposable module in $\mathbf{CM}_{gr}(M)$ which is free at height one primes. Then the following are equivalent.*

(a) *$t\colon \mathrm{End}_S(M) \to S$ is surjective.*

(b) *S is a graded summand of* $\mathrm{End}_S(M)$ *(notation:* $S \mid \mathrm{End}_S(M)$*).*

(c) *S is a summand of* $\mathrm{End}_S(M)$ *as ungraded modules.*

Suppose M in $(\mathbf{CM}_{gr}S)_0$ is free at height one primes. Then in Section 3 of Part I we showed that the ranks of the free $S_{\mathfrak{p}}$-modules $M_{\mathfrak{p}}$ are the same for all prime ideals $\mathfrak{p}$ of height at most one and we defined this integer to be the rank of M. Further we showed that $t(\mathrm{Id}_M) = \mathrm{rank}\ M \cdot 1$ in S. Finally if M is indecomposable, we defined M to be a splitting trace module if $t(\mathrm{End}_S(M)) = S$. We now point out the graded analogues of results obtained in Part I, Section 3, about splitting trace modules. We begin with the following characterization of splitting trace modules.

PROPOSITION 2.3. *Let M in $(\mathbf{CM}_{gr}S)_0$ be an indecomposable module which is free at height one primes. Then we have the following.*

(a) *M is a splitting trace module if rank M and* ch k *are relatively prime.*

(b) *If k is algebraically closed, then M is a splitting trace module if and only if rank M and* ch k *are relatively prime.*

Proof. Essentially the same as the proof of Proposition 7 in [4].

Although it is not needed in our discussion of almost split sequences, we point out the following interesting property of splitting trace modules.

PROPOSITION 2.4. *Let M and N be two indecomposable modules in $(\mathbf{CM}_{gr}S)_0$ which are free at height one primes.*

(a) *If $S \mid \mathrm{Hom}_S(N, M)$, then $N \cong M$ and M is a splitting trace module.*

(b) *If M is a splitting trace module, then $M \cong N$ if and only if $S \mid \mathrm{Hom}_S(N, M)$.*

Proof. Essentially the same as the proof of Proposition 5.10 in [4].

We now return to the question of constructing almost split sequences from the fundamental sequence $0 \to \mathrm{Hom}_R(S, R(-2)) \to E \to S \to k \to 0$. The module $\mathrm{Hom}_R(S, R(-2))$ is usually called the dualizing module and is denoted by ω.

THEOREM 2.5. *Suppose M is an indecomposable nonfree module in $(\mathbf{CM}_{gr}S)_0$ and let $0 \to \omega \to E \to S \to k \to 0$ be the fundamental exact sequence in* $(\mathrm{mod}\ _{gr}S)_0$. *Further, let* $\mathrm{Hom}_S(M, S) = M^*$. *Then the induced sequence* $0 \to \mathrm{Hom}_S(M^*, \omega) \to \mathrm{Hom}_S(M^*, E) \to \mathrm{Hom}_S(M^*, S) = M \to 0$ *is exact in* $(\mathrm{mod}\ _{gr}S)_0$ *and has the following properties.*

(a) *It splits if and only if M is not a splitting trace module.*

(b) *If M is a splitting trace module, then it is an almost split sequence in* $(\mathbf{CM}_{gr}S)_0$.

3. Projective Curves

Throughout this section X is a connected projective Cohen–Macaulay curve over an infinite field k. It is well known that there is a graded k-algebra S satisfying the hypothesis of Section 1 of this part which is a homogeneous coordinate ring of X. This section is devoted to showing how the results about almost split sequences in $(\mathbf{CM}\mathit{gr}S)_0$ give results about almost split sequences in the category of Cohen–Macaulay coherent sheaves over X. We would like to thank David Eisenbud for his help with some of the material in this section.

Let $\mathcal{O}_x$ be the structure sheaf on X and let mod $\mathcal{O}_x$ be the category of coherent sheaves on X. We denote the usual sheafification functor $(\text{mod}\ \mathit{gr}S)_0 \to \text{mod}\ \mathcal{O}_x$ by $M \mapsto \tilde{M}$. It is clear that if M is in $(\mathbf{CM}\mathit{gr}S_0)$, then $\tilde{M}$ is in $\mathbf{CM}\mathcal{O}_x$, the full subcategory of mod $\mathcal{O}_x$ whose objects are the Cohen–Macaulay coherent sheaves. Then we have the following.

Proposition 3.1. *The functor $(\mathbf{CM}\mathit{gr}S)_0 \to \mathbf{CM}\mathcal{O}_x$ given by $M \to \tilde{M}$ for all M in $(\mathbf{CM}\mathit{gr}S)_0$ is an equivalence of categories.*

Proof. Associated with each $\mathcal{F}$ in mod $\mathcal{O}_x$ is the graded S-module (not necessarily finitely generated) $\Gamma_*(\mathcal{F})$. Also for each M in $(\text{mod}\ \mathit{gr}S)_0$ there is a homogeneous degree 0 morphism $M \to \Gamma_*(\tilde{M})$ which is functorial in M. It is well known that $M \to \Gamma_*(\tilde{M})$ is an isomorphism if and only if depth $M \geqslant 2$. Thus if M is in $(\mathbf{CM}\mathit{gr}S)_0$ then $M \to \Gamma_*(\tilde{M})$ is an isomorphism. This implies that the functor $(\mathbf{CM}\mathit{gr}S)_0 \to \mathbf{CM}\mathcal{O}_x$ is fully faithful. For let A and B be in $(\mathbf{CM}\mathit{gr}S)_0$. Then $\text{Hom}_S(A, B)$ is in $(\mathbf{CM}\mathit{gr}S)_0$ and so we have the homogeneous degree zero isomorphism $\text{Hom}_S(A, B) \to \Gamma_*(\widetilde{\text{Hom}_S(A, B)})$. But $\widetilde{\text{Hom}_S(A, B)}$ is the sheaf $\mathbf{Hom}_{\mathcal{O}_x}(\tilde{A}, \tilde{B})$. Thus we obtain the homogeneous degree zero isomorphism $\text{Hom}_S(A, B) \to \Gamma_*(\mathbf{Hom}_{\mathcal{O}_x}(\tilde{A}, \tilde{B})$. This gives our desired isomorphism, $\text{Hom}_S(A, B)_0 \to \Gamma(\mathbf{Hom}_{\mathcal{O}_x}(\tilde{A}, \tilde{B})) = \text{Hom}_{\mathcal{O}_x}(\tilde{A}, \tilde{B})$.

We now show that the functor $(\mathbf{CM}\mathit{gr}S)_0 \to \mathbf{CM}\mathcal{O}_x$ is dense. Let $\mathcal{F}$ be in $\mathbf{CM}\mathcal{O}_x$. Then there is an M in mod $\mathit{gr}S$ such that $\tilde{M} \cong \mathcal{F}$. Since $\mathcal{F}$ is Cohen–Macaulay, $M_\mathfrak{p}$ is Cohen–Macaulay for all homogeneous prime ideals $\mathfrak{p} \neq \mathfrak{m}$. From this it follows that $M_\mathfrak{q}$ is $R_\mathfrak{q}$-free for all prime ideals of height one in R. Therefore the kernel and cokernel of the homogeneous degree 0 morphism $M \to \text{Hom}_R(\text{Hom}_R(M, R), R)$ have finite length. Therefore $\tilde{M} \simeq \text{Hom}_R(\text{Hom}_R(M, R), R)$, which shows that $\mathcal{F} \simeq \text{Hom}_R(\text{Hom}_R(M, R), R)$. Since $\text{Hom}_R(\text{Hom}_R(M, R), R)$ is a Cohen–Macaulay module, this shows that the functor $(\mathbf{CM}\mathit{gr}S)_0 \to \mathbf{CM}\mathcal{O}_X$ given by $M \to \tilde{M}$ is dense and hence an equivalence of categories.

As an easy consequence of this proposition we have the following.

Corollary 3.2. (a) *For each $\mathcal{F}$ in $\mathbf{CM}\mathcal{O}_x$ we have that $\Gamma_*(\mathcal{F})$ is in* $(\mathbf{CM}\mathit{gr}S)_0$.

(b) *The functors* $\Gamma_*: \mathbf{CM}\mathcal{O}_x \to (\mathbf{CM}\mathit{gr}S)_0$ *and* $(\mathbf{CM}\mathit{gr}S)_0 \to \mathbf{CM}\mathcal{O}_x$ *given by* $M \to \tilde{M}$ *are inverse equivalences.*

This equivalence $(\mathbf{CM}\mathit{gr}S)_0 \to \mathbf{CM}\mathcal{O}_x$ enables us to translate the results in the previous two sections of this part about $(\mathbf{CM}\mathit{gr}S)_0$ to $\mathbf{CM}\mathcal{O}_x$. For convenience of notation we denote by ω_x the sheaf $\tilde{\omega} = \mathrm{Hom}_R(S, R(-2))$. The sheaf ω_x is the dualizing sheaf on X. The fundamental sequence $0 \to \omega \to E \to S \to k \to 0$ in $(\mathrm{mod}\,\mathit{gr}S)_0$ gives rise to an exact sequence $0 \to \omega_x \to E_x \to \mathcal{O}_x \to 0$ in $\mathbf{CM}\mathcal{O}_x$ which is easily seen to be an almost split sequence in $\mathbf{CM}\mathcal{O}_x$. Clearly the exact sequences $0 \to \omega_x(n) \to E_x(n) \to \mathcal{O}_x(n) \to 0$ are also almost split sequences in $\mathbf{CM}\mathcal{O}_x$ for all n in $\mathbb{Z}$. Hence for each indecomposable locally free sheaf $\mathcal{F} \cong \mathcal{O}_x(n)$ for some n, there is an almost split sequence $0 \to \omega_x \otimes \mathcal{F} \to E_x \otimes \mathcal{F} \to \mathcal{O}_x \otimes \mathcal{F} \to 0$ in $\mathbf{CM}\mathcal{O}_x$.

Suppose now that $\mathcal{F}$ in $\mathbf{CM}\mathcal{O}_x$ is indecomposable and not isomorphic to $\mathcal{O}_x(n)$ for any n. Then there is an almost split sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F} \to 0$ in $\mathbf{CM}\mathcal{O}_x$ if and only if there is an almost split sequence $0 \to A \to B \to \Gamma_*(\mathcal{F}) \to 0$ in $(\mathbf{CM}\mathit{gr}S)_0$. Now we know by Theorem 1.8 and Lemma 1.10 that there is an almost split sequence $0 \to A \to B \to \Gamma_*(\mathcal{F}) \to 0$ in $(\mathbf{CM}\mathit{gr}S)_0$ if and only if $\Gamma_*(\mathcal{F})$ is free at height one primes. But it is easily seen that $\Gamma_*(\mathcal{F})$ is free at height one primes if and only if $\mathcal{F}$ is locally free. Therefore there is an almost split sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F} \to 0$ in $\mathbf{CM}\mathcal{O}_x$ if and only if $\mathcal{F}$ is locally free.

Now suppose that $\mathcal{F}$ is locally free. Then we know that there is an almost split sequence $0 \to \mathrm{Hom}_R(\mathrm{Hom}_S(\Gamma_*\mathcal{F}, S), R(-2)) \to E \to \Gamma_*(\mathcal{F}) \to 0$. Now $\mathrm{Hom}_R(\mathrm{Hom}_S(\Gamma_*\mathcal{F}, S), R(-2)) \simeq \mathrm{Hom}_S(\mathrm{Hom}_S(\Gamma_* \mathcal{F}, S), \omega)$. Therefore we obtain the almost split sequence $0 \to \mathbf{Hom}_{\mathcal{O}_x}(\mathrm{Hom}_{\mathcal{O}_x}(\mathcal{F}, \mathcal{O}_x), \omega_x) \to \mathcal{E} \to \mathcal{F} \to 0$ in $\mathbf{CM}\mathcal{O}_x$. Since $\mathcal{F}$ is locally free, it follows that $\mathbf{Hom}_{\mathcal{O}_x}(\mathrm{Hom}_{\mathcal{O}_x}(\mathcal{F}, \mathcal{O}_x), \omega_x) \cong \mathcal{F} \otimes_{\mathcal{O}_x} \omega_x$. Thus we have the almost split sequence

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_x} \omega_x \to \mathcal{E} \to \mathcal{F} \to 0.$$

Summarizing this discussion we have the following.

THEOREM 3.3. *Let* $\mathcal{F}$ *in* $\mathbf{CM}\mathcal{O}_x$ *be indecomposable. Then we have the following.*

(a) $\mathcal{F}$ *is locally free if and only if there is an almost split sequence* $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F} \to 0$ *in* $\mathbf{CM}\mathcal{O}_x$.

(b) *If* $\mathcal{F}$ *is locally free, then the almost split sequence for* $\mathcal{F}$ *has the form* $0 \to \mathcal{F} \otimes_{\mathcal{O}_x} \omega_x \to \mathcal{E} \to \mathcal{F} \to 0$.

We now point out some properties of the almost split sequences in $\mathbf{CM}\mathcal{O}_x$.

PROPOSITION 3.4. *Let* $0 \to \omega_x \otimes_{\mathcal{O}_x} \mathcal{F} \to \mathcal{E} \to \mathcal{F} \to 0$ *be an almost split sequence in* $\mathbf{CM}\mathcal{O}_x$. *Then* $\mathcal{F}$ *is not isomorphic to* $\mathcal{O}(n)$ *for any n if and only if* $0 \to \Gamma((\omega_x \otimes_{\mathcal{O}_x} \mathcal{F})(n)) \to \Gamma(\mathcal{E}(n)) \to \Gamma(\mathcal{F}(b)) \to 0$ *is exact for all n in* $\mathbb{Z}$.

Proof. Since $\mathcal{F} \not\simeq \mathcal{O}(n)$ for any n, we have that $0 \to \operatorname{Hom}_{\mathcal{O}_x}(\mathcal{O}(n), \omega_x \otimes \mathcal{F}) \to \operatorname{Hom}_{\mathcal{O}_x}(\mathcal{O}(n), \mathcal{E}) \to \operatorname{Hom}_{\mathcal{O}_x}(\mathcal{O}(n), \mathcal{F}) \to 0$ is exact for all n. This gives the desired result.

Next we recall that X is a Gorenstein curve if and only if ω_x is locally free. Therefore if $\mathcal{F}$ is locally free, we have that $\omega_x \otimes_{\mathcal{O}_x} \mathcal{F}$ is locally free if and only if ω_x is locally free. As a consequence of these remarks, we have the following.

PROPOSITION 3.5. *Let* $0 \to \omega_x \otimes_{\mathcal{O}_x} \mathcal{F} \to \mathcal{E} \to \mathcal{F} \to 0$ *be an almost split sequence in* $\mathbf{CM}\mathcal{O}_x$. *Then the following are equivalent.*

(a) *X is a Gorenstein curve.*

(b) *$\omega_x \otimes_{\mathcal{O}_x} \mathcal{F}$ is locally free.*

(c) *$\mathcal{E}$ is locally free.*

Having interpreted the results of Section 1 of this part for sheaves, we now turn our attention to translating the results of Section 2 of this part to sheaves.

Let $\mathcal{F}$ be a locally free sheaf on X. The fact that X is connected gives us that the ranks of the free $\mathcal{O}_{X_x}$-modules $\mathcal{F}_x$ are the same for all x. This number is called the rank of $\mathcal{F}$ and is denoted by rank $\mathcal{F}$. It is easily seen that rank $\mathcal{F} = \operatorname{rank} \Gamma_*(\mathcal{F})$.

As an immediate consequence of Theorem 2.5 we have the following.

THEOREM 3.6. *Let p be the characteristic of k (p can be zero). Suppose* $\mathcal{F}$ *is an indecomposable locally free sheaf and let* $0 \to \omega_x \to \mathcal{E}_x \to \mathcal{O}_x \to 0$ *be an almost split sequence.*

(a) *If rank* $\mathcal{F}$ *is not divisible by p, then the exact squence*

$$0 \to \omega_x \otimes_{\mathcal{O}_x} \mathcal{F} \to \mathcal{E}_x \otimes \mathcal{F} \to \mathcal{F} \to 0$$

does not split and the converse holds if k is algebraically closed.

(b) *If p does not divide rank* $\mathcal{F}$, *then* $0 \to \omega_x \otimes_{\mathcal{O}_x} \mathcal{F} \to \mathcal{E}_x \otimes \mathcal{F} \to \mathcal{F} \to 0$ *is an almost split sequence.*

We have the following as an immediate consequence of Theorem 3.6.

COROLLARY 3.7. *Suppose* $\operatorname{ch} k = 0$. *Then the almost split sequence* $0 \to \omega_x \to \mathcal{E}_x \to \mathcal{O}_x \to 0$ *has the following property. Let* $\mathcal{F}$ *be an* *indecomposable locally free sheaf. Then the exact sequence* $0 \to \omega_x \otimes_{\mathcal{O}_x} \mathcal{F} \to \mathcal{E}_x \otimes_{\mathcal{O}_x} \mathcal{F} \to \mathcal{F} \to 0$ *is an almost split sequence.*

Suppose that $k = \mathbb{C}$ and X is an elliptic curve. Then Corollary 3.7 combined with Atiyah's classification of locally free sheaves on X [1] gives an effective way of computing the almost split sequences in $\mathbf{CM}\mathcal{O}_x$. Since $\omega_x = \mathcal{O}_x$ for elliptic curves, we have an almost split sequence $0 \to \mathcal{O}_x \to \mathcal{F}_1 \to \mathcal{O}_x \to 0$. Thus for each indecomposable locally free sheaf $\mathcal{F}$, we have the almost split sequence $0 \to \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_x} \mathcal{F}_2 \to \mathcal{F} \to 0$. Since Atiyah constructs $\mathcal{F}_2$, which is indecomposable, and describes the decomposition of $\mathcal{F} \otimes_{\mathcal{O}_x} \mathcal{F}_2$ into indecomposable summands for all $\mathcal{F}$, we have all almost split sequences $0 \to \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_x} \mathcal{F}_2 \to \mathcal{F} \to 0$. From this description it follows directly that the components of the AR-quiver for the locally free indecomposable sheaves over X are of the form $A_\infty : \rightleftarrows \cdot \rightleftarrows \cdot \rightleftarrows \cdot \ldots$. This result has also been observed by A. Schofield, who has also obtained results along these lines when X is not necessarily elliptic.

In analogy with the situation for graded modules, we say that an indecomposable locally free sheaf $\mathcal{F}$ on X is a splitting trace sheaf if $\mathcal{O}_x$ is summand of $\mathbf{Hom}_{\mathcal{O}_x}(\mathcal{F}, \mathcal{F})$. We then have the following.

PROPOSITION 3.8. *Let $\mathcal{F}$ be an indecomposable locally free sheaf on X and let p be the characteristic of the field k of X. Then we have the following.*

(a) *$\mathcal{F}$ is a splitting trace sheaf if rank $\mathcal{F}$ and p are relatively prime.*

(b) *If k is algebraically closed, then $\mathcal{F}$ is a splitting trace sheaf if and only if rank M and p are relatively prime.*

Proof. Trivial consequence of Proposition 2.3.

We also have the following easy consequence of Proposition 2.4.

PROPOSITION 3.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be two indecomposable locally free sheaves.*

(a) *If $\mathcal{O}_x \mid \mathbf{Hom}_{\mathcal{O}_x}(\mathcal{F}, g)$, then $\mathcal{F} \cong \mathcal{G}$ and $\mathcal{F}$ is a splitting trace sheaf.*

(b) *If $\mathcal{F}$ is a splitting trace sheaf, then $\mathcal{F} \cong \mathcal{G}$ if and only if $\mathcal{O}_x \mid \mathbf{Hom}_{\mathcal{O}_x}(\mathcal{F}, g)$.*

REFERENCES

1. M. F. ATIYAH, Vector bundles over an elliptic curve, *Proc. London Math. Soc* (*3*) 7, No. 27 (1957), 414–452.
2. M. AUSLANDER, Existence theorems for almost split sequence, *in* "Proceedings, Conference on Ring Theory II, Oklahoma," pp. 1–44, Dekker, New York, 1977.

3. M. AUSLANDER, Functors and morphisms determined by objects *in* "Proceedings, Conference on Representation Theory, Philadelphia, 1976," pp. 1–244, Dekker, New York, 1978.
4. M. AUSLANDER, Rational singularities and almost split sequence, *Trans. Amer. Math. Soc.* **293**, No. 2 (1986), 511–531.
5. M. AUSLANDER, Isolated singularities and almost split sequences, *in* "Proceedings, ICRA IV," pp. 194–241, Lecture Notes in Mathematics, Vol. 1178, Springer-Verlag, New York/Berlin, 1986.
6. M., AUSLANDER AND I. REITEN, Stable equivalence of dualizing *R*-varieties, I, *Adv. in Math.* **12** (1974), 306–366.
7. M. AUSLANDER AND I. REITEN, Representation theory of artin algebras. III. Almost split sequences, *Comm. Algebra* **3** (1975), 239–294.
8. M. AUSLANDER AND I. REITEN, The Cohen–Macaulay type of Cohen–Macaulay rings, preprint.
9. R. GORDON AND E. GREEN, Graded artin algebras, *J. Algebra* **76**, No. 1 (1982), 111–137.
10. R. HARTSHORNE, Complete intersections and connectedness, *Amer. J. Math.* **84** (1962), 497–508.
11. I. REITEN, Representations of algebras, *in* "Proceedings, Workshop, Puebla, Mexico, 1980," Lecture Notes in Mathematics, Vol. 944 pp. 29–104, Springer-Verlag, New York/Berlin, 1980.
12. I. REITEN, An introduction to the representation theory of artin algebras, *Bull. London Math. Soc.* **17** (1985), 209–233.

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