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Contributions to the theory of graphic sequences

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Abstract

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In this article we present a new version of the Erdős–Gallai theorem concerning graphicness of the degree sequences. The best conditions of all known on the reduction of the number of Erdős–Gallai inequalities are given. Moreover, we prove a criterion of the bipartite graphicness and give a sufficient condition for a sequence to be graphic which does not require checking of any Erdős–Gallai inequality.

1. Introduction

All graphs will be finite and undirected without loops or multiple edges. A sequence $\underset{\sim}{d}$ of nonnegative integers is called graphic, if there exists a graph whose degree sequence is $\underset{\sim}{d}$. Unless otherwise specified, we assume that the sequence $\underset{\sim}{d}$ has the following form:

$$\underset{\sim}{d} = (d_1, d_2, \ldots, d_p), \quad d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p \geqslant 0. \tag{1}$$

The well-known theorem of Erdős and Gallai [5] gives the necessary and sufficient conditions for a sequence to be graphic. There are English [2] and French [1] versions of this theorem. In this article we present a new (Russian) version, which is not equivalent to the original Erdős–Gallai theorem.

Hammer, Ibaraki, Simeone and Li [8, 10] have shown the superfluity of Erdős–Gallai inequalities (EGI), which must be checked in order to determine the graphicness of a sequence. In fact, they proved that EGI must be checked up to certain index. Eggleton [4] also undertook the research concerning reduction of EGI. His result reduces the number of EGI to the cardinality of the degree set. In Theorems 4–5, we get the best conditions of all known ones on the reduction of the number of EGI.

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It is intuitively clear that if a sequence without zeros has an even sum and its length is large enough in comparison to the value of the maximum element, then this sequence is graphic. In Theorem 6, we give a precise wording of this observation. The result enables for a very wide class of sequences to recognize their graphicness without checking any EGI.

On the basis of the theorem of Hammer and Simeone [7] about split degree sequences, we transfer our results on the task of the bipartite graphicness (Theorems 7–8). Another criteria of the bipartite graphicness can be found in [3, 6].

2. Reduction of the number of Erdős–Gallai inequalities

In [5] Erdős and Gallai found the necessary and sufficient conditions for a sequence to be graphic.

Theorem 1 (Erdős and Gallai [5]). *A sequence $\underset{\sim}{d}$ of the form* (1) *is graphic iff its sum is an even integer and for any $k = 1, 2, \ldots, p-1$ it holds*

$$\sum_{i=1}^{k} d_i \leqslant k(k-1) + \sum_{i=k+1}^{p} \min\{d_i, k\}. \tag{EGI}$$

As it turned out [8, 10], the inequalities of Erdős and Gallai (EGI) are not independent—it is sufficient to check EGI only for strong indices (Theorem 2). The element d_k (and the index k too) in a sequence of the form (1) is called strong, if $d_k \geqslant k$. The maximum strong index in $\underset{\sim}{d}$ is denoted by $k_m = k_m(\underset{\sim}{d})$.

Theorem 2 (Hammer, Ibaraki, Simeone and Li [8, 10]). *A sequence $\underset{\sim}{d}$ of the form* (1) *is graphic iff its sum is an even integer and for every strong index k* (EGI) *holds.*

In connection with Theorem 2 we make the following remark. In the references [8, 10], this theorem was stated for those indices k for which $d_k \geqslant k-1$, i.e., under a stronger condition. Let us prove the correctness of Theorem 2. Consider the case, when the conditions $d_k \geqslant k$ and $d_k \geqslant k-1$ are different. This takes place, if $d_j < j$ and $d_j \geqslant j-1$ for some index j. Then $d_j = j-1$ and it is obvious that the next indices after j do not satisfy the inequality $d_k \geqslant k-1$. Thus, there is one and only element d_j, which expresses the difference between the conditions under consideration. Now we shall prove that the $(j-1)$th EGI implies the jth EGI, provided that $d_j = j-1$:

$$\sum_{i=1}^{j-1} d_i \leqslant (j-1)(j-2) + \sum_{i=j}^{p} \min\{d_i, j-1\}, \quad ((j-1)\text{th EGI})$$

$$\sum_{i=1}^{j} d_i - (j-1) \leqslant j(j-1) - 2(j-1) + (j-1)$$
$$+ \sum_{i=j+1}^{p} \min\{d_i, j-1\} \quad \text{(rearranging)}.$$

Since $d_i \leq d_j = j - 1$ for $i \geq j + 1$, then $\min\{d_i, j-1\} = \min\{d_i, j\}$ for the same i, i.e.,

$$\sum_{i=1}^{j} d_i \leq j(j-1) + \sum_{i=j+1}^{p} \min\{d_i, j\}, \quad (j\text{th EGI})$$

as required.

Let $n_j = n_j(\underset{\sim}{d})$ denote the number of all elements of $\underset{\sim}{d}$ which are equal to j $(j \geq 0)$.

Theorem 3. *A sequence $\underset{\sim}{d}$ of the form* (1) *having an even sum is graphic iff for every strong index k it holds*

$$r_k \leq k(p-1), \tag{2}$$

where $r_k = \sum_{i=1}^{k} (d_i + i n_{k-i})$.

Proof. Now we prove that for strong k, (EGI) and (2) are equivalent. Let k be fixed and s be the maximum index such that $d_s \geq k$. The existence of s follows from the fact that k is strong. It is easily checked that

$$p = s + \sum_{j=0}^{k-1} n_j \quad \text{and} \quad \sum_{j=0}^{k-1} j n_j = \sum_{i=s+1}^{p} d_i. \tag{3}$$

(Throughout the paper, it is assumed that $\sum_{i=s+1}^{p} = 0$ for $s = p$.)

Using (3), we get

$$\begin{aligned}
k(p-1) &- \sum_{i=1}^{k} i n_{k-i} \\
&= k(p-1) - \left(k \sum_{j=0}^{k-1} n_j - \sum_{j=0}^{k-1} j n_j\right) \\
&= k\left(s + \sum_{j=0}^{k-1} n_j - 1\right) - k \sum_{j=0}^{k-1} n_j + \sum_{j=0}^{k-1} j n_j \\
&= k(s-1) + \sum_{j=0}^{k-1} j n_j = k(k-1) + k(s-k) + \sum_{i=s+1}^{p} d_i \\
&= k(k-1) + \sum_{i=k+1}^{s} \min\{d_i, k\} + \sum_{i=s+1}^{p} \min\{d_i, k\} \\
&= k(k-1) + \sum_{i=k+1}^{p} \min\{d_i, k\},
\end{aligned}$$

since $d_{k+1} \geq \cdots \geq d_s \geq k > d_{s+1} \geq \cdots \geq d_p$. The result now follows from Theorem 2. □

The simplest examples show that the inequalities (2) do not hold for nonstrong indices k, i.e., Theorem 3 cannot be stated analogously to Theorem 1. If k is a strong index, then the inequalities (2) will be referred to as EGI too.

Johnson [9], with the help of the Tutte–Berge theorem, has proved that for any graphic sequence $\underset{\sim}{d}$ of the form (1) and for any even integer c satisfying $d_p \geq c \geq 0$, the sequence $\underset{\sim}{d} \cup (c)$ is graphic. A more general result can be easily deduced from Theorem 3.

Corollary 1. *If a sequence $\underset{\sim}{d}$ of the form* (1) *is graphic, a sequence $\underset{\sim}{c} = (c_1, c_2, \ldots, c_q)$ has an even sum and $k_m(\underset{\sim}{d}) \geq c_i$ for any $i = 1, 2, \ldots, q$, then the sequence $\underset{\sim}{d} \cup \underset{\sim}{c}$ is graphic.*

Proof. Let $\underset{\sim}{e}$ be the non-increasing rearrangement of the sequence $\underset{\sim}{d} \cup \underset{\sim}{c}$ in such a way, that $c_1, c_2, \ldots, c_q$ are on the right side from the element $d_{k_m(\underset{\sim}{d})}$. This is possible, as $d_{k_m(\underset{\sim}{d})} \geq k_m(\underset{\sim}{d}) \geq c_i$, $i = 1, 2, \ldots, q$. Obviously, $k_m(\underset{\sim}{e}) = k_m(\underset{\sim}{d})$. For fixed strong k, we have

$$\begin{aligned} r_k(\underset{\sim}{e}) &= r_k(\underset{\sim}{d}) + \sum_{i=1}^{k} i n_{k-i}(\underset{\sim}{c}) \\ &\leq k(p-1) + kq = k(p+q-1). \end{aligned}$$

By Theorem 3, the sequence $\underset{\sim}{e}$ is graphic. This completes the proof. □

Corollary 1 really extends the mentioned result of Johnson by the reason that $d_p \leq k_m$. Indeed, $\underset{\sim}{d}$ is graphic and hence $d_p < p$, i.e., the element d_p is not strong. This implies the existence of nonstrong d_{k_m+1}. Clearly, $d_{k_m+1} < k_m + 1$ and $d_p \leq d_{k_m+1}$. Thus $d_p \leq k_m$, as required.

In the next, we shall reduce the number of the inequalities in Theorem 3 to the number of the different strong elements. This reduction relates to Theorems 1–2, since (EGI) and (2) are equivalent for strong indices. The strong index k is called right strong, if $d_k > d_{k+1}$ or $k = k_m$.

Theorem 4. *A sequence $\underset{\sim}{d}$ of the form* (1) *having an even sum is graphic iff the inequalities* (2) *hold for every right strong index k.*

Proof. Necessity follows from Theorem 3. To prove the 'if' part of the theorem, consider any nonzero sequence $\underset{\sim}{d}$.

Let s be the minimum right strong index. This means that $d_1 = d_2 = \cdots = d_s$. The inequality $r_s \leq s(p-1)$ holds by the condition of the theorem. Let $1 \leq k \leq s$. Let us prove that (2) holds for k. We have

$$r_s = \sum_{i=1}^{s} d_i + \sum_{i=1}^{s} i n_{s-i} = s d_1 + \sum_{i=1}^{s} i n_{s-i} \leq s(p-1).$$

Hence

$$d_1 \leq p - 1 - \sum_{i=1}^{s} \frac{i}{s} n_{s-i}.$$

Then

$$\begin{aligned} r_k &= kd_1 + \sum_{i=1}^{k} in_{k-i} \\ &\leqslant k\left(p - 1 - \sum_{i=1}^{s} \frac{i}{s} n_{s-i}\right) + \sum_{i=1}^{k} in_{k-i} \\ &\leqslant k(p-1) - k \sum_{i=s-k+1}^{s} \frac{i}{s} n_{s-i} + \sum_{i=1}^{k} in_{k-i} \\ &= k(p-1) - \sum_{i=1}^{k} \left(\frac{k}{s}(s-k+i) - i\right) n_{k-i}. \end{aligned}$$

It remains to notice that $(k/s)(s-k+i) - i \geqslant 0$ for any $i = 1, 2, \ldots, k$.

Now let s, t be right strong indices and besides $d_{s+1} = d_{s+2} = \cdots = d_t$. Prove the inequality (2) for k, which satisfies $s < k \leqslant t$.

$$\begin{aligned} r_t &= \sum_{i=1}^{t} d_i + \sum_{i=1}^{t} in_{t-i} \\ &= r_s + \sum_{i=s+1}^{t} d_i + (1 \cdot n_{t-1} + \cdots + (t-s)n_s) + (t-s)(n_{s-1} + \cdots + n_0) \\ &= r_s + (t-s)d_k + \sum_{j=s}^{t-1} (t-j)n_j + (t-s)\sum_{j=0}^{s-1} n_j \leqslant t(p-1). \end{aligned}$$

Hence

$$d_k \leqslant \frac{t(p-1)}{t-s} - \frac{r_s}{t-s} - \sum_{j=s}^{t-1} \frac{t-j}{t-s} n_j - \sum_{j=0}^{s-1} n_j.$$

We have

$$\begin{aligned} r_k &= r_s + (k-s)d_k + k\sum_{j=s}^{j-1} (k-j)n_j + (k-s)\sum_{j=0}^{s-1} n_j \\ &\leqslant r_s + \frac{t(p-1)(k-s)}{t-s} - \frac{k-s}{t-s} r_s - \sum_{j=s}^{t-1} \frac{(t-j)(k-s)}{t-s} n_j \\ &\quad - (k-s)\sum_{j=0}^{s-1} n_j + \sum_{j=s}^{k-1} (k-j)n_j + (k-s)\sum_{j=0}^{s-1} n_j \\ &\leqslant \left(1 - \frac{k-s}{t-s}\right) r_s + \frac{t(p-1)(k-s)}{t-s} - \sum_{j=s}^{k-1} \frac{(t-j)(k-s)}{t-s} n_j \\ &\quad + \sum_{j=s}^{k-1} (k-j)n_j \leqslant \frac{(t-s)-(k-s)}{t-s} s(p-1) + \frac{t(p-1)(k-s)}{t-s} \\ &\quad - \sum_{j=s}^{k-1} \left(\frac{(t-j)(k-s)}{t-s} - k + j\right) n_j = k(p-1) - \sum_{j=s}^{k-1} \frac{(t-k)(j-s)}{t-s} n_j. \end{aligned}$$

Clearly $(t-k)(j-s)/(t-s) \geqslant 0$ for $s \leqslant j \leqslant k-1$. Therefore $r_k \leqslant k(p-1)$, proving the theorem. □

Eggleton [4] announced the result on the reduction of EGI to the number $\|\underset{\sim}{d}\|$ of the different elements of the sequence $\underset{\sim}{d}$ (i.e., to the cardinality of the degree set), and in the case $\|\underset{\sim}{d}\| > 1$ the last EGI can be omitted too. Theorem 4 improves this result and on the basis of this theorem a further advance in the reduction of the number of EGI is possible.

Theorem 5. *Let $s < k < t$ be strong indices of a sequence $\underset{\sim}{d}$ of the form* (1) *and $d_{s+1} = d_t + 1$. Then inequalities $r_s \leqslant s(p-1)$ and $r_t \leqslant t(p-1)$ imply $r_k \leqslant k(p-1)$.*

Proof. Let us prove the theorem for k satisfying the equalities $d_{s+1} = \cdots = d_k = d_{k+1} + 1 = \cdots = d_t + 1$. Then for the rest values of k, Theorem 5 will follow from Theorem 4. Let

$$d_k + \sum_{j=0}^{k} n_j \leqslant p - 1.$$

Then

$$\begin{aligned} r_k &= \sum_{i=1}^{k} d_i + \sum_{i=1}^{k} i n_{k-i} \\ &= \sum_{i=1}^{s} d_i + \sum_{j=s+1}^{k} d_j + \sum_{i=1}^{s} i n_{s-i} + \sum_{j=s}^{k} (k-j) n_j + (k-s) \sum_{j=0}^{s-1} n_j \\ &\leqslant r_s + (k-s) d_k + (k-s) \sum_{j=0}^{k} n_j \\ &\leqslant s(p-1) + (k-s)\Big(d_k + \sum_{j=0}^{k} n_j\Big) \\ &\leqslant k(p-1). \end{aligned}$$

Now let

$$d_k + \sum_{j=0}^{k} n_j \geqslant p.$$

Then

$$\begin{aligned} r_k &= \sum_{i=1}^{k} d_i + \sum_{i=1}^{k} i n_{k-i} \\ &= \sum_{i=1}^{t} d_i - \sum_{j=k+1}^{t} d_j + \sum_{i=1}^{t} i n_{t-i} - \sum_{j=k+1}^{t} (t-j) n_j - (t-k) \sum_{j=0}^{k} n_j \\ &\leqslant r_t - (t-k)(d_k - 1) - (t-k) \sum_{j=0}^{k} n_j \\ &\leqslant t(p-1) + (t-k)\Big(1 - d_k - \sum_{j=0}^{k} n_j\Big) \\ &\leqslant k(p-1), \end{aligned}$$

as required. □

Under the conditions of Theorem 5, we say that the elements $d_{s+1}, \ldots, d_t$ form a threshold of height 1. Considered from the viewpoint of increasing of the threshold height, Theorem 5 cannot be improved. To take an example, let

$$\underset{\sim}{d} = (8\,8\,6\,4\,4\,4\;\;2\,2\,2).$$

The elements $d_3 = 6$ and $d_4 = 4$ form the threshold of height 2. It is checked directly that inequality (2) holds for $k = 2,4$ and does not hold for $k = 3$, i.e., the statement of Theorem 5 fails in this case.

3. Graphicness of restricted sequences

For sequences with an even sum whose elements are restricted in comparison with $p - n_0$, Theorem 3 allows to prove their graphicness.

To put it more exactly, let a, b be integers and $a \geqslant b > 0$. $K(a, b)$ denotes the class of sequences of the form (1) having an even sum and satisfying $a \geqslant d_1$, $d_p \geqslant b$. It is required to find the minimum p_m such that if $\underset{\sim}{d} \in K(a, b)$ and $p = |\underset{\sim}{d}| \geqslant p_m$, then $\underset{\sim}{d}$ is graphic.

Theorem 6. *If* $\underset{\sim}{d} \in K(a, b)$ *and*

$$p = |\underset{\sim}{d}| \geqslant (a + b + 1)^2/4b, \tag{4}$$

then the sequence $\underset{\sim}{d}$ *is graphic.*

Proof. Let k be a strong index of the sequence $\underset{\sim}{d}$. If $k - 1 < b$, then $n_j = 0$ for all $j = 0, 1, \ldots, k - 1$. Hence

$$r_k = \sum_{i=1}^{k} (d_i + i n_{k-i}) \leqslant ka \leqslant k(p - 1),$$

as $a < (a + b + 1)^2/4b \leqslant p$.

Now let $k - 1 \geqslant b$. By the definition of the strong element, $d_{k_m} \geqslant k_m \geqslant k$. Therefore,

$$\sum_{j=b}^{k-1} n_j \leqslant p - k_m. \tag{5}$$

Using (4) and (5), we have

$$\begin{aligned} r_k &\leqslant ka + (k - b) \sum_{j=b}^{k-1} n_j \leqslant ka + (k - b)(p - k_m) \\ &= k(p - 1) + k(a + 1 - k_m) + bk_m - bp \\ &\leqslant k(p - 1) + k_m(a + b + 1) - k_m^2 - b(a + b + 1)^2/4b \\ &= k(p - 1) - (k_m - (a + b + 1)/2)^2 \leqslant k(p - 1). \end{aligned}$$

Here we used the inequalities $k \leqslant k_m$, $b \leqslant k_m$ and $a+1-k_m>0$, as $a \geqslant d_1 \geqslant \cdots \geqslant d_a$ and, consequently, $k_m \leqslant a$.

Thus the inequalities (2) hold for all strong indices. By Theorem 3, the sequence $\underset{\sim}{d}$ is graphic. The proof is complete. □

The bound (4) cannot be lowered over the set of all classes $K(a, b)$. To prove this, we shall construct 2-parametric series of classes $K(a, b)$ such that every class contains a non-graphic sequence $\underset{\sim}{d}$ with an even sum and of length $p=(a+b+1)^2/4b-1$.

Consider in the class $K(a, b)=K(8st-2t-1, 2t)$, $s \geqslant 1$, $t \geqslant 1$, the sequence

$$\underset{\sim}{d}=((8st-2t-1)^{l_1}, (2t)^{l_2}),$$

where $l_1=4st$, $l_2=8s^2t-4st-1$. Here we used the well-known exponential form of a sequence

$$\underset{\sim}{d}=(d_1^{l_1}, d_2^{l_2}, \ldots, d_q^{l_q}), \tag{6}$$

where $d_i^{l_i}$ means that element d_i occurs exactly l_i times. Clearly $|\underset{\sim}{d}|=8s^2t-1=(a+b+1)^2/4b-1$ and the element $d_{4st}=8st-2t-1$ is strong for the sequence $\underset{\sim}{d}$ represented in the form (1). Calculate r_{4st}:

$$\begin{aligned} r_{4st} &= 4st(8st-2t-1)+(4st-2t)n_{2t} \\ &= 32s^3t^2-8st+2t. \end{aligned}$$

We have $r_{4st}>4st(|\underset{\sim}{d}|-1)=32s^3t^2-8st$. Thus the $(4st)$th inequality in (2) does not hold and hence $\underset{\sim}{d}$ is not graphic.

Corollary 2. *If a sequence $\underset{\sim}{d}$ of the form* (1) *has an even sum and*

$$d_1 \leqslant 2(p-n_0)^{\frac{1}{2}}-2, \tag{7}$$

then $\underset{\sim}{d}$ is graphic.

Proof. If $\underset{\sim}{d}$ consists of the zeros, then $\underset{\sim}{d}$ is graphic. Otherwise, we construct the sequence $\underset{\sim}{d}'$ by means of deleting of all zero elements from $\underset{\sim}{d}$. Since $\underset{\sim}{d}' \in K(d_1, 1)$ and from (7) it follows that $p'=p-n_0 \geqslant \frac{1}{4}(d_1+2)^2$, then $\underset{\sim}{d}'$ is graphic by Theorem 6. Hence the sequence $\underset{\sim}{d}$ is graphic. This completes the proof. □

The bound (7) cannot be improved. In order to show this, consider the following sequence in the form (6):

$$\underset{\sim}{d}=((2p^{\frac{1}{2}}-1)^{l_1}, 1^{l_2}),$$

where $l_1=p^{\frac{1}{2}}+1$, $l_2=p-p^{\frac{1}{2}}-1$ for $p=4n^2$. The sequence $\underset{\sim}{d}$ has an even sum and $d_1=2p^{\frac{1}{2}}-1$. For any $n \geqslant 1$, this sequence is not graphic. Indeed, let us suppose to the contrary that $\underset{\sim}{d}$ is realised by some graph G. Every vertex of the degree $2p^{\frac{1}{2}}-1$ in G would be adjacent at least to $p^{\frac{1}{2}}-1$ pendant vertices, i.e.,

$$|\{u \in V(G)/\deg u=1\}| \geqslant (p^{\frac{1}{2}}-1)(p^{\frac{1}{2}}+1)=p-1.$$

Then $n_1=p-p^{\frac{1}{2}}-1 \geqslant p-1$ and $p^{\frac{1}{2}}=2n \leqslant 0$, a contradiction.

To complete this section, we exhibit an example illustrating Theorem 6. Assume that some class of sequences has been generated and all elements of the sequences range between 1000 and 1500. Then any sequence from this class having an even sum and the length at least 1564 is graphic. With using the Erdős–Gallai theorem, any number of checks of EGI may be required and with using the improved Erdős–Gallai theorem (Theorem 2)—up to 1500 checks.

4. Bipartite graphic sequences

An unordered pair $(\underset{\sim}{d}, \underset{\sim}{c})$, where

$$\underset{\sim}{c} = (c_1, c_2, \ldots, c_q), \quad q \geq 1, c_1 \geq c_2 \geq \cdots \geq c_q \geq 0, \tag{8}$$

is called bipartite graphic, if there exists a bipartite graph G such that the degree sequences of the parts of G coincide with $\underset{\sim}{d}$ and $\underset{\sim}{c}$. The following criterion of the bipartite graphicness is based on Theorem 3 and the theorem of Hammer and Simeone [7] about split degree sequences.

Theorem 7. *Let $\underset{\sim}{d}$ and $\underset{\sim}{c}$ be sequences of the form* (1) *and* (8), *respectively, and*

$$\sum_{i=1}^{p} d_i = \sum_{i=1}^{q} c_i. \tag{9}$$

Then the pair $(\underset{\sim}{d}, \underset{\sim}{c})$ is bipartite graphic iff for any $k = 1, 2, \ldots, p$,

$$\sum_{i=1}^{k} (d_i + in_{k-i}(\underset{\sim}{c})) \leq kq. \tag{10}$$

Proof. Necessity. Consider a bipartite realization (G, A, B) of the pair $(\underset{\sim}{d}, \underset{\sim}{c})$. In the part A (which corresponds to $\underset{\sim}{d}$), choose vertices $u_1, u_2, \ldots, u_k$ with the degrees $d_1, d_2, \ldots, d_k$ $(1 \leq k \leq p)$. Denote by m_{ij} $(1 \leq i \leq k,\ 0 \leq j \leq k-1)$ the number of all vertices of the degree j in B which are adjacent to u_i. Then

$$\begin{aligned}
\sum_{i=1}^{k} d_i &\leq k\Big(|B| - \sum_{j=0}^{k-1} n_j(\underset{\sim}{c})\Big) + \sum_{i=1}^{k}\sum_{j=0}^{k-1} m_{ij} \\
&\leq kq - \sum_{j=0}^{k-1} kn_j(\underset{\sim}{c}) + \sum_{j=0}^{k-1} jn_j(\underset{\sim}{c}) \\
&= kq - \sum_{j=0}^{k-1} (k-j)n_j(\underset{\sim}{c}) \\
&= kq - \sum_{i=1}^{k} in_{k-i}(\underset{\sim}{c}).
\end{aligned}$$

To prove sufficiency, we make use of the idea from [12]. Form the sequence

$$\underset{\sim}{e} = (d_1 + p - 1, \ldots, d_p + p - 1, c_1, \ldots, c_q) = (e_1, e_2, \ldots, e_{p+q}).$$

The inequalities $d_p + p - 1 \geqslant p \geqslant c_1$ mean that $\underset{\sim}{e}$ has been ordered by non-increasing of its elements and that $\{1, 2, \ldots, p\}$ is the set of the strong indices of $\underset{\sim}{e}$. In order to satisfy the first inequality $d_p + p - 1 \geqslant p$, we assume without loss of generality that $d_p \geqslant 1$. Let us prove the second inequality.

Lemma. (9) *and* (10) *imply the inequality* $p \geqslant c_1$.

Proof. Denote $s = |\{j = 1, 2, \ldots, q \mid c_j > p\}|$. Using (9), we transform (10) for $k = p$:

$$
\begin{aligned}
pq &\geqslant \sum_{i=1}^{p} (d_i + in_{p-i}(\underset{\sim}{c})) = \sum_{i=1}^{q} c_i + \sum_{i=1}^{p} in_{p-i}(\underset{\sim}{c}) \\
&= \left(\sum_{j=1}^{s} c_j + \sum_{j=0}^{p} jn_j\right) + \sum_{j=0}^{p-1} (p-j)n_j \\
&\geqslant \left(s(p+1) + \sum_{j=0}^{p} jn_j\right) + \sum_{j=0}^{p} (p-j)n_j \\
&= s(p+1) + p\sum_{j=0}^{p} n_j(\underset{\sim}{c}) = s(p+1) + p(q-s) = pq + s.
\end{aligned}
$$

Hence it follows $s \leqslant 0$. Consequently, $s = 0$, as required. □

How we go on the proof of the theorem. Recall that the graph G is called split, if there is a partition of its vertex set $V(G) = A \cup B$ into the complete subgraph $\langle A \rangle$ and the empty subgraph $\langle B \rangle$.

It is not difficult to check that $\underset{\sim}{e}$ satisfies (2). By Theorem 3, the sequence $\underset{\sim}{e}$ is graphic. From the theorem of Hammer and Simeone [7] about split degree sequences it follows that $\underset{\sim}{e}$ is realized by a split graph whose complete part consists of p vertices with the degrees $e_1, e_2, \ldots, e_p$. Removing from the complete part all edges, we get a bipartite realization of the pair $(\underset{\sim}{d}, \underset{\sim}{c})$. The proof of Theorem 7 is complete. □

In the proof of Theorem 7, it can be used Theorem 4 instead of Theorem 3. This enables to reduce the number of the inequalities (10) to the number of the different elements of $\underset{\sim}{d}$ (or $\underset{\sim}{c}$, as the pair $(\underset{\sim}{d}, \underset{\sim}{c})$ is unordered). Let $\underset{\sim}{d}$ has the exponential form (6): $\underset{\sim}{d} = (f_1^{l_1}, \ldots, f_s^{l_s})$, where $f_1 > \cdots > f_s$ are the different elements of $\underset{\sim}{d}$ (the degree set of $\underset{\sim}{d}$) and l_i is the multiplicity of f_i.

Theorem 8. *Under the conditions of Theorem* 7, *the pair* $(\underset{\sim}{d}, \underset{\sim}{c})$ *is bipartite graphic iff* (10) *holds for every* $k = l_1, l_1 + l_2, \ldots, (l_1 + l_2 + \cdots + l_s)$.

A further improvement of Theorem 7 is easily obtained on the basis of Theorem 5.

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