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Theory and Applications

Complexity of projected images of convex subdivisions ☆

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Abstract

Let S be a subdivision of $\mathbb{R}^d$ into n convex regions. We consider the combinatorial complexity of the image of the $(k-1)$-skeleton of S orthogonally projected into a k-dimensional subspace. We give an upper bound of the complexity of the projected image by reducing it to the complexity of an arrangement of polytopes. If $k = d - 1$, we construct a subdivision whose projected image has $\Omega(n^{\lfloor (3d-2)/2 \rfloor})$ complexity, which is tight when $d \leqslant 4$. We also investigate the number of topological changes of the projected image when a three-dimensional subdivision is rotated about a line parallel to the projection plane.

Keywords: Computational Geometry; Combinatorial Complexity; Convex Subdivision; Algorithms; Projection

1. Introduction

Projection and projected images often play important roles in algorithms for computational geometry. For example, a Voronoi diagram in d-dimensional space is obtained as a projected image of a convex polytope in $(d+1)$-dimensional space (a comprehensive survey of Voronoi diagrams is given by Aurenhammer [2].) Visibility problems and occlusion problems in computer vision and computer graphics are also directly related to the computation of projected images of objects [16].

In this paper, we deal with the projected images of (skeletons of) convex subdivisions under orthogonal projections. Let S be a subdivision of $\mathbb{R}^d$ into n convex regions.

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The closure of each convex region is decomposed into *faces*. A face of dimension j is called a j-face. The region itself is a d-face. A $(d-1)$-dimensional face is often called a facet, and a one-dimensional face is often called an edge. S is decomposed into faces of convex regions in a natural manner, and forms a cell complex [10]. The total number of faces of S is called the complexity of S. It is well known that the complexity of a convex subdivision into n convex regions of $\mathbb{R}^d$ is $O(n^{\lfloor (d+1)/2 \rfloor})$ [10, 12].

The union of all the j-faces for $j \leqslant h$ is called the h-skeleton of S. Let H be a k-dimensional subspace of $\mathbb{R}^d$. The image $Pr(S; H)$ of the $(k-1)$-skeleton of S orthogonally projected into H makes a convex subdivision of H. It is easy to see that a region in $Pr(S; H)$ is an intersection of the projected images of k-faces of S.

We use $pr(S; H)$ to denote the complexity of $Pr(S; H)$. Since two projected faces may intersect and create new faces, $pr(S; H)$ is often larger than the complexity of S. We define $pr_k(S) = \min\{pr(S; H)\colon H \in G(k, \mathbb{R}^d)\}$, where $G(k, \mathbb{R}^d)$, (the *Grassmann variety*) is the set of k-dimensional subspaces of $\mathbb{R}^d$. We define $opr_{d,k}(n) = \max\{pr_k(S)\}$ and $pr_{d,H}(n) = \max\{pr(S; H)\}$, where the maximum is taken over all convex subdivisions S of $\mathbb{R}^d$ into n regions. Since $pr_{d,H}(n)$ is independent of the choice of the k-dimensional subspace H, we denote it by $fpr_{d,k}(n)$. We call $fpr_{d,k}(n)$ (resp. $opr_{d,k}(n)$) the complexity of a projected image along a fixed direction (resp. along the optimal direction). In this paper, we investigate the theoretical bounds of $opr_{d,k}(n)$ and $fpr_{d,k}(n)$.

The reason we consider the complexities as functions of n is that S is often given by an input of size n. A Voronoi diagram of n points is a typical example. Another familiar example occurs in the diagnosis problem, which determines the name of a possible *disease* by using d health-check items. Given n non-intersecting convex sets (each set represents a disease) in d-dimensional space, we construct a convex subdivision of the space such that each region contains exactly one convex set. Given a point (*a patient*), we locate the point in the convex subdivision, and supply the name of the associated disease. Here, the number n of regions (i.e. diseases) is the only given discrete parameter of the problem.

We demonstrate the importance of the complexity of the projected images by showing an application to the analysis of point location data structures. Point location in a space subdivision is a major research problem in computational geometry [8, 9, 18, 23]. The efficiency of a data structure usually depends on the complexity of the images of the subdivision S projected into lower-dimensional spaces. For simplicity, we consider three-dimensional point location data structures.

Let us first recall the point location data structure of Dobkin and Lipton [9] (with slight modification). It precomputes the projected image $Pr(S; H)$ of the 1-skeleton of S onto the x-y plane H. For each region r of $Pr(S; H)$, the three-dimensional regions of S that are above r are stored in a list. When a query point p given, the region r containing the projected image of p in the subdivision $Pr(S; H)$ is found by using a planar point location algorithm. Next, the three-dimensional region of S containing p is searched for by using the list associated with r. If we use an optimal planar point location data structure (for example, [22]), the above method locates a point in S in $O(\log n)$ query time, and the space complexity required for the data structure is

$O(n\, fpr_{3,2}(n))$. The space complexity is improved to $O(fpr_{3,2}(n))$ by Cole [8] and Tan, Hirata and Inagaki [23] through the use of *similar list searching* and *persistent search tree* [22], respectively. Furthermore, we can choose a good projection plane H instead of the x-y plane (by a polynomial time exhaustive search on the *aspect graph* [16], or more cheaply by a sufficient number of random choices) and hence reduce the space complexity of the three-dimensional point location data structure to $O(opr_{3,2}(n))$.

Another popular method is the following: Project the vertices of S onto the z-axis L, and cut the z-axis into intervals at those projected points (in our terminology, compute $Pr(S, L)$). We consider a planar subdivision $S(h)$ obtained by cutting S with the horizontal hyperlane $z = h$. In each such interval I of $Pr(S; L)$, the combinatorial structure of $S(h)$ is independent of the choice of h. If we have a planar point location data structure that depends only on the combinatorial structure of the plane subdivision (roughly speaking), we obtain a point location data structure in S by giving such a planar point location data structure for each interval of $Pr(S; L)$. This scenario naively gives $O(\log^2 n)$ query time and $O(opr_{3,1}(n)n)$ space data structure if we use Lee–Preparata's planar point location [13] algorithm. (Note that $fpr_{3,1}(n) = O(opr_{3,1}(n))$). The space complexity was improved by Preparata and Tamassia [18] and Goodrich and Tamassia [11] to $O(opr_{3,1}(n) \log^2 n)$ and $O(opr_{3,1}(n) \log n)$ respectively.

Although the first method gives the optimal query time, we can only implement it if the system can afford to use $O(opr_{3,2}(n))$ memory space. If $opr_{3,2}(n) \gg opr_{3,1}(n) \log n$, and if we can use only $O(opr_{3,1}(n) \log n)$ memory space, we should choose the second method. We need to estimate $opr_{3,1}(n)$ and $opr_{3,2}(n)$ in order to decide which method should we use. The zero-skeleton of S is the set of the vertices of S, whose cardinality $K = \Theta(n^2)$. Therefore, $opr_{3,1}(n) = \Theta(n^2)$. Because of Euler's relation, the complexity of the one-skeleton of S is $\Theta(K)$. It is easy to see that $pr(S; H) = O(K^2)$ for a fixed projection plane H. Since K is $\Theta(n^2)$, a naive upper bound of $opr_{3,2}(n)$ is $O(n^4)$. On the other hand, it is not trivial to show a better lower bound than $\Omega(n^2)$ for $opr_{3,2}(n)$. In this paper, we first give a $\Theta(n^3)$ bound for $opr_{3,2}(n)$.

Projection is also related to the visualization of a convex subdivision of three-dimensional space. For example, imagine the visualization of a 3D-Voronoi diagram, which is a popular tool for simulating natural objects in solid state physics [24], amorphous state physics [4, 25], astrophysics, and other fields (see Okabe, Boots and Sugihara [15] for more applications of Voronoi diagrams).

A popular method of visualizing a three-dimensional object is to show an animation of the parallel view (the projected image) of the object while moving the view point (or equivalently, moving the object) continuously. From the point of view of computational geometry, visualization through animation is related to "dynamic computational geometry" [3], which estimates the number of changes in the combinatorial structure (often called *topological changes*) of $Pr(S; H)$ when we rotate S. We show a $\Theta(n^4)$ bound for the number of topological changes of the projected image when a three-dimensional subdivision is rotated about a line in the projection plane.

Next, we consider higher-dimensional cases, and give an upper bound on $fpr_{d,k}(n)$ and a lower bound on $fpr_{d,d-1}(n)$.

Finally, we discuss algorithms for computing the projected images.

2. Three-dimensional case

In this section, S is a convex subdivision of the three-dimensional space $\mathbb{R}^3$ into n polytopes, and H is a two-dimensional subspace of $\mathbb{R}^3$. The number of edges of S is denoted by K, $Pr(S; H)$ is the projected image of the 1-skeleton of S onto a plane H.

Theorem 2.1. *The number of vertices in* $Pr(S; H)$ *is* $O(nK)$.

Proof. The regions of S are numbered as Q_i, for $i = 1, 2, \ldots, n$. For each region Q_i, the boundary of the convex hull of the projected image of Q_i is called the *cap boundary* of Q_i and denoted by $C(Q_i)$ (Fig. 1), and the number of edges of $C(Q_i)$ is denoted by k_i. Every edge of $C(Q_i)$ is a projection of an edge of Q_i, and each edge contributes to at most two $C(Q_i)$, so $\sum_{i=1}^{n} k_i \leqslant 2K$. For each edge e in S, let $H(e)$ be the plane containing e and perpendicular to H. Because of the convexity, there exists a region Q containing e on its boundary that does not intersect $H(e)$. Evidently, the projected image of e is an edge of $C(Q)$. Therefore, every projected image of an edge is contained in the convex boundary of a projected image of a suitable region. Since $C(Q_i)$ and $C(Q_j)$ have at most $2\min(k_i, k_j)$ intersection points, there are at most $2\sum_{i,j=1}^{n}\min(k_i, k_j) \leqslant 2n\sum_{i=1}^{n} k_i \leqslant 4nK$ intersections in $Pr(S; H)$. □

Next, we give a lower bound. The proof of Theorem 2.1 shows that $Pr(S; H)$ is an arrangement of convex polygons $C(Q_i)$ $(i = 1, 2, \ldots, n)$, each of which contains $O(n)$ edges. Although two convex n-gons can intersect at $2n$ points, they normally intersect at many fewer points. Thus, only very special arrangements can have $\Omega(n^3)$ intersections, which means that each of $\Omega(n^2)$ pairs of polygons intersects at $\Omega(n)$ points. We

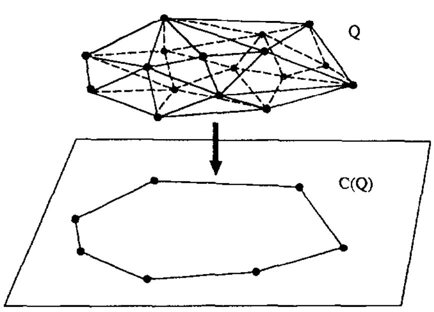

Fig. 1. Cap boundary.

show that such an arrangement can be attained as a projected image of a convex subdivision.

More formally, we give a lower bound for the complexity of the projection of the skeleton of a three-dimensional convex subdivision with K edges and n regions. Note that $K \leqslant n^2$, from the Dehn–Sommerville equation [10]. We specify a plane H in the space.

Theorem 2.2. *For any given K with $n^2 \geqslant K \geqslant n$, there exists a convex subdivision S of $\mathbb{R}^3$ satisfying the following conditions:* (1) *S has n convex polytopes,* (2) *S has* $O(K)$ *edges, and* (3) *the projection $Pr(S; H)$ of S on H has $\Omega(nK)$ vertices.*

Proof. We assume that H is the x-y plane. For a given arbitrary number $18n \leqslant k \leqslant n^2$, we set $m = k/6n$, and $s = (n - 2m)/2$. It is easy to see that $s > n/3$. Let P be a regular m-gon on H whose center is the origin. We construct an S with $O(k)$ edges, such that $Pr(S; H)$ has (1) $O(n)$ regular m-gons each of which is obtained by scaling P with (mutually different) scaling factors infinitesimally near to 1, and (2) $O(n)$ regular m-gons obtained by rotating the m-gons of (1) through π/m. Since each pair of an m-gon of (1) and one of (2) intersect at $2m$ points, there exist $O(n^2m) = O(nK)$ intersections in $Pr(S; H)$. We realize S as a Voronoi diagram [2] as follows:

We consider a unit circle C defined by $x^2 + y^2 = 1$ in the plane $z = \zeta$. $A_1 = \{(\cos 2\pi i/m, \sin 2\pi i/m, \zeta): i = 1, 2, \ldots, m\}$ is the set of vertices of a regular m-gon circumscribed by the circle C. Further, we consider a point set $B_1 = \{(0, 0, \zeta + j\delta): j = 0, 1, \ldots, s - 1\}$ consisting of s points on the z axis near the center of C (Fig. 2). Here, δ is a sufficiently small constant. We define the Voronoi diagram V_1 of $A_1 \cup B_1$. Recall that the Voronoi region of a point p is the region which is the intersection of half-spaces (containing p) defined by the perpendicular bisector of p and other points.

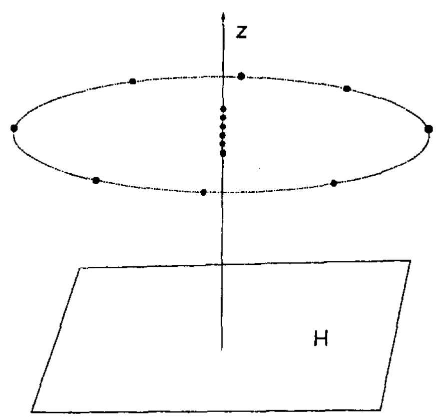

Fig. 2.

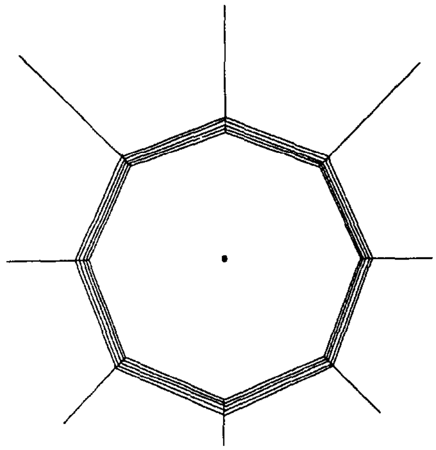

Fig. 3.

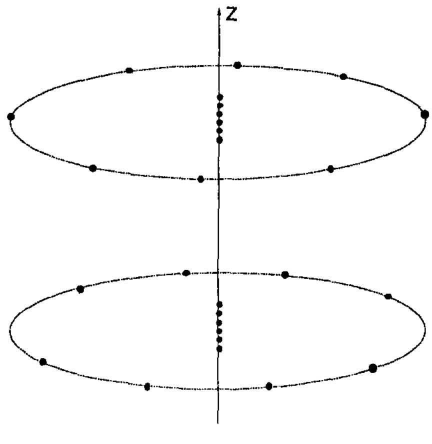

Fig. 4.

Thus, it is easy to see that the cap boundary (convex hull of the projected image) of each Voronoi region of a point of B_1 in V_1 is a regular m-gon that has the origin as its center. Further, each of these polygons can be transformed into another by a scaling transformation. If δ is small enough, the scaling factor can become larger than $\cos \pi/m$ (and smaller than 1) for each pair (Fig. 3).

Let A_2 (resp. B_2) be the point set obtained by first translating A_1 (resp. B_1) by $(0, 0, \tau)$ and then rotating it around the z-axis through an angle π/m.

The Voronoi diagram V_2 of $A_2 \cup B_2$ is congruent to V_1, although its projected image is rotated through π/m.

Now, we set $\tau \geqslant 4s\delta$, and consider the Voronoi diagram V of $P = A_1 \cup A_2 \cup B_1 \cup B_2$ (Fig. 4 shows the point set P).

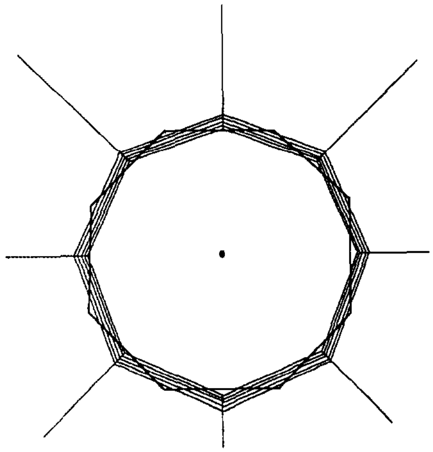

Fig. 5.

The cap boundary of the Voronoi region of a point of B_1 intersects at $2m$ points with that of any point of B_2 (Fig. 5). Since there are s^2 pairs of such cap boundaries, the total number of intersections is at least $2ms^2 \geqslant \frac{1}{27}nk$.

On the other hand, the Voronoi diagram V has n regions and $K = \frac{k}{9} + O(n)$ edges. Thus, we obtain the theorem. □

Corollary 2.3. $fpr_{3,2}(n) = \Theta(n^3)$.

More generally, the following holds.

Theorem 2.4. $opr_{3,2}(n) = \Theta(n^3)$.

Proof. We construct a convex subdivision S of the space into n convex regions such that its projected image onto any plane has $\Omega(n^3)$ vertices. We use the Voronoi diagram V of the point set P constructed in Theorem 2.2. We define a ball B of radius r that has the origin as its center. If r is large enough, B contains all the points of P. We consider four points u_1, u_2, u_3, and u_4 on the boundary sphere of B such that they form a regular tetrahedron. Further, we place u_1 on the z-axis. For each u_i, we define v_i by $v_i = 2u_i$ (i.e., u_i becomes the midpoint of the origin and v_i).

We define the Voronoi diagram $\tilde{V}$ of the union of P and $\{v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4\}$. If we discard the Voronoi regions of $v_i (i = 1, 2, 3, 4)$, we get a convex subdivision W of the regular tetrahedra defined by the bisecting planes of u_i and v_i $(i = 1, 2, 3, 4)$. We transform W by a linear transformation $(x, y, z) \to (x, y, \varepsilon z)$ into a convex subdivision W_ε, where ε is an arbitrary positive real number.

We define an angle $\theta(\varepsilon)$ such that the complexity of the projected image of W_ε to a projection plane is $\Omega(n^3)$ if the face angle between the projection plane and the x-y

plane is less than $\theta(\varepsilon)$. It is easy to see that if we choose ε small enough, we can make $\theta(\varepsilon)$ greater than a constant, say $\pi/8$.

Thus, there exists a family of a constant number of convex subdivisions obtained by rotating W_ε so that for any projection direction, at least one of them has a projected image of complexity $\Omega(n^3)$.

Each such subdivision is a subdivision inside a tetrehedron. Thus, we have a set of tetrahedra. We can easily obtain a convex subdivision of the space into a constant number of regions such that each of the given tetrahedra is embedded as a region of it by translation. In fact, let us take a point p in a tetrahedron, and define four points q_1, q_2, q_3, and q_4 to be the mirror images of p with respect to facets. The tetrahedron is then the Voronoi region of p in the Voronoi diagram of those five points. Let c be the number of tetrahedra in the set. We place the tetrahedra so that the distance between each pair is large enough. We define $5c$ points as above. Thus it is clear that the Voronoi diagram of the $5c$ points has each tetrahedron as a region.

Filling the interior of each tetrahedron with the rotated copy of W_ε, we obtain a convex subdivision of space into $\mathrm{O}(n)$ regions, so that the projected image along any direction has complexity $\Omega(n^3)$. □

3. Rotation and topological change

In this section, we investigate the topological change of $Pr_{3,H}(S)$ when S is rotated about a line (rotation axis). It is easy to obtain a $\Theta(n^2)$ complexity of the topological changes of projection image of a rotated convex polytope with n faces, since the topological change occurs when a projected vertex crosses a projected edge. However, $Pr_{3,H}(S)$ is a more complicated object than the projected image of a polytope.

We say that a rotation is parallel if the rotation axis is parallel to the projection plane. Otherwise, it is called a skew rotation. Let θ be the rotation angle ($0 \leqslant \theta \leqslant \pi$), and let S_θ be the associated rotated subdivision. For simplicity, we assume that no two facets of S are coplanar and that no two edges of S are located on the same plane perpendicular to the rotation axis. This assumption can be removed by a perturbation method. Then, a topological change occurs at values of θ where three projected edges (except ones sharing an endpoint in S) meet on H. Given any triplet of lines, the condition that the projected images of the lines intersect at a point is written as a trigonometric equation that is at most cubic with respect to $\sin\theta$ and $\cos\theta$. If the lines are located in a general position, the equation has at most six (two if the rotation is parallel) roots. Since there are $K = \mathrm{O}(n^2)$ edges, a naive bound of the number of topological changes is $\mathrm{O}(K^3) = \mathrm{O}(n^6)$. We improve this bound for a parallel rotation.

Theorem 3.1. *The number of topological changes is $\Theta(n^4)$ for a parallel rotation.*

Proof. First, we prove the upper bound. By applying a basis transformation, we can see that the combinatorial structure of the projected image of S_θ on H is the same as that of S on $H_{-\theta}$, where $H_{-\theta}$ is the plane obtained by rotating H through $-\theta$ about the rotation axis. We can assume that the rotation axis is on the projection plane, since the combinatorial structure of the projected image of S on H_θ is stable with respect to a translation of the rotation axis. Hence, we assume that the rotation axis is the x-axis, and that the projection plane H_θ is obtained by rotating the x-y plane about the x- axis through an angle θ.

Let p, q, and r be three points in $\mathbb{R}^3$. These points can be projected onto the same point on H_θ for an angle θ only if they are (1) located on a plane parallel to the y-z plane, and (2) colinear.

For an edge e of S, the projected image of e on the x-axis is an interval $I(e)$. Let e_1, e_2, and e_3 be edges of S, and let $I = I(e_1) \cap I(e_2) \cap I(e_3)$. Obviously, the projected images of these three edges never intersect at a point of $I = \emptyset$.

For $t \in I$, we consider the plane $\mathrm{Cut}(t)$ intersecting the x-axis orthogonally at $(t, 0, 0)$. Because of (2), the intersecting points of e_1, e_2, e_3 with $\mathrm{Cut}(t)$ must be colinear if these points are projected to the same point. In the interval I, the intersecting points of e_i ($i = 1, 2, 3$) with $\mathrm{Cut}(t)$ are written as vectors whose entries are linear functions with respect to t. The colinearity condition of three vectors $u(t)$, $v(t)$, and $w(t)$ in a plane is a quadratic equation $\det(u(t) - v(t), u(t) - w(t)) = 0$. Hence, there are at most two values of t such that these three points are colinear.

Therefore, the number of topological changes is bounded (within a constant factor) by the number of intersecting triplets among $O(n^2)$ intervals $\mathscr{I} = \{I(e)\colon e \in \mathrm{edge}(S)\}$ on a line, where $\mathrm{edge}(S)$ is the set of edges in S. Let $S(t)$ be the intersection of S with $\mathrm{Cut}(t)$. Obviously, $S(t)$ is a planar convex subdivision with $O(n)$ regions; thus it has $O(n)$ vertices. Thus, at most $O(n)$ intervals of $\mathscr{I}$ contain any given point. We count the intersecting triplets by sweeping the intervals from left to right. Let $x_1, \ldots, x_N$ be the set of endpoints of the intervals, where $N = O(n^2)$. Let k_i be the number of intervals that have x_i at their left endpoint. When we sweep through $x = x_i$, k_i intervals are newly inserted. Thus, $k_i n^2$ triples are newly created. Since $k_i = O(n)$ and $\sum_{i=1}^{N+1} k_i = O(n^2)$, the total number of triplets is $O(n^4)$.

Next, we consider the lower bound. We use the Voronoi diagram V defined in Theorem 2.2. The general idea of our construction is as follows: Let l be a line that is an infinitesimal translate of the x-axis (the rotation axis) on the projected plane. Assume that the example in Theorem 2.2 is placed sufficiently far from the projected plane. $\Omega(n^3)$ vertices (counted in the proof of Theorem 2.2) in the projected image survive if we give a rotation from $-a(n) < \theta < b(n)$, where $a(n)$ and $b(n)$ are small angles dependent on n. Since the location is very far, the projection of the trajectory of these $\Omega(n^3)$ vertices along this rotation intersects l. Thus, if we can create n copies of l in the projected image, we have $\Omega(n^4)$ topological changes (Fig. 6).

Let us give the precise construction. We adopt the definitions (C, A_i, B_i, ζ, τ, etc.) used in the proof of Theorem 2.2. We assume that the distance ζ between C and the origin is large enough, and that the distance τ between C and the other circle (on which A_2 is located) is small compared with ζ. Let us assume that the rotation axis is the x-axis. We define a set Y of n points on the y-axis, such that the maximal distance

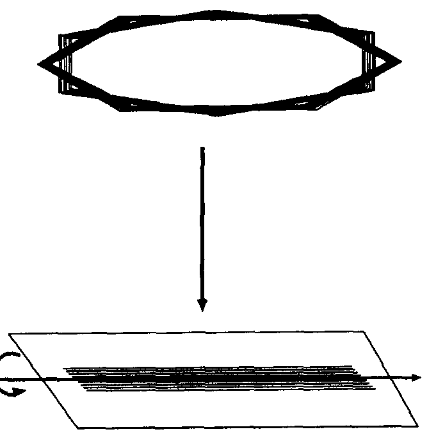

Fig. 6. Lower bound construction for topological change.

γ between a point of Y and the origin O is small enough. Let $\bar{V}$ be the Voronoi diagram of the point set $A_1 \cup A_2 \cup B_1 \cup B_2 \cup Y$. We consider the subdivision S that arises when the x-y plane is added to $\bar{V}$. Evidently, S is a convex subdivision consisting of $O(n)$ regions. Since ζ is large enough, almost all regions of V (the Voronoi diagram of $A_1 \cup A_2 \cup B_1 \cup B_2$) survive in S (actually, only the lower envelope of V is changed). We call this part $\tilde{V}$. There exists a maximal angle ϕ such that the topological structure of $Pr(\tilde{V})$ is not changed if we rotate $\tilde{V}$ by any angle between $-\phi$ and ϕ. This angle ϕ is independent of the distance ζ. On the other hand, S contains the set $\mathscr{L}$ of $n-1$ lines parallel to the x-axis, which are intersections of the x-y plane and the Voronoi boundary of the points of Y. The maximal distance between them is bounded by 2γ. We set ζ such that $\zeta \tan \phi > 2\gamma$. Then, during the rotation of S from $-\phi$ to ϕ, each of the $\Theta(n^3)$ vertices of $Pr(\tilde{V})$ meets the projected image of each of the $n-1$ segments of $\mathscr{L}$ at an angle. Thus, there are $\Omega(n^4)$ topological changes. $\square$

Any rotation is written as a product of the three rotations about the x-axis, y-axis, and z-axis. Obviously, the rotation about the z-axis causes no topological change in $Pr(S; H)$. Thus, a skew rotation for a given angle is represented as a product of parallel rotations. However, the bound in Theorem 3.1 might fail for the number of topological changes if the rotation is skew. It remains an open problem to obtain a nontrivial bound for the number of topological changes for skew rotation.

We remark that if we count the number of possible different topologies with respect to all three-dimensional rotations (that is, the orthogonal group of the space), as $O(K^6)$ upper bound can be obtained by using the argument of Plantinga and Dyer [16]. If we naively substitute $K = O(n^2)$ into this, we get a complexity $O(n^{12})$.

4. Higher-dimensional extension

In this section, we investigate convex subdivisions in higher-dimensional spaces and give upper bounds and lower bounds for the complexities of the projected images.

Let S be a convex subdivision of $\mathbb{R}^d$ into n polytopes. It is well known [10] that the worst-case complexity of S is $\Theta(n^{\lfloor (d+1)/2 \rfloor})$. We consider the projected image $Pr(S; L)$ of the $(k-1)$-skeleton of S onto a k-dimensional subspace L.

Any face of $Pr(S; L)$, except a k-dimensional face, is written as an intersection of projected images of faces of the $(k-1)$-skeleton of S. Let $A(e)$ be the set of the faces of the $(k-1)$-skeleton of S whose projected images contain a given face e of $Pr(S; L)$ in their relative interior. If there exists an element u of $A(e)$ such that e can be expressed as the intersection of the projected faces of $A - \{u\}$, e is called a degenerate face. The projection is called nondegenerate if there is no degenerate face in $Pr(S; L)$. It is easy to show the following lemma.

Lemma 4.1. *The complexity of $Pr(S; L)$ is asymptotically bounded by the number of vertices in $Pr(S; L)$ plus the number of original faces if the projection is nondegenerate.*

To obtain an upper bound of the complexity of $Pr(S; L)$, we can assume without loss of generality that the projection is nondegenerate. We use the following lemma given by Aronov, Bern and Eppstein [1].

Lemma 4.2. *The complexity of an arrangement of N convex polytopes with M facets in k-dimensional space is* $O(N^{\lceil k/2 \rceil} M^{\lceil k/2 \rceil})$.

Proof. Although a proof can be found in Aronov, Bern and Eppstein [1], we give a brief sketch of the proof for the reader's convenience. It suffices to count the number of vertices of the arrangement. Suppose that v is the interaction of k facets $h(1), \ldots, h(k)$. Let $P(i)$ be the convex polytope containing $h(i)$ as its facet. For simplicity, we assume that $P(i)$ are pairwise different (the other cases can be handled similarly). Then, it is easy to see that v is a vertex of the intersection $P(1) \cap \cdots \cap P(k)$. Let m_i be the number of facets in $P(i)$. The complexity of this intersection is $(\sum_{i=1}^{k} m_i)^{\lfloor k/2 \rfloor}$. If we sum up above number over all possible combinations of k polytopes, the complexity becomes $O(N^{\lceil k/2 \rceil} M^{\lceil k/2 \rceil})$. $\square$

Theorem 4.3. *Let f_i be the number of faces of dimension i of S. The complexity of $Pr(S; L)$ is* $O((f_{k+1})^{\lceil k/2 \rceil} (f_{k-1})^{\lfloor k/2 \rfloor})$.

Proof. Let P be a $(k+1)$-face of S. The cap boundary (the convex hull of the projected image) $C(P)$ of P is a convex polytope in $\mathbb{R}^k$. Let f be an arbitrary $(k-1)$-face of S. There exists a hyperplane h, which is orthogonal to the projection subspace L and contains f. It is easy to see that at least one $(k+1)$-face P bounded by f is located in one of the half-spaces defined by h. Obviously, $Pr(f)$ is contained in $C(P)$. In fact,

$Pr(f)$ is a facet of $C(P)$. Let $f(P)$ be the number of facets of $C(P)$. Since a $(k-1)$-face is contained in at most two cap boundaries, the sum of $f(P)$ over all of P is $O(f_{k-1})$ (recall that f_i denotes the number of i-faces in S).

Thus, $Pr(S; L)$ is an arrangement (in $\mathbb{R}^k$) of f_{k+1} convex polytopes, and the sum of the number of their facets is $O(f_{k-1})$. Thus, the theorem follows from Lemma 4.2. □

Corollary 4.4. *We have the following table for the upper bounds of* $fpr_{d,k}(n)$:

(1) *If* k *is even and* $k \geqslant \lfloor d/2 \rfloor + 2$, $O(n^{k(d-k+1)})$.
(2) *If* k *is even and* $k \leqslant \lfloor d/2 \rfloor$, $O(n^{k\lceil d/2 \rceil})$.
(3) *If* k *is even and* $k = \lfloor d/2 \rfloor + 1$, $O(n^{k(d-k+1)-k/2})$.
(4) *If* k *is odd and* $k \geqslant \lfloor d/2 \rfloor + 2$, $O(n^{k(d-(k+1)-1})$.
(5) *If* k *is odd and* $k \leqslant \lfloor d/2 \rfloor$, $O(n^{k\lceil d/2 \rceil})$.
(6) *If* k *is odd and* $k = \lfloor d/2 \rfloor + 1$, $O(n^{k(d-k+1)-(k+1)/2})$.

Proof. Since $f_i = O(n^{\min\{\lceil d/2 \rceil, d+1-i\}})$, the proposition follows from Theorem 4.3. □

We are especially interested in the projection of codimension one, (i.e. $k = d-1$).

Corollary 4.5. $fpr_{d,d-1}(n) = O(n^{2d-3})$ *if* d *is even, and* $fpr_{d,d-1}(n) = O(n^{2d-2})$ *if* d *is odd. Moreover,* $fpr_{d,d-1}(n) = O(n^d)$ if $d \leqslant 4$.

Proof. If $d > 4$, the corollary follows cases (1) and (4) of Corollary 4.4. If $d = 3$ and 4, cases (3) and (6) of Corollary 4.4 give $O(n^3)$ and $O(n^4)$ bounds respectively. □

This upper bound is better than the naive $O(n^{3(d-1)})$ bound of $fpr_{d,d-1}(n)$ by a factor of n^d or n^{d-1}. A naive lower bound of $fpr_{d,d-1}(n)$ is $\Omega(n^{d-1})$. We give a better lower bound below.

Theorem 4.6. $fpr_{d,d-1}(n) = \Omega(n^{\lfloor (3d-3)/2 \rfloor})$.

Proof. The lower bound construction is similar to that of Theorem 2.2. $Pr(S; H)$ is an arrangement of n convex polytopes. In our special $Pr(S; H)$, we have a set of cn convex polytopes containing the origin in their interior, and we cluster it into $d-1$ clusters containing $cn/(d-1)$ polytopes (c is a constant). Furthermore, if we choose an arbitrary polytope from each cluster, the component containing the origin of the interaction of the chosen $d-1$ polytopes contains $\Omega(n^{\lfloor (d-1)/2 \rfloor})$ vertices which are not contained in any intersection of $d-2$ polytopes. Thus, we have $\Omega(n^{\lfloor (d-1)/2 \rfloor})$ vertices for each of $(cn/(d-1))^{d-1}$ combinations, and consequently $\Omega(n^{\lfloor (3d-3)/2 \rfloor})$ pairwise distinct vertices in total. Instead of the regular m-gon in the proof of Theorem 2.2, we use the dual of the cyclic polytope [10] for the construction; indeed, we decompose the generating points of a cyclic polytope into $d-1$ clusters, and make $cn/(d-1)$ copies of the dual of the convex hull of each cluster. The following is the precise construction:

Let us consider the moment curve Γ: $x(t) = (t, t^2, t^3, \ldots, t^{d-1})$ of R^{d-1}. We consider a set $M = \{x(\tau_i)$: $i = 1, 2, \ldots, n\}$ of n points on Γ. We assume that $\tau_i < \tau_j$ if $i < j$. The convex hull of M is denoted by $C(M)$. It is well known that $C(M)$ has $\Omega(n^{\lfloor (d-1)/2 \rfloor})$ facets. The subset M_i, of M is defined by the set $\{x(\tau_j): j \equiv i \,(\text{mod } d-1)\}$. We cluster M into $d-1$ subsets $M_1, M_2, \ldots, M_{d-1}$.

Let us investigate the facets in detail. An index set $I = \{i_1, i_2, \ldots, i_{d-1}\}$ of size $d-1$ is called *special* if $I \subset \{1, 2, \ldots, n\}$ and $i_j = i_{j-1} + 1$ if j is even. Furthermore, we set $i_{d-1} = n$ if $d-1$ is odd. We define the function $f_I(t) = \Pi_{j \subset I}(\tau_j - t)$. Since the degree of $f_I(t)$ is $d-1$, from a similar argument to the one on p. 101 of Edelsbrunner [10], $f_I(t) = (u, x(t)) - v_0$ on Γ for suitable vectors u and v_0. We consider the function $F_I(x) = (u, x) - v_0$ defined on $\mathbb{R}^{d-1}$. It is easy to see that this function is nonnegative on M if I is special, and zero on $x(\tau_j)$ if $j \in I$. Thus, this function defines a hyperplane spanned by $\{x(\tau_j): j \in I\}$, and the hyperplane defines a facet of $C(M)$.

Thus, if we choose an arbitrary point p_j from M_j for each odd $j < n$, there exists at least a facet of $C(M)$ containing all those points on it. Hence, there are $\Omega(n^{\lfloor (d-1)/2 \rfloor})$ facets of $C(M)$, each of which is spanned by a point set containing exactly one point of each subset M_i $(i = 1, 2, \ldots, d-1)$.

Let $D(M)$ be the set of dual hyperplanes of M in $\mathbb{R}^{d-1}$, and let $D(C(M))$ be the dual of $C(M)$.

We choose a point x in the interior of $D(C(M))$. For each hyperplane h in $D(M)$, the point opposite to x with respect to h is denoted by $x(h)$. The point set $\{x(h): h \in D(M_i)\}$ is denoted by $\hat{M}_i$. Note that the Voronoi region of x in the Voronoi diagram of $\hat{M}_i$ is $D(C(M_i))$.

Let g be the d-th axis of $\mathbb{R}^d$. We choose the points $x_i = (0, 0, \ldots, 0, a_{i,d})$ $i = 1, 2, \ldots, d-1$ such that the distance between each pair of these points is sufficiently large. We consider the hyperplane L_i orthogonal to g and containing x_i. Now, embed a copy of the point set $\hat{M}_i$ in L_i so that x is translated to x_i. Precisely speaking, we map a point $p = (p_1, \ldots, p_{d-1})$ to $(p_1, p_1, \ldots, p_{d-1}, a_{i,d})$. Now, we have a set $S_{i,1}$ of $n/(d-1)$ points on L_i. Next, we generate a set $S_{i,2}$ of $n/(d-1)$ points on g that are infinitesimally near to x_i. Let us denote by V_i for the Voronoi diagram generated by the set $S_i = S_{i,1} \cup S_{i,2}$ of these $2n/(d-1)$ points. The Voronoi region of a point on g is called a central region. Two adjacent central regions are separated by a facet parallel to L_i, and the boundary of this facet is a magnified copy of $D(C(M_i))$ with a scaling factor infinitesimally near to 1. We call this facet a critical facet.

V is the Voronoi diagram of the union of point sets $\cup_{i=1}^{d-1} S_i$. Since S_i are far enough from each other, almost all critical facets of V_i remain facets in V. Those critical facets are classified into $d-1$ clusters so that critical facets corresponding to Voronoi regions of.points of S_i from a cluster for each $i = 1, 2, \ldots, d-1$.

We select $d-1$ critical facets, selecting a critical facet from each cluster. The intersection of the projected images of these $d-1$ critical facets has the same combinatorial structure as $D(C(M))$. There are $\Omega(n^{d-1})$ such combinations of $d-1$ central regions, and each combination creates $\Omega(n^{\lfloor (d-1)/2 \rfloor})$ vertices associated with the *special* indices defined above. A vertex corresponding to a *special* index is the intersection of the projected images of $d-1$ ridges of V (a ridge is a $(d-2)$ dimensional face). Thus, without loss of generality, we can assume the vertex is different from any other such vertex. Thus, there are $\Omega(n^{\lfloor (d-1)/2 \rfloor})$ vertices in the projected image of V. $\square$

From Theorem 4.4, the upper bound of Corollary 4.3 is tight if $d \leqslant 4$.

5. Algorithmic aspect

The proof of Theorem 4.3 gives an algorithm for computing $Pr(S; H)$, for a subspace H of codimension 1, that runs in $O(n^{2\lfloor (d-1)/2 \rfloor + d - 1})$ time when the optimal convex hull algorithm of Chazelle [6] or Seidel [21] is used. With more precise analysis, this algorithm runs in $O(n^3)$ or $O(n^4 \log n)$ time if the dimension is three or four, respectively. Let N be the number of faces of the projected image. Because $N = \Theta(n^3)$ if $d = 3$ and $N = \Theta(n^4)$ if $d = 4$, the above algorithms are optimal if $d = 3$ and nearly (within a $\log n$ factor) optimal if $d = 4$, since we need at least $\Omega(N)$ time to compute the projected images.

However, the output size N is usually much smaller than the worst-case size; thus, an efficient *output-sensitive* algorithm is desirable. The plane sweep method solves the problem in $O(N \log n)$ time if $d = 3$. Further, if we use the optimal segment-intersection reporting algorithm [7], an $O(N + K \log n)$ time algorithm can be designed, where K is the number of edges in S.

In the four-dimensional case, we give a space-sweep algorithm, which computes the projected image in $O(N \log n)$ time. As shown in the proof of Theorem 4.3, the projected image is an arrangement of n convex polyhedra in three-dimensional space. The number of faces of the polyhedra is $O(n^2)$ in total, although we showed in the previous section that the number N of faces of the arrangement is $\Theta(n^4)$. Let us consider the space sweep method for computing $Pr(S; H)$, where H is a three-dimensional subspace containing the x-axis. We consider the sweep plane $x = t$ orthogonal to the x-axis, and translate it from $t = -\infty$ to $t = \infty$. The intersection $\Sigma(t)$ of $Pr(S; H)$ with the plane $x = t$ is an arrangement of convex polygons. The complexity of $\Sigma(t)$ is $O(\min(n^3, N))$. For each edge e of $\Sigma(t)$, we compute the value of t at which the edge vanishes. For all such edges, we keep these values in a priority queue. We update this priority queue during the sweep. If the sweep comes to the abscissa of a vertex of S, more than one element of the priority queue may be updated. However, the total number of priority queue operations is $O(N)$ during the sweep. Therefore, the sweep method gives an $O(N \log n)$ time algorithm for computing $Pr(S; H)$.

For higher-dimensional cases, an output-sensitive algorithm for computing a convex hull in $O(n^2 + h \log n)$ time has been developed by Seidel [20], (h is the number of faces on the convex hull), and improved to $[O(n^{2-\frac{2}{1+\lfloor d/2 \rfloor}+\delta} + h \log n)]$ by Matoušek and Schwarzkopf [14]. Let k_i be the number of vertices of $Pr(S; H)$ that lie on the projected images of a $(d-2-i)$-dimensional face of S. If we apply Seidel's output-sensitive convex hull algorithm, we obtain a slightly output-sensitive algorithm. The time complexity is at most $O(n^{d+1} + \sum_{i=0}^{d-2} n^i k_i \log n)$. It is easy to see that $[k_i = O(n^{2\lfloor (d-1)/2 \rfloor + d - 1 - i})]$ and can be much smaller in practice.

6. Concluding remarks

We give nontrivial upper and lower bounds on the complexity of the projected images of convex subdivisions of d dimensional space into lower dimensional subspaces (especially, subspaces of codimension 1). The lower bounds on $fpr_{d,d-1}(n)$ is tight if $d \leqslant 4$. It remains an open problem to establish tight bounds for higher dimensional cases.

The lower bound examples are constructed by using Voronoi diagrams. Indeed, the lower bound of $fpr_{d,d-1}(n)$ (Theorem 4.6) holds for Voronoi diagrams. However, it remains open whether the lower bound of $opr_{3,2}$ (Theorem 2.4) holds for a Voronoi diagram or not. Moreover, it may be interesting to investigate the cases where the convex subdivision satisfies certain *fatness* conditions.

If $d = 3$, the theoretical complexity (neglecting a constant factor) of the projected image is independent of the choice of the projecting plane (Theorem 2.4). Nevertheless, it is a practically important open problem to devise an efficient algorithm for finding the projecting plane such that the complexity of the projected image is minimized.

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