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Implicit summation formulae for Hermite and related polynomials

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Abstract

In this article, we derive some implicit summation formulae for Hermite and related polynomials by using different analytical means on their respective generating functions.

Keywords: Hermite polynomials; Summation formulae; Multi-variable Hermite polynomials

1. Introduction and preliminaries

The 2-variable Hermite–Kampé de Fériet polynomials (2VHKdFP) $H_n(x, y)$ [2, p. 341 (23)] are defined as:

$$H_n(x, y) = n! \sum_{r=0}^{[n/2]} \frac{x^{n-2r} y^r}{(n-2r)!r!}. \tag{1.1}$$

In terms of classical Hermite polynomials $H_n(x)$ or $He_n(x)$ [1], it is easily seen from the definition (1.1) that

$$H_n(2x, -1) = H_n(x) \tag{1.2a}$$

and

$$H_n\left(x, -\frac{1}{2}\right) = He_n(x). \tag{1.2b}$$

Also, there exists the following close relationship [2, p. 341 (21)]:

$$H_n(x, y) = (-i)^n y^{n/2} H_n\left(\frac{ix}{2\sqrt{y}}\right) = i^n (2y)^{n/2} He_n\left(\frac{x}{i\sqrt{2y}}\right) \tag{1.3}$$

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with the classical Hermite polynomials. The usage of a second variable (parameter) y in the 2VHKdFP $H_n(x, y)$ is found to be convenient from the viewpoint of their applications. Indeed, from an entirely different viewpoint and considerations, Hermite polynomials of several variables are introduced and investigated by Erdélyi et al. [8, p. 283].

Recently, Dattoli [4] introduced the incomplete 2-variable 2-index 1-parameter Hermite polynomials (i2V2I1PHP) $h_{m,n}(x, y \mid \tau)$ specified by the series [4, p. 447 (1a)] (see also [6])

$$h_{m,n}(x, y \mid \tau) = m!n! \sum_{r=0}^{\min(m,n)} \frac{\tau^r x^{m-r} y^{n-r}}{r!(m-r)!(n-r)!}. \tag{1.4}$$

They are defined through the generating function

$$\exp(xu + yv + \tau uv) = \sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} h_{m,n}(x, y \mid \tau) \tag{1.5}$$

and are linked to the i2V2IHP

$$h_{m,n}(x, y) = m!n! \sum_{r=0}^{\min(m,n)} \frac{x^{m-r} y^{n-r}}{r!(m-r)!(n-r)!} \tag{1.6}$$

by the relation

$$h_{m,n}(x, y \mid \tau) = \tau^{(m+n)/2} h_{m,n}\left(\frac{x}{\sqrt{\tau}}, \frac{y}{\sqrt{\tau}}\right). \tag{1.7}$$

Also, the i2V2I1PHP $h_{m,n}(x, y \mid \tau)$ are linked to the associated Laguerre polynomials $L_n^{(\alpha)}(x)$ [1] by the relations

$$\begin{aligned} h_{m,n}(x, y \mid \tau) &= m!\tau^m x^{n-m} L_m^{(n-m)}\left(-\frac{xy}{\tau}\right), \quad n > m, \\ h_{m,n}(x, y \mid \tau) &= n!\tau^n y^{m-n} L_n^{(m-n)}\left(-\frac{xy}{\tau}\right), \quad m > n. \end{aligned} \tag{1.8}$$

If we take $\tau = -1$ and replace x and y by z and $\bar{z}$ respectively in relations (1.8), we get

$$\begin{aligned} h_{m,n}(z, \bar{z} \mid -1) &= (-1)^m m!(\bar{z})^{n-m} L_m^{(n-m)}(z\bar{z}) = L_{m,n}(I; z, \bar{z}), \\ h_{m,n}(z, \bar{z} \mid -1) &= (-1)^n n!(z)^{m-n} L_n^{(m-n)}(z\bar{z}) = L_{m,n}(I; z, \bar{z}), \end{aligned} \tag{1.9}$$

where $L_{m,n}(I; z, \bar{z})$ is a special case of Laguerre 2D polynomials $L_{m,n}(U; z, \bar{z})$ [12], which play an important role for the different representations of quasi-probabilities in quantum optics.

Again, taking $m = n$ and replacing τ by $-y$ in relations (1.8), we get

$$h_{n,n}(x, y \mid -y) = (-y)^n n! L_n(x), \tag{1.10}$$

where $L_n(x)$ are the ordinary Laguerre polynomials [1].

Also, we note the following special cases of i2V2I1PHP $h_{m,n}(x, y \mid \tau)$:

$$h_{m,0}(x, y \mid \tau) = x^m, \qquad h_{0,n}(x, y \mid \tau) = y^n, \tag{1.11}$$

$$h_{m,n}(x, y \mid 0) = x^m y^n, \tag{1.12}$$

$$h_{m,n}(x, 0 \mid \tau) = \begin{cases} n!\binom{m}{n} x^{m-n} \tau^n & \text{if } m \geqslant n, \\ 0 & \text{if } m < n, \end{cases} \tag{1.13}$$

and

$$h_{m,n}(0, y \mid \tau) = \begin{cases} m!\binom{n}{m} y^{n-m} \tau^m & \text{if } n \geqslant m, \\ 0 & \text{if } n < m. \end{cases} \tag{1.14}$$

The i2V2I1PHP $h_{m,n}(x, y \mid \tau)$ are particular cases of the more general family of complete multi-dimensional Hermite polynomials often exploited in applications concerning entangled harmonic oscillator states [7]. The possibility of developing the theory of complete 2D Hermite polynomials from the point of view of the incomplete forms is analyzed in Ref. [4].

It happens very often that the solution of a given problem in physics or applied mathematics requires the evaluation of infinite sums, involving special functions. Problems of this type arise, e.g., in the computation of the higher-order moments of a distribution or to evaluate transition matrix elements in quantum mechanics. In [5], Dattoli has shown that summation formulae of special functions, often encountered in applications ranging from electromagnetic processes to combinatorics, can be written in terms of Hermite polynomials with more than one variable.

Motivated by the work going in this direction, in this paper, we derive the implicit summation formulae for Hermite polynomials $H_n(x)$ and for i2V2I1PHP $h_{m,n}(x, y \mid \tau)$ by using different analytical means on their respective generating functions.

Just to give an idea of the procedure adopted here, we consider the generating function [10, p. 196]

$$\exp\bigl(2x(t+u)-(t+u)^2\bigr)=\sum_{k,l=0}^{\infty}\frac{t^k u^l}{k!\,l!}H_{k+l}(x), \tag{1.15}$$

which on separating the power series in r.h.s. into their even and odd terms by using the elementary identity

$$\sum_{n=0}^{\infty}\Phi(n)=\sum_{n=0}^{\infty}\Phi(2n)+\sum_{n=0}^{\infty}\Phi(2n+1) \tag{1.16}$$

becomes

$$\begin{aligned}\exp\bigl(2x(t+u)-(t+u)^2\bigr)={}&\sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!}\left(\sum_{l=0}^{\infty}\frac{u^{2l}}{(2l)!}H_{2k+2l}(x)+\sum_{l=0}^{\infty}\frac{u^{2l+1}}{(2l+1)!}H_{2k+2l+1}(x)\right)\\&+\sum_{k=0}^{\infty}\frac{t^{2k+1}}{(2k+1)!}\left(\sum_{l=0}^{\infty}\frac{u^{2l}}{(2l)!}H_{2k+2l+1}(x)+\sum_{l=0}^{\infty}\frac{u^{2l+1}}{(2l+1)!}H_{2k+2l+2}(x)\right).\end{aligned} \tag{1.17}$$

Now replacing t by it and u by iu in Eq. (1.17) and equating the real and imaginary parts of the resultant equation, we get the summation formulae

$$\exp\bigl((u+t)^2\bigr)\cos\bigl(2x(u+t)\bigr)=\sum_{k,l=0}^{\infty}\frac{(-1)^{k+l}t^{2k}u^{2l}}{(2k)!(2l)!}H_{2k+2l}(x)+\sum_{k,l=0}^{\infty}\frac{(-1)^{k+l}t^{2k+1}u^{2l+1}}{(2k+1)!(2l+1)!}H_{2k+2l+2}(x) \tag{1.18}$$

and

$$\exp\bigl((u+t)^2\bigr)\sin\bigl(2x(u+t)\bigr)=\sum_{k,l=0}^{\infty}\frac{(-1)^{k+l}t^{2k}u^{2l+1}}{(2k)!(2l+1)!}H_{2k+2l+1}(x)+\sum_{k,l=0}^{\infty}\frac{(-1)^{k+l}t^{2k+1}u^{2l}}{(2k+1)!(2l)!}H_{2k+2l+1}(x). \tag{1.19}$$

Further, taking $t \to 0$ in formulae (1.18) and (1.19), we obtain the well-known results for the Hermite polynomials $H_n(x)$ [9, p. 252]

$$\exp\bigl(u^2\bigr)\cos(2xu)=\sum_{l=0}^{\infty}\frac{(-1)^l u^{2l}}{(2l)!}H_{2l}(x) \tag{1.20}$$

and

$$\exp\bigl(u^2\bigr)\sin(2xu)=\sum_{l=0}^{\infty}\frac{(-1)^l u^{2l+1}}{(2l+1)!}H_{2l+1}(x). \tag{1.21}$$

In the following sections, we will see how the above results can be extended to more generalized forms of Hermite polynomials.

2. Implicit formulae involving Hermite polynomials $H_n(x)$

First we prove the following results involving Hermite polynomials $H_n(x)$:

Theorem 2.1. *The following implicit summation formula for Hermite polynomials $H_n(x)$ holds true*:

$$H_{k+l}(y) = \sum_{n,m=0}^{k,l} \binom{k}{n}\binom{l}{m}\big(2(y-x)\big)^{n+m} H_{k+l-n-m}(x). \tag{2.1}$$

Proof. We rewrite the generating function (1.15) as:

$$\exp\big(-(t+u)^2\big) = \exp\big(-2x(t+u)\big)\sum_{k,l=0}^{\infty} \frac{t^k u^l}{k!l!} H_{k+l}(x).$$

Replacing x by y in the above equation and equating the resultant equation to the above equation, we find

$$\sum_{k,l=0}^{\infty} \frac{t^k u^l}{k!l!} H_{k+l}(y) = \exp\big(2(y-x)(t+u)\big)\sum_{k,l=0}^{\infty} \frac{t^k u^l}{k!l!} H_{k+l}(x)$$

or

$$\sum_{k,l=0}^{\infty} \frac{t^k u^l}{k!l!} H_{k+l}(y) = \sum_{N=0}^{\infty} \frac{[2(y-x)]^N (t+u)^N}{N!} \sum_{k,l=0}^{\infty} \frac{t^k u^l}{k!l!} H_{k+l}(x), \tag{2.2}$$

which on using formula [11, p. 52 (2)]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n y^m}{n!m!}, \tag{2.3}$$

in the r.h.s. becomes

$$\sum_{k,l=0}^{\infty} \frac{t^k u^l}{k!l!} H_{k+l}(y) = \sum_{n,m=0}^{\infty} \frac{[2(y-x)]^{n+m} t^n u^m}{n!m!} \sum_{k,l=0}^{\infty} \frac{t^k u^l}{k!l!} H_{k+l}(x). \tag{2.4}$$

Now, replacing k by $k-n$, l by $l-m$ and using the lemma [11, p. 100 (1)]

$$\sum_{k,n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} A(n,k-n), \tag{2.5}$$

in the r.h.s. of Eq. (2.4), we find

$$\sum_{k,l=0}^{\infty} \frac{t^k u^l}{k!l!} H_{k+l}(y) = \sum_{k,l=0}^{\infty}\sum_{n,m=0}^{k,l} \frac{[2(y-x)]^{n+m} t^k u^l}{n!m!(k-n)!(l-m)!} H_{k+l-n-m}(x). \tag{2.6}$$

Finally, on equating the coefficients of like powers of t and u in Eq. (2.6), we are led to the assertion (2.1) of Theorem 2.1. □

Remark 1. By taking $l=0$ in Theorem 2.1, we immediately deduce the following consequence of Theorem 2.1.

Corollary 2.1. *The following implicit summation formula for Hermite polynomials $H_n(x)$ holds true*:

$$H_k(y) = \sum_{n=0}^{k} \binom{k}{n} 2^n (y-x)^n H_{k-n}(x). \tag{2.7}$$

Remark 2. On replacing y by $x+y$ in Eq. (2.7) we obtain [9, p. 255 (5.6.4)]

$$H_n(x+y)=\sum_{m=0}^{n}\binom{n}{m}(2y)^{n-m}H_m(x). \tag{2.8}$$

Theorem 2.2. *The following implicit summation formula for Hermite polynomials $H_n(x)$ holds true*:

$$\frac{(x)^{2k+\delta}}{(2k+\delta)!}H_{2k+\delta}(y)=\sum_{m=0}^{k}\frac{(y^2-x^2)^m(y)^{2k+\delta-2m}}{m!(2k+\delta-2m)!}H_{2k+\delta-2m}(x),\quad \delta\in\{0,1\}. \tag{2.9}$$

Proof. We rewrite Eqs. (1.20) and (1.21) as:

$$\exp\bigl(-t^2\bigr)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+\delta}}{(2k+\delta)!}H_{2k+\delta}(x)=\begin{Bmatrix}\cos(2xt), & \text{when } \delta=0\\ \sin(2xt), & \text{when } \delta=1\end{Bmatrix}.$$

Replacing t by yz in the above equation, we find

$$\exp\bigl(-y^2z^2\bigr)\sum_{k=0}^{\infty}\frac{(-1)^k (yz)^{2k+\delta}}{(2k+\delta)!}H_{2k+\delta}(x)=\begin{Bmatrix}\cos(2xyz), & \text{when } \delta=0\\ \sin(2xyz), & \text{when } \delta=1\end{Bmatrix}. \tag{2.10}$$

Again, replacing x by y and y by x in Eq. (2.10) and equating the resultant equation to Eq. (2.10), we find, after expanding the exponential in series

$$\sum_{k=0}^{\infty}\frac{(-1)^k(xz)^{2k+\delta}}{(2k+\delta)!}H_{2k+\delta}(y)=\sum_{k,m=0}^{\infty}\frac{(x^2-y^2)^m z^{2m}(-1)^k(yz)^{2k+\delta}}{m!(2k+\delta)!}H_{2k+\delta}(x),\quad \delta\in\{0,1\}. \tag{2.11}$$

Now, replacing k by $k-m$ and using Eq. (2.5) in the r.h.s. of the above equation and equating the coefficients of like powers of z, we get the assertion (2.9) of Theorem 2.2. □

Remark 1. Taking $k=1$ in Eq. (2.9), we get

$$\frac{(x)^{\delta+2}}{(\delta+2)!}H_{\delta+2}(y)=\frac{(y)^{\delta+2}}{(\delta+2)!}H_{\delta+2}(x)+\frac{y^{\delta}(y^2-x^2)}{\delta!}H_{\delta}(x),\quad \delta\in\{0,1\}. \tag{2.12}$$

Remark 2. Setting $x=\sqrt{2}y$ in Eq. (2.9), we get

$$\frac{(2)^{k+\frac{\delta}{2}}}{(2k+\delta)!}H_{2k+\delta}(y)=\sum_{m=0}^{k}\frac{(-1)^m}{m!(2k+\delta-2m)!}H_{2k+\delta-2m}(\sqrt{2}y). \tag{2.13}$$

Now, we prove the following results involving product of Hermite polynomials $H_n(x)$:

Theorem 2.3. *The following implicit summation formula involving product of Hermite polynomials $H_n(x)$ holds true*:

$$\frac{(\frac{x}{y})^l}{k!l!}H_{k+l}(X)=\sum_{n=0}^{k}\sum_{m=0}^{l-n}\frac{(\frac{x^2}{y^2}-1)^{m/2}(2(1-\frac{x}{y}))^n}{m!n!(k-n)!(l-m-n)!}H_m\left(X\sqrt{\left(\frac{x-y}{x+y}\right)}\right)H_{k+l-m-2n}(X). \tag{2.14}$$

Proof. After replacing x by X, we rewrite the generating function (1.15) as:

$$\exp\bigl(2X(u+t)-(u+t)^2\bigr)=\sum_{k,l=0}^{\infty}\frac{t^k u^l}{k!l!}H_{k+l}(X),$$

which on replacing u by yz and t by YZ becomes

$$\exp(2XYZ) = \exp\bigl(-2Xyz + (yz + YZ)^2\bigr) \sum_{k,l=0}^{\infty} \frac{(YZ)^k (yz)^l}{k!l!} H_{k+l}(X). \tag{2.15}$$

Again, replacing x by y and y by x in Eq. (2.15) and equating the resultant equation to Eq. (2.15), we find

$$\sum_{k,l=0}^{\infty} \frac{(YZ)^k (xz)^l}{k!l!} H_{k+l}(X) = \exp\bigl(2Xz(x-y) - z^2\bigl(x^2-y^2\bigr)\bigr) \exp\bigl(2z(y-x)YZ\bigr) \sum_{k,l=0}^{\infty} \frac{(YZ)^k (yz)^l}{k!l!} H_{k+l}(X). \tag{2.16}$$

By setting $T := z\sqrt{(x^2-y^2)}$ and $x' := X\sqrt{(\frac{x-y}{x+y})}$ in Eq. (2.16), the first exponential in the r.h.s. becomes the generating function of Hermite polynomials $H_n(x)$:

$$\exp\bigl(2x'T - T^2\bigr) = \sum_{m=0}^{\infty} \frac{T^m}{m!} H_m(x'). \tag{2.17}$$

Now, from Eqs. (2.16) and (2.17), we have

$$\begin{aligned}\sum_{k,l=0}^{\infty} \frac{(YZ)^k (xz)^l}{k!l!} H_{k+l}(X) &= \sum_{m=0}^{\infty} H_m\left(X\sqrt{\left(\frac{x-y}{x+y}\right)}\right) \frac{z^m (x^2-y^2)^{m/2}}{m!} \sum_{n=0}^{\infty} \frac{(2(y-x))^n z^n (YZ)^n}{n!} \\ &\quad \times \sum_{k,l=0}^{\infty} \frac{(YZ)^k (yz)^l}{k!l!} H_{k+l}(X).\end{aligned} \tag{2.18}$$

Finally, replacing l by $l-m-n$, k by $k-n$ and using Eq. (2.5) in the r.h.s. of Eq. (2.18) and then equating the coefficients of like powers of (YZ) and z, we get formula (2.14). □

Remark 1. By taking $l-n=N$ and $k=0$ in Theorem 2.3, we immediately deduce the following consequence of Theorem 2.3.

Corollary 2.2. *The following implicit summation formula involving product of Hermite polynomials $H_n(x)$ holds true*:

$$\left(\frac{x}{y}\right)^N H_n(X) = \sum_{m=0}^{N} \binom{N}{m} \left(\frac{x^2}{y^2} - 1\right)^{m/2} H_m\left(X\sqrt{\left(\frac{x-y}{x+y}\right)}\right) H_{N-m}(X). \tag{2.19}$$

Theorem 2.4. *The following implicit summation formula involving product of Hermite polynomials $H_n(x)$ holds true*:

$$\frac{(\frac{x}{y})^r (\frac{X}{Y})^s}{r!s!} H_r(y) H_s(Y) = \sum_{m=0}^{[r/2]} \sum_{n=0}^{[s/2]} \frac{(1-\frac{x^2}{y^2})^m (1-\frac{X^2}{Y^2})^n}{m!n!(r-2m)!(s-2n)!} H_{r-2m}(x) H_{s-2n}(X). \tag{2.20}$$

Proof. Consider the product of Hermite polynomials generating functions (2.17) in the following form:

$$\exp\bigl(-\bigl(2xt + 2XT + t^2 + T^2\bigr)\bigr) = \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s} t^r T^s}{r!s!} H_r(x) H_s(X), \tag{2.21}$$

which on replacing t by yz and T by YZ becomes

$$\exp\bigl(-\bigl(2xyz + 2XYZ + y^2z^2 + Y^2Z^2\bigr)\bigr) = \sum_{r,s=0}^{\infty} \frac{(-1)^{r+s} (yz)^r (YZ)^s}{r!s!} H_r(x) H_s(X). \tag{2.22}$$

Next, replacing x by y, y by x, X by Y and Y by X in Eq. (2.22) and equating the resultant equation to Eq. (2.22), we find, after expanding the exponentials in series

$$\sum_{r,s=0}^{\infty} \frac{(-1)^{r+s}(xz)^r(XZ)^s}{r!s!} H_r(y)H_s(Y)$$
$$= \sum_{r,m=0}^{\infty} \frac{(y^2-x^2)^m z^{2m}(-1)^r(yz)^r}{m!r!} H_r(x) \sum_{s,n=0}^{\infty} \frac{(Y^2-X^2)^n Z^{2n}(-1)^s(YZ)^s}{n!s!} H_s(X). \tag{2.23}$$

Finally, replacing r by $r-2m$, s by $s-2n$ and using the lemma [11, p. 100 (3)]

$$\sum_{r,m=0}^{\infty} A(m,r) = \sum_{r=0}^{\infty}\sum_{m=0}^{[r/2]} A(m, r-2m),$$

in the r.h.s. of Eq. (2.23) and then equating the coefficients of like powers of z and Z, we get formula (2.20). □

3. Implicit formulae involving i2V2I1PHP $h_{m,n}(x, y \mid \tau)$

We prove the following results involving i2V2I1PHP $h_{m,n}(x, y \mid \tau)$:

Theorem 3.1. *The following implicit summation formula involving i2V2I1PHP $h_{m,n}(x, y \mid \tau)$ holds true*:

$$\left(\frac{x}{y}\right)^m \left(\frac{X}{Y}\right)^n h_{m,n}(y, Y \mid \tau) = \sum_{M=0}^{\min(m,n)} M!\binom{m}{M}\binom{n}{M}\left(\tau\left(\frac{xX-yY}{yY}\right)\right)^M h_{m-M,n-M}(x, X \mid \tau). \tag{3.1}$$

Proof. We rewrite the generating function (1.5) as:

$$\exp(xt + XT + \tau tT) = \sum_{m,n=0}^{\infty} \frac{t^m T^n}{m!n!} h_{m,n}(x, X \mid \tau), \tag{3.2}$$

which on replacing t by yz and T by YZ becomes

$$\exp(xyz + XYZ + \tau yzYZ) = \sum_{m,n=0}^{\infty} \frac{(yz)^m(YZ)^n}{m!n!} h_{m,n}(x, X \mid \tau). \tag{3.3}$$

Next, replacing x by y, y by x, X by Y and Y by X in Eq. (3.3) and equating the resultant equation to Eq. (3.3), we find, after expanding the exponential in series

$$\sum_{m,n=0}^{\infty} \frac{(xz)^m(XZ)^n}{m!n!} h_{m,n}(y, Y \mid \tau) = \sum_{M=0}^{\infty} \frac{(\tau zZ)^M(xX-yY)^M}{M!} \sum_{m,n=0}^{\infty} \frac{(yz)^m(YZ)^n}{m!n!} h_{m,n}(x, X \mid \tau). \tag{3.4}$$

Now, we replace m by $m-M$ and n by $n-M$ in the r.h.s. of Eq. (3.4) to obtain

$$\sum_{m,n=0}^{\infty} \frac{(xz)^m(XZ)^n}{m!n!} h_{m,n}(y, Y \mid \tau)$$
$$= \sum_{m,n=0}^{\infty} (yz)^m(YZ)^n \sum_{M=0}^{\min(m,n)} \frac{1}{M!(m-M)!(n-M)!}\left(\tau\left(\frac{xX-yY}{yY}\right)\right)^M h_{m-M,n-M}(x, X \mid \tau),$$

which on equating the coefficients of like powers of z and Z yields formula (3.1). □

Theorem 3.2. *The following implicit summation formula involving i2V2I1PHP $h_{m,n}(x, y \mid \tau)$ holds true*:

$$h_{m,n}(y, x \mid \tau) = \sum_{M,N=0}^{m,n} \binom{m}{M}\binom{n}{N}(y-x)^M(x-y)^N h_{m-M,n-N}(x, y \mid \tau). \tag{3.5}$$

Proof. From generating function (1.5), we have

$$\exp(\tau uv)=\exp(-xu-yv)\sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}h_{m,n}(x,y\mid\tau). \tag{3.6}$$

Now, replacing x by y and y by x in Eq. (3.6) and equating the resultant equation to Eq. (3.6), we find, after expanding the exponentials in series

$$\sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}h_{m,n}(y,x\mid\tau)=\sum_{M=0}^{\infty}\frac{(y-x)^M u^M}{M!}\sum_{N=0}^{\infty}\frac{(x-y)^N v^N}{N!}\sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}h_{m,n}(x,y\mid\tau). \tag{3.7}$$

Again replacing m by $m-M$, n by $n-N$ and using Eq. (2.5) in the r.h.s. of Eq. (3.7), we find

$$\sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}h_{m,n}(y,x\mid\tau)=\sum_{m,n=0}^{\infty}\sum_{M,N=0}^{m,n}\frac{(y-x)^M(x-y)^N}{M!N!(m-M)!(n-N)!}h_{m-M,n-M}(x,y\mid\tau)u^m v^n, \tag{3.8}$$

which on equating the coefficients of like powers of u and v, yields the assertion (3.5) of Theorem 3.2. □

Remark 1. Taking $\tau=0$ in Eq. (3.5) and using Eq. (1.12), we get

$$\sum_{M,N=0}^{m,n}\binom{m}{M}\binom{n}{N}\left(\frac{y-x}{x}\right)^M\left(\frac{x-y}{y}\right)^N\left(\frac{x}{y}\right)^m\left(\frac{y}{x}\right)^n=1. \tag{3.9}$$

By using relations (1.8), the summation formulae (3.1) and (3.5) can be expressed in terms of the associated Laguerre polynomials $L_n^{(\alpha)}(x)$. Also, summation formulae (3.1) and (3.5) can be expressed in terms of the Laguerre 2D polynomials $L_{m,n}(I;Z,\bar{Z})$ and Laguerre polynomials $L_n(x)$ by using relations (1.9) and (1.10), respectively.

4. Concluding remarks

In the previous sections, we have derived several implicit summation formulae for Hermite polynomials $H_n(x)$ and for i2V2I1PHP $h_{m,n}(x,y\mid\tau)$ by using different analytical means on their respective generating functions. This process can be extended to establish summation formulae for more generalized forms of Hermite polynomials.

To give an example, we consider the 4-variable 2-index 1-parameter Hermite polynomials (4V2I1PHP) $H_{m,n}(x,z;y,w\mid\tau)$ defined as [3]:

$$H_{m,n}(x,z;y,w\mid\tau)=m!n!\sum_{s=0}^{\min(m,n)}\frac{\tau^s}{s!(m-s)!(n-s)!}H_{m-s}(x,z)H_{n-s}(y,w) \tag{4.1}$$

with the generating function

$$\exp\bigl(xu+zu^2+yv+wv^2+\tau uv\bigr)=\sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}H_{m,n}(x,z;y,w\mid\tau). \tag{4.2}$$

If we rewrite the generating function (4.2) as:

$$\exp\bigl(zu^2+wv^2+\tau uv\bigr)=\exp(-xu-yv)\sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}H_{m,n}(x,z;y,w\mid\tau),$$

and then following the lines of the proof of Theorem 3.2, we get the following implicit summation formula involving 4V2I1PHP $H_{m,n}(x,z;y,w\mid\tau)$:

$$H_{m,n}(y,z;x,w\mid\tau)=\sum_{M,N=0}^{m,n}\binom{m}{M}\binom{n}{N}(y-x)^M(x-y)^N H_{m-M,n-N}(x,z;y,w\mid\tau). \tag{4.3}$$

From Eqs. (4.2) and (1.5), we note that

$$H_{m,n}(x,0;y,0\mid\tau)=h_{m,n}(x,y\mid\tau). \tag{4.4}$$

Therefore, by taking $z=w=0$ in formula (4.3) and using Eq. (4.4), we get formula (3.5).

Also, from Eq. (4.1), we have

$$H_{m,n}(x,z;y,w\mid 0)=H_m(x,z)H_n(y,w). \tag{4.5}$$

Thus, by taking $\tau=0$ in formula (4.3) and using Eq. (4.5), we get the following summation formula involving product of 2VHKdFP $H_n(x,y)$

$$H_m(y,z)H_n(x,w)=\sum_{M,N=0}^{m,n}\binom{m}{M}\binom{n}{N}(y-x)^M(x-y)^N H_{m-M}(x,z)H_{n-N}(y,w), \tag{4.6}$$

which on taking $z=w=-1$ and replacing x by $2x$, y by $2y$ and using relation (1.2a), yields the following summation formula involving product of Hermite polynomials $H_n(x)$:

$$H_m(y)H_n(x)=\sum_{M,N=0}^{m,n}\binom{m}{M}\binom{n}{N}2^{M+N}(y-x)^M(x-y)^N H_{m-M}(x)H_{n-N}(y). \tag{4.7}$$

Similarly, taking $z=w=-\frac{1}{2}$ in Eq. (4.6) and using relation (1.2b), we get the following summation formula involving product of Hermite polynomials $He_n(x)$:

$$He_m(y)He_n(x)=\sum_{M,N=0}^{m,n}\binom{m}{M}\binom{n}{N}(y-x)^M(x-y)^N He_{m-M}(x)He_{n-N}(y). \tag{4.8}$$

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