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Functorial rings of quotients—III: the maximum in archimedean f-rings

Anthony W. Hager[a], Jorge Martinez[b,*]

[a] Department of Mathematics, Wesleyan University, Middletown, CT 06459, USA

[b] Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

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Abstract

The category of discourse is **Arf**, consisting of archimedean f-rings with identity and ℓ-homomorphisms which preserve the identity. Based on a notion of Wickstead, an f-ring A is said to be *strongly* ω_1*-regular* if for each countable subset $D \subseteq A$ of pairwise disjoint elements there is an $s \in A$ such that $d^2 s = d$, for each $d \in D$, and $xs = 0$, for each $x \in A$ which annihilates each $d \in D$. It is shown that strong ω_1-regularity is monoreflective in **Arf**; indeed, A is strongly ω_1-regular if and only if it is laterally σ-complete and has bounded inversion, if and only if A is von Neumann regular and laterally σ-complete. Recently the authors have characterized the category of laterally σ-complete archimedean ℓ-groups with weak unit as the epireflective class generated by the class of all laterally complete archimedean ℓ-groups. This, together with the above characterization of strong ω_1-regularity, leads to a description of the subcategory upon which the maximal functorial ring of quotients $\mu(Q)$ in **Arf** reflects. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

The seed for this investigation is already contained in our first two forays into the subject of functorial rings of quotients [10,11]. We have learned since then that, in a category like **Arf**, whose objects are the archimedean f-rings with identity, and morphisms are the ℓ-homomorphisms which are also ring homomorphisms that preserve the identity, general principles predict that there is a monoreflection which is the largest

* Corresponding author. Tel.: +1-352-392-0281-245; fax: +1-352-392-8357.

E-mail address: martinez@math.ufl.edu (J. Martinez).

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functorial ring of quotients. The reader is referred to [12,15]; very little of the categorical edifice of those papers need be recalled here, but we shall incorporate what is necessary for good reading. Suffice it to say for now that the relevant notion of the latter two papers is that, under certain "completion" operators, there always exists a largest monoreflection.

Along with the ideas referred to above, there is now a significant body of information about **W**, whose objects are the archimedean ℓ-groups with a designated weak order unit, and morphisms are the ℓ-homomorphisms which preserve the designated units, and about the monoreflections of **W**. Indeed, the reader should expect **W** to hover in the background throughout this article. The information about **W** most relevant to our concerns here is that Σ, the subcategory of all laterally σ-complete objects in **W**, is monoreflective [14], and that Σ is the least epireflective class containing the class of all laterally complete objects in **W** [19].

We proceed now to introduce the concepts which will be needed in the sequel, and to clarify those already mentioned in the preceding two paragraphs. It is understood that all rings in this article are commutative and possess an identity. A *semiprime ring* is one having no nilpotent elements but 0; as is well known, A is semiprime precisely when the intersection of all its prime ideals is zero.

A *lattice-ordered group* is a group G which has an underlying lattice structure so that $a \leqslant b$ implies that $a + c \leqslant b + c$, for each $c \in G$; all lattice-ordered groups are also assumed to be abelian. The lattice-ordered group G is *archimedean* if, for each $0 \leqslant a, b \in G$, $na \leqslant b$, for each positive integer n, implies that $a = 0$. It is well known that an archimedean lattice-ordered group is abelian anyway [5, 11.1.3]. We use the familiar abbreviation "ℓ-group" for "lattice-ordered group"; indeed, the prefix ℓ for words such as "homomorphism", "subgroup", etc., shall indicate that the lattice structure is preserved as well as the addition, multiplication, etc. An *f-ring* A is a ring which is (additively) an ℓ-group, so that whenever $a \wedge b = 0$, it follows that $a \wedge bc = 0$, for all $c \leqslant 0$ in A. An archimedean f-ring with identity is necessarily commutative, by [5, 12.3.2], as well as semiprime [5, 12.3.8]. Note as well that in a semiprime f-ring A $ab = 0$ precisely when $|a| \wedge |b| = 0$, which implies that the concepts of *minimal prime ideal* and *minimal prime convex ℓ-subgroup* agree in A.

We shall try, within reason, to explain all relevant concepts; for any unexplained terminology on ℓ-groups and f-rings we refer the reader to [1,5,8].

1. Strong forms of von Neumann regularity

Definition and Remark 1.1. (a) Let A be a ring. Recall that A is said to be *von Neumann regular* if for each $a \in A$ there is a $b \in A$ such that $a^2 b = a$. Following Wickstead [30], we say that A is *strongly regular* if for each subset D of A of pairwise annihilating elements and any partition $D = D_1 \cup D_2$ of D, there is a $b \in A$ such that $d^2 b = d$, for each $d \in D_1$, and $db = 0$, for each $d \in D_2$. Wickstead uses this notion to characterize the semiprime rings which are complete rings of quotients. We explain, presently.

Next, we need the concept of a ring of quotients, in the sense of Lambek [25]. Recall that if $A \leqslant B$ (that is, if A is a subring of B preserving the identity) we say that B is a *ring of quotients* of A if for each $b_1, b_2 \in B$, with $b_2 \neq 0$, there is an $a \in A$ so that $ab_1, ab_2 \in A$, with $ab_2 \neq 0$. Among the rings of quotients of A there is a maximum, denoted QA, which we refer to as the *complete ring of quotients* of A. We shall also say that A is a *complete ring of quotients* if $A = QA$.

In [30], Wickstead studies the connection between complete rings of quotients and self-injectivity. This is not relevant to our purposes. However, the strong regularity *per sé* is interesting, in that it is closely related to lateral completeness. Theorem 1.5 will give an account of this. Some preliminaries need go first, however.

(b) Let G be an ℓ-group. Recall that G is *laterally complete* (resp. *laterally σ-complete*) if every subset (resp. every *countable* subset) of pairwise disjoint elements of G has a supremum.

Recall that if $X \subseteq G$, then

$$X^{\perp} = \{g \in G\colon |g| \wedge |x| = 0 \text{ for each } x \in X\}.$$

The convex ℓ-subgroup C of G is a *polar* if $C = X^{\perp}$, for a suitable $X \subseteq G$; equivalently, if $C^{\perp\perp} = C$. It is well known that the set of polars of G is a complete boolean algebra under inclusion. We write $a^{\perp}$ for $\{a\}^{\perp}$.

Recall that G is *projectable* if $G = a^{\perp\perp} + a^{\perp}$, for each $a \in G$. G is *orthocomplete* if it is both laterally complete and projectable.

(c) Recall the following terminology: if G is an ℓ-subgroup of H we say that G is *essential in H* (or that H is *an essential extension of G*) if for each convex ℓ-subgroup C of H, $C \cap G = \{0\}$ implies that $C = \{0\}$. The embedding is *dense* if for each $0 < h \in H$ there is a $0 < g \in G$ such that $g \leqslant h$. It is clear that dense ℓ-subgroups are essential.

From a categorical point of view, H is an essential extension of G if and only if each ℓ-homomorphism $f: H \to K$ which is one-to-one when restricted to G is, in fact, one-to-one.

(d) If A is an f-ring, say that it has the *bounded inversion property* if $a \geqslant 1$ implies that a is invertible. The concept of bounded inversion probably first appeared in [22]; it is often attributed to Melvin Henriksen. Every full ring of real-valued continuous functions $C(X)$ on a topological space X has the bounded inversion property. Von Neumann regularity implies the bounded inversion property.

Associated with the classes of ℓ-groups and f-rings introduced in 1.1 are operators, which are hulls and, in some cases, functors. Let us review some of the ones which will rear their heads in this development.

Remark 1.2. (a) It is well known that each ℓ-group G has a unique *lateral completion* lG: lG is laterally complete, $G \leqslant lG$ is dense, and no proper ℓ-subgroup H of lG containing G is laterally complete (see [8, Section 39]. It is also well known that if G is an f-ring then so is lG; this appears as Theorem 4.6 of [6]. Similar remarks apply to the *lateral σ-completion* $G \leqslant l(\omega_1)G$; if one takes the time to read [6], and the material leading up to Theorem 4.6 of that paper, this becomes evident. In the archimedean case the reader is referred to [16]. Among other things found there (and elsewhere) is the fact, due to Bernau, that any laterally σ-complete archimedean ℓ-group is projectable, and hence orthocomplete.

For use further on, we introduce the following notation. If G is projectable and $a, b \in G$, then $a[b]$ stands for the projection of a on $b^{\perp\perp}$. In an f-ring which is projectable, note that $1[b]$ is always an idempotent.

(b) Likewise, there is an *orthocompletion* $G \leqslant oG$: oG is orthocomplete, contains G as an essential ℓ-subgroup, and no proper ℓ-subgroup H of oG contains G and is orthocomplete. As explained in [26], if G is a semiprime f-ring then oG is an f-ring; indeed, oG is an ℓ-subring of QG. Moreover, G is a complete ring of quotients if and only if G is orthocomplete and every regular element of G is invertible. (In a ring R, $0 \neq x \in R$ is *regular* if it is not a divisor of zero.) Note, in view of the comments in (a), that for archimedean ℓ-groups, $l = o$.

Finally, if we denote the classical ring of quotients of G by qG, then it is shown in [26], Theorem 2.4, that $QG = qoG$, for any semiprime f-ring G. If G is archimedean then the operators can be reversed, $QG = oqG$ [26, Corollary 2.7.1]. For the sake of completeness we record the well known fact from the theory of commutative rings that, for any R, $qR = R$ if and only if each regular element of R is invertible.

(c) Regarding the bounded inversion property, we recall from [10] that for each f-ring A we may form the ring of quotients

$$bA = \{a/d : a, d \in A \text{ and } d \geqslant 1\}.$$

bA is an ℓ-subring of qA, and $A \leqslant bA$ is the reflection of the category of semiprime f-rings in the subcategory of rings with the bounded inversion property. Although much more will be said about reflections in 4.1, let us specify what we mean for bA: for each ring ℓ-homomorphism $f : A \to B$, assuming that B satisfies the bounded inversion property, there is a unique ring ℓ-homomorphism $\hat{f} : bA \to B$ extending f, and, in fact, $\hat{f}(a/d) = f(a)f(d)^{-1}$, whenever $d \geqslant 1$.

As observed in [26], it is unknown whether $oqA = qoA$, for each semiprime f-ring A, although there is strong evidence to believe it is not so. The operators b and o do commute, however.

Theorem 1.3. *Let A be a semiprime f-ring. Then*
(a) *If A satisfies the bounded inversion property, then so does oA.*
(b) *If A is orthocomplete then bA is too.*
(c) $boA = obA$.
(d) *$obA = A$ if and only if A is orthocomplete and satisfies the bounded inversion property.*

Proof. (a) We recall what is necessary about the so-called Banaschewski representation (see [26]). oA may be constructed as follows: begin with a set of prime ℓ-ideals $\mathscr{P}$ for which $\cap\mathscr{P}=\{0\}$. Endow $\mathscr{P}$ with the hull-kernel topology. For each dense open set $\mathscr{V}$ in $\mathscr{P}$ we consider the ring of functions $A_{\mathscr{V}}$ consisting of all $f:\mathscr{V}\to\bigcup_{\mathfrak{p}\in\mathscr{V}} A/\mathfrak{p}$ such that $f(\mathfrak{p})\in A/\mathfrak{p}$ and for each $\mathfrak{p}\in\mathscr{V}$ there is an open neighborhood of $\mathfrak{p}$, $\mathscr{U}\subseteq\mathscr{V}$ and an $a\in A$, such that $f(\mathfrak{m})=\mathfrak{m}+a$, for each $\mathfrak{m}\in\mathscr{U}$. One then considers the $A_{\mathscr{V}}$, over all dense open $\mathscr{V}$ and forms the direct limit with restrictions as bonding maps. The limit is oA.

Now to the proof; assume A has the bounded inversion property. Suppose that $1\leqslant f\in oA$. Then over a dense open set $\mathscr{V}$ of $\mathscr{P}$, take f described as in the previous paragraph; we refer to the notation introduced there. Clearly, when $f(\mathfrak{m})=\mathfrak{m}+a$, with $a\in A$, for all $\mathfrak{m}$ in the neighborhood $\mathscr{U}$ of $\mathfrak{p}$, we may assume that $a\geqslant 1$; else replace it with $a\vee 1$. Thus, each such a is invertible in A. We can then define, in $A_{\mathscr{V}}$, the element g in the obvious way: for each $\mathfrak{p}\in\mathscr{V}$, $g(\mathfrak{m})=\mathfrak{m}+a^{-1}$, for each $\mathfrak{m}\in\mathscr{U}$. It is easy to check that this defines $g\in A_{\mathscr{V}}$ unambiguously, and that by doing this, compatibly, over the direct limit, the result is $g\in oA$. Evidently, g is the inverse of f.

(b) We imitate the proof of Theorem 2.5 in [26]. Start with $\{a_\lambda/d_\lambda\colon \lambda\in\Lambda\}$, pairwise disjoint, and with each $d_\lambda\geqslant 1$. As in the cited proof, one may replace each d_λ with $d_\lambda[a_\lambda]+(1-1[a_\lambda])$. Then $a_\lambda/d_\lambda=a_\lambda/(d_\lambda[a_\lambda]+(1-1[a_\lambda]))$, for each $\lambda\in\Lambda$. Taking suprema: $a=\bigvee_\lambda a_\lambda$, $x=\bigvee_\lambda d_\lambda[a_\lambda]$, and $e=\bigvee_\lambda 1[a_\lambda]$, it follows that $d=x+(1-e)\geqslant 1$ and that $a/d=\bigvee_\lambda a_\lambda/d_\lambda$. This shows that bA is laterally complete.

For f-rings with 1 it suffices, to prove projectability, to show that 1 has a projection on each $x^{\perp\perp}$, along $x^{\perp}$. But here this is clear, as $(a/d)^{\perp}=a^{\perp}$, for each fraction in qA, and A was assumed to be projectable. This proves (b).

(c) The embeddings $A\leqslant oA$ and $A\leqslant bA$ are always essential. By (a), $A\leqslant obA$ is an essential embedding into an orthocomplete f-ring with the bounded inversion property. Thus, obA contains a copy of oA, and, owing to the functorial feature of b, this embedding extends to an embedding $boA\leqslant obA$. On the other hand, since b is a reflection for which $X\leqslant bX$ is always essential, applying b to the extension $A\leqslant oA$ yields the essential embedding $bA\leqslant boA$ into an orthocomplete f-ring. Then, by the minimality of the orthocompletion, the extension $boA\leqslant obA$ must be surjective. □

We will prove in the next section that $bo=Q$ for archimedean f-rings. Without archimedeaneity this is false.

Example 1.4. *$boA\neq QA$, in general. Putting it differently, if $boA=A$, A need not be a complete ring of quotients.*

Consider the ring of formal power series in one variable $\mathbb{Z}[[T]]$, with integer coefficients. It is lexicographically ordered via

$$1\gg T\gg T^2\gg\cdots\gg T^n\gg\cdots.$$

In this situation the elements which are $\geqslant 1$ are those power series whose constant coefficients are $\geqslant 1$. As is well known, the inverse of such a power series is one with rational coefficients. The point is that in $b\mathbb{Z}[[T]]$ all elements are power series with rational coefficients. This is *not* $q\mathbb{Z}[[T]]$; for example, $b\mathbb{Z}[[T]]$ does not include the Maclaurin series

$$T^{-1} + 1 + T + \cdots .$$

The next theorem includes Wickstead's result that, for any semiprime ring $QA = A$ is equivalent to strong regularity. For f-rings there are a few new wrinkles.

Theorem 1.5. *For a semiprime f-ring A the following are equivalent*:
(a) *A is a complete ring of quotients.*
(b) *A is strongly regular.*
(c) *A is von Neumann regular and laterally complete.*
(d) *A is orthocomplete and each regular element is invertible.*

Proof. The equivalence of (a) and (b) is due to Wickstead. That (a) and (d) are equivalent is in [26], as has already been observed. We prove that (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

(b) $\Rightarrow$ (c): Obviously, strong regularity implies von Neumann regularity. Now, suppose that $\{a_\lambda\colon \lambda \in \Lambda\}$ is a pairwise disjoint set of positive elements in A. Expand this to a maximal pairwise disjoint set by adjoining $\{b_i\colon i \in I\}$. Then there is an $s \in A$ such that $sa_\lambda^2 = a_\lambda$, for each $\lambda \in \Lambda$, and $sb_i = 0$, for each $i \in I$. Without loss of generality, $s \geqslant 0$. Next, choose $t \geqslant 0$ such that $ts^2 = s$. The reader will then easily verify that $a \equiv t^2 s = \bigvee_\lambda a_\lambda$.

(c) $\Rightarrow$ (d): In any von Neumann regular ring the regular elements are invertible; this is well known. All that is left to do then is to prove that A is projectable. Now, if $a \in A$, then, as A is von Neumann regular, there is an idempotent $e \in A$ so that $Aa = Ae$. Thus, $A = Aa + A(1-e)$, and it is easy to see that $Aa = a^{\perp\perp}$, while $A(1-e) = a^{\perp}$. □

2. Representations and ringification

This is an expository section, in which we document the background for this paper on the Yosida and Henriksen–Johnson representations, and the concept of ringification of archimedean ℓ-groups. Our main references for the discussion are [20,21]. The reader might also have a look at [23].

Definition and Remark 2.1. Before getting started, we distinguish, between a *convex ℓ-subgroup* of an abelian ℓ-group, which is an ℓ-subgroup that is also order-convex, and an ℓ-ideal of an f-ring, which is a convex ℓ-subgroup and a ring ideal.

(a) Suppose that G is an archimedean ℓ-group with designated weak unit $u > 0$; that is, (G,u) is a **W**-object. (To say that u is a *weak unit* is to say that $u \wedge g = 0$ implies that $g = 0$.) $Y(G,u)$ stands for the *Yosida space* of u; that is $Y(G,u)$ consists of all the convex ℓ-subgroups which are maximal with respect to not containing u, and it bears the hull-kernel topology. $Y(G,u)$ is, thus, a compact Hausdorff space.

When the object is an f-ring A and the designated unit is the identity 1—that is, when A is an **Arf**-object—there is a canonical isomorphic copy of $Y(A,1)$, namely the space of all maximal ℓ-ideals m(A). For each maximal ℓ-ideal M there is a unique convex ℓ-subgroup $M' \in Y(A,1)$ containing M. The map $M \mapsto M'$ is a homeomorphism of $Y(A,1)$ onto m(A).

(b) Let X be a compact Hausdorff space. $D(X)$ denotes the set of all continuous functions f on X with values in the two-point compactification of the reals $\mathbb{R}$, with the additional stipulation that $f^{-1}\mathbb{R}$ is dense. It is well known that, in general, $D(X)$ is not a group or a ring under pointwise addition and multiplication, respectively. It is a lattice under pointwise suprema and infima.

One uses the term *ℓ-group in $D(X)$* for an ℓ-group $G \subseteq D(X)$, when in G each sum $k = g + h$ satisfies $k(x) = g(x) + h(x)$ on a dense subset of points of X. One uses a similar convention with the term *f-ring in $D(X)$*.

Here is a formulation of the representation theorems:

(i) The Yosida representation: *Suppose that G is an archimedean ℓ-group with weak unit $u > 0$. Then there is an ℓ-isomorphism ϕ from G onto an ℓ-group G' in $D(Y(G,u))$, carrying u onto the constant function* 1, *so that G' separates the points of $Y(G,u)$.*

(ii) The Henriksen–Johnson representation: *Suppose that A is an archimedean f-ring. Then there is a ring ℓ-isomorphism ϕ from A onto an f-ring A' in* $D($m$(A))$, *carrying the identity to the constant function* 1, *so that A' separates the points of* m(A).

Note that a subset $S \subseteq D(X)$ is said to *separate the points of X* if for each pair of distinct points x and y in X there is an $f \in S$ such that $f(x) \neq f(y)$. It is easy to show that the separation of points makes the spaces above unique (up to homeomorphism). Thus, the Henriksen–Johnson representation is a special case of the Yosida representation, and (ii) essentially says that the latter preserves the ring structure.

Now for the second part of this section: the passage from the category **W** to **Arf**.

Remark 2.2. There is an obvious "forgetful" functor from **Arf** to **W**, which ignores the ring structure on an archimedean f-ring. Let us denote this functor by φ. On the other hand, suppose that A is a **W**-object with positive designated unit u_A. Then there is an archimedean f-ring $r^f A$ and a **W**-embedding $r^f_A : A \to r^f A$ (so that $r^f_A(u_A) = 1$), such that if $f : A \to B$ is a **W**-morphism into the archimedean f-ring B, then there is a unique ring ℓ-homomorphism $\bar{f} : r^f A \to B$ so that $\bar{f} \cdot r^f_A = f$; this is proved in [20], and there is also a very readable discussion of this subject in the introduction to [21].

In fact, $r^f \cdot \varphi$ is the identity functor. Of course, r^f is the left adjoint of φ. What is implied by the above, and was explicitly shown by Conrad [7, 2.2] is that each **W**-object A admits at most one f-ring multiplication making the designated unit u_A the identity. In addition, a **W**-morphism between **Arf**-objects is multiplicative. Effectively, then **Arf** is a monoreflective (full) subcategory of **W**, and we shall think of it as such. The reflection r^f and each embedding $A \leqslant r^f A$ are referred to as *ringification*.

One more note: we have already remarked, in 1.2(a), that the lateral completion of an f-ring is an f-ring. In symbols, with **fRng** denoting the category of f-rings, we have l**fRng** $\subseteq$ **fRng**. The "reverse" fails. If **L** stands for the class of laterally complete objects in **W**, then $r^f\mathbf{L} \nsubseteq \mathbf{L}$. An example may be found in [21].

3. Strong ω_1-regularity and archimedean f-rings

We define the central concept of this article.

Definition and Remark 3.1. We say that A is *strongly* ω_1*-regular* if for each countable subset D of A of pairwise annihilating elements there is an $s \in A$ so that $d^2 s = d$, for each $d \in D$ and $xs = 0$ whenever $xD = 0$. It should be evident that strong regularity implies strong ω_1-regularity, which, in turn, implies von Neumann regularity.

The reader may observe that, given the concept of strong regularity, our notion of strong ω_1-regularity does not strike one as the most natural one to consider. We offer no notion of "ω_1-regularity", making it fair to ask why one should wish to carry the modifier "strong". Another possible definition of strong ω_1-regularity might be the following: for each countable subset D of A, and each partition of $D = D_1 \cup D_2$, there exists ...; the reader may complete the sentence. The one we have introduced is ostensibly stronger; more importantly, it does what we want. Whether the two are equivalent is another story; we do not know, but we have not really dwelt on the question either.

We now state the main theorem of this section; the proof depends on a lemma which follows some observations.

Theorem 3.2. *For a semiprime f-ring A the following are equivalent*:
(a) *A is strongly ω_1-regular.*
(b) *A is von Neumann regular and laterally σ-complete.*
If A is also archimedean then these are equivalent to
(c) *A is laterally σ-complete and has the bounded inversion property.*

Remark 3.3. That (a) implies (b) in Theorem 3.2 follows from the proof that (b) implies (c) in Theorem 1.5.

We should recall some facts prior to the proof of Theorem 3.2.

Definition and Remark 3.4. (a) The topological spaces which occur in this article will be *Tychonoff*: that is, Hausdorff, with the additional property that for each closed set K and each point p not in K, there is a continuous real-valued function g such that $g(p) = 1$ and $g(K) = \{0\}$.

Recall that, if $f \in C(X)$

$$\mathrm{coz}(f) = \{x \in X\colon\ f(x) \neq 0\}$$

is called the *cozeroset* of f. A space is Tychonoff if and only if it is Hausdorff and the cozerosets form a base for the open subsets. For this and for any unexplained topological terminology the reader should consult [9].

We shall also use the notation $\mathrm{coz}(f)$, as above, for a function $f \in D(X)$.

(b) Recall that a space X is *basically disconnected* if the closure of each cozeroset of X is open. It is shown in [13] that for any **W**-object (G, u) which is laterally σ-complete, $Y(G, u)$ is basically disconnected. This is true, in particular, for any laterally σ-complete archimedean f-ring A. It is well known that any compact basically disconnected space is *zero dimensional*; that is, that the clopen sets form a base for the open sets. If X is compact and zero dimensional, each cozeroset can be written as a countable disjoint union of clopen sets (see [27, 4.5]).

The following lemma is needed in the proof of Theorem 3.2. It is interesting in its own right.

Lemma 3.5. *Suppose that A is a laterally σ-complete archimedean f-ring which satisfies the bounded inversion property. Then A is von Neumann regular.*

Proof. According to 3.4(b), $\mathrm{m}(A)$ is basically disconnected. It suffices to prove the regularity condition for positive elements. So suppose that $a > 0$. We identify A with its image in $D(\mathrm{m}(A))$ under the Henriksen–Johnson representation. Write $\mathrm{coz}(a)$ as a disjoint union of clopen sets U_n $(n < \omega)$. Let e_n denote the characteristic function of U_n; it is shown in [13, 3.2 and 2.2(b)], that $ae_n \in A$. Next, put $c_n = ae_n + 1 - e_n$. Since U_n is compact, each c_n exceeds a nonzero constant function, and by the bounded inversion property must be invertible. Let $d_n = c_n^{-1}$, and form

$$d \equiv \bigvee_n d_n e_n;$$

this supremum exists on account of the lateral σ-completeness. We observe that d inverts a on each U_n and is zero off the closure of $\mathrm{coz}(a)$. The reader should then have no trouble verifying that $a^2 d = a$. □

Proof of Theorem 3.2. We have already observed how to get (a) ⇒ (b). As for the converse, it suffices (by taking positive and negative parts) to verify strong ω_1-regularity for positive elements. So, if $\{a_n\colon\ n < \omega\}$ is a set of pairwise disjoint elements, we may form $a = \bigvee_n a_n$. Next, pick $s \in A$ so that $a^2 s = a$ and set $c = s^2 a$. It must be shown that $a_n^2 c = a_n$, for each $n < \omega$, and that $xc = 0$, whenever $xa_n = 0$, for all $n < \omega$. We check the first part and leave the other to the reader.

If $a_n^2 c \neq a_n$, there is a minimal prime ideal such that $a_n(1 - a_n c) \notin P$, whence $a_n \notin P$. But since a_n is disjoint to $\bigvee_{m \neq n} a_m$, it follows that $a = a_n \mod P$. Furthermore,

$as \equiv 1 \bmod P$, whence

$$a_n c \equiv asas \equiv 1 \bmod P,$$

which is a contradiction.

Now assume that A is archimedean. As we have already observed, von Neumann regularity implies the bounded inversion property, it is clear that (b) implies (c). The converse is Lemma 3.5. □

To conclude the section, let us record a corollary of our results hitherto, characterizing in yet a different way the archimedean complete f-rings of quotients.

Corollary 3.6. Suppose that A is an archimedean f-ring. Then it is a complete ring of quotients if and only if it is laterally complete and has the bounded inversion property. Moreover,

$$bl = lb = Q,$$

for archimedean f-rings.

Proof. The necessity is clear from Theorem 1.5. The sufficiency follows from Lemma 3.5. The final statement is a consequence of Theorem 1.3 and the observation that $o = l$ for archimedean ℓ-groups. □

4. The strongly ω_1-regular reflection

The objective of this section is to show that the subcategory **SR** of **Arf** whose objects are the strongly ω_1-regular f-rings is monoreflective, and to describe the monoreflection in **SR**. We begin with the appropriate categorical preliminaries.

We assume that the reader is intuitively familiar with the notion of a category and associated concepts, such as monomorphism, epimorphism, functor, etc. Our basic reference will be [24]. Every subcategory here is assumed to be *full*: that is, if **B** is a subcategory of **A**, and $f : B \to C$ is an **A**-morphism between objects in **A**, then f is, in fact, in **B**.

We now review the basic features of epireflections.

Definition and Remark 4.1. (a) Let **A** be a category, and **B** be a subcategory of **A**. **B** is *reflective* in **A** (or a *reflective subcategory of* **A**) if for each **A**-object A there is a **B**-object rA and a morphism $r_a : A \to rA$ so that, for each **B**-object B and morphism $f : A \to B$ there is a unique morphism $f^* : rA \to B$ such that $f^* \cdot r_A = f$.

Intuitively, this should be a familiar concept; for amplification the reader should consult [24, Chapter X]. For now let us review some of the basic language associated with reflections. First, if **B** is reflective in **A**, then a functor emerges: we denote it by r, having already said in the definition what it does to objects; if $g: A_1 \to A_2$ is a morphism then $r(g)$ is the unique morphism (guaranteed by the definition) such that

$$r(g) \cdot r_{A_1} = r_{A_2} \cdot g.$$

The functor r is the actual *reflection*. In the language of adjoint functors, $r: \mathbf{A} \to \mathbf{B}$ is the left adjoint of the inclusion functor $j: \mathbf{B} \to \mathbf{A}$ [24, Section 27]. In this perspective r (with object subscripts) may also be viewed as a natural transformation from $1_{\mathbf{A}}$, the identity functor, to the composite $j \cdot r$. Each r_A is called a *reflection map*. r is called an *epireflection* (resp. *monoreflection*) if each reflection map is epic (resp. monic). Every monoreflection is an epireflection (24, 36.3]. If $r: \mathbf{A} \to \mathbf{B}$ is an epireflection (resp. monoreflection) we say that $\mathbf{B}$ is an *epireflective* (resp. *monoreflective*) subcategory of $\mathbf{A}$.

There is a well established literature on epireflections [24, Section 27]; we prefer to postpone a review of it until the next section.

(b) When considering composition of monoreflections there is a useful principle, which appears in [14, Proposition 2.7(b)]. Let us suppose that r and s are monoreflections on the subcategories $\mathbf{B}_r$ and $\mathbf{B}_s$, respectively. According to [14, Proposition 2.7(b)], if $s\mathbf{B}_r \subseteq \mathbf{B}_r$, then $s \cdot r$ monoreflects on $\mathbf{B}_r \cap \mathbf{B}_s$, with reflection maps $s_{rA} \cdot r_A$. Note that if the inclusion $r\mathbf{B}_s \subseteq \mathbf{B}_s$ also holds then r and s commute, because $r \cdot s$ and $s \cdot r$ reflect on $\mathbf{B}_r \cap \mathbf{B}_s$, and must therefore agree.

Next, let us recall from [18] some aspects of the laterally σ-complete reflection $\sigma: \mathbf{W} \to \Sigma$.

Proposition 4.2 (Hager and Martinez [18, Corollary 2.4]). *Let A be an* **Arf**-*object. Then the extension $A \leqslant \sigma A$ is an* **Arf**-*morphism. σ, restricted to* **Arf**, *monoreflects into the subcategory Σ_r of laterally σ-complete f-rings.*

It will also be useful to have the following description of σA, from [18, Section 4].

Remark 4.3. For the moment, let X be any Tychonoff space. $B(X)$ will denote the algebra of all real-valued Baire (measurable) functions. (Recall that a subset of X is *Baire measurable* if it lies in the boolean σ-algebra generated by the zerosets of X. Then $f \in \mathbb{R}^X$ is *Baire* if the inverse image of any open interval in $\mathbb{R}$ is Baire measurable.)

Next, suppose that A is an archimedean f-ring, and let $Y = \mathrm{m}(A)$. We consider the subalgebra $B_{\omega,A}(Y)$ of $B(Y)$ defined as follows: $f \in B_{\omega,A}(Y) \Leftrightarrow f \in B(Y)$ and there is a countable partition of Y, $Y = \bigcup_n Y_n$, into Baire measurable sets Y_n, and there exists a sequence a_n of elements of A such that $f|_{Y_n} = a_n|_{Y_n}$, for each $n < \omega$. Then

$$\sigma A = \varinjlim B_{\omega,A}(U),$$

where U ranges over the members of $(A^{-1}\mathbb{R})_\delta$, the filter base of all countable intersections of sets in $A^{-1}\mathbb{R}$. In this direct limit the bonding maps are restrictions: with $U, V \in (A^{-1}\mathbb{R})_\delta$, and $V \subseteq U$, $B_{\omega,A}(U) \to B_{\omega,A}(V)$ is the restriction of a function to V.

Let us denote by **BI** the monoreflective subcategory of **Arf** consisting of all f-rings with the bounded inversion property; the reflection $b:\mathbf{Arf} \to \mathbf{BI}$ has been described in 1.2(c). We are now able to state the main result of this section. The proof will be accomplished through two lemmas.

Theorem 4.4. *In* **Arf**, $b \cdot \sigma = \sigma \cdot b$, *and*

$$\rho \equiv b \cdot \sigma$$

reflects onto the subcategory $\mathbf{SR} = \mathbf{BI} \cap \Sigma_r$.

The keys to the proof of Theorem 4.4 are the two lemmas which follow, together with 4.1(b).

Lemma 4.5. $\sigma\mathbf{BI} \subseteq \mathbf{BI}$; *that is, if A satisfies the bounded inversion property then so does σA.*

Proof. We use the characterization of σA outlined in 4.3. Suppose $f \geqslant 1$ in σA. Then, modulo the equivalence relation of identification in a direct limit, we may view the situation as follows: there exists a partition of $\mathrm{m}(A)$ by Baire sets Y_n $(n < \omega)$ and $a_n \in A$ $(n < \omega)$ such that $f|_{Y_n} = a_n|_{Y_n}$. Without loss of generality (by changing to $a_n \vee 1$, if necessary), we may assume that each $a_n \geqslant 1$. Since A possesses the bounded inversion property, we have $b_n = a_n^{-1}$ in A, for each positive integer n. Define g by $g|_{Y_n} \equiv b_n|_{Y_n}$; according to the remarks of 4.3 it should be evident that $g \in \sigma A$ and also that $fg = 1$. □

Lemma 4.6. $b\Sigma_r \subseteq \Sigma_r$; *that is, if A is a laterally σ-complete* **Arf***-object then bA is again laterally σ-complete.*

Proof. This is a special case of the proof of Theorem 1.3(b), restricted to countable pairwise disjoint sets. □

Proof of Theorem 4.4. That $\mathbf{SR} = \mathbf{BI} \cap \Sigma_r$ is part of Theorem 3.2. Apply the preceding lemmas and the comment in 4.1(b). □

5. SR from complete rings of quotients

The goal in this section is to incorporate the material from [17], describing Σ as the epireflective subcategory generated by the class of laterally complete **W**-objects, in order to obtain **SR** as the epireflective subcategory generated by the complete rings of quotients in **Arf**.

First, some general comments about epireflective subcategories are in order; we refer the reader to [24, Section 37]. We model what follows on our account in [19, Theorem 2.1].

Definition and Remark 5.1. Let **A** be a category which is cowellpowered, has products, each morphism f has an essentially unique factorization $f = m \cdot e$, with e epic and m extremal monic, and suppose that the composition of extremal monics is extremal monic. (We leave the interested reader to consult [24] for most of these technical conditions. We explain the label "extremal" below.) The characterization of epireflective subcategories that we need here is contained in [24, Theorem 37.2]. We also incorporate into the formulation below the notion of a least epireflective class containing any collection of objects; for this we refer the reader to [24, 37.5 and 37.6].

The subcategory **B** *of* **A** *is epireflective if and only if it is closed under products and extremal subobjects. Moreover, each class of objects* **C** *is contained in a least epireflective subcategory* $\mathrm{R}(\mathbf{C})$ *of* **A**. $B \in \mathrm{R}(\mathbf{C})$ *if and only if* B *is an extremal subobject of a product of objects in* **C**.

Let us explain *extremal subobjects*. Let $m : A \to B$ be a monomorphism. We say that m is *extremal* if for any factorization $m = h \cdot e$, with e epic, it follows that e is an isomorphism. If m is an extremal monomorphism then we also say that A is an *extremal subobject of* B. The problem with extremal subobjects is that, typically, they are difficult to characterize. In any case, here is at least one common way one might encounter them. If $h \cdot m = 1_A$, with $m : A \to B$, then m is extremal. Such morphisms are called *sections*.

Note that if a class **C** of **A**-objects is already product closed, then

$$\mathrm{R}(\mathbf{C}) = \{A\text{: } A \text{ is an extremal subobject of some } C \in \mathbf{C}\}.$$

The following is one of the main accomplishments of [19].

Theorem 5.2. *Every laterally σ-complete* **W**-*object is an extremal subobject of a laterally complete one. Thus,* $\mathrm{R}(\mathbf{L}) = \Sigma$, *where* **L** *denotes the (product closed) class of all laterally complete* **W**-*objects.*

A careful reading of the proof of Theorem 5.2 will reveal this:

Theorem 5.3. *Every strongly ω_1-regular* **Arf**-*object is an extremal subobject of a complete ring of quotients in* **Arf**. *Thus,* $\mathrm{R}(\mathbf{Q}) = \mathbf{SR}$, *where* **Q** *stands for the (product closed) class of archimedean complete rings of quotients.*

Proof. It is well known that **Q** is product closed; see [25]. In view of Theorems 1.5 and 3.2 it is clear that an extremal subobject of a complete ring of quotients is laterally σ-complete and has bounded inversion, and therefore strongly ω_1-regular. To complete the proof then it is the first claim that must be established.

Now suppose that A is strongly ω_1-regular. The relevant portion of the proof of Theorem 5.2 goes like this. One looks for P-sets of $\mathrm{m}(A)$; recall that a *P-set T* of a Tychonoff space X is one which has the feature that any real-valued continuous function f which vanishes on T also vanishes on a neighborhood of T. Let T be such a P-set. The point is that, first, the restriction map $g \mapsto g|_T$ is a surjective ℓ-homomorphism of A onto the **W**-object $A|_T$ of all such restrictions. In our context, since A is a ring, so is $A|_T$. It is not hard to see that $A|_T$ also has the bounded inversion property.

Next, one passes to L_T, the lateral completion of $A|_T$; this is now also an f-ring. Owing to Corollary 3.6, L_T inherits the bounded inversion property. Finally, it is shown in [19] that there are enough nonempty P-sets so that the induced map $A \to \prod_T L_T$ is an embedding, and extremal. Obviously, $\prod_T L_T$ satisfies the bounded inversion property as well.

To summarize, each strongly ω_1-regular **Arf**-object A is an extremal subobject in **W** of a laterally complete archimedean f-ring B with the bounded inversion property. By Theorem 1.5, B is a complete ring of quotients. The only comment that need to be added to this is that all the morphisms in these arguments preserve multiplication, so that A is, indeed an extremal subobject of B in **Arf**. □

6. The maximum *roq*-functor on Arf

In [19] the maximum monoreflection in **W** beneath the lateral completion is described. Here we use similar ideas to get at the maximal *roq*-functor in **Arf**.

We begin with a review of maximum monoreflections beneath a completion. We give the definition of a completion, for the record, but refer the reader to the discussions in [12,15], and (of course) also to [24] for any unexplained technical terms.

Definition and Remark 6.1. Suppose that **A** is a category. A *completion* is an operator γ which assigns to each **A**-object A an object γA along with a monomorphism $\gamma_A : A \to \gamma A$, so that if $\gamma_A = f \cdot g$, with $g : A \to B$ and $f : B \to \gamma A$, and f monic and g epic, then there is an isomorphism $h : \gamma B \to \gamma A$ such that $h \cdot \gamma_B = f$. We say that a completion γ is *idempotent* if each $\gamma_{\gamma A}$ is an isomorphism. Obviously, monoreflections are idempotent completion operators.

Next, we define the partial ordering for monoreflections. Suppose that r and t are monoreflections of **A** into subcategories $\mathbf{B}_r$ and $\mathbf{B}_t$, respectively. We say that $r \leqslant t$ if, for each **A**-object A, there is a monomorphism $m_A : rA \to tA$ so that $m_A \cdot r_A = t_A$. Since a monoreflection is necessarily an epireflection, it follows that $r \leqslant t$ and $t \leqslant r$ imply that $r = t$, and that $\leqslant$ defines a partial ordering. Observe that $r \leqslant t$ if and only if $\mathbf{B}_t \subseteq \mathbf{B}_r$.

Suppose now that γ is a completion operator. We consider the monoreflections r such that, for each **A**-object A, there is a monomorphism $m_A : rA \to \gamma A$ such that $\gamma_A = m_A \cdot r_A$. Evidently, this extends the definition in the preceding paragraph. The maximum monoreflection in this class (if it exists) will be denoted by $\mu(\gamma)$. We refer to it as the *maximum monoreflection beneath* γ. The following theorem is proved in [12], as Theorem 4.2. We are purposely vague about the hypotheses on the category; it is up to the reader to check the reference. The hypotheses are satisfied in both **W** and **Arf**. In any event, the existence of these maxima is not an issue in this paper:

Suppose that **A** *is a category subject to certain constraints. Then, for any completion* γ, $\mu(\gamma)$ *exists.*

Ref. [19] is dedicated to describing $\mu(l)$ in **W**. We are about to do the same with $\mu(Q)$ in **Arf**.

Next, we summarize the first section of [19], on maximum monoreflections under an essential completion.

Definition and Remark 6.2. (a) First, recall that a monomorphism $m: A \to B$ of the category **A** is *essential* if $g: B \to C$ is monic whenever $g \cdot m$ is monic. If m is essential we also say that B is an *essential extension of* A, or that A is an *essential subobject* of B. This general definition specializes to the one given earlier (1.1(c)) for **W**.

A monoreflection r is called *essential* if each reflection map r_A is essential. The subcategory into which r reflects is said to be *essentially reflective*. Likewise, a completion γ is *essential* if each γ_A is essential. There is no reason for a first factor of an essential extension to be essential. We say that a completion (resp. monoreflection) γ is *restrictably essential* if, for each **A**-object A and each factorization $\gamma_A = f \cdot g$, with f monic, g is necessarily essential.

(b) Let us assume that γ is a restrictably essential idempotent completion, and denote by **C** the class of γ-closed objects; that is, $A \in \mathbf{C}$ if and only if γ_A is an isomorphism. Evidently, **C** is the range of γ. We also assume that **A** possesses a maximum essential monoreflection, which we denote by ε; the range of ε is denoted by **E**. Suppose as well that **A**, has a maximum monoreflection, denoted by β. We now recite [17, Theorem 1.1]:

Let $\bar{\gamma}$ *stand for the monoreflection into* $\mathrm{R}(\mathbf{C})$ *and* $\mu(\gamma)$ *for the monoreflection into* $\mathrm{R}(\mathbf{C} \cup \mathbf{E})$. *Assume in addition that* **A** *has intersections. Then we have*

(i) $\mathrm{R}(\mathbf{C} \cup \mathbf{E})$ *is the least essentially epireflective subcategory containing* **C**.
(ii) $\mu(\gamma)$ *is the largest monoreflection beneath* γ.
(iii) $\mu(\gamma)A = \varepsilon A \cap \bar{\gamma}A \leqslant \beta A$.
(iv) *In the lattice of monoreflections on* **A**, *we have* $\mu(\gamma) = \varepsilon \wedge \bar{\gamma}$.

Note that $\mathrm{R}(\mathbf{C})$ is, indeed, monoreflective, as each **A**-object A may be embedded in the object γA in $\mathbf{C} \subseteq \mathrm{R}(\mathbf{C})$.

We are almost to the highlight of this section. Let us first recall for the reader what the maximum essential monoreflection looks like in **Arf**.

Remark 6.3. Suppose that G is a **W**-object. We denote by c^3G the following direct limit:

$$c^3G = \lim_{\rightarrow} C(U),$$

where U ranges over $(G^{-1}\mathbb{R})_\delta$, and once again the bonding maps are restrictions. (Note: as before, we identify G with its image under the Yosida representation.) It is shown in [4], Theorem 9.2, that c^3 is the maximum essential monoreflection in **W**. c^3G is often called the *closure of A under countable composition*. The notion of countable composition was introduced in [22]; the closure itself first occurs in [2,3].

Now, since c^3G is always a ring, we get that c^3 majorizes the ringification functor r^f. Thus, restricted to **Arf**, c^3 is the maximum essential monoreflection in that category. As set out in 6.2(b), we use ε for c^3, acting on **Arf**, and the range of ε is denoted by **E**. Observe that, per 6.2(b),

$$\mu(l) = \varepsilon \wedge \sigma \quad \text{(see [19])}.$$

The main result of this section reads as follows. Recall that **Q** stands for the class of complete rings of quotients in **Arf**.

Theorem 6.4. *In* **Arf**,

$$\mu(Q) = \varepsilon \wedge \rho$$

and $\mu(Q)$ *reflects into* R(**E**∪**Q**), *which is the least essentially epireflective subcategory containing all complete rings of quotients in* **Arf**. $\mu(Q)$ *is also the largest roq-functor in* **Arf**.

Proof. To apply our result from [17] (6.2(b) above) all we need do is verify that $A \leqslant QA$ is a restrictably essential, idempotent completion. That it is a completion and idempotent is well known. On the other hand, suppose B is any ring of quotients of A, and f is any ring homomorphism out of B whose restriction to A is one-to-one. If $f(b) = 0$, while $b \neq 0$, then (by definition of ring of quotients) there is an $a \in A$ so that $ab \in A$ and is *not zero*. But then $f(ab) = f(a)f(b) = 0$, which contradicts our assumption.

Knowing that the operator Q is a restrictably essential, idempotent completion allows us to invoke 6.2(b). Let us recite: **Q** is the range of Q; **Arf** certainly has intersections; and as we pointed out in 6.3, ε is the largest essential monoreflection. □

In view of Theorems 1.3 and 4.4 it is reasonable to wonder whether the functors b and $\mu(l)$ commute. They do not. The example which witnesses this involves integer-valued functions. We do not know whether an example exists which is a vector lattice.

Example 6.5. *$\mu(Q) = \mu(l) \cdot b$, but $b \cdot \mu(l) \neq \mu(Q)$.*

First, the reason why $\mu(Q) = \mu(l) \cdot b$: now, for each **Arf**-object A, we have

$$(\mu(l) \cdot b)A = \varepsilon bA \cap \sigma bA = \varepsilon A \cap \rho A = \mu(Q)A,$$

because $\varepsilon bA = \varepsilon A$, since ε is the largest essential monoreflection and, thus, $b < \varepsilon$.

However, suppose that $A = C(\beta\omega, \mathbb{Z})$, the ring of all continuous integer-valued functions on the Stone–Čech compactification of the discrete natural numbers. This is the ring of all sequences of integers of finite range. It is easy to see that $\varepsilon A = C(\beta\omega)$. On the other hand, σA is complicated, but consists of integer valued Baire functions on $\beta\omega$. This implies that any $f \in \varepsilon A \cap \sigma A$ must be a bounded sequence of integers; the point is that $\varepsilon A \cap \sigma A = A$, and therefore $b(\mu(l)A) = bA$. Note that bA is the ring of rational sequences of finite range. Hence,

$$\mu(Q)A = \mu(l)bA = \varepsilon bA \cap \sigma bA,$$

finally, note that if g is the function defined by $g(n) = 1/n$ and by being identically zero on $\beta\omega \setminus \omega$, then $g \in \sigma bA$, and, therefore, $g \in \mu(l)bA$, but $g \notin bA$.

We conclude this article with some applications of Theorem 6.4 and the information about the strongly ω_1-regular reflection. Let us begin with an easy corollary of Theorem 6.4.

Corollary 6.6. *Suppose that A is an* **Arf**-*object. Then $\mu(Q)A = A$ if and only if A is an extremal subobject of $B \times C$, where C is a complete ring of quotients in* **Arf** *and $B = \varepsilon B$.*

Proof. This is immediate from Theorem 6.4, after one reflects that both **Q** and **E** are closed under products. □

In many ways Corollary 6.6 is unsatisfactory. In fact, we know of no explicit algebraic characterization of the objects A for which $\mu(Q)A = A$.

We consider one application which was also highlighted in [18]. For each compact (Hausdorff) and zero-dimensional space X, let $S(X)$ be the algebra generated by all the idempotents in $C(X)$, or, if the reader prefers, the algebra of finite linear combinations of characteristic functions of clopen subsets of X. We will describe $\mu(Q)S(X)$.

Proposition 6.7. *Suppose that X is a compact, zero-dimensional space. Then*

$$\mu(Q)S(X) = \{f \in C(X) \colon f \text{ has countable image}\}.$$

Proof. Begin by noting that $S(X)$ has the bounded inversion property. Thus, by Theorem 4.4, $\sigma S(X) = \rho S(X)$. [18, Corollary 4.10(c)] tells us that

$$\sigma S(X) = \{f \in B(X) \colon f \text{ has countable image}\}.$$

As $\varepsilon S(X) = C(X)$, the proposition follows from Theorem 6.4. □

So when is $\mu(Q)S(X)=C(X)$? There is a tidy answer to that; let us give it. First, we record the following definition.

Definition 6.8. Let X be a compact space. Recall that X is *scattered* if every closed subspace Y has a point which is isolated in Y. It is well known that a compact scattered space is necessarily zero dimensional. By a result of Rudin [28], a compact metric space is scattered precisely when it is countable.

Proposition 6.9. *With X compact and zero-dimensional, $\mu(Q)S(X)=C(X)$ if and only if X is scattered.*

Proof. [19, Corollary 5.8(c)] states that $\mu(l)S(X)=C(X)$ if and only if X is scattered. Moreover, $S(X)\in \mathbf{BI}$; therefore $\mu(l)S(X)=\mu(Q)S(X)$, by appealing to Example 6.5. □

Note added in proof (31 July 2001). We have just realized that there is an overlap between our Theorem 1.5 and Theorem 5.1 in [29]. We apologize for the oversight.

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