Symmetrical Subgroups of Artin Groups

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We show that the subgroup fixed by a group of symmetries of an Artin system \((A,S)\) is itself an Artin group under the hypothesis that the Deligne complex associated to \(A\) admits a suitable CAT(0) metric. Such a metric is known to exist for all Artin groups of type FC, which include all the finite type Artin groups as well as many infinite types. We also recover the previously known analogous result for an arbitrary Coxeter system \((W,S)\).

1. INTRODUCTION

An Artin system \((A,S)\) consists of a finite set \(S\) and a group \(A\) with the presentation

\[
\langle S | (st)^{m_{st}} = (ts)^{m_{st}}, \text{for } s, t \in S \text{ such that } m_{st} \neq \infty \rangle,
\]

where \((st)^{m_{st}}\) denotes the word \(ststs...\) of length \(m_{st}\), and where \(m_{st}\) is defined for each pair \(s, t \in S\) such that \(m_{ss} = 1\) and \(m_{st} = m_{ts} \in \{2, 3, 4, ... \}\) for \(s \neq t\). Each Artin system is associated with the Coxeter system \((W,S)\) in which \(W\) is the group with presentation

\[
\langle S | (st)^{m_{st}} = 1, \text{for } s, t \in S \text{ such that } m_{st} \neq \infty \rangle.
\]

Note that adding the relations \(s^2 = 1\) for each \(s \in S\) to the presentation (1) gives a presentation for the Coxeter group \(W\) as a quotient of the Artin group \(A\). An Artin system is said to be of finite type when the associated Coxeter group is finite.

A convenient way of encoding the defining information (namely the values \(m_{st}\)) for Artin and Coxeter systems alike is given by the Coxeter graph \(\Gamma\), which is the labelled graph with vertex set \(S\) and an edge labelled \(m_{st}\) between \(s\) and \(t\) whenever \(m_{st} \neq 1\) or 2. (When drawing \(\Gamma\), the labels equal to 3 are usually suppressed.)
Definition 1. We define a symmetry of the Artin system \((A, S)\) to be any group automorphism \(g: A \to A\) which preserves the generating set, that is \(g(S) = S\).

It is easily seen that a symmetry of \((A, S)\) naturally induces an automorphism of the associated Coxeter group (a symmetry of \((W, S)\)), and furthermore manifests itself as a graph automorphism of the Coxeter graph \(\Gamma\) which respects the labelling of edges. Conversely, any labelled graph automorphism of \(\Gamma\) defines uniquely a symmetry of the Artin system \((A, S)\) or equivalently of the Coxeter system \((W, S)\).

Let \(G\) denote a (necessarily finite) group of such symmetries, which we consider to act on each of \(A\), \(W\) and \(\Gamma\), and write \(A^G\) and \(W^G\) for the fixed subgroups respectively of \(A\) and \(W\) under the action of \(G\). Let \(S\) denote the set of orbits \(T\) of the action of \(G\) on \(S\) for which the subgroup \(W_T\) of \(W\), generated by \(T\), is finite. It was proved in [21] for finite Coxeter groups, and in [13, 19] for the general case, that the fixed subgroup \(W^G\) is generated as a Coxeter group by the set \(w^G_S = \{w_T | T \in S\}\), where \(w_T\) denotes the longest element of the finite Coxeter system \((W_T, T)\). If one writes \(A^G\) for the element of \(A\) represented by any reduced expression for \(w_T\), then this result suggests the following conjecture which is the subject of this paper.

Conjecture 2. Suppose that \(G\) is a group of symmetries of the Artin system \((A, S)\). Then \(A^G\) is generated by the set \(A^G_S = \{A_T | T \in S\}\), and \((A^G, A^G_S)\) is isomorphic to the Artin system associated with \((W^G, w^G_S)\).

In Section 3 we describe explicitly, in terms of \((A, S)\) and \(G\), an Artin System \((A, S)\) which is ultimately shown (Theorem 14) to be isomorphic to that associated with \((W^G, w^G_S)\). One easily checks that mapping \(T \in S\) to \(A_T \in A^G_S\) defines a homomorphism \(\phi_G: A \to A\). The content of Conjecture 2 is that the homomorphism \(\phi_G\) is injective with image \(A^G\). There are two distinct cases to be considered:

(i) Finite Type Systems. In Section 3 we prove that Conjecture 2 holds true for \((A, S)\) of finite type (Theorem 11). We remark that this has also been proven independently by Michel [17], and Dehornoy and Paris [12]. In [17], Michel shows that this follows from the result for Coxeter groups via the existence of normal forms [7] for elements of a finite type Artin group. Our proof is independent of the Coxeter group result, relying instead upon the semi-lattice ordering of the Artin monoid \(A^+\) (which determines a lattice order on the group \(A\) in the finite type case), and in this respect is closer to that of [12]. We obtain the analogue of Conjecture 2 for Artin monoids of any type (Lemma 10) and recover from this an alternative route to the general result of [13, 19] for Coxeter groups (Theorem 14, Section 4).
(ii) Infinite Type Systems. In the case where \((A, S)\) is of infinite type we represent the map \(\phi_G\) geometrically via a simplicial map on the corresponding Deligne complexes. We are then able to establish that Conjecture 2 holds under the hypothesis that the Deligne complex \(D\) for \((A, S)\) admits a suitable CAT(0) metric. By “suitable” we mean that the simplicial automorphism induced on \(D\) by any symmetry \(g\) of \((A, S)\) must be an isometry with respect to that metric. There are two natural candidates for a CAT(0) metric, both of which are suitable in this sense. The most natural of these is the Moussong metric [18], which is defined so that certain subcomplexes of links of vertices in the complex which are combinatorially equivalent to spherical Coxeter complexes are indeed isometric to spheres. It is conjectured that the Moussong metric is always CAT(0). This is known for the 2-dimensional Artin groups [10], which do not provide any interesting cases of Conjecture 2 above.

Note added in proof. Ruth Charney has recently pointed out that the Moussong metric is CAT(0) for what she calls “locally reducible” Artin systems, those in which each finite type special subgroup is a direct product of infinite cyclic and tw-generator Artin groups, and that these do provide further interesting cases where our main result, Theorem 23, applies.

However, the Deligne complex may also be viewed as a cube complex, and Charney and Davis [10] have shown that the associated Euclidean cubical metric is CAT(0) if and only if the Artin group is of type FC.

Definition 3. An Artin System \((A, S)\) is said to be of type FC if the associated Coxeter system \((W, S)\) satisfies the following condition:

\((FC)\) If \(T \subseteq S\) and every pair of elements of \(T\) generates a finite subgroup of \(W\), then \(T\) generates a finite subgroup of \(W\).

Obviously this class includes all finite type Artin groups. It also includes the family of right-angled Artin groups or graph groups, that is those for which \(m_s = 2\) or \(\infty\) for every pair \(s \neq t \in S\). The main result of this paper may now be stated as follows:

Theorem 4. Suppose that \((A, S)\) is a type FC Artin system and that \(G\) is a group of symmetries of \((A, S)\). Then the subgroup \(A^G\) of elements fixed by \(G\) is an Artin group with generating set \(A_S\). More precisely, \((A^G, A_S)\) is isomorphic to the Artin system \((A, S)\) described in Section 3.

We note that the use of CAT(0) curvature in our proof of Theorem 4 is motivated by ideas in the paper of Charney [9] on injectivity of the Artin monoid.

2. THE BASICS OF ARTIN SYSTEMS

Given an Artin system \((A, S)\) we associate to it the Artin monoid \((A^+, S)\), or just \(A^+\), which is defined to be the monoid (semigroup with 1)
given by the presentation (1) considered as a monoid presentation. There is a canonical monoid homomorphism $A^+ \to A$ taking $s$ to $s$ for each $s \in S$. This is known to be injective for $(A, S)$ of finite type [6, 15], in which case one may identify $A^+$ as a submonoid of $A$. The word problem for Artin groups of finite type is generally solved by showing that each element of $A$ can be written in the form $x^{-1}y$ for $x, y \in A^+$ in some canonical way [6, 15, 8].

We turn now to the Artin monoid itself. It was shown in [6] that $A^+$ is a cancellative monoid (that is, if $uxv = uyv$ then $x = y$, for $u, v, x, y \in A^+$). For $x, y \in A^+$ we say that $x$ divides $y$, written $x \leq y$, precisely when $y = xw$ for some $w \in A^+$. Since the relations of the presentation (1) respect word-length, we may also define a length function $l$ on $A^+$ where $l(x)$ is the length of any word representing $x$. Then the relation, $\leq$, defines a partial order on $A^+$ which is both left invariant and left cancellative (that is to say $ax \leq ay$ if and only if $x \leq y$, for $a, x, y \in A^+$), and $\ell$ is a strict order preserving homomorphism of $A^+$ onto $\mathbb{N}$. An element $\mu \in A^+$ is said to be a common multiple of the subset $X$ of $A^+$ if $x \leq \mu$ for all $x \in X$. Common divisors are defined analogously. In [6], Brieskorn and Saito prove that the poset $(A^+, \leq)$ is a semi-lattice. That is to say

(i) Every finite subset of $A^+$ which has a common multiple has a (necessarily unique) least common multiple (that is, one which divides all other common multiples), and

(ii) Every finite subset of $A^+$ has a greatest common divisor.

Note that, in our case, (ii) may be deduced from (i) by using the length function and the fact that the generating set $S$ is finite.

For convenience we define the lattice completion of $(A^+, \leq)$ by introducing the symbol $\infty$ and setting $x \leq \infty$ for all $x \in A^+$. We adopt the notation commonly used when discussing lattices by writing $x \lor y$ to denote the least common multiple (or “join”) in $(A^+ \cup \{\infty\}, \leq)$ of the two elements $x$ and $y$ of $A^+$. Thus $x \lor y = \infty$ precisely when $x$ and $y$ have no common multiple in $A^+$. It follows from the Reduction Lemma of [6] that, for any pair of generators $s, t \in S$,

$$s \lor t = \begin{cases} \langle st \rangle^m & \text{if } m \neq \infty, \\ \infty & \text{if } m = \infty. \end{cases}$$

More generally, let $T$ be an arbitrary subset of the generating set $S$. We write $A_T$ for the least common multiple of $T$, writing $A_T = \infty$, of course, if there is no common multiple at all in $A^+$. An Artin system $(A, S)$ is of finite type if and only if $A_S \neq \infty$ [6, Satz 5.6], and if and only if $(A^+, \leq)$ is a lattice (that is, every finite subset of $A^+$ has a least common multiple), [6, Proposition 5.5]. For $(A, S)$ of finite type, the element $A = A_S \in A^+$ is
known as the *fundamental* element, and may be represented by any positive word which is a minimal length representative for the longest element in the finite Coxeter group \((W, S)\) (described for example in [4]).

Now, let \((A, S)\) be an Artin system with Coxeter graph \(\Gamma\), and let \((W, S)\) be the associated Coxeter system. Given any subset \(T\) of \(S\) we define the *special* subgroup \(A_T\) (respectively \(W_T\)) to be the subgroup of \(A\) (respectively \(W\)) generated by the set \(T\). Let \(\Gamma_T\) denote the labelled subgraph of \(\Gamma\) consisting of the vertex set \(T\) and all edges of \(\Gamma\) both of whose vertices lie in \(T\). Then it has been shown by van der Lek [16] that \((A_T, T)\) is an Artin system with Coxeter graph \(\Gamma_T\). (The corresponding result, that \((W_T, T)\) is a Coxeter system of type \(\Gamma_T\), is a well-known fact [4].)

Define the poset \(P_f = \{ T \subseteq S \mid A_T \text{ exists} \}\) ordered by inclusion. Clearly \(P_f\) is order isomorphic to the poset of finite type special subgroups of \((A, S)\) also ordered by inclusion.

### 3. Definition of \(\phi_G\) and the Finite Type Case

Let \(G\) denote a group of symmetries of the Artin system \((A, S)\). We define a new Artin system \((A, S)_G\), or just \((A, S)\), as follows. Let \(S\) be the subset of \(SG\) consisting of those orbits \(T\) for which \(W_T\) exists, that is \(W_T\) is finite. This is just the set of minimal \(G\)-invariant elements of \(P_f\). For each pair of distinct \(G\)-orbits \(B, C \in S\) we specify \(m_{BC}\) as follows:

- If each \(b \in B\) commutes with every element of \(C\) then \(m_{BC} = 2\).
- If \(\Gamma_{B \cup C}\) is a disjoint union of isomorphic copies of one of the following graphs,
  
  (i) \[ \begin{array}{c} \bullet \\ m \end{array} \] 
  then \(m_{BC} = m,\)
  
  (ii) \[ \begin{array}{c} \bullet \\ \bullet \end{array} \] 
  then \(m_{BC} = 4,\)
  
  (iii) \[ \begin{array}{c} \bullet \\ \bullet \end{array} \] 
  then \(m_{BC} = 4,\)
  
  (iv) \[ \begin{array}{c} \bullet \\ \bullet \end{array} \] 
  then \(m_{BC} = 6.\)

(In each of the above figures, the two sets of differently shaded vertices indicate distinct \(G\)-orbits.)

- In all other cases \(m_{BC} = \infty.\)

We illustrate this definition with two examples where \(G\) is cyclic of order 2 generated by a symmetry \(g\). Figure 1 shows, in each case, the Coxeter graph of \((A, S)\) with symmetry \(g\) indicated as a graph automorphism...
FIG. 1. Artin systems derived from a symmetry $g$.

Together with the Coxeter graph for the resulting system $(A, S)$. In the first instance, Fig. 1a, $(A, S)$ is the Artin system of finite type $E_6$, and $(A, S)$ the finite type $F_4$. A complete list of symmetries of finite type Coxeter graphs may be easily drawn up, see [14], for example. For the second example, Fig. 1b, we have chosen $g$ to be a symmetry of a type FC Artin system. The statement of Theorem 4 therefore applies to this example.

**Lemma 5.** If $m_{BC} = \infty$ then $A_{B, C}$ is of infinite type.

**Proof.** Suppose that $\Gamma' = \Gamma_{B, C}$ is a finite type Coxeter graph. The group $G$ acts on $\Gamma'$ with precisely two vertex orbits, $B$ and $C$. Now, either each $b \in B$ commutes with every element of $C$, in which case $m_{BC} = 2$, or there is some connected component $I_0$ of $\Gamma'$ which intersects both vertex orbits. In the latter case each connected component of $\Gamma'$ must be isomorphic to $I_0$ (under the graph automorphism $g \in G$ which takes some vertex of that component into $I_0$), and the subgroup $H < G$ of elements which stabilise $I_0$ acts on $I_0$ with precisely two vertex orbits. Since $\Gamma_0$ is of finite type and connected, it must, by the classification of finite irreducible Coxeter systems (see [4]), be one of the graphs shown in (i)-(iv) above (the $H$-actions being the obvious ones of order 1, 2 and 2 for cases (i)-(iii), respectively, and either the rotation of order 3 or the full dihedral group action of order 6 in case (iv)). In each of these cases $m_{BC}$ has been specified to be a finite integer. 

The converse of this statement is easily seen, but follows indeed from the much stronger statement of Lemma 6 below. Given $x$ and $y$ elements of $A^+$, let $(x, y)^m$ denote the product $xyxy\ldots$ consisting of $m$ factors. So, for example $(x, y)^3 = x^3y$. 

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$(A, S)$ with symmetry $g$

$(A, S)_{(g)}$

(a)

(b)
Lemma 6. For $B, C \in S$, one has $A_B \vee A_C = \langle A_B, A_C \rangle^{m_{BC}} = \langle A_C, A_B \rangle^{m_{BC}}$ if $m_{BC} \neq \infty$, and $A_B \vee A_C = \infty$ otherwise.

Proof. If $A_B$ and $A_C$ possess a least common multiple in $A^+$, then it is $A_{B \cup C}$ which exists, by Lemma 5, only when $m_{BC} \neq \infty$. We now verify the lemma in each case where $m_{BC}$ can be finite. This is easily done in the case that some (and hence, by symmetry, every) element of $B$ commutes with each element of $C$, for then $A_B A_C = A_C A_B$ is easily seen to be a least common multiple (by resorting to the algorithm of [6] if necessary).

We now consider the cases that $T_{B \cup C}$ is one of the four connected graphs of (i)-(iv). Case (i) is trivial, and cases (ii) and (iv) follow immediately from [6, Lemma 5.8]. For the graph of case (iii), label the vertices $s_1, \ldots, s_4$ from left to right so that (without loss of generality) $B = \{s_1, s_4\}$ and $C = \{s_2, s_3\}$. This is the Coxeter graph of type $A_4$ which has fundamental element $\langle s_1 s_3, s_2 s_4 \rangle^5$ (by [6, Lemma 5.8] again), and one can easily check that

$$\langle A_B, A_C \rangle^4 = \langle s_1 s_4, s_2 s_3 s_2 \rangle^4 = A_{A_4} = \langle A_C, A_B \rangle^4.$$ 

More generally, let $T_{B \cup C}$ be the disjoint union of subgraphs $T_i = T_{B_i \cup C_i}$ for $B_i \subset B$ and $C_i \subset C$, where the $T_i$ are isomorphic copies of one of the graphs (i)-(iv). Then $A_B = A_{B_1} \cdots A_{B_k}$ and $A_C = A_{C_1} \cdots A_{C_k}$, and $A_{B \cup C} = A_{B_1 \cup C_1} \cdots A_{B_k \cup C_k} = \langle A_{B_1}, A_{C_1} \rangle^{m_{BC}} \cdots \langle A_{B_k}, A_{C_k} \rangle^{m_{BC}}$, using the results established above for commuting sets and the connected graphs respectively. This last expression may now be rearranged to give $\langle A_B, A_C \rangle^{m_{BC}}$ by using the commutativity relations among the $A_B$'s and $A_C$'s, for $1 \leq i, j \leq k$. A similar argument with the roles of $B$ and $C$ reversed gives the second equality, completing the proof.

Definition 7. Given a group $G$ of symmetries of the Artin system $(A, S)$ we define the Artin system $(A, S)$ as above, and define a monoid homomorphism

$$\phi_G^+: A^+ \rightarrow A^+$$

such that $\phi_G^+(T) = A_T$ for each $G$-orbit $T \in S$. We define the Artin group homomorphism

$$\phi_G : A \rightarrow A$$

to be the one canonically induced by $\phi_G^+$. Thus for $T \in S$, $\phi_G(T)$ is the image of $A_T$ under the canonical homomorphism $A^+ \rightarrow A$, also written $A_T$. As in the introduction, $A_S$ will denote the set of $A_T$ for $T \in S$. 


Moreover, which is to say monoids \( (x, y) \). Therefore \( x \) implies that \( z \) and since otherwise the result follows trivially. Thus write particular, \( x \) is injective.\( x \) satisfies the hypothesis of the following theorem. Part (b) of this theorem is proved in \([14, \text{Theorem 1.3}]\) and part (a) is hinted at by Proposition 4.3 of the same paper.

**Theorem 8.** Given a homomorphism \( \phi^+ : A_1^+ \to A_2^+ \) between Artin monoids \( (A_1^+, S_1) \) and \( (A_2^+, S_2) \), we allow \( \phi^+ \) to extend to the lattices \( (A_1^+ \cup \{ \infty \}, \leq) \) by setting \( \phi^+(\infty) = \infty \). Suppose that \( \phi^+ \) respects len's, which is to say (Definition 1.1 of \([14]\)) that the following conditions hold

(i) \( \phi^+(s) \neq 1 \) for each generator \( s \in S_1 \), and

(ii) \( \phi^+(s) \vee \phi^+(t) = \phi^+(s \vee t) \) for all \( s, t \in S_1 \).

Then

(a) \( \phi^+(x) \vee \phi^+(y) = \phi^+(x \vee y) \) for all \( x, y \in A_1^+ \), and

(b) \( \phi^+(x) \leq \phi^+(y) \) if and only if \( x \leq y \), for \( x, y \in A_1^+ \). In particular, \( \phi^+ \) is injective.

**Proof.** For \( x \in A_1^+ \) we write \( \bar{x} \) to denote its image \( \phi^+(x) \).

(a) Note that \( \bar{x} \vee \bar{y} \leq \bar{x \vee y} \) holds always. Suppose, by way of contradiction, that there exist \( x, y \in A_1^+ \) for which \( \bar{x} \vee \bar{y} \neq \bar{x \vee y} \). In particular \( \bar{x} \vee \bar{y} \neq \infty \). We may suppose also that \( x \) and \( y \) are chosen so as to minimise \( \ell(\bar{x} \vee \bar{y}) \). To obtain a contradiction, it will suffice to show that \( \bar{x} \vee \bar{y} \leq \bar{x} \vee \bar{y} \). We may assume that both \( x \) and \( y \) have length at least 1, since otherwise the result follows trivially. Thus write \( x = sx' \) and \( y = ty' \) for \( s, t \in S_1 \). Putting \( z = \bar{x} \vee \bar{y} \) we have \( z = \bar{x} \vee \bar{u} \) for some \( u \in A_1^+ \). Also, since both \( \bar{x} \) and \( \bar{t} \) divide \( z \), it follows, by condition (ii), that \( z = (\bar{x} \vee \bar{t}) \bar{z'} \) for some \( \bar{z'} \in A_1^+ \). Write \( s \vee t = st \) for \( \bar{z} \in A_1^+ \). Then by cancellation, we have \( \bar{v}_1 = \bar{v}_2 \) which is therefore a multiple of \( w = \bar{v}_1 \vee \bar{v}_2 \). Since \( \ell(w) \leq \ell(\bar{v}_1) \) it follows, by the choice of \( \bar{z} \), that \( w = \bar{v}_1 \vee \bar{v}_2 \). Now, writing \( \bar{x} \vee \bar{t} = \bar{x} \vee \bar{t} \) for \( \bar{z} \in A_1^+ \), it follows by cancellation that \( \bar{x} \leq \bar{z'} \) (since \( w \leq \bar{v}_2 \)). Moreover,

\[
x = sx' \leq sx = (s \vee t) x.
\]

Similarly, we find \( \beta \in A_1^+ \) such that \( \beta \leq \bar{z'} \) and \( y \leq (s \vee t) \beta \). Thus we have \( \bar{x} \vee \bar{y} \leq \bar{z'} \), and since \( \ell(\bar{x} \vee \bar{y}) \leq \ell(\bar{z'}) \leq \ell(z) \), it follows, again by the choice of \( \bar{z} \), that \( \bar{x} \vee \bar{y} = \bar{x} \vee \bar{z'} \). On the other hand, \( x \vee y \leq (s \vee t)(x \vee \beta) \). Therefore \( \bar{x} \vee \bar{y} \leq \bar{x} \vee \bar{z'} \) as required.

(b) Observe, first of all, that if \( \phi^+(x) = 1 \) then \( x = 1 \), for condition (i) implies that \( \ell(\bar{x}) \leq \ell(\bar{x}) \) in all cases. Suppose that \( \bar{x} \leq \bar{y} \) for some \( x, y \in A_1^+ \). Then the result of part (a) above implies that \( \bar{x} \vee \bar{y} = \bar{x} \vee \bar{y} = \bar{y} \). Then \( x \vee y \neq \infty \), and we may write \( x \vee y = zw \) for some \( w \in A_1^+ \). But then,
by cancellation, \( w = 1 \) and hence \( x \leq y \). Injectivity of \( \phi^+ \) follows immediately.

**Corollary 9.** Let \( G \) denote a group of symmetries of the Artin system \((A, S)\). If \((A, S)\) is of finite type, then \((A, S)_G\) is also of finite type.

**Proof.** This follows immediately from Theorem 8(a) applied to the homomorphism \( \phi^+_G\), since an Artin system \((A, S)\) is of finite type if and only if \( x \vee y \neq \infty \) for all \( x, y \in A^+ \) (see [6, Proposition 5.5, Satz 5.6]).

**Lemma 10.** Let \( G \) be a group of symmetries of the Artin system \((A, S)\). Then the monoid homomorphism \( \phi^+_G : A^+ \to A^+ \) is injective with image \((A^+)^G\), the submonoid of elements fixed by \( G \).

**Proof.** The injectivity follows by Lemma 6 and Theorem 8(b), and since \( g(A_b) = A_{g(b)} = A_b \) for each \( g \in G \) and each \( G \)-orbit \( B \in S \), it is clear that \( \text{im} \phi^+_G \subseteq (A^+)^G \).

We now show that \((A^+)^G \subseteq \text{im} \phi^+_G\). Take any \( x \in A^+ \) such that \( g(x) = x \) for all \( g \in G \). Since 1 \( \in \text{im} \phi^+_G \) we may suppose that \( t(x) \geq 1 \) and hence that \( s \leq x \) for some \( s \in S \). By symmetry, it follows that \( t \leq x \) for each \( t \) in the \( G \)-orbit \( T \) containing \( s \). Therefore \( A_T = \phi^+_G(T) \) divides \( x \), and we may write \( x = \phi^+_G(T) \cdot x' \) for some \( x' \in A^+ \). By cancellation of \( g(A_T) = A_T \) one also has that \( g(x') = x' \) for all \( g \in G \), and the result now follows by an induction on the wordlength.

**Theorem 11.** Let \( G \) be a group of symmetries of the Artin system \((A, S)\), and suppose that \((A, S)\) is of finite type. Then the group homomorphism \( \phi_G : A \to A \) is injective with image \( A^G \), the subgroup of elements fixed by \( G \). That is \((A^G, A_b)\) is an Artin system isomorphic to \((A, S)_G\).

**Proof.** Note that, by Corollary 9, both systems \((A, S)\) and \((A, S)_G\) are of finite type. Hence we shall identify the monoids \( A^+ \) and \( A^* \) as submonoids of these groups respectively. In this proof we use the result of Charney [8] that in an Artin group \( A \) of finite type every element has a unique orthogonal form \( x^{-1}y \) with \( x, y \in A^+ \) (identified as a submonoid of \( A \)) such that \( x \) and \( y \) have no nontrivial common divisor.

Suppose that \( x^{-1}y \) represents an arbitrary element of \( A \) in orthogonal form \((so x, y \in A^+)\). Then, if \( \phi_G(x^{-1}y) = 1 \) one has \( \phi^+_G(x) = \phi^+_G(y) \) and therefore, by Lemma 10 (or Theorem 8), \( x = y \). Thus \( \phi_G \) is injective. On the other hand, if \( g \in G \) then \( g(x^{-1}y) \) is the unique orthogonal form for \( g(x^{-1}y) \). Thus if \( x^{-1}y \) is fixed by \( G \) then both \( x \) and \( y \) are also fixed by \( G \) and, by Lemma 10, one has \( x^{-1}y \in \text{im} \phi_G \). That is, \( A^G \subseteq \text{im} \phi_G \). The reverse inclusion is clear since \( \text{im} \phi_G \) is generated by \( \text{im} \phi^+_G = (A^+)^G \).
4. SYMMETRICAL SUBGROUPS OF COXETER GROUPS

At this point we are able to recover the analogous result to that of Theorem 11 for a group $G$ of symmetries acting on an arbitrary Coxeter system $(W,S)$. The theorem which follows was originally due to Steinberg [21] for finite Coxeter groups. One may refer to Hée [13] or Mühlherr [19] for the general case. The proof given here is different to these in that it does not use any of the language of root systems, depending rather on the preceding Lemma 10 (which comes from a purely combinatorial study of the Artin monoid) together with the Tits bijection between elements of the Coxeter group and square free elements of the Artin monoid.

An element of $A^+$ is said to be square free if it is not represented by any word which contains a square ($s^2$ for $s \in S$) as a subword (see also [6, 3.4]). Denote by $QFA^+$ the set of all square free elements. Note that each element $T \subseteq S$ is known to be square free, [6, Lemma 5.4]. One has a canonical homomorphism $\pi: A^+ \rightarrow W$ which maps generators to corresponding generators. By a theorem of Tits [22, Théoreme 3] it is known that the restriction $\pi_{QF}: QFA^+ \rightarrow W$ is a bijection such that, for each $w \in W$, $\pi_{QF}^{-1}(w)$ is the unique minimal length representative for $w$ in $A^+$. An easy consequence of this is the following result of [6] which reformulates a well-known basic property of Coxeter groups.

**Lemma 12** [6, Lemma 3.4]. If $x \in QFA^+$, but $sx \notin QFA^+$ for some $s \in S$, then $s \leq x$.

**Proof.** Writing $\hat{x} = \pi_{QF}^{-1}(\pi(sx))$ one has $\ell(\hat{x}) \leq \ell(sx) - 2 = \ell(x) - 1$, since $\hat{x}$ is a minimal length representative of $\pi(sx)$ while $sx$ is not, and since representatives in $A^+$ for the same element of $W$ must have lengths congruent mod 2. Therefore, one has $\ell(s\hat{x}) \leq \ell(x)$ and $\pi(s\hat{x}) = \pi(s^2x) = \pi(x)$, and, since $x \in QFA^+$ is the unique minimal representative for $\pi(x)$, it follows that $x = s\hat{x}$. □

The following lemma is a consequence of Lemmas 3.1 and 3.2 of [6], but follows more directly from the semi-lattice structure of the monoid also established in that paper. For $T \subset S$ we write $A^+_T$ for the submonoid of $A^+$ generated by the set $T$.

**Lemma 13.** Let $T \subset S$, and suppose that $s \in T$, $v \in A^+_T$ and $x \in A^+$. If $s \leq vx$, but $s \nleq v$, then $t \leq x$ for some $t \in T$.

**Proof.** Since both $s$ and $v$ divide $vx$ there exists $s \vee v$ in $A^+$. Note, however, that $A^+_T$ is the image in $A^+$ of a homomorphism trivially satisfying the hypothesis of Theorem 8 and, by virtue of that theorem, it follows that $s \vee v$ lies in $A^+_T$. Therefore we may write $s \vee v = ev$ for some $y \in A^+_T$.
which is nontrivial, since \( s \not\equiv v \), and hence divisible by some \( t \in T \). Now, since \( s \not\equiv v \not\equiv x \), cancellation gives \( y \not\equiv x \) completing the proof.

A group \( G \) of symmetries of an Artin system \((A, S)\) may equally be considered as a group of symmetries of the associated Coxeter system \((W, S)\). Write \((W, S)\) for the Coxeter system associated with \((A, S)\), defined with respect to the group \( G \). It is shown in [6, Lemma 5.1(ii)] that \( \Delta_T = \text{rev} \Delta_T \) for \( T \in P \) where “rev” is the anti-isomorphism of \( A^+ \) which one obtains by rewriting any word (representing an element) in reverse order. Consequently, for \( T \in S \), one has \( \phi_G^+(T^2) = \Delta_T.\text{rev} \Delta_T \) which maps to 1 under \( \pi \). Therefore \( \phi_G^+ \) canonically induces a group homomorphism

\[
\phi_W: W \rightarrow W.
\]

**Theorem 14** [Hée [13]; Mühlherr [19]; Steinberg [21]). Let \((W, S)\) be an arbitrary Coxeter system with \( G \) a group of symmetries. Then the homomorphism \( \phi_W: W \rightarrow W \) is injective with image the subgroup \( W^G \) of elements fixed under the action of \( G \). That is to say \((W^G, w_G)\) constitutes a Coxeter system isomorphic to \((W, S)\).

**Proof.** We base our proof on the correspondence between Coxeter group elements and square free elements of the Artin monoid. We have already shown in Lemma 10 that \( \phi_W^+: A^+ \rightarrow A^+ \) is an injective homomorphism with image \((A^+)^G\). Note also that \( \text{QFA}^+ \) is a \( G \)-invariant set, so that elements of \( W^G \) correspond precisely to elements of \( \text{QFA}^+ \). The theorem is therefore a consequence of the following Lemma 15.

**Lemma 15.** \( \phi_W^+(\text{QFA}^+) = \text{im} \phi_W^+ \cap \text{QFA}^+ \).

**Proof.** Note that if \( x \notin \text{QFA}^+ \) then \( x = uBBv \) for some \( u, v \in A^+ \) and \( B \in S \). But then \( \phi_W^+(x) \) is not square free either since it is represented by a word containing a subword \( A_B.\text{rev} A_B \). For the converse, we show by induction on the word length that the image of every square free element \( x \) is square free. This is trivially true if \( l(x) = 0 \).

Suppose that \( l(x) \geq 1 \) and write \( x = Bx' \) for some \( B \in S \) and \( x' \in \text{QFA}^+ \). Write \( z = \phi_W^+(x') \) so that \( \phi_W^+(x) = A_Bz \). By induction we may suppose that \( z \in \text{QFA}^+ \), and we know that \( A_B \in \text{QFA}^+ \). Suppose by way of contradiction that \( \phi_W^+(x) \) is not square free. Then we may write \( A_B = uv \) for \( u, v \in A_B^+ \) and \( s \in S \), such that \( vz \) is square free, but \( sz \) is not. Then, by Lemma 12, \( s \equiv vz \) and, since \( s \equiv v \) (for otherwise \( A_B \) would not be square free, a contradiction), it follows from Lemma 13 that \( t \equiv z \) for some \( t \in B \). But since \( z \) is \( G \)-fixed it follows by symmetry that \( b \leq y \) for each generator \( b \) in the \( G \)-orbit \( B \). Thus \( A_B \equiv z \), or rather \( \phi_W^+(B^2) \equiv \phi_W^+(x) \). By Theorem 8(b) it
now follows that \( B^2 \leq x \) which contradicts \( x \) being square free, completing the proof.

The proof of Theorem 14 is similar to that of [14, Proposition 2.3] which treats the case of a so-called LCM-homomorphism. In that paper it was subsequently shown that an Artin group homomorphism of this kind is realised in a very natural way as the map induced on fundamental groups by a simplicial inclusion of the corresponding Salvetti/W complex. [14, Theorem 3.4]. The Salvetti complex, first defined in [20], is a finite simplicial complex associated to \((A, S)\) on which \( W \) acts freely by simplicial isomorphisms, and whose orbit space \( Z \) under this action has fundamental group \(
abla_1(Z) \cong A\). It is conjectured that \( Z \) is always a \( K(A, 1) \)-space, and this is known for all Artin groups of type FC, [11].

The proof of Theorem 3.4 of [14] applies equally to the case of homomorphisms \( G : A \rightarrow A \) considered here, given that we have established Theorem 14 above. Combining this with Theorem 4 we obtain:

**Theorem 16.** Let \((A, S)\) be an Artin group of type FC and \( G \) a group of symmetries of \((A, S)\). Then there exists a \( \pi_1 \)-injective simplicial inclusion \( \Psi : Z \rightarrow Z \) between aspherical complexes (the Salvetti/W complexes for \( A \) and \( A \) respectively) which realises the homomorphism \( \phi_G : A \rightarrow A \) as the map induced on fundamental groups. Moreover, there is a naturally defined action of the group \( G \) by simplicial automorphisms of \( Z \) with fixed point set precisely \( \Psi(Z) \).

Tits proved that any Coxeter group \((W, S)\) admits a faithful representation \( \rho : W \rightarrow GL(V) \) as a linear reflection group on \( V = \mathbb{R}^{|S|} \) with fundamental chamber a simplicial cone \( C \) ([4], Chap. V). The Salvetti/W complex for \((A, S)\) is known to be homotopy equivalent to the manifold \( M = \tilde{M}/W \) where \( \tilde{M} \) is the complement (restricted to \( V + iI \subseteq V \otimes \mathbb{C} \), where \( I = \bigcup_{w \in W} \rho(w)(C) \) is the Tits cone) of the complexified reflection hyperplane arrangement associated to such a representation \( \rho \) (see [10, 11]). Any group \( G \) of symmetries of \((W, S)\) (or \((A, S)\)) acts naturally on \( V \) by linear automorphisms such that \( g \cdot \rho(w) \cdot g^{-1} = \rho(g(w)) \) for \( g \in G \) and \( w \in W \), and the Coxeter system \((W^G, w_S) \cong (W, S)\) acts as a reflection group on \( V^G \), the subspace of \( V \) fixed by \( G \). The submanifold \( M^G \) of \( M \) which is fixed by the corresponding action of \( G \) on \( M \) is then precisely the quotient of the complexified hyperplane complement associated to this representation of \( W \) as a reflection subgroup of \( GL(V^G) \). As in Theorem 16 above, the inclusion \( M^G \subset M \) realises \( \phi_G : A \rightarrow A \) as the map induced on fundamental groups. Therefore, by Theorem 4, \( M^G \) is a \( \pi_1 \)-injective submanifold of \( M \) when \((A, S)\) is type FC, and conjecturally so in general. This interpretation of Conjecture 2 suggests a generalisation to braid groups associated with (finite) complex reflection groups. Results of this kind have already been obtained by D. Bessis, in his thesis and in [3].

We turn now to the proof of Theorem 4 and in what follows we shall utilise a relationship, analogous to those just discussed, which exists
between the Deligne complexes associated to \((A, S)\) and \((A, S)\). These complexes, first introduced by Deligne [15], were studied extensively in [10] where Charney and Davis showed that the hyperplane complement \(M\), referred to above, aspherical if and only if the associated Deligne complex\(^1\) is contractible. We shall find it convenient to deal with the Deligne complex because, in certain cases, it may be endowed with a CAT(0) (non-positively curved) metric in such a way that symmetries of the Artin system manifest themselves as isometries of the complex.

5. THE MAP BETWEEN DELIGNE COMPLEXES

Let \((A, S)\) be an Artin system. Recalling the definition of the poset 
\[ P_f = \{ T \subseteq S \mid A_T \text{ exists} \}, \]
we define the partially ordered set of finite type special cosets
\[ AP_f = \{ xA_T \mid x \in A, T \in P_f \}, \]
where the ordering is by set inclusion. The Deligne complex \(D\) associated to \((A, S)\) is defined to be the flag complex of the poset \(AP_f\). That is, \(D\) is an abstract simplicial complex whose \(n\)-simplices are the totally ordered sets of \(n+1\) elements in \(AP_f\). We shall identify the vertices of \(D\) with the corresponding elements of \(AP_f\). We shall also view the abstract complex \(D\) in terms of its topological realisation as a union of real simplices endowed with the piecewise linear topology, and make no distinction in our notation between the abstract complex and the topological space.

Left multiplication of cosets defines an action of \(A\) on \(AP_f\) by order preserving bijections. This induces a natural action of \(A\) on \(D\) by simplicial isomorphisms, under which the vertex stabilisers are subgroups conjugate to special subgroups of finite type, and the stabiliser of a simplex is just the stabiliser of its minimal vertex. The subcomplex \(K\) which consists of all simplices which contain the vertex \(\{1\}\) (the trivial coset) is clearly a fundamental domain for the action of \(A\) on \(D\). Recall that we may identify \(P_f\) with the poset of finite type special subgroups, a partially ordered subset of \(AP_f\). Under this identification, the subcomplex \(K\) is precisely the flag complex associated to \(P_f\).

Let \(G\) be a group of symmetries of \((A, S)\). Then there is a natural action of \(G\) on the Deligne complex \(D\) via simplicial automorphisms, defined such that \(g(xA_T) = g(x)A_{gT}\) for all \(g \in G\) and \(xA_T \in AP_f\). The fixed point set of \(D\) under \(G\) is a subcomplex which we denote \(DG\), and its intersection with

\(^1\)The complex referred to here was called in [10] the modified Deligne complex. However, the more direct terminology, which is consistent with [9], seems preferable and should not cause any confusion.
K we denote $K^G$. Let $A^K^G$ denote the subcomplex of $D^G$ which is the union of the $a.K^G$ for $a \in A^G$. We note that it is not immediately clear whether in fact $A^K^G = D^G$. However, this is the case when $(A, S)$ is of finite type. In this case, Altobelli [1] defines a family \{\{m(xA_T) \mid x \in A, T \subseteq S\}\} of “minimal” coset representatives which are canonical in the sense that for any symmetry $g$ of $(A, S)$ one has $g(m(xA_T)) = m(g(x)A_{g(T)})$. It follows that, for $(A, S)$ of finite type, $g(xA_T) = xA_T$ if and only if $g(T) = T$ and $g(m(xA_T)) = m(xA_T)$, and hence $A^K^G = D^G$.

Now, let $P_f, D$, and $K$ denote respectively the poset of finite type special subgroups, the Deligne complex, and its fundamental region, for the Artin system $(A, S)$. Clearly one has a well-defined order preserving map $p: P_f \to P_f$ such that $p(T)$ is the disjoint union of those subsets of $S$ which comprise $T \subseteq S$. The group homomorphism $\phi_G: A \to A$ then naturally induces a simplicial map between the corresponding Deligne complexes:

**Definition 17.** We write $\Phi_G : D \to D$ for the continuous simplicial map which is defined on vertices such that $\Phi_G(a.A_T) = \phi_G(a).A_{p(T)}$, for all $a \in A$ and $T \subseteq P_f$.

Note that, for $Q, R \subseteq S$ and $a, b \in A$, one has $aA_Q \subseteq bA_R$ if and only if both $Q \subseteq R$ and $a^{-1}b \in A_R$. Since $\phi_G$ is a homomorphism, it follows that $\Phi_G$ is defined via an order preserving map $AP_f \to AP_f$ and hence is a well-defined simplicial map.

**Lemma 18.** The image of $\Phi_G$ lies in $A^K^G$ and, moreover,

(i) The restriction of $\Phi_G$ to $K$ is an injective map with image $K^G$.

(ii) The simplicial map $\Phi_G$ is injective if and only if $\Phi_G$ is injective, and has image $A^K^G$ if and only if $\Phi_G$ has image $A^K^G$.

(iii) If $(A, S)$ is of finite type, then $\Phi_G$ is an injective simplicial map with image $D^G$.

**Proof.** It is easily checked that the map $p: P_f \to P_f$ is injective and that its image consists precisely of those subsets of $S$ which are $G$-invariant. Statement (i) then follows immediately. That $\text{im } \Phi_G \subseteq A^K^G$ now follows also, by the fact that $\text{im } \phi_G \subseteq A^G$ which is the case since each $A_T$ for $T \subseteq S$ is fixed by $G$ (cf. the proof of Lemma 10). Statement (ii) follows from (i), the fact that the Deligne complex of an Artin group always has a free orbit (the orbit of \{1\}), and Theorem 11 which ensures that $A_{p(T)} \cap \text{im } \phi_G \subseteq \phi_G(A_T)$ for each $T \in P_f$ (needed for the proof that $\Phi_G$ is injective if $\phi_G$ is). Finally, (iii) follows from (ii) together with Theorem 11 and the fact that in this case $A^K^G = D^G$. 


It is convenient to consider the Deligne complex as a cube complex in the following manner. Given a pair $\eta \leq \mu$ of comparable elements of $AP_f$, we define the cube between $\eta$ and $\mu$ to be the subcomplex of $D$ spanned by the set of vertices $\{\xi \in AP_f \mid \eta \leq \xi \leq \mu\}$. If $U, V \in P_f$ and $U \subseteq V$, then we write $C(U, V)$ for the cube between $A_U$ and $A_V$ and observe that this is indeed a combinatorial cube. It is easily seen that every other cube is a translate of one of these, namely $x.C(U, V)$, the cube between $xA_U$ and $xA_V$, for some $x \in A$ and $U \subseteq V$ in $P_f$.

For $\xi \in AP_f$ we define $\overline{N}(\xi)$ to be the union of all cubes containing the vertex $\xi$ of $D$. This is a closed set whose interior $N(\xi)$ we refer to as the open neighbourhood of $\xi$. We note that $N(\xi)$ is simply the union of the interiors of all cubes containing $\xi$ together with the vertex $\xi$ itself. In particular, the open neighbourhood of a vertex contains no other vertex of the complex $D$. Note also that $N(xA_T) = x.N(A_T)$.

**Lemma 19.** The open neighbourhoods of distinct translates, under $A$, of a vertex of the Deligne complex $D$ are mutually disjoint. That is, given $T \in P_f$ and $x \in A$, either $x \in A_T$ or $N(xA_T) \cap N(A_T) = \emptyset$.

**Proof.** Consider the translates of the vertex $A_T$ for fixed $T \in P_f$. The statement of the lemma follows from the fact that any cube $x.C(U, V)$ contains at most one coset of $A_T$, namely $xA_T$ if $U \subseteq T \subseteq V$, and hence its interior lies in at most one of the open neighbourhoods under consideration.

The collection $C(D) = \{N(\xi) \mid \xi \in AP_f\}$ of open neighbourhoods of vertices forms an open covering of the Deligne complex $D$ which is invariant under the action of $A$. Similarly, one defines an $A$-invariant open covering $C(D)$ of $D$. We define $C(D^G) = \{N(\xi) \cap D^G \mid \xi \in AP_f \cap D^G\}$, an $A^G$-invariant collection of open sets in the subspace topology on $D^G$. Observe that $C(D^G)$ is also an open covering of $D^G$ as follows. Any cube in $D$ which contains a $G$-fixed point $q$ in its interior must be $G$-invariant. Hence the minimal (or maximal) vertex of that cube will be fixed under $G$ and its open neighbourhood will contain $q$. If a $G$-fixed point is not in the interior of some cube, then it must be a fixed vertex itself. Thus every element of $D^G$ lies in the open neighbourhood of some fixed vertex of $D$.

Now, for a given $T \in P_f$, the Deligne complex $DT$ associated to the finite type special subgroup $(A_T, T)$ may be viewed as a subcomplex of $D$ in the obvious way, and is just the union of the cubes $x.C(U, T)$ for $x \in A_T$ and $U \subseteq T$. The closed set $\overline{N}(A_T)$ is the union of all cubes $x.C(U, V)$ such that $x \in A_T$ and $U \subseteq T \subseteq V$. Each of the cubes $C(U, V)$ decomposes as a direct product $C(U, T) \times C(T, V)$ of cubes, with $C(T, V)$ fixed by $A_T$. Writing $K_T$ for the union of cubes $C(T, V)$ with minimal vertex $T$, one now has

$$\overline{N}(A_T) = DT \times K_T.$$
Moreover, \( \tilde{N}(A_T) \) has stabiliser \( A_T \) which acts by fixing the factor \( K_T \) pointwise.

**Lemma 20.** (i) For \( x \in A \) and \( T \in P_f \), the open set \( N(xA_T) \) in \( C(D) \) maps homeomorphically, under \( \Phi_G \), onto the open set \( N(\phi_G(x)A_{p(T)}) \cap D^G \) in \( C(D^G) \).

(ii) Each open set in \( C(D^G) \) which intersects \( \text{im} \Phi_G \) is evenly covered by open sets from \( C(D) \).

**Proof.** (i) Since \( \Phi_G \) is equivariant with respect to the actions of \( A \) on \( D \) and \( \phi_G(A) \subseteq A^G \) on \( D \), we may assume without loss of generality that \( x=1 \). Fixing \( T \in P_f \), it will suffice to show that \( \Phi_G \) maps the closed set \( \tilde{N}(A_T) = D_T \times K_T \) homeomorphically onto \( \tilde{N}(A_{p(T)}) \cap D^G = (D_{p(T)})^G \times (K_{p(T)})^G \).

Since \( K_T \) is a subcomplex of \( K \), it follows by Lemma 18(i) that \( \Phi_G \) maps \( K_T \) isomorphically onto \( (K_{p(T)})^G \). On the other hand, \( (A_T, T) \) is precisely the Artin system derived from the action of \( G \) on the finite type special sub-system \( (A_{p(T)}, p(T)) \) of \( (A, S) \), and, by Lemma 18(iii), one sees that \( \Phi_G \) maps \( D_T \) isomorphically onto \( (D_{p(T)})^G \).

(ii) We note first that, by part (i), any open set from \( C(D^G) \) which is of the form \( N(\phi_G(\xi)) \cap D^G \) for \( \xi \in \mathbb{P}_f \) is evenly covered by the open sets \( N(q) \) for which \( \phi_G(q) = \phi_G(\xi) \), for these sets are mutually disjoint by Lemma 19. Now take an arbitrary open set from \( C(D^G) \), that is \( N(xA_T) \cap D^G \) where \( xA_T \) is a vertex of \( D^G \), which contains the image of some point \( q \in D \). The proof is completed by showing that \( xA_T \) is the image of some vertex of \( D \). If \( q \) is itself a vertex of \( D \) then \( \phi_G(q) = xA_T \) immediately. Otherwise, we may assume \( q \) lies in the interior of some cube \( yC(Q, R) \) whose image spans a cube containing \( xA_T \). That is, \( \phi_G(y) A_{p(Q)} \subseteq xA_T \subseteq \phi_G(y) A_{p(R)} \), from which it follows that \( xA_T = \phi_G(y) A_T \). Finally we note that, since \( xA_T \) is \( G \)-fixed, \( T = p(T) \) for some \( T \in P_f \). Therefore \( xA_T \) is the image of the vertex \( yA_T \) of \( D \).}

**Lemma 21.** Let \( D^G_0 \) denote the path component of \( K^G \) in \( D^G \). Then \( \Phi_G \) defines a covering projection \( D \to D^G_0 \).

**Proof.** It follows immediately from Lemma 20 that \( \Phi_G \) is a covering projection onto its image. We show that \( \text{im} \Phi_G = D^G_0 \).

Note that \( D \), and hence its image under \( \Phi_G \), is path connected. Thus \( \text{im} \Phi_G \subseteq D^G_0 \). By Lemma 20(ii), each neighbourhood in the open cover \( C(D^G) \) is either evenly covered, so contained in \( \text{im} \Phi_G \), or is disjoint from \( \text{im} \Phi_G \) altogether. Thus both \( \text{im} \Phi_G \) and its complement in \( D^G \) are unions of open sets. That is, \( \text{im} \Phi_G \) disconnects \( D^G \) and so must be the whole path component \( D^G_0 \).
In order to establish Theorem A, it is now enough to show that \( D^G \) is path connected and simply connected, for then \( \Phi_G \) must be a homeomorphism onto \( D^G \). In order to obtain such global information we introduce a geometrical hypothesis, namely that \( D \) admits a metric of non-positive curvature.

6. THE CAT(0) HYPOTHESIS AND THE MAIN THEOREM

Suppose that \((X, d)\) is a complete geodesic metric space. By a Theorem of Bridson [5] this is the case when \( X \) is the Deligne complex \( D \) for an Artin group and \( d \) is any piecewise Euclidean metric on \( D \) which is invariant under the group action, for then \( D \) has only finitely many isometry types of cells. We now define what it means for \((X, d)\) to be a CAT(0) space. For further details one may refer to [2, 5].

If \( T = T(a, b, c) \) is a geodesic triangle in \((X, d)\) with vertices \( a, b, c \), then a comparison triangle for \( T(a, b, c) \) is any triangle \( T' = T(a', b', c') \) in the Euclidean plane having the same corresponding sidelengths as \( T \). We say that \((X, d)\) is a CAT(0) space, or that \( d \) is a CAT(0) metric, if every geodesic triangle \( T \) in \( X \) is no “fatter” than its comparison triangle \( T' \). More precisely, for any vertex \( x \) of \( T \) and any point \( t \) lying on the side of \( T \) opposite to \( x \), the distance \( d(x, t) \) is no greater than the distance in the Euclidean plane between the corresponding vertex \( x' \) of \( T' \) and the corresponding point \( t' \) on the side of \( T' \) opposite to \( x' \). We will use the following well-known fundamental properties of a CAT(0) space.

**Proposition 22.** If \((X, d)\) is a CAT(0) space, then

(i) there exists a unique geodesic between any two points in \((X, d)\),

and

(ii) \( X \) is contractible.

Statement (i) is an easy consequence of the definition. To see that statement (ii) is true, one may exhibit a uniformly continuous contraction of \( X \) down to a given basepoint \( x_0 \) by mapping each \( x \in X \), at time \( t \), to the point a distance \( t \cdot d(x_0, x) \) along the geodesic from \( x \) back to \( x_0 \).

We are now able to state and prove the main theorem of this paper.

**Theorem 23.** Let \( G \) be a group of symmetries of the Artin system \((A, S)\), and suppose that the Deligne complex \( D \) associated to \((A, S)\) admits a piecewise Euclidean CAT(0) metric with respect to which every \( g \in G \) is an isometry. Then
(i) the subcomplex $D^G$ fixed by $G$ is a geodesically convex subset of $D$, and hence a CAT(0) subspace, and

(ii) the Artin group $(A, S)$ is isomorphic under the map $\phi_G$ to the fixed subgroup $A^G$ of $A$, and its Deligne complex $D$ is isomorphic under the map $\Phi_G$ to the fixed subcomplex $D^G$ of $D$.

Proof. (i) Take any two points $p$ and $q$ in $D^G$, and let $\gamma$ denote the unique geodesic in $D$ from $p$ to $q$. Now, since each $g \in G$ is an isometry with respect to the CAT(0) metric on $D$, it follows that $g(\gamma)$ is also a geodesic from $p$ to $q$, and by uniqueness it must be the same geodesic as $\gamma$. Moreover, the isometry $g$ cannot reparametrise $\gamma$, so it must fix every point in $\gamma$. Since this holds for every $g \in G$, it follows that $\gamma$ lies wholly in $D^G$, proving that $D^G$ is a geodesically convex subset of $D$, and therefore inherits a CAT(0) path metric.

(ii) Since $D^G$ is therefore both path connected and contractible it follows from Lemma 21 that $\Phi_G$ is an injective simplicial map with image $D^G$. By Lemma 18(ii) one equally has that $\phi_G: A \to A$ is an injective group homomorphism with image $A^G$.

One may easily define the so-called cubical metric, $d_C$, a piecewise Euclidean metric on $D$ such that each cube $x.C(\emptyset, T)$, for $x \in A$ and $T \in P_f$, is isometric to a right Euclidean cube of dimension $|T|$. It is immediately clear that any symmetry of $(A, S)$ induces an isometry of $D$ with respect to this metric. In [10], Charney and Davis show that $(D, d_C)$ is a CAT(0) space if and only if $(A, S)$ is an Artin system of FC type. Thus the main result of this paper, stated as Theorem 4 in the introduction, now follows from Theorem 23 above.

Another very natural piecewise Euclidean metric which may be put on the Deligne complex is the Moussong metric, $d_M$, introduced in [18], and, as before, one may easily check that any symmetry also induces an isometry of $(D, d_M)$. In [10] it is conjectured that $(D, d_M)$ is always a CAT(0) space, and if this were indeed the case then one would have a proof of Conjecture 2, that our main result holds for arbitrary Artin systems.

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