Available online at www.sciencedirect.com
ScienceDirect
Journal of Combinatorial Theory, Series B 97 (2007) 627–646
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

Cubic symmetric graphs of order a small number times a prime or a prime square ☆

Yan-Quan Feng [a], Jin Ho Kwak [b]

[a] *Department of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China*
[b] *Combinatorial and Computational Mathematics Center, Pohang University of Science and Technology, Pohang 790-784, Republic of Korea*

Received 15 April 2002
Available online 29 December 2006

Abstract

A graph is *s-regular* if its automorphism group acts regularly on the set of its s-arcs. In this paper, the s-regular elementary abelian coverings of the complete bipartite graph $K_{3,3}$ and the s-regular cyclic or elementary abelian coverings of the complete graph K_4 for each $s \geqslant 1$ are classified when the fibre-preserving automorphism groups act arc-transitively. A new infinite family of cubic 1-regular graphs with girth 12 is found, in which the smallest one has order 2058. As an interesting application, a complete list of pairwise non-isomorphic s-regular cubic graphs of order $4p$, $6p$, $4p^2$ or $6p^2$ is given for each $s \geqslant 1$ and each prime p.

Keywords: Symmetric graphs; s-Regular graphs; Regular coverings

1. Introduction

In this paper we consider an undirected finite connected graph without loops or multiple edges. For a graph X, we denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, edge set and automorphism group, respectively. For $u, v \in V(X)$, denote by uv the edge incident to u and v in X, and by $N_X(u)$ the *neighborhood* of u in X, that is, the set of vertices adjacent to u in X. A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection

☆ This work was supported by the National Natural Science Foundation of China (Grant No. 10571013), the Key Project of Chinese Ministry of Education (106029), the Specialized Research Fund for the Doctoral Program of Higher Education in China (20060004026), and Com^{2}MaC-KOSEF in Korea (R11-1999-054).

E-mail addresses: yqfeng@center.njtu.edu.cn (Y.-Q. Feng), jinkwak@postech.ac.kr (J.H. Kwak).

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2006.11.001

$p: V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\tilde{v})}: N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. A covering $\widetilde{X}$ of X with a projection p is said to be *regular* (or K-*covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that the graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h (for the purpose of this paper, all functions are composed from left to right). If K is cyclic or elementary abelian then $\widetilde{X}$ is called a *cyclic* or an *elementary abelian covering* of X, and if $\widetilde{X}$ is connected K becomes the covering transformation group. The *fibre* of an edge or a vertex is its preimage under p. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while every covering transformation maps a fibre onto itself. All of fibre-preserving automorphisms form a group called the *fibre-preserving group*.

An s-*arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leqslant i \leqslant s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i < s$; in other words, a directed walk of length s which never includes a backtracking. A graph X is said to be s-*arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. A graph X is said to be *edge-transitive* if $\mathrm{Aut}(X)$ is transitive on $E(X)$ and *half-transitive* if X is vertex-transitive, edge-transitive, but not arc-transitive. A subgroup of the automorphism group of a graph X is said to be s-*regular* if it acts regularly on the set of s-arcs of X. In particular, if the subgroup is the full automorphism group $\mathrm{Aut}(X)$ of X then X is said to be s-*regular*. Thus, if a graph X is s-regular then $\mathrm{Aut}(X)$ is transitive on the set of s-arcs and the only automorphism fixing an s-arc is the identity automorphism of X. It may be easily seen that if X is edge-transitive but not vertex-transitive then X is necessarily bipartite, and if X has regular valency then the two parts of bipartition have equal cardinality. Such a graph will be referred to as a *semisymmetric* graph.

Clearly, a cycle is s-arc-transitive for any $s \geqslant 0$. Tutte [49,50] showed that every finite connected cubic symmetric graph is s-regular for some $s \geqslant 1$, and this s is at most five. Many people have investigated the automorphism groups of cubic symmetric graphs, for example see [7,8,11,43]. Djoković and Miller [11] constructed an infinite family of 2-regular cubic graphs, and Conder and Praeger [8] constructed two infinite families of s-regular cubic graphs for $s = 2$ or 4. Also, several different types of infinite families of tetravalent 1-regular graphs have been constructed in [27,33,39,45,47]. The first cubic 1-regular graph was constructed by Frucht [21] and later Miller [43] constructed an infinite family of cubic 1-regular graphs of order $2p$, where $p \geqslant 13$ is a prime congruent to 1 modulo 3. By Cheng and Oxley's classification of symmetric graphs of order $2p$ [5], Miller's construction is actually the all cubic 1-regular graphs of order $2p$. Marušič and Xu [42] showed a way to construct a cubic 1-regular graph Y from a tetravalent half-transitive graph X with girth 3 by letting the triangles of X be the vertices in Y with two triangles being adjacent when they share a common vertex in X. Using the Marušič and Xu's result, Miller's construction can be generalized to graphs of order $2n$, where $n \geqslant 13$ is odd such that 3 divides $\varphi(n)$, the Euler function (see [1,41]). It may be shown that all cubic 1-regular Cayley graphs on the dihedral groups are exactly those graphs generalized by Miller's construction. Recently, more 1-regular cubic graphs were constructed by the authors [15–17]. Also, as shown in [40] or [41], one can see an importance of a study for cubic 1-regular graphs in connection with chiral (that is regular and irreflexible) maps on a surface by means of tetravalent half-transitive graphs or in connection with symmetries of hexagonal molecular graphs on the torus.

Covering techniques have long been known as a powerful tool in topology and graph theory. Regular covering of a graph is currently an active topic in algebraic graph theory. Malnič et al. [37] classified the cubic semisymmetric cyclic coverings of the bipartite graph $K_{3,3}$ when the fibre-preserving group contains an edge-, but not vertex-transitive subgroup. By using the covering technique, cubic semisymmetric graphs of order $6p$ and $2p^3$ were classified in [29,38]. Some general methods of elementary abelian coverings were developed in [13,35,36]. By using the method developed in [36], Malnič and Potočnik [32] classified the vertex-transitive elementary abelian coverings of the Petersen graph when the fibre-preserving group is vertex-transitive. To investigate the s-regular cyclic or elementary abelian coverings of a graph, we like to assume that the fibre-preserving group is arc-transitive. As a part of this paper, we classify the s-regular cyclic or elementary abelian coverings of the complete graph K_4 and the s-regular elementary abelian coverings of the complete bipartite graph $K_{3,3}$ for each $1 \leqslant s \leqslant 5$. The s-regular cyclic coverings of the complete bipartite graph $K_{3,3}$ are classified for each $1 \leqslant s \leqslant 5$ in [18]. For the hypercube Q_3, its s-regular cyclic or elementary abelian coverings are given in [19,20]. As an application of those classifications, this paper provides a classification of s-regular cubic graphs of order $4p$, $4p^2$, $6p$ and $6p^2$ for each $1 \leqslant s \leqslant 5$ and each prime p. In particular, we find a new infinite family of cubic 1-regular graphs, in which the smallest one has order 2058. Each graph in this infinite family has girth 12 and a solvable automorphism group so that it does not belong to any family of cubic 1-regular graph discussed in the previous paragraph. As mentioned in the previous paragraph, there are infinitely many cubic 1-regular graphs of order $2p$ (see [5]). In this paper, we show that besides a few sporadic graphs, there are an infinite family of cubic 1-regular graphs of order $6p$ or $6p^2$ respectively, and another infinite family of cubic 2-regular graphs of order $6p$ (see Theorems 5.2 and 5.3). Surprisingly, there are only five connected cubic symmetric graphs of order $4p$ or $4p^2$ (see Theorem 6.2). A similar work for order $8p$ or $8p^2$ was done in [20].

2. Preliminaries related to coverings

Let X be a graph and K a finite group. By a^{-1} we mean the reverse arc to an arc a. A *voltage assignment* (or, *K-voltage assignment*) of X is a function $\phi : A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of ϕ are called *voltages*, and K is the *voltage group*. The *graph $X \times_\phi K$* derived from a voltage assignment $\phi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e, g) of $X \times_\phi K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $p : X \times_\phi K \to X$, which is called the *natural projection*. By defining $(u, g')^g := (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\phi K)$, K becomes a subgroup of $\mathrm{Aut}(X \times_\phi K)$ which acts semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a *K-covering*. For each $u \in V(X)$ and $uv \in E(X)$, the vertex set $\{(u, g) \mid g \in K\}$ is the fibre of u and the edge set $\{(u, g)(v, \phi(a)g) \mid g \in K\}$ is the fibre of uv, where $a = (u, v)$. Conversely, each regular covering $\widetilde{X}$ of X with a covering transformation group K can be derived from a K-voltage assignment. Given a spanning tree T of the graph X, a voltage assignment ϕ is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [23] showed that every regular covering $\widetilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X. It is clear that if ϕ is reduced, the derived graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

Let $\widetilde{X}$ be a K-covering of X with a projection p. If $\alpha \in \mathrm{Aut}(X)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\widetilde{\alpha} p = p\alpha$, we call $\widetilde{\alpha}$ a *lift* of α, and α the *projection* of $\widetilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$, respectively. In particular, if the covering graph $\widetilde{X}$ is connected, then the covering transformation group K is the lift of the trivial group, that is, $K = \{\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X}):\ p = \widetilde{\alpha} p\}$. Clearly, if $\widetilde{\alpha}$ is a lift of α, then $K\widetilde{\alpha}$ are the all lifts of α.

Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. The problem whether an automorphism α of X lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\bar{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by

$$\big(\phi(C)\big)^{\bar{\alpha}} = \phi\big(C^\alpha\big),$$

where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages on C and C^α, respectively. Note that if K is abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X.

The next proposition is a special case of [30, Theorem 4.2].

Proposition 2.1. *Let $X \times_\phi K \to X$ be a connected K-covering derived from a T-reduced voltage assignment ϕ. Then, an automorphism α of X lifts if and only if $\bar{\alpha}$ extends to an automorphism of K.*

For more results on the lifts of automorphisms of X, we refer the reader to [2,3,10,31,34]. Let X be a graph and let N be a subgroup of $\mathrm{Aut}(X)$. Denote by $\underline{X}$ the quotient graph corresponding to the orbits of N, that is the graph having the orbits of N as vertices with two orbits adjacent in $\underline{X}$ whenever there is an edge between those orbits in X. In view of Theorem 9 of [28] (see also [46]), we have

Proposition 2.2. *Let X be a connected symmetric graph of prime valency and G an s-arc-transitive subgroup of* $\mathrm{Aut}(X)$ *for some $s \geqslant 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-arc-transitive subgroup of* $\mathrm{Aut}(\underline{X})$ *where $\underline{X}$ is the quotient graph of X corresponding to the orbits of N. Furthermore, X is a regular covering of $\underline{X}$ with the covering transformation group N.*

The next proposition was first mentioned in [12, Lemma 2.2] with no restriction on the voltage group K. However, it is not true if K is not abelian. For a correction, see [14].

Proposition 2.3. [12, Lemma 2.2] *Let $X \times_\phi K$ be a connected regular covering of a graph X with the covering transformation group K, and let $\alpha \in$* $\mathrm{Aut}(X)$ *be an automorphism one of whose liftings $\tilde{\alpha}$ centralizes K. If K is abelian then $\phi(C^\alpha) = \phi(C)$ for any cycle C of X.*

Two coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ of X with projections p_1 and p_2 respectively, are said to be *equivalent* if there exists a graph isomorphism $\widetilde{\alpha} : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\widetilde{\alpha} p_2 = p_1$. We quote the following proposition.

Proposition 2.4. [25,48] *Two connected regular coverings $X \times_\phi K$ and $X \times_\psi K$, where ϕ and ψ are T-reduced, are equivalent if and only if there exists an automorphism $\sigma \in$* $\mathrm{Aut}(K)$ *such that $\phi(u, v)^\sigma = \psi(u, v)$ for any cotree arc (u, v) of X.*

The next proposition is due to Burnside.

Proposition 2.5. [26, Chapter IV, Theorem 2.6] *Let G be a finite group and let P be a Sylow p-subgroup of G. Let $N_G(P)$ denote the normalizer of P in G and $C_G(P)$ the centralizer of P in G. If $N_G(P) = C_G(P)$, then G has a normal subgroup N such that $G/N \cong P$.*

3. Graph constructions and isomorphisms

Denote by $\mathbb{Z}_n$ the cyclic group of order n, as well as the ring of integers modulo n. Let $\mathbb{Z}_n^*$ be the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, and for a prime p let $\mathbb{Z}_p^m$ be the elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (m times), as well as the m-dimensional row vector space over the field $\mathbb{Z}_p$. In this section, we construct some examples of cubic symmetric graphs to use later for classifications of cubic symmetric graphs of order $4p$, $4p^2$, $6p$ or $6p^2$. The first example is the well-known generalized Petersen graphs, while others come from the cyclic or the elementary abelian coverings of the complete graph K_4 or the bipartite graph $K_{3,3}$.

Example 3.1. Let $n \geqslant 3$ and $k \in \mathbb{Z}_n \backslash \{0\}$. The *generalized Petersen graph* $P(n,k)$ is the graph with vertex-set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{x_i x_{i+1}, x_i y_i, y_i y_{i+k} \mid i \in \mathbb{Z}_n\}$.

In the following examples, let $V(K_4) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and $V(K_{3,3}) = \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be the vertex sets of K_4 and $K_{3,3}$ as illustrated in Figs. 3 and 1, respectively. The first letter of the names of the graphs in those examples is E or C which stands for elementary abelian or cyclic, and the second one is C or B which stands for the complete graph K_4 or the bipartite graph $K_{3,3}$. For example, EB stands for an elementary abelian covering graph of the bipartite graph $K_{3,3}$.

Example 3.2. Let p be a prime and let $\mathbb{Z}_p^3$ be the 3-dimensional row vector space over the field $\mathbb{Z}_p$. Take the standard basis vectors: $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. The graph EC_{p^3} is defined to have vertex set $V(EC_{p^3}) = V(K_4) \times \mathbb{Z}_p^3$ and edge set

$$\begin{aligned} E(EC_{p^3}) = \big\{&(\mathbf{a},x)(\mathbf{b},x),\ (\mathbf{a},x)(\mathbf{c},x),\ (\mathbf{a},x)(\mathbf{d},x),\ (\mathbf{b},x)(\mathbf{c},x+e_1),\ (\mathbf{c},x)(\mathbf{d},x+e_2),\\ &(\mathbf{d},x)(\mathbf{b},x+e_3) \mid x \in \mathbb{Z}_p^3\big\}. \end{aligned}$$

Later in Theorem 6.1, it will be shown that the graph EC_{p^3} can be described as a $\mathbb{Z}_p^3$-covering of the complete graph K_4 and it is a cubic 2-regular graph.

Example 3.3. Let $n = p$ or p^2 such that $p-1$ is divisible by 3 and let λ be an element of order 3 in the multiplicative group $\mathbb{Z}_n^*$ of $\mathbb{Z}_n$. The graphs CB_n and $\overline{CB}_n$ are defined to have the same vertex set $V(CB_n) = V(\overline{CB}_n) = V(K_{3,3}) \times \mathbb{Z}_n$ and edge sets

$$\begin{aligned} E(CB_n) = \big\{&(\mathbf{u},i)(\mathbf{x},i),\ (\mathbf{u},i)(\mathbf{y},i),\ (\mathbf{u},i)(\mathbf{z},i),\ (\mathbf{v},i)(\mathbf{y},i),\ (\mathbf{v},i)(\mathbf{x},i+\lambda+1),\\ &(\mathbf{v},i)(\mathbf{z},i+1),\ (\mathbf{w},i)(\mathbf{x},i-1),\ (\mathbf{w},i)(\mathbf{y},i-\lambda-1),\ (\mathbf{w},i)(\mathbf{z},i) \mid i \in \mathbb{Z}_n\big\},\\ E(\overline{CB}_n) = \big\{&(\mathbf{u},i)(\mathbf{x},i),\ (\mathbf{u},i)(\mathbf{y},i),\ (\mathbf{u},i)(\mathbf{z},i),\ (\mathbf{v},i)(\mathbf{y},i),\ (\mathbf{v},i)\big(\mathbf{x},i+\lambda^2+1\big),\\ &(\mathbf{v},i)(\mathbf{z},i+1),\ (\mathbf{w},i)(\mathbf{x},i-1),\ (\mathbf{w},i)\big(\mathbf{y},i-\lambda^2-1\big),\ (\mathbf{w},i)(\mathbf{z},i) \mid i \in \mathbb{Z}_n\big\}, \end{aligned}$$

respectively. It is easy to see that both CB_n and $\overline{CB}_n$ are cubic and bipartite. Note that there are only two elements of order 3 in $\mathbb{Z}_n^*$, that is, λ and λ^2. The graph $\overline{CB}_n$ is obtained by replacing λ with λ^2 in each edge of CB_n. It will be shown that $CB_n \cong \overline{CB}_n$ in Lemma 3.7. Thus, the graph CB_n is independent of the choice of λ. It will be shown that the graphs CB_p and CB_{p^2} $(p > 3)$ are 1-regular in Lemma 5.1.

Example 3.4. Let p be a prime and let $\mathbb{Z}_p^2$ be the 2-dimensional row vector space over the field $\mathbb{Z}_p$. Take the standard basis vectors: $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The graphs EB_{p^2} and $\overline{EB}_{p^2}$ are defined to have the same vertex set $V(EB_{p^2}) = V(\overline{EB}_{p^2}) = V(K_{3,3}) \times \mathbb{Z}_p^2$ and edge sets

$$\begin{aligned} E(EB_{p^2}) = \big\{&(\mathbf{u},x)(\mathbf{x},x),\ (\mathbf{u},x)(\mathbf{y},x),\ (\mathbf{u},x)(\mathbf{z},x),\ (\mathbf{v},x)(\mathbf{y},x),\ (\mathbf{v},x)(\mathbf{x},x+e_2),\\ &(\mathbf{v},x)(\mathbf{z},x+e_1), (\mathbf{w},x)(\mathbf{x},x-e_1), (\mathbf{w},x)(\mathbf{z},x), (\mathbf{w},x)(\mathbf{y},x-e_2) \mid x \in \mathbb{Z}_p^2\big\},\\ E(\overline{EB}_{p^2}) = \big\{&(\mathbf{u},x)(\mathbf{x},x), (\mathbf{u},x)(\mathbf{y},x),\ (\mathbf{u},x)(\mathbf{z},x),\ (\mathbf{v},x)(\mathbf{y},x),\ (\mathbf{v},x)(\mathbf{x},x+e_2),\\ &(\mathbf{v},x)(\mathbf{z},x+e_1),\ (\mathbf{w},x)(\mathbf{z},x),\ (\mathbf{w},x)(\mathbf{x},x+e_2),\\ &(\mathbf{w},x)(\mathbf{y},x-e_1+e_2) \mid x \in \mathbb{Z}_p^2\big\}, \end{aligned}$$

respectively. It will be shown that $EB_{p^2} \cong \overline{EB}_{p^2}$ in Lemma 3.7. Both graphs are bipartite and 2-regular as covering graphs of the complete bipartite graph $K_{3,3}$ (see Theorem 4.1).

Example 3.5. Let $p = 3$ or p be a prime such that $p - 1$ is divisible by 3. Let $\mathbb{Z}_p^3$ be the 3-dimensional row vector space over the field $\mathbb{Z}_p$. Take the standard basis vectors: $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. If $p = 3$ let $\lambda = 1$ and if $p > 3$ let λ be an element of order 3 in the multiplicative group $\mathbb{Z}_p^*$ of $\mathbb{Z}_p$. The graphs EB_{p^3} and $\overline{EB}_{p^3}$ are defined to have the same vertex set $V(EB_{p^3}) = V(\overline{EB}_{p^3}) = V(K_{3,3}) \times \mathbb{Z}_p^3$ and edge sets

$$\begin{aligned} E(EB_{p^3}) = \big\{&(\mathbf{u},x)(\mathbf{x},x),\ (\mathbf{u},x)(\mathbf{y},x),\ (\mathbf{u},x)(\mathbf{z},x),\ (\mathbf{v},x)(\mathbf{y},x),\ (\mathbf{w},x)(\mathbf{z},x),\\ &(\mathbf{v},x)(\mathbf{x},x+e_3),\ (\mathbf{v},x)(\mathbf{z},x+e_1),\ (\mathbf{w},x)(\mathbf{x},x+e_2),\\ &(\mathbf{w},x)\big(\mathbf{y},x-e_1-\lambda e_2+(1+\lambda)e_3\big) \mid x \in \mathbb{Z}_p^3\big\},\\ E(\overline{EB}_{p^3}) = \big\{&(\mathbf{u},x)(\mathbf{x},x),\ (\mathbf{u},x)(\mathbf{y},x),\ (\mathbf{u},x)(\mathbf{z},x),\ (\mathbf{v},x)(\mathbf{y},x),\ (\mathbf{w},x)(\mathbf{z},x),\\ &(\mathbf{v},x)(\mathbf{x},x+e_3),\ (\mathbf{v},x)(\mathbf{z},x+e_1),\ (\mathbf{w},x)(\mathbf{x},x+e_2),\\ &(\mathbf{w},x)\big(\mathbf{y},x-e_1-\lambda^2 e_2+\big(1+\lambda^2\big)e_3\big) \mid x \in \mathbb{Z}_p^3\big\}, \end{aligned}$$

respectively. The graph $\overline{EB}_{p^3}$ is obtained by replacing λ with λ^2 in each edge of EB_{p^3}. It will be shown that $EB_{p^3} \cong \overline{EB}_{p^3}$ in Lemma 3.7. Thus, the graph EB_{p^3} is independent of the choice of λ because λ and λ^2 are the only two elements of order 3 in $\mathbb{Z}_p^*$ for $p > 3$. The graph EB_{3^3} is 3-regular and EB_{p^3} ($p > 3$) is 1-regular as covering graphs of the complete bipartite graph $K_{3,3}$ (see Theorem 4.1).

Example 3.6. Let p be a prime and let $\mathbb{Z}_p^4$ be the 4-dimensional row vector space over the field $\mathbb{Z}_p$. Take the standard basis vectors: $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$ and $e_4 = (0, 0, 0, 1)$. The graph EB_{p^4} is defined to have vertex set $V(EB_{p^4}) = V(K_{3,3}) \times \mathbb{Z}_p^4$ and edge set

$$\begin{aligned} E(EB_{p^4}) = \big\{&(\mathbf{u},x)(\mathbf{x},x),\ (\mathbf{u},x)(\mathbf{y},x),\ (\mathbf{u},x)(\mathbf{z},x),\ (\mathbf{v},x)(\mathbf{y},x),\ (\mathbf{v},x)(\mathbf{x},x+e_3),\\ &(\mathbf{w},x)(\mathbf{z},x),\ (\mathbf{v},x)(\mathbf{z},x+e_1),\ (\mathbf{w},x)(\mathbf{x},x+e_2),\ (\mathbf{w},x)(\mathbf{y},x+e_4) \mid x \in \mathbb{Z}_p^4\big\}. \end{aligned}$$

The graph EB_{p^4} is 3-regular as a covering graph of the complete bipartite graph $K_{3,3}$ (see Theorem 4.1).

Among the examples of cubic symmetric graphs constructed above, some of them are isomorphic.

Lemma 3.7. *$EB_{p^2} \cong \overline{EB}_{p^2}$, $EB_{p^3} \cong \overline{EB}_{p^3}$, $CB_p \cong \overline{CB}_p$ and $CB_{p^2} \cong \overline{CB}_{p^2}$.*

Proof. To show $EB_{p^2} \cong \overline{EB}_{p^2}$, we define a map α from $V(EB_{p^2})$ to $V(\overline{EB}_{p^2})$ by

$$\begin{aligned}
&(\mathbf{u}, ke_1 + \ell e_2) \mapsto \bigl(\mathbf{u}, (k+\ell)e_1 - ke_2\bigr),\\
&(\mathbf{v}, ke_1 + \ell e_2) \mapsto \bigl(\mathbf{w}, (k+\ell+1)e_1 - (k+1)e_2\bigr),\\
&(\mathbf{w}, ke_1 + \ell e_2) \mapsto \bigl(\mathbf{v}, (k+\ell-1)e_1 - ke_2\bigr),\\
&(\mathbf{x}, ke_1 + \ell e_2) \mapsto \bigl(\mathbf{x}, (k+\ell)e_1 - ke_2\bigr),\\
&(\mathbf{y}, ke_1 + \ell e_2) \mapsto \bigl(\mathbf{y}, (k+\ell)e_1 - ke_2\bigr),\\
&(\mathbf{z}, ke_1 + \ell e_2) \mapsto \bigl(\mathbf{z}, (k+\ell)e_1 - ke_2\bigr),
\end{aligned}$$

where $k, \ell \in \mathbb{Z}_p$. Clearly,

$$\begin{aligned}
N_{EB_{p^2}}\bigl((\mathbf{w}, ke_1 + \ell e_2)\bigr) &= \bigl\{\bigl(\mathbf{x}, (k-1)e_1 + \ell e_2\bigr), \bigl(\mathbf{y}, ke_1 + (\ell-1)e_2\bigr), (\mathbf{z}, ke_1 + \ell e_2)\bigr\},\\
N_{\overline{EB}_{p^2}}\bigl((\mathbf{w}, ke_1 + \ell e_2)^\alpha\bigr) &= N_{\overline{EB}_{p^2}}\bigl(\bigl(\mathbf{v}, (k+\ell-1)e_1 - ke_2\bigr)\bigr)\\
&= \bigl\{\bigl(\mathbf{x}, (k+\ell-1)e_1 + (1-k)e_2\bigr), \bigl(\mathbf{y}, (k+\ell-1)e_1 - ke_2\bigr),\\
&\qquad \bigl(\mathbf{z}, (k+\ell)e_1 - ke_2\bigr)\bigr\}.
\end{aligned}$$

Now, one can easily show that

$$\bigl[N_{EB_{p^2}}\bigl((\mathbf{w}, ke_1 + \ell e_2)\bigr)\bigr]^\alpha = N_{\overline{EB}_{p^2}}\bigl((\mathbf{w}, ke_1 + \ell e_2)^\alpha\bigr).$$

Similarly, one can show that

$$\bigl[N_{EB_{p^2}}\bigl((\mathbf{e}, ke_1 + \ell e_2)\bigr)\bigr]^\alpha = N_{\overline{EB}_{p^2}}\bigl((\mathbf{e}, ke_1 + \ell e_2)^\alpha\bigr)$$

for $\mathbf{e} = \mathbf{u}$ or $\mathbf{v}$. This implies that α is an isomorphism from EB_{p^2} to $\overline{EB}_{p^2}$ because the graphs are bipartite, that is, $EB_{p^2} \cong \overline{EB}_{p^2}$.

If $p = 3$, then $EB_{p^3} = \overline{EB}_{p^3}$ by definition. Hence, to prove the last three isomorphic relations, we assume that $n = p$ or p^2 such that $p-1$ is divisible by 3. For such an n, by [17, Lemma 3.4], λ is an element of order 3 in $\mathbb{Z}_n^*$ if and only if $\lambda^2 + \lambda + 1 \equiv 0 \pmod n$. With the aid of this property, one can prove that the following two maps β and γ are actually isomorphisms to show $EB_{p^3} \cong \overline{EB}_{p^3}$ and $CB_n \cong \overline{CB}_n$ for $n = p$ or $n = p^2$.

For $i, j, k \in \mathbb{Z}_p$, a map β from $V(EB_{p^3})$ to $V(\overline{EB}_{p^3})$ is defined by

$$\begin{aligned}
&(\mathbf{u}, ie_1 + je_2 + ke_3) \mapsto \bigl(\mathbf{u}, -ie_1 + \bigl(k - \lambda^2 i\bigr)e_2 + (j - \lambda i)e_3\bigr),\\
&(\mathbf{v}, ie_1 + je_2 + ke_3) \mapsto \bigl(\mathbf{w}, -ie_1 + \bigl(k - \lambda^2 i\bigr)e_2 + (j - \lambda i)e_3\bigr),\\
&(\mathbf{w}, ie_1 + je_2 + ke_3) \mapsto \bigl(\mathbf{v}, -ie_1 + \bigl(k - \lambda^2 i\bigr)e_2 + (j - \lambda i)e_3\bigr),\\
&(\mathbf{x}, ie_1 + je_2 + ke_3) \mapsto \bigl(\mathbf{x}, -ie_1 + \bigl(k - \lambda^2 i\bigr)e_2 + (j - \lambda i)e_3\bigr),\\
&(\mathbf{y}, ie_1 + je_2 + ke_3) \mapsto \bigl(\mathbf{z}, -ie_1 + \bigl(k - \lambda^2 i\bigr)e_2 + (j - \lambda i)e_3\bigr),\\
&(\mathbf{z}, ie_1 + je_2 + ke_3) \mapsto \bigl(\mathbf{y}, -ie_1 + \bigl(k - \lambda^2 i\bigr)e_2 + (j - \lambda i)e_3\bigr),
\end{aligned}$$

and for $i \in \mathbb{Z}_n$, a map γ from $V(CB_n)$ to $V(\overline{CB}_n)$ is defined by

$$\begin{aligned}
&(\mathbf{u}, i) \mapsto (\mathbf{u}, \lambda i), \qquad (\mathbf{v}, i) \mapsto (\mathbf{w}, \lambda i), \qquad (\mathbf{w}, i) \mapsto (\mathbf{v}, \lambda i),\\
&(\mathbf{x}, i) \mapsto (\mathbf{x}, \lambda i), \qquad (\mathbf{y}, i) \mapsto (\mathbf{z}, \lambda i), \qquad (\mathbf{z}, i) \mapsto (\mathbf{y}, \lambda i). \qquad \square
\end{aligned}$$

4. Elementary abelian coverings of $K_{3,3}$

In [18], the authors classified the s-regular cyclic coverings of the complete bipartite graph $K_{3,3}$. In this section, we classify the s-regular elementary abelian coverings of $K_{3,3}$.

Theorem 4.1. *Let $\widetilde{X}$ be a connected regular covering of the complete bipartite graph $K_{3,3}$, whose covering transformation group is elementary abelian but not cyclic and whose fibre-preserving group is arc-transitive. Then, $\widetilde{X}$ is* 1-, 2- *or* 3-*regular. Furthermore,*

1. *$\widetilde{X}$ is 1-regular if and only if $\widetilde{X}$ is isomorphic to one of EB_{p^3} for a prime p such that $p-1$ is divisible by* 3 *(defined in Example* 3.5*).*
2. *$\widetilde{X}$ is 2-regular if and only if $\widetilde{X}$ is isomorphic to one of EB_{p^2} for a prime p (defined in Example* 3.4*).*
3. *$\widetilde{X}$ is 3-regular if and only if $\widetilde{X}$ is isomorphic to EB_{3^3} or one of EB_{p^4} for a prime p (defined in Example* 3.6*).*

Remark. From Theorem 4.1, one can construct an infinite family of cubic 1-regular graphs of type EB_{p^3}, where $p-1$ is divisible by 3. Since 7 is the smallest such a prime, the smallest cubic 1-regular graph in this family has order 2058. By Example 3.5, $(\mathbf{u},0)$, $(\mathbf{y},0)$, $(\mathbf{w},e_1+\lambda e_2-(1+\lambda)e_3)$, $(\mathbf{z},e_1+\lambda e_2-(1+\lambda)e_3)$, $(\mathbf{u},e_1+\lambda e_2-(1+\lambda)e_3)$, $(\mathbf{x},e_1+\lambda e_2-(1+\lambda)e_3)$, $(\mathbf{w},e_1+(\lambda-1)e_2-(1+\lambda)e_3)$, $(\mathbf{y},-e_2)$, $(\mathbf{u},-e_2)$, $(\mathbf{z},-e_2)$, $(\mathbf{w},-e_2)$ and $(\mathbf{x},0)$ consist of the vertices of a 12-cycle in EB_{p^3}. Thus, one may prove that this infinite family of cubic 1-regular graph has girth 12.

Proof. Let $\widetilde{X}=K_{3,3}\times_\phi \mathbb{Z}_p^m$ $(m\geqslant 2)$ be a covering graph of the graph $K_{3,3}$ satisfying the hypotheses in the theorem, where p is a prime and $\phi=0$ on the spanning tree T which is depicted by dark lines in Fig. 1. We assign voltages z_1, z_2, z_3 and z_4 to the cotree arcs as shown in Fig. 1 where $z_i\in\mathbb{Z}_p^m$ $(i=1,2,3,4)$. Note that the vertices of $K_{3,3}$ is labelled by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$. By the hypotheses, the fibre-preserving group, say $\widetilde{L}$, of the covering graph $K_{3,3}\times_\phi\mathbb{Z}_p^m$ acts arc-transitively on $K_{3,3}\times_\phi\mathbb{Z}_p^m$. Hence, the projection of $\widetilde{L}$, say L, is arc-transitive on the base graph $K_{3,3}$. Clearly, L is also vertex transitive on $K_{3,3}$. Since $K_{3,3}\times_\phi\mathbb{Z}_p^m$ is assumed to be connected, $\langle z_1,z_2,z_3,z_4\rangle=\mathbb{Z}_p^m$.

By the act-transitivity of L, $|L|=|\widetilde{L}/\mathbb{Z}_p^m|$ is divisible by 18. Since $\mathrm{Aut}(K_{3,3})\cong(S_3\times S_3)\rtimes\mathbb{Z}_2$, L and $\mathrm{Aut}(K_{3,3})$ have the same normal Sylow 3-subgroup, say P_3. Thus, we may assume $P_3=\langle\alpha_1\rangle\times\langle\alpha_2\rangle$, where $\alpha_1=(\mathbf{uvw})$ and $\alpha_2=(\mathbf{xyz})$. It is easy to see that P_3 has two orbits, which are

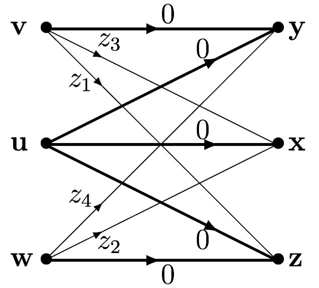

Fig. 1. The complete bipartite graph $K_{3,3}$ with voltage assignment ϕ.

Table 1
Fundamental cycles and their images with corresponding voltages

C	$\phi(C)$	C^{α_1}	$\phi(C^{\alpha_1})$	C^{α_2}	$\phi(C^{\alpha_2})$	C^{β}	$\phi(C^{\beta})$	C^{γ}	$\phi(C^{\gamma})$	C^{δ}	$\phi(C^{\delta})$
vzuy	z_1	**wzvy**	$-z_1-z_4$	**vxuz**	z_3-z_1	**ywxv**	$z_2-z_3-z_4$	**wyuz**	z_4	**yvxw**	$z_3+z_4-z_2$
wxuz	z_2	**uxvz**	z_1-z_3	**wyux**	z_4-z_2	**zuxw**	$-z_2$	**vxuy**	z_3	**zuxv**	z_1-z_3
vxuy	z_3	**wxvy**	$z_2-z_3-z_4$	**vyuz**	$-z_1$	**yuxv**	$-z_3$	**wxuz**	z_2	**yuxw**	z_4-z_2
wyuz	z_4	**uyvz**	z_1	**wzux**	$-z_2$	**zvxw**	$z_3-z_1-z_2$	**vzuy**	z_1	**zwxv**	$z_1+z_2-z_3$

actually the two bipartite subsets of the graph $K_{3,3}$. Since L is vertex-transitive on $V(K_{3,3})$ and $\mathrm{Aut}(K_{3,3})$ has no element of order 8, there exists either an involution, say β, or an element of order 4, say δ, in L, which interchanges the two bipartite subsets of the graph $K_{3,3}$. It is also easy to see that β is a product of three transpositions and δ is a product of one cycle of length 4 and a transposition. Clearly, we may assume $\beta = (\mathbf{ux})(\mathbf{vy})(\mathbf{wz})$ and by considering the conjugates of δ under the elements α_1 and α_2, one may assume $\delta = (\mathbf{vywz})(\mathbf{ux})$. Thus, $\alpha_1, \alpha_2 \in L$, and either $\beta \in L$ or $\delta \in L$. Set $\gamma = \delta^2$. Then, α_1, α_2, β, γ and δ are automorphisms of $K_{3,3}$. It is easy to prove that $\langle \alpha_1, \alpha_2, \beta\rangle \cong \mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$, $\langle \alpha_1, \alpha_2, \beta, \gamma\rangle \cong \mathbb{Z}_3^2 \rtimes \mathbb{Z}_2^2$ and $\langle \alpha_1, \alpha_2, \beta, \delta\rangle \cong (S_3 \times S_3) \rtimes \mathbb{Z}_2$, which are 1-regular, 2-regular and 3-regular, respectively.

Denote by $i_1 i_2 \cdots i_s$ the cycle having the consecutively adjacent vertices $i_1, i_2, \ldots, i_s$. There are four fundamental cycles **vzuy**, **wxuz**, **vxuy** and **wyuz** in $K_{3,3}$, which are generated by the four cotree arcs $(\mathbf{v}, \mathbf{z})$, $(\mathbf{w}, \mathbf{x})$, $(\mathbf{v}, \mathbf{x})$ and $(\mathbf{w}, \mathbf{y})$, respectively. Each cycle maps to a cycle of the same length under the actions of α_1, α_2, β, γ and δ. We list all these cycles and their voltages in Table 1, in which C denotes a fundamental cycle of $K_{3,3}$ and $\phi(C)$ denotes the voltage on C.

Consider the mapping $\bar{\alpha}_1$ from the set of voltages on the four fundamental cycles of $K_{3,3}$ to the elementary abelian group $\mathbb{Z}_p^m$, defined by $\phi(C)^{\bar{\alpha}_1} = \phi(C^{\alpha_1})$, where C ranges over the four fundamental cycles. Similarly, we can define $\bar{\alpha}_2$, $\bar{\beta}$, $\bar{\gamma}$ and $\bar{\delta}$. By Proposition 2.1, either $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\beta}$ or $\bar{\alpha}_1$, $\bar{\alpha}_2$ and $\bar{\delta}$ can be extended to automorphisms of $\mathbb{Z}_p^m$. We denote by α_1^*, α_2^*, β^* and δ^* these automorphisms, respectively. Then, α_1^* and α_2^* always exist and either β^* or δ^* exists.

Since α_1^* and α_2^* always exist, by Table 1, $z_1^{\alpha_1^*} = -z_1 - z_4$ and $z_1^{\alpha_2^*} = z_3 - z_1$. If $z_1 = 0$ then $z_3 = z_4 = 0$, contrary to the fact that $\langle z_1, z_2, z_3, z_4\rangle = \mathbb{Z}_p^m$ $(m \geqslant 2)$. Thus, $z_1 \neq 0$ and similarly, all $z_2, z_3, z_4 \neq 0$. Now, we consider three cases: $K = \mathbb{Z}_p^2$, $\mathbb{Z}_p^3$ or $\mathbb{Z}_p^m$ $(m \geqslant 4)$.

Case I. $K = \mathbb{Z}_p^m$ $(m \geqslant 4)$.

In this case, $\langle z_1, z_2, z_3, z_4\rangle = K$ implies that $m = 4$ and z_1, z_2, z_3, z_4 are linearly independent. By Table 1, it is easy to check that $\phi((\mathbf{vzuy})^\alpha)$, $\phi((\mathbf{wxuz})^\alpha)$, $\phi((\mathbf{vxuy})^\alpha)$ and $\phi((\mathbf{wyuz})^\alpha)$ are linearly independent for $\alpha = \alpha_1, \alpha_2, \beta$ or δ. Thus, $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}, \bar{\delta}$ can be extended to automorphisms of $\mathbb{Z}_p^4$ and by Proposition 2.1, α_1, α_2, β and δ lift to automorphisms of $K_{3,3} \times_\phi \mathbb{Z}_p^m$. Since $\mathrm{Aut}(K_{3,3}) = \langle \alpha_1, \alpha_2, \beta, \delta\rangle$, $\mathrm{Aut}(K_{3,3})$ lifts and so $\mathrm{Aut}(\widetilde{X})$ contains a 3-regular subgroup. By Djoković and Miller [11, Theorem 3], no s-regular group for $s > 3$ contains a 3-regular subgroup and so $\widetilde{X}$ is 3-regular. By Proposition 2.4 with the linear independence of z_1, z_2, z_3 and z_4, one may assume that $z_1 = e_1$, $z_2 = e_2$, $z_3 = e_3$ and $z_4 = e_4$ are the standard basis of the vector space $\mathbb{Z}_p^4$. By Example 3.6, $\widetilde{X} \cong EB_{p^4}$.

Case II. $K = \mathbb{Z}_p^2$.

Suppose that z_1 and z_3 are linearly dependent. Then, one may assume that $z_3 = kz_1$ for some $k \in \mathbb{Z}_p$ and $z_3 \neq 0$ implies that $k \neq 0$. By Table 1, $z_3^{\alpha_1^*} = kz_1^{\alpha_1^*}$ and $z_3^{\alpha_2^*} = kz_1^{\alpha_2^*}$ imply that $z_2 = (1-k)z_4$ and $kz_3 = (k-1)z_1$. Substituting $z_3 = kz_1$ to the second equation, one has $k^2 - k + 1 = 0$ in the field $\mathbb{Z}_p$. We know that β or δ lifts. If β lifts then $z_3^{\beta^*} = kz_1^{\beta^*}$ implies that $(k-1)z_3 = kz_2 - kz_4 = k(1-k)z_4 - kz_4 = -k^2 z_4$. It follows that $z_4 = k^{-2}(1-k)z_3$, where k^{-2} is the inverse of k^2 in $\mathbb{Z}_p^*$ $(k \neq 0)$. Thus, $K = \langle z_1, z_2, z_3, z_4\rangle = \langle z_1\rangle$, which is impossible because $K = \mathbb{Z}_p^2$. If δ lifts then $z_3^{\delta^*} = kz_1^{\delta^*}$ implies that $kz_3 = (k-1)z_2 + (1-k)z_4 = -(1-k)^2 z_4 + (1-k)z_4 = -(k^2 - k + 1)z_4 + z_4 = z_4$ and we have the same contradiction as above. Thus, z_1 and z_3 are linearly independent.

Since K has dimension 2, one may assume that $z_2 = kz_1 + \ell z_3$ for some $k, \ell \in \mathbb{Z}_p$. Then, $z_2^{\alpha_1^*} = kz_1^{\alpha_1^*} + \ell z_3^{\alpha_1^*}$ and $z_2^{\alpha_2^*} = kz_1^{\alpha_2^*} + \ell z_3^{\alpha_2^*}$ imply $(1 + k - k\ell)z_1 + (\ell - \ell^2 - 1)z_3 + (k + \ell)z_4 = 0$ and $z_4 = -\ell z_1 + (k + \ell)z_3$. It follows that $(1 + k - \ell^2 - 2k\ell)z_1 + (k^2 + \ell + 2k\ell - 1)z_3 = 0$. By the linear independence of z_1 and z_3, one has

$$1 + k - \ell^2 - 2k\ell = 0, \tag{1}$$

$$k^2 + \ell + 2k\ell - 1 = 0. \tag{2}$$

By (1) + (2), $(k + \ell)(k - \ell + 1) = 0$. Thus, $k = -\ell$ or $k = \ell - 1$.

Suppose that δ lifts. Then, $z_2^{\delta^*} = kz_1^{\delta^*} + \ell z_3^{\delta^*}$ implies that $z_1 + (k + \ell)z_2 - (k + 1)z_3 - (k + \ell)z_4 = 0$. Substitute $z_2 = kz_1 + \ell z_3$ and $z_4 = -\ell z_1 + (k + \ell)z_3$ to this equation and consider the coefficient of z_3. The linear independence of z_1 and z_3 implies the following equation

$$k^2 + k\ell + k + 1 = 0. \tag{3}$$

Recall that $k = -\ell$ or $\ell - 1$. If $k = -\ell$ then $\ell = 1$ by Eq. (3). Thus, $k = -\ell = -1$ and by Eq. (1), $1 = 0$, a contradiction. If $k = \ell - 1$ then $2\ell^2 - 2\ell + 1 = 0$ by Eq. (3) and $3\ell^2 - 3\ell = 0$ by Eq. (1). This implies $\ell^2 - \ell - 1 = 0$ and so $3 = 0$ again by $2\ell^2 - 2\ell + 1 = 0$. It follows that $p = 3$. In this case, $2\ell^2 - 2\ell + 1 \neq 0$ for each $\ell = 0, 1, 2$, a contradiction. Thus, δ cannot lift.

Note that β or δ lifts. Then, β lifts because δ cannot. Substituting $z_2 = kz_1 + \ell z_3$ and $z_4 = -\ell z_1 + (k+\ell)z_3$ to the image of $z_2 = kz_1 + \ell z_3$ under β, one has $(k^2 + k + k\ell)z_1 - (k^2 + k)z_3 = 0$. Thus, $k^2 + k = 0$ and $k^2 + k + k\ell = 0$. It follows that $k\ell = 0$, that is $k = 0$ or $\ell = 0$. If $k = -\ell$ then $k = \ell = 0$, implying that $z_2 = 0$, a contradiction. Thus, $k = \ell - 1$ and since $k = 0$ or $\ell = 0$, one has $\ell = 0$ and $k = -1$ or $k = 0$ and $\ell = 1$.

For $\ell = 0$ and $k = -1$, one has $z_2 = -z_1$ and $z_4 = -z_3$. Note that z_1 and z_3 are linearly independent. By Table 1, it is easy to check that $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\bar{\beta}$ and $\bar{\gamma}$ can be extended to automorphisms of $\mathbb{Z}_p^2$, but $\bar{\delta}$ cannot. By Proposition 2.1, α_1, α_2, β lift, but δ cannot. Since $\langle \alpha_1, \alpha_2, \beta, \gamma\rangle$ is 2-regular, $\mathrm{Aut}(\widetilde{X})$ contains a 2-regular subgroup, say B, lifted by $\langle \alpha_1, \alpha_2, \beta, \gamma\rangle$. We claim that $\widetilde{X}$ is actually 2-regular. Suppose to the contrary that $\widetilde{X}$ is s-regular for some $s \geqslant 3$. By Djoković and Miller [11, Theorem 3], if an s-regular group for $s \geqslant 3$ contains a 2-regular subgroup then $s = 3$. Thus, $\widetilde{X}$ is 3-regular. Let $A = \mathrm{Aut}(\widetilde{X})$. Then, $|A : B| = 2$ and $B \lhd A$. By Conder and Dobcsányi [6], there is only one connected cubic symmetric graphs of order 24 or 54 respectively, both of which are 2-regular. Thus, one may assume $p \geqslant 5$. Note that $K = \mathbb{Z}_p^2$ is normal in B and so characteristic in B because K is a Sylow p-subgroup of B, which implies that $K \lhd A$. Clearly, the quotient graph of X corresponding to the orbits of K is $K_{3,3}$. By Proposition 2.2, A/N is a 3-regular subgroup of $K_{3,3}$ and so $A/N = \mathrm{Aut}(K_{3,3})$. This means that $\mathrm{Aut}(K_{3,3})$ lifts, contrary to the fact that δ cannot lift. Thus, $\widetilde{X}$ is 2-regular. By Proposition 2.4, one may assume that $z_1 = e_1$, $z_3 = e_2$, $z_2 = -e_1$ and $z_4 = -e_2$, where e_1 and e_2 are the standard basis of the vector space $\mathbb{Z}_p^2$. By Example 3.4, $\widetilde{X} \cong EB_{p^2}$.

For $k = 0$ and $\ell = 1$, one has $z_2 = z_3$ and $z_4 = z_3 - z_1$. By Proposition 2.4 and Example 3.4, $\widetilde{X} \cong \overline{EB}_{p^2}$ and by Lemma 3.7, $\widetilde{X} \cong EB_{p^2}$.

Case III. $K = \mathbb{Z}_p^3$.

The proof in the first paragraph of Case II means that z_1 and z_3 are linearly independent. Suppose that z_2 is a linear combination of z_1 and z_3. Considering the image of this combination under α_2^*, one has that z_4 is also a combination of z_1 and z_3. Thus, K can be generated by z_1 and z_3, which is impossible because $K = \mathbb{Z}_p^3$. Thus, z_1, z_2 and z_3 are linearly independent.

Let $z_4 = iz_1 + jz_2 + kz_3$ for some $i, j, k \in \mathbb{Z}_p$. By $z_4^{\alpha_2^*} = iz_1^{\alpha_2^*} + jz_2^{\alpha_2^*} + kz_3^{\alpha_2^*}$, $z_2 = (i+k)z_1 + jz_2 - iz_3 - jz_4 = (i+k-ij)z_1 + (j-j^2)z_2 - (i+jk)z_3$. The linear independence of z_1, z_2 and z_3 implies the following equations

$$i + k - ij = 0, \tag{4}$$

$$j^2 - j + 1 = 0, \tag{5}$$

$$i + jk = 0. \tag{6}$$

Suppose that β cannot lift. Then, δ lifts because β or δ lifts. Substitute $z_4 = iz_1 + jz_2 + kz_3$ to its image under δ^* and consider the coefficients of z_1 and z_2. The linear independence of z_1, z_2 and z_3 implies the following equations

$$i^2 + ik + j - 1 = 0, \tag{7}$$

$$ij + jk - i - k - 1 = 0. \tag{8}$$

Since α_2 lifts, Eqs. (4)–(6) hold. By (4)–(6) + (8), $i = -1$ and by (7), $j = k$. By (8), $j^2 - 2j = 0$ and by (5), $j = k = -1$. Since $j^2 - 2j = 0$, one has $3 = 0$ and so $p = 3$. However, in this case $\overline{\beta}$ can be extended to an automorphism of $\mathbb{Z}_3^3$ and so β lifts, a contradiction. Thus, β must lift.

Substitute $z_4 = iz_1 + jz_2 + kz_3$ to its image under β^* and consider the coefficients of z_1 and z_2. The linear independence of z_1, z_2 and z_3 implies the following equations

$$i^2 = 1, \tag{9}$$

$$i - j - ij + 1 = 0. \tag{10}$$

If $p = 2$, then (5) has no solution. If $p = 3$ then (5) implies that $j = 2$. By (10), $i = 2$ and by (4), $k = 2$. Thus, $z_4 = 2(z_1 + z_2 + z_3)$. It is easy to prove that $\bar{\alpha}_1, \bar{\alpha}_2, \overline{\beta}$ and $\bar{\delta}$ can be extended to automorphisms of $\mathbb{Z}_p^3$. By Proposition 2.1, $\mathrm{Aut}(K_{3,3})$ lifts and so $\mathrm{Aut}(\widetilde{X})$ contains a 3-regular subgroup. By Djoković and Miller [11, Theorem 3], $\widetilde{X}$ is 3-regular, and by Proposition 2.4 and Example 3.5, $\widetilde{X} \cong EB_{3^3}$.

We now let $p > 3$. By (9), one has $i = 1$ or -1. Suppose $i = 1$. Then by (10), $j = 1$ and by (5), $1 = 0$, a contradiction. Thus, $i = -1$. By (6), $jk = 1$ and by multiplying k to Eq. (5), one has $k = 1 - j$. It follows that $z_4 = -z_1 + jz_2 + (1-j)z_3$, where $j^2 - j + 1 = 0$ by (5). Clearly, $j \neq -1$. Thus, $j^3 + 1 = (j+1)(j^2 - j + 1) = 0$ implies that $-j$ is an element of order 3 in the multiplicative group $\mathbb{Z}_p^*$. Thus, $p - 1$ is divisible by 3. Set $\lambda = -j$. By Proposition 2.4, Example 3.5 and Lemma 3.7, $\widetilde{X} \cong EB_{p^3}$. We claim that EB_{p^3} is actually 1-regular. By Table 1 and Proposition 2.1, it is easy to prove that α_1, α_2 and β lift, but γ cannot. Thus, $\mathrm{Aut}(\widetilde{X})$ contains a 1-regular subgroup, say B, lifted by $\langle \alpha_1, \alpha_2, \beta \rangle$. Clearly, $K = \mathbb{Z}_p^3$ is a normal Sylow p-subgroup of B and $|B| = 18p^3$.

Suppose that an s-regular subgroup of $\mathrm{Aut}(K_{3,3})$ lifts for some $s \geqslant 2$. Then, the subgroup contains an involution, say α, which fixes the arc $(\mathbf{u}, \mathbf{x})$ in $K_{3,3}$. Clearly, $\alpha = (\mathbf{vw})(\mathbf{yz})$, $(\mathbf{vw})$ or $(\mathbf{yz})$. Since α lifts and $\gamma = (\mathbf{vw})(\mathbf{yz})$ cannot lift, $\alpha = (\mathbf{vw})$ or $(\mathbf{yz})$. But in this case it is easy to show that $\alpha\beta^{-1}\alpha\beta = (\mathbf{vw})(\mathbf{yz}) = \gamma$, where $\beta = (\mathbf{ux})(\mathbf{vy})(\mathbf{wz})$. Since α and β lift, γ lifts, a contradiction. This implies that no s-regular subgroup of $\mathrm{Aut}(K_{3,3})$ lifts for any $s \geqslant 2$.

Let $A = \mathrm{Aut}(\widetilde{X})$ and suppose to the contrary that $\widetilde{X}$ is s-regular for some $s \geqslant 2$. By Tutte [49, 50], $s \leqslant 5$ and so $|A| \mid 6p^3 \cdot 48$. Thus, K is a Sylow p-subgroup of A and since $p-1$ is divisible by 3, we have $p \geqslant 7$. The normality of K in B implies that $B \leqslant N_A(K)$, where $N_A(K)$ is the normalizer of K in A. Since $\widetilde{X}$ is at most 5-regular, $|A : N_A(K)| \mid 16$. By Sylow's theorem, the number of Sylow p-subgroups of A is $np+1$ and $np+1 = |A : N_A(K)|$. Thus, $np+1 \mid 16$. Since $p \geqslant 7$, we have $np+1 = 1$, or $p = 7$ and $n = 1$. If $np+1 = 1$ then $K \lhd A$. By Proposition 2.2, A/K is an s-regular subgroup of $K_{3,3}$. This is impossible because otherwise the s-regular subgroup A/K $(s \geqslant 2)$ of $\mathrm{Aut}(K_{3,3})$ lifts. Thus, $p = 7$ and $n = 1$. Let $H = N_A(K)$. Then, $|A : H| = 8$. By considering the right multiplication action of A on the set of right cosets of H in A, $|A/H_A| \mid 8!$, where H_A is the largest normal subgroup of A in H. Let L be a Sylow 7-subgroup of H_A. Then the normality of L in H_A implies that L is characteristic in H_A. Thus, $L \lhd A$ because $H_A \lhd A$. Since $|A : H| = 8$, the Sylow 7-subgroups of A are not normal in A, forcing that $|L| \neq 7^3$. Thus, $|A/H_A| \mid 8!$ implies that $7^2 \mid |H_A|$ and so $L \cong \mathbb{Z}_7^2$. However, the quotient graph corresponding to the orbits of L on $V(\widetilde{X})$ is a cubic s-regular graph of order $6 \times 7 = 42$ for some $s \geqslant 2$, which is impossible according to Conder and Dobcsányi [6]. Thus, EB_{p^3} is 1-regular. □

5. The cubic symmetric graphs of order $6p$ or $6p^2$

In this section, we shall classify the s-regular cubic graphs of order $6p$ or $6p^2$ for each prime p. First, we introduce a part of the classification of the s-regular cyclic coverings of $K_{3,3}$ in [18] for our classification. By Conder and Dobcsányi [6], there is a unique cubic symmetric graph of order 18 which is 3-regular. This graph is called the Pappus graph, denoted by 9_3 (see [9]). By the proof of [18, Theorem 1.1], the graph 9_3 is a $\mathbb{Z}_3$-covering of $K_{3,3}$ admitting a lift of a 2-regular automorphism subgroup of $K_{3,3}$ isomorphic to $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$. Thus, $\mathrm{Aut}(9_3) \cong (\mathbb{Z}_3 \cdot ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4)) \cdot \mathbb{Z}_2$, where $\cdot$ means group extension. Let p be a prime such that $p-1$ is divisible by 3 and let $n = p$ or p^2. By [17, Lemma 3.4], λ is an element of order 3 in $\mathbb{Z}_n^*$ if and only if $\lambda^2 + \lambda + 1 = 0$ in $\mathbb{Z}_n$. Clearly, $\mathbb{Z}_n^*$ has exactly two elements of order 3, that is, λ and λ^2. In view of [18, Theorem 1.1], Example 3.3 and Lemma 3.7 imply the following.

Lemma 5.1. *Let $\widetilde{X}$ be a connected regular covering of the complete bipartite graph $K_{3,3}$ whose covering transformation group is $\mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$ for a prime p and whose fibre-preserving group is arc-transitive. Then, $\widetilde{X}$ is* 1- *or* 3-*regular. Furthermore,*

(1) *$\widetilde{X}$ is* 1-*regular if and only if $\widetilde{X}$ is isomorphic to one of CB_n (defined in Example* 3.3*), where $n = p$ or p^2 such that $p-1$ is divisible by* 3.
(2) *$\widetilde{X}$ is* 3-*regular if and only if $\widetilde{X}$ is isomorphic the Pappus graph* 9_3.

There is a connected cubic symmetric graph of order 30 denoted by L_{30}, which is called the Levi graph. For its construction, see [24, Fig. 14.13] and by Tutte [49], L_{30} is 5-regular and $\mathrm{Aut}(L_{30}) \cong S_6 \rtimes \mathbb{Z}_2$. There is a connected cubic symmetric graph of order 102 discovered by Smith and Biggs and investigated by Biggs [4]. We denote by SB_{102} this graph. For its construction, see [4, Fig. 3] and by Biggs [4], SB_{102} is 4-regular and $\mathrm{Aut}(SB_{102}) \cong \mathrm{PSL}(2, 17)$.

Theorem 5.2. *Let X be a connected cubic symmetric graph of order $6p$ for a prime p. Then, X is* 1-, 3-, 4- *or* 5-*regular. Furthermore,*

(1) *X is* 1*-regular if and only if X is isomorphic to one of CB_p* (*defined in Example* 3.3)*, where $p-1$ is divisible by* 3.
(2) *X is* 3*-regular if and only if X is isomorphic to the Pappus graph* 9_3.
(3) *X is* 4*-regular if and only if X is isomorphic to the Smith and Biggs graph* SB_{102}.
(4) *X is* 5*-regular if and only if X is isomorphic to the Levi graph* L_{30}.

Proof. Let $A = \mathrm{Aut}(X)$. Since X is symmetric, by Tutte [50], X is at most 5-regular. Thus, $|A|$ is a divisor of $6p \cdot 48$. For $p = 3$, 5 or 17, by Conder and Dobcsányi [6], there is only one connected cubic symmetric graph of order $6p$, that is, the 3-regular Pappus graph 9_3, the 5-regular Levi graph L_{30} and the 4-regular Smith and Biggs graph SB_{102}, and for each prime $p = 2, 11, 23, 29, 41, 47, 53, 59$ or 71, there is no connected cubic symmetric graph of order $6p$. Similarly, for each prime $p = 7$, 13, 19, 31, 37, 43, 61 or 67, there is only one connected cubic symmetric graph of order $6p$ which is the 1-regular graph CB_p by Lemma 5.1. Thus, one may assume that $p \geqslant 73$.

Let P be a Sylow p-subgroup of A and $N_A(P)$ the normalizer of P in A. By Sylow's theorem, the number of Sylow p-subgroups of A is $np+1$ and $|A : N_A(P)| = np+1$. If $np+1 > 1$ then $np+1 = 96$, 144 or 288 because $|np+1| \mid 6 \cdot 48$ and $np+1 \geqslant 74$. Since $95 = 5 \times 19$, $143 = 11 \times 13$ and $287 = 7 \times 41$, one has $p = 5, 7, 11, 13, 19$ or 41, contrary to the hypothesis that $p \geqslant 73$. Thus, $np+1 = 1$ and so $P \lhd A$. Clearly, the quotient graph X corresponding to the orbits of P is a cubic symmetric graph with 6 vertices and so it is the bipartite graph $K_{3,3}$. By Propositions 2.2, X is a regular covering of $K_{3,3}$ with the covering transformation group $P \cong \mathbb{Z}_p$ and since $P \lhd A$, the symmetry of X means that the fibre-preserving group is arc-transitive. By Lemma 5.1, $X \cong CB_p$. □

Theorem 5.3. *Let X be a connected cubic symmetric graph of order $6p^2$ for a prime p. Then, X is* 1*- or* 2*-regular. Furthermore,*

(1) *X is* 1*-regular if and only if X is isomorphic to one of CB_{p^2}* (*defined in Example* 3.3)*, where $p-1$ is divisible by* 3.
(2) *X is* 2*-regular if and only if X is isomorphic to one of EB_{p^2}* (*defined in Example* 3.4).

Proof. Let $A = \mathrm{Aut}(X)$. Since X is at most 5-regular, one has $|A| \mid 6p^2 \cdot 48$. For each prime $p = 2, 3, 5$ or 11, by Conder and Dobcsányi [6] and Theorem 4.1, there exists only one connected cubic symmetric graph of order $6p^2$ which is the cubic 2-regular graph EB_{p^2}, and for $p = 7$ there are two connected cubic symmetric graphs of order 6×7^2 which are the 1-regular graph CB_{7^2} (Lemma 5.1) and the 2-regular graph EB_{7^2}. Thus, one may assume $p \geqslant 13$. Let P be a Sylow p-subgroup of A and $N_A(P)$ the normalizer of P in A. Then, $P \cong \mathbb{Z}_{p^2}$ or $\mathbb{Z}_p^2$ and $|A : N_A(P)| = np+1$ for some integer n. We claim that $P \lhd A$.

Suppose to the contrary that P is not normal in A. Then, $np+1 \geqslant 14$ because $p \geqslant 13$. If $N_A(P) = P$ then $C_A(P) = N_A(P) = P$ because P is abelian, where $C_A(P)$ is the centralizer of P in A. By Proposition 2.5, A has a normal subgroup N such that $A/N \cong P$, and by Proposition 2.2, the quotient graph corresponding to the orbits of N has odd order and valency 3, a contradiction. Thus, let $N_A(P) \neq P$ and so $np+1 \mid 3 \cdot 2^5$ or $3^2 \cdot 2^4$. It follows that np is one of the following: $143 = 11 \times 13$, $95 = 5 \times 19$, 71, 47, $35 = 5 \times 7$, 31, 23, 17 or $15 = 3 \times 5$. Since $p \geqslant 13$, there are three possible cases:

I. $p = 17, 23, 31, 47$ or 71 and $n = 1$,
II. $p = 19$ and $n = 5$, or
III $p = 13$ and $n = 11$.

Case I. $p = 17, 23, 31, 47$ or 71 and $n = 1$.

Let $H = N_A(P)$. By considering the right multiplication action of A on the set of right cosets of H in A, we have $|A/H_A| \mid (p+1)!$, where H_A is the largest normal subgroup of A in H. This forces $p \mid |H_A|$ because $|A|$ is divisible by p^2. Let L be a Sylow p-subgroup of H_A. Clearly, L is characteristic in H_A and so $L \lhd A$. By the normality of L, we have $L \leqslant P$. Since the Sylow p-subgroups of A are not normal, one has $p^2 \nmid |H_A|$ and so $L \cong \mathbb{Z}_p$. By Proposition 2.2, the quotient graph $\underline{X}$ of X corresponding to the orbits of L is a connected cubic symmetric graph of order $6p$, and A/L is a subgroup of $\mathrm{Aut}(\underline{X})$. If $p = 23, 31, 47$ or 71, Theorem 5.2 implies that $\underline{X}$ is 1-regular. Thus, $|\mathrm{Aut}(\underline{X})| = 18p$. By Sylow theorem, $|\mathrm{Aut}(\underline{X})| = 18p$ has normal Sylow p-subgroups. Since P/L is a Sylow p-subgroup of A/L and $A/L \leqslant \mathrm{Aut}(\underline{X})$, one has $P/L \lhd A/L$, implying that $P \lhd A$, a contradiction. Thus, $p = 17$. It follows that $|L| = 17$ and $|V(\underline{X})| = 102$. By Theorem 5.2, $\underline{X} \cong SB_{102}$ is 4-regular and $\mathrm{Aut}(\underline{X}) \cong \mathrm{PSL}(2, 17)$. Let X be s-regular. Then, $|A| = 18p^2 \cdot 2^{s-1}$ for some $s \geqslant 1$ and so $|A : N_A(P)| = 18$ implies that $|N_A(P) : P|$ is a 2-power. Since $N_A(P) \neq P$, $|N_A(P) : P|$ is divisible by 2. It follows that $|A|$ has a divisor 4, implying that X is at least 2-regular. By Proposition 2.2, A/L is an s-regular subgroup of $\mathrm{Aut}(SB_{102})$ for some $s \geqslant 2$. Since $\mathrm{Aut}(SB_{102}) \cong \mathrm{PSL}(2, 17)$ and $\mathrm{PSL}(2, 17)$ has no subgroup with index less than 5, it must be $A/L = \mathrm{Aut}(SB_{102})$, that is, X is 4-regular. Let $C_A(L)$ be the centralizer of L in A. Since Sylow 17-subgroups of A are abelian, $L \neq C_A(L)$. Then, $C_A(L)/L \neq 1$ and so $C_A(L)/L$ is a non-trivial normal subgroup of A/L. Since A/L is simple, one has $C_A(L) = A$. By Proposition 2.2, X is a 4-regular cyclic covering of SB_{102} with the covering transformation group $L \cong \mathbb{Z}_{17}$.

Using the notation in Biggs [4], we know that the graph SB_{102} has vertices $p_i, q_i, r_i, s_i, x_i, y_i$ ($i = 1, 2, \ldots, 17$) such that for each suffix i, the vertex x_i is jointed to p_i, q_i, y_i, the vertex y_i is jointed to r_i, s_i, the vertex p_i is jointed to p_{i-1}, p_{i+1}, the vertex q_i is jointed to q_{i-4}, q_{i+4}, the vertex r_i is jointed to r_{i-2}, r_{i+2}, and the vertex s_i is jointed to s_{i-8} and s_{i+8}, all suffixes being taken modulo 17. Let N_i denote the set of vertices having distance i from x_1. For convenience, we depict SB_{102} as in Fig. 2 (also, see [4, Fig. 3]), where we omit the edges which join N_3 to N_4, N_4 to N_5, N_5 to N_6, and N_6 to N_7. Note that for each $u \in N_i$ ($1 \leqslant i \leqslant 6$) there is a unique neighbor of u in N_{i+1} and N_{i-1}, respectively. Furthermore, for any given 4-arc there is a unique 9-cycle passing through the given 4-arc.

For each vertex v in N_7 we choose one edge which is incident to the vertex v, so that we obtain altogether $|N_7| = 8$ edges, say e_i ($1 \leqslant i \leqslant 8$). One may assume that $X = SB_{102} \times_\phi L$, where $\phi = 0$ on the spanning tree T such that an edge of T either joins N_i to N_{i+1} for some $0 \leqslant i \leqslant 5$, or is one of e_i ($i = 1, 2, \ldots, 8$). Consider the 9-arc $C = (x_1, p_1, p_{17}, x_{17}, q_{17}, q_{13}, q_9, q_5, q_1, x_1)$ which is actually a cycle of length 9. Since SB_{102} is 4-regular, there is an $\alpha \in \mathrm{Aut}(X)$ such that $(x_1, p_1, p_{17}, x_{17})^\alpha = (x_{17}, p_{17}, p_1, x_1)$. Since C is the unique 9-cycle passing through the 4-arc $(x_1, p_1, p_{17}, x_{17})$ or the 4-arc $(x_{17}, p_{17}, p_1, x_1)$, α fixes the cycle C and so reverses C. Assume $\phi((q_{17}, q_{13})) = k$. Then, $\phi(C) = k$ and $\phi(C^{-1}) = -k$, where C^{-1} is the inverse cycle of C. Since $C_A(L) = A$, the liftings of α commutes with each element in L. By Proposition 2.3, $k = -k$ and since p is odd, $k = 0$. Similarly, one can show that $\phi = 0$ on every arc of X. Thus, X is p copies of SB_{102}, contrary to the connectivity of X.

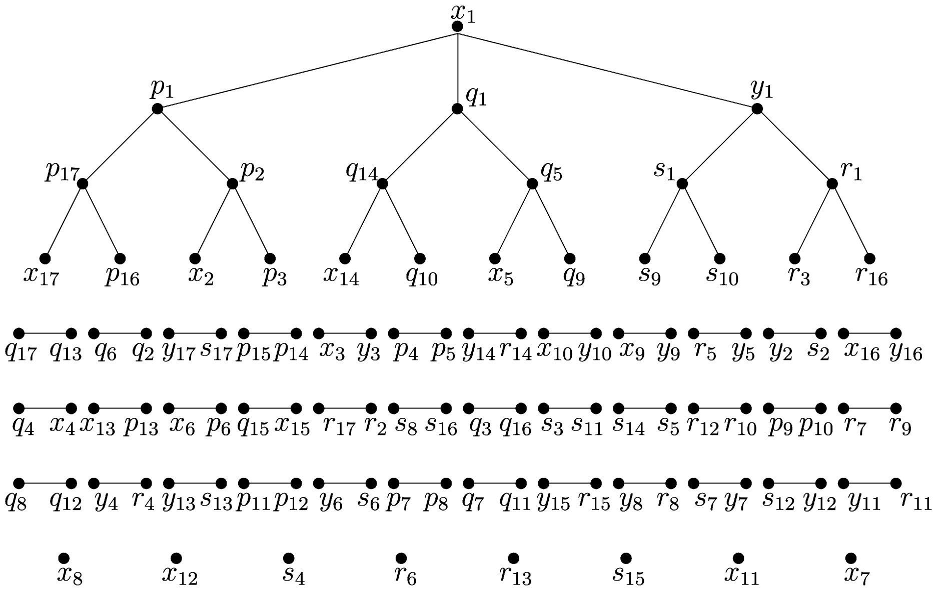

Fig. 2. The Smith and Biggs graph SB_{102}.

Case II. $p = 19$ and $n = 5$.

In this case, $|A : N_A(P)| = 96 = 3 \cdot 2^5$. Thus, 2^5 is a divisor of $|A|$ and since X is at most 5-regular, $|A| = 6p^2 \cdot 3 \cdot 2^4$, that is, X is 5-regular. Let q be a prime. By Gorenstein [22, pp. 12–14], if there exists a simple $\{2, 3, q\}$-group then $q = 5, 7, 13$ or 17. Thus, A is solvable. Let N be a minimal normal subgroup of A and $\underline{X}$ the quotient graph of X corresponding to the orbits of N. Then, N is an elementary abelian r-group, where $r = 2, 3$ or 19. If $r = 2$, Proposition 2.2 implies that $\underline{X}$ has odd order and valency 3, a contradiction. If $r = 3$ then $\underline{X}$ is a connected cubic 5-regular graph of order $2 \times 19^2 = 722$, and if $r = 19$ then $\underline{X}$ is a connected cubic 5-regular graph of order $6 \times 19 = 114$. By Conder and Dobcsányi [6], both are impossible.

Case III. $p = 13$ and $n = 11$.

In this case, $|A : N_A(P)| = 144 = 9 \cdot 2^4$. Since $|A| = 6p \cdot 2^{s-1}$ for some $1 \leqslant s \leqslant 5$, $|N_A(P) : P|$ must be a 2-power. Since $P \neq N_A(P)$, $|N_A(P) : P|$ is divisible by 2. It follows that $|A| = 6p^2 \cdot 3 \cdot 2^4$ and so X is 5-regular. By Gorenstein [22, pp. 12–14], the only simple $\{2, 3, 13\}$-group is PSL(3, 3) which has order $2^4 \cdot 3^3 \cdot 13$. Since $3^3 \nmid |A|$, A is solvable and we have a contradiction by a similar argument to the Case II.

So far, we have proved that $P \lhd A$. By Proposition 2.2, X is a $\mathbb{Z}_{p^2}$- or $\mathbb{Z}_p^2$-covering of the bipartite graph $K_{3,3}$ and the normality of P implies that the fibre-preserving group is the automorphism group $\mathrm{Aut}(X)$ of X, so that it is arc-transitive. By Lemma 5.1 and Theorem 4.1, $X \cong CB_{p^2}$ or EB_{p^2}, as required. □

6. Regular coverings of K_4 and related classifications

In this section, we first classify the cyclic or elementary abelian coverings of the complete graph K_4. The proof is similar but easier to that in Section 4. As an application of this classification, a list of s-regular cubic graphs of order $4p$ or $4p^2$ for each s and each prime p is given.

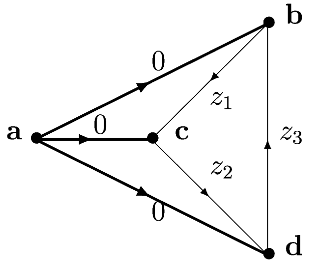

Fig. 3. The complete graph K_4 with voltage assignment ϕ.

Table 2
Fundamental cycles and their images with voltages on K_4

C	$\phi(C)$	C^α	$\phi(C^\alpha)$	C^β	$\phi(C^\beta)$	C^γ	$\phi(C^\gamma)$
abc	z_1	**bad**	z_3	**acd**	z_2	**acb**	$-z_1$
acd	z_2	**bdc**	$-z_1-z_2-z_3$	**adb**	z_3	**abd**	$-z_3$
adb	z_3	**bca**	z_1	**abc**	z_1	**adc**	$-z_2$

Theorem 6.1. *Let K be a cyclic or an elementary abelian group and let $\widetilde{X}$ be a connected K-covering of the complete graph K_4 whose fibre-preserving group is arc-transitive. Then, X is 2-regular. Moreover,*

(1) *if K is cyclic then $\widetilde{X}$ is isomorphic to the complete graph K_4, the 3-dimensional hypercube Q_3, or the generalized Petersen graph $P(8,3)$;*
(2) *If K is elementary abelian but not cyclic, then $\widetilde{X}$ is isomorphic to one of EC_{p^3} for a prime p (defined in Example 3.2).*

Proof. Let $\widetilde{X} = K_4 \times_\phi K$ be a connected K-covering of the graph K_4 satisfying the hypotheses, where $\phi = 0$ on the spanning tree T as illustrated by dark lines in Fig. 3. Identify the vertex set of K_4 with $\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$ and we assign voltages z_1, z_2 and z_3 in K to the cotree arcs $(\mathbf{b},\mathbf{c})$, $(\mathbf{c},\mathbf{d})$ and $(\mathbf{d},\mathbf{b})$, respectively. The connectivity of $\widetilde{X}$ means that $\langle z_1, z_2, z_3\rangle = K$.

Clearly, if $K = 1$ then $\widetilde{X} = K_4$. Assume $K \neq 1$ and set $\alpha = (\mathbf{ab})(\mathbf{cd})$, $\beta = (\mathbf{bcd})$ and $\gamma = (\mathbf{bc})$. Then the arc-transitivity of the fibre-preserving group implies that α and β lift. Let C be a fundamental cycle in K_4. Then, C is **abc**, **acd** or **adb**, and one may easily obtain Table 2.

The mapping $\bar{\alpha}$ from the set of voltages on the three fundamental cycles of K_4 to the voltage group K is defined by $\phi(C)^{\bar{\alpha}} = \phi(C^\alpha)$, where C ranges over these three fundamental cycles. Similarly, one can define $\overline{\beta}$ and $\overline{\gamma}$. Since α and β lift, by Proposition 2.1, $\bar{\alpha}$ and $\overline{\beta}$ can be extended to automorphisms of K, say α^* and β^*, respectively. Then, $z_1^{\beta^*} = z_2$ and $z_2^{\beta^*} = z_3$ imply that z_1, z_2 and z_3 have the same order. We now consider two cases according to K being cyclic or elementary abelian.

Case I. $K = \mathbb{Z}_n$ $(n > 1)$.

In this case, K can be generated by each of z_1, z_2 and z_3 because they have the same order. Thus, one may assume $z_1 = 1$. Let $1^{\beta^*} = k$. By considering the images of z_1, z_2 and z_3 under β^*, one has $z_2 = k$, $z_3 = k^2$ and $k^3 = 1$ in the ring $\mathbb{Z}_n$. Let $1^{\alpha^*} = \ell$. By considering the images of z_1 and z_3 under α^*, one has $\ell = k^2$ and $\ell k^2 = 1$. Thus, $k = \ell$ and so $k = 1$. It follows that $z_1 = z_2 = z_3 = 1$. By $z_2^{\alpha^*} = -z_1 - z_2 - z_3$, one has $4 = 0$ and so $n = 2$ or 4. In both cases, it is easy to check that $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ can be extended to automorphisms of $\mathbb{Z}_n$, and so $\widetilde{X}$ is at least 2-regular. Note that $\widetilde{X}$ has order 8 or 16. By Miller [43, Table 3.1], there is only one connected cubic symmetric graph of order 8 or 16 respectively, which are the 3-dimensional hypercube Q_3 and the generalized Petersen graph $P(8,3)$ by Nedela and Škoviera [44]. These two graphs are 2-regular.

Case II. $K = \mathbb{Z}_p^m$ $(m \geqslant 2)$.

Since $\langle z_1, z_2, z_3\rangle = \mathbb{Z}_p^m$, let $K = \mathbb{Z}_p^2$ or $\mathbb{Z}_p^3$. If $K = \mathbb{Z}_p^2$ then $z_1^{\beta^*} = z_2$, $z_2^{\beta^*} = z_3$ and $z_3^{\beta^*} = z_1$ imply that z_1 and z_2 are linearly independent. Write $z_3 = kz_1 + \ell z_2$ for some $k, \ell \in \mathbb{Z}_p$. Substituting $z_3 = kz_1 + \ell z_2$ to its image under β^*, the linear independence of z_1 and z_2 implies that $\ell k = 1$ and $k + \ell^2 = 0$ in the field $\mathbb{Z}_p$. It follows that $k = \ell^{-1} \neq 0$ and $\ell^3 = -1$. Substitute $z_3 = kz_1 + \ell z_2$ to its image under α^* and consider the coefficient of z_2. The linear independence of z_1 and z_2 implies that $\ell(k - \ell - 1) = 0$. Since $\ell \neq 0$, one has $k - \ell - 1 = 0$ and $k = \ell^{-1}$ means that $\ell^2 + \ell - 1 = 0$. Multiplying ℓ to this equation, one has $\ell^2 - \ell - 1 = 0$ because $\ell^3 = -1$. Thus, $2\ell = 0$ and so $2 = 0$, implying that $p = 2$. But, in this case the equation $\ell^2 - \ell - 1 = 0$ has no solution. Thus, $K = \mathbb{Z}_p^3$. Since $\langle z_1, z_2, z_3\rangle = \mathbb{Z}_p^m$, z_1, z_2 and z_3 are linearly independent. It is easy to prove that $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ can be extended to automorphisms of $\mathbb{Z}_p^3$ and so $\mathrm{Aut}(\widetilde{X})$ contains a 2-regular subgroup, say B, lifted by $\langle \alpha, \beta, \gamma\rangle$. We claim that $\widetilde{X}$ is actually 2-regular. Otherwise $\widetilde{X}$ is at least 3-regular. By Djoković and Miller [11, Theorem 3], $\widetilde{X}$ is 3-regular. Let $A = \mathrm{Aut}(\widetilde{X})$. Then, $|A : B| = 2$ and $B \lhd A$. If $p = 2$ or 3, by Conder and Dobcsányi [6], there is only one connected cubic symmetric graph of order $4p^3$, which is 2-regular. Thus, one may assume $p \geqslant 5$ and so K is a Sylow p-subgroup of A and B. Since $K \lhd B$, K is characteristic in B and so normal in A. By Proposition 2.2, $\mathrm{Aut}(K_4)$ contains the s-regular subgroup A/K and so K_4 is at least 3-regular, a contradiction. Thus, $\widetilde{X}$ is 2-regular. By Proposition 2.4 and Example 3.2, $\widetilde{X} \cong EC_{p^3}$. □

To classify the cubic symmetric graphs of order $4p$ or $4p^2$, we introduce a graph of order 28 which was discovered by Coxeter and investigated by Tutte [51]. Denote by C_{28} this graph. For its construction, see Biggs [4, Fig. 2(ii)] and by Biggs [4], C_{28} is 3-regular and $\mathrm{Aut}(C_{28}) \cong \mathrm{PGL}(2,7)$.

Theorem 6.2. *Let X be a connected cubic symmetric graph of order $4p$ or $4p^2$ for a prime p. Then X is isomorphic to the 2-regular hypercube Q_3 of order 8, the 2-regular generalized Petersen graphs $P(8,3)$ or $P(10,7)$ of order 16 or 20 respectively, the 3-regular Dodecahedron of order 20 or the 3-regular Coxeter graph C_{28} of order 28.*

Proof. Let X be a connected cubic symmetric graph of order $4m$ where $m = p$ or p^2. Set $A = \mathrm{Aut}(X)$. Since X is at most 5-regular, $|A| \mid 192m$.

Let $p \leqslant 13$. By Conder and Dobcsányi [6], if there exists a connected cubic symmetric graph of order $4p$ or $4p^2$ then the order must be 8, 20, 28 or 16. Moreover, there is only one connected cubic symmetric graph for each order 8, 16 and 28, and there are two for the order 20. Clearly, the cubic symmetric graph of order 8 is the 3-dimensional hypercube Q_3 which is 2-regular. By Miller [43, Table 3.1] and Nedela and Škoviera [44], the cubic graph of order 16 is the generalized Petersen graph $P(8,3)$ which is 2-regular, and the cubic graphs of order 20 are the Dodecahedron and the generalized Petersen graph $P(10,7)$, of which the first one is 2-regular and the second one is 3-regular. By Biggs [4], the connected cubic graph of order 28 is the Coxeter graph C_{28} which is 3-regular. To prove the theorem, we only need to show that no connected cubic symmetric graph of order $4p$ or $4p^2$ exists for $p \geqslant 17$. Suppose to the contrary that X is such a graph.

Let P be a Sylow p-subgroup of A and $N_A(P)$ the normalizer of P in A. By Sylow's theorem, the number of Sylow p-subgroups of A is $np+1$ and $np+1 = |A : N_A(P)|$, where n is an integer. If $np + 1 = 1$ then $P \lhd A$. By Proposition 2.2, X is a cyclic or an elementary abelian covering of K_4 with the covering transformation groups of order p or p^2 for $p \geqslant 17$. By Theorem 6.1, no such a covering exists. Thus, we assume $np + 1 > 1$ and so P is not normal in A. Since $p \geqslant 17$, one has $np + 1 \geqslant 18$ and so $|A| \mid 192m$ implies that $np + 1 \mid 192$. It follows that $np + 1 = 192$, 96, 48, 24, 64 or 32, and so $np = 191$, 95, 47, 23, 63 or 31. Since $p \geqslant 17$, one has that either $p = 191, 47, 31$ or 23 and $n = 1$, or $p = 19$ and $n = 5$. By Conder and Dobcsányi [6], for each of such primes there is no connected cubic symmetric graph of order $4p$. This implies that X has order $4p^2$.

Assume that $n = 1$ and $p = 191, 47, 31$ or 23. Let $H = N_A(P)$. By considering the right multiplication action of A on the set of right cosets of H in A, we have $|A/H_A| \mid (p+1)!$, where H_A is the largest normal subgroup of A in H. This implies that $p \mid |H_A|$. Let L be a Sylow p-subgroup of H_A. Clearly, L is characteristic in H_A and so $L \lhd A$. Since Sylow p-subgroups of A are not normal, $p^2 \nmid |H_A|$. Thus, $L \cong \mathbb{Z}_p$ and the quotient graph of X corresponding to the orbits of L is a connected cubic symmetric graph of order $4p$ where $p = 191, 47, 31$ or 23, but we have shown that no such a graph exists, a contradiction.

Now, assume that $p = 19$ and $n = 5$. Then, $|A : N_A(P)| = 96$ and so $|A|$ is divisible by $4 \cdot 19^2 \cdot 3 \cdot 2^3$, which implies that X is at least 4-regular. Let q be a prime. By Gorenstein [22, pp. 12–14], a simple $\{2, 3, q\}$-group exists if and only if $q = 5, 7, 13$ or 17. Thus, A is solvable. Let N be a minimal normal subgroup of A and let $\underline{X}$ be the quotient graph of X corresponding to the orbits of N. Then, N is an elementary abelian r-group, where $r = 2$, 3 or 19. Clearly, $r \neq 3$ because any subgroup of order 3 is a stabilizer of some vertex in A. If $r = 2$ or 19, by Proposition 2.2, $\underline{X}$ is a connected cubic s-regular graph of order 722, 76 or 4 for some $s \geqslant 4$. However, by Conder and Dobcsányi [6], there is no such cubic s-regular graph, a contradiction. □

References

[1] B. Alspach, D. Marušič, L. Nowitz, Constructing graphs which are 1/2-transitive, J. Aust. Math. Soc. A 56 (1994) 391–402.
[2] D. Archdeacon, R.B. Richter, J. Širan, M. Škoviera, Branched coverings of maps and lifts of map homomorphisms, Australas. J. Combin. 9 (1994) 109–121.
[3] D. Archdeacon, P. Gvozdnjak, J. Širan, Constructing and forbidding automorphisms in lifted maps, Math. Slovaca 47 (1997) 113–129.
[4] N. Biggs, Three remarkable graphs, Canad. J. Math. 25 (1973) 397–411.
[5] Y. Cheng, J. Oxley, On weakly symmetric graphs of order twice a prime, J. Combin. Theory Ser. B 42 (1987) 196–211.
[6] M.D.E. Conder, P. Dobcsányi, Trivalent symmetric graphs on up to 768 vertices, J. Combin. Math. Combin. Comput. 40 (2002) 41–63.
[7] M.D.E. Conder, P. Lorimer, Automorphism groups of symmetric graphs of valency 3, J. Combin. Theory Ser. B 47 (1989) 60–72.
[8] M.D.E. Conder, C.E. Praeger, Remarks on path-transitivity on finite graphs, European J. Combin. 17 (1996) 371–378.
[9] H.S.M. Coxeter, W.O.J. Moser, Generators and Relations for Discrete Groups, third ed., Springer, New York, 1972.
[10] D.Ž. Djoković, Automorphisms of graphs and coverings, J. Combin. Theory Ser. B 16 (1974) 243–247.
[11] D.Ž. Djoković, G.L. Miller, Regular groups of automorphisms of cubic graphs, J. Combin. Theory Ser. B 29 (1980) 195–230.
[12] S.F. Du, D. Marušič, A.O. Waller, On 2-arc-transitive covers of complete graphs, J. Combin. Theory Ser. B 74 (1998) 376–390.

[13] S.F. Du, J.H. Kwak, M.Y. Xu, Lifting of automorphisms on the elementary abelian regular coverings, Linear Algebra Appl. 373 (2003) 101–119.

[14] S.F. Du, J.H. Kwak, M.Y. Xu, 2-Arc-transitive regular covers of complete graphs having the covering transformation group $\mathbb{Z}_p^3$, J. Combin. Theory Ser. B 93 (2005) 73–93.

[15] Y.-Q. Feng, J.H. Kwak, Constructing an infinite family of cubic 1-regular graphs, European J. Combin. 23 (2002) 559–565.

[16] Y.-Q. Feng, J.H. Kwak, An infinite family of cubic one-regular cubic graphs with insolvable automorphism groups, Discrete Math. 269 (2003) 281–286.

[17] Y.-Q. Feng, J.H. Kwak, One-regular cubic graphs of order a small number times a prime or a prime square, J. Aust. Math. Soc. 76 (2004) 345–356.

[18] Y.-Q. Feng, J.H. Kwak, s-Regular cyclic coverings of the complete bipartite graph $K_{3,3}$, J. Graph Theory 45 (2004) 101–112.

[19] Y.-Q. Feng, J.H. Kwak, s-Regular cubic graphs as coverings of the three-dimensional hypercube Q_3, European J. Combin. 24 (2003) 719–731.

[20] Y.-Q. Feng, J.H. Kwak, K.S. Wang, Classifying cubic symmetric graphs of order $8p$ or $8p^2$, European J. Combin. 26 (2005) 1033–1052.

[21] R. Frucht, A one-regular graph of degree three, Canad. J. Math. 4 (1952) 240–247.

[22] D. Gorenstein, Finite Simple Groups, Plenum Press, New York, 1982.

[23] J.L. Gross, T.W. Tucker, Generating all graph coverings by permutation voltage assignment, Discrete Math. 18 (1977) 273–283.

[24] F. Harary, Graph Theory, Addison–Wesley, Reading, MA, 1969.

[25] S. Hong, J.H. Kwak, J. Lee, Regular graph coverings whose covering transformation groups have the isomorphism extension property, Discrete Math. 168 (1996) 85–105.

[26] B. Huppert, Endliche Gruppen I, Springer, Berlin, 1979.

[27] J.H. Kwak, J.-M. Oh, Infinitely many finite one-regular graphs of any even valency, J. Combin. Theory Ser. B 90 (2004) 185–191.

[28] P. Lorimer, Vertex-transitive graphs: Symmetric graphs of prime valency, J. Graph Theory 8 (1984) 55–68.

[29] Z.P. Lu, C.Q. Wang, M.Y. Xu, On semisymmetric cubic graphs of order $6p$, Sci. China Ser. A 47 (2004) 1–17.

[30] A. Malnič, Group actions, coverings and lifts of automorphisms, Discrete Math. 182 (1998) 203–218.

[31] A. Malnič, D. Marušič, Imprimitive graphs and graph coverings, in: D. Jungnickel, S.A. Vanstone (Eds.), Coding Theory, Design Theory, Group Theory: Proc. M. Hall Memorial Conf., J. Wiley and Sons, New York, 1993, pp. 221–229.

[32] A. Malnič, P. Potočnik, Invariant subspaces, duality, and covers of the Petersen graph, European J. Combin. 27 (2006) 971–989.

[33] A. Malnič, D. Marušič, N. Seifter, Constructing infinite one-regular graphs, European J. Combin. 20 (1999) 845–853.

[34] A. Malnič, R. Nedela, M. Škoviera, Lifting graph automorphisms by voltage assignments, European J. Combin. 21 (2000) 927–947.

[35] A. Malnič, D. Marušič, P. Potočnik, On cubic graphs admitting an edge-transitive solvable group, J. Algebraic Combin. 20 (2004) 99–113.

[36] A. Malnič, D. Marušič, P. Potočnik, Elementary abelian covers of graphs, J. Algebraic Combin. 20 (2004) 71–97.

[37] A. Malnič, D. Marušič, P. Potočnik, C.Q. Wang, An infinite family of cubic edge- but not vertex-transitive graphs, Discrete Math. 280 (2004) 133–148.

[38] A. Malnič, D. Marušič, C.Q. Wang, Cubic edge-transitive graphs of order $2p^3$, Discrete Math. 274 (2004) 187–198.

[39] D. Marušič, A family of one-regular graphs of valency 4, European J. Combin. 18 (1997) 59–64.

[40] D. Marušič, R. Nedela, Maps and half-transitive graphs of valency 4, European J. Combin. 19 (1998) 345–354.

[41] D. Marušič, T. Pisanski, Symmetries of hexagonal graphs on the torus, Croat. Chemica Acta 73 (2000) 69–81.

[42] D. Marušič, M.Y. Xu, A $\frac{1}{2}$-transitive graph of valency 4 with a nonsolvable group of automorphisms, J. Graph Theory 25 (1997) 133–138.

[43] R.C. Miller, The trivalent symmetric graphs of girth at most six, J. Combin. Theory Ser. B 10 (1971) 163–182.

[44] R. Nedela, M. Škoviera, Which generalized Petersen graphs are Cayley graphs?, J. Graph Theory 19 (1995) 1–11.

[45] J.-M. Oh, K.W. Hwang, Construction of one-regular graphs of valency 4 and 6, Discrete Math. 278 (2004) 195–207.

[46] C.E. Praeger, Imprimitive symmetric graphs, Ars Combin. 19 (1985) 149–163.

[47] N. Seifter, V.I. Trofimov, Automorphism groups of covering graphs, J Combin. Theory Ser. B 71 (1977) 67–72.

[48] M. Škoviera, A contribution to the theory of voltage graphs, Discrete Math. 61 (1986) 281–292.
[49] W.T. Tutte, A family of cubical graphs, Proc. Cambridge Philos. Soc. 43 (1947) 459–474.
[50] W.T. Tutte, On the symmetry of cubic graphs, Canad. J. Math. 11 (1959) 621–624.
[51] W.T. Tutte, A non-Hamiltonian graph, Canad. Math. Bull. 3 (1960) 1–5.