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Some Properties of the Relative Rearrangement

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We develop some new properties of the relative rearrangement. Some of these properties generalize well-known results known as the Hardy–Littlewood inequality. © 1988 Academic Press, Inc.

INTRODUCTION

The notion of relative rearrangement was introduced first by J. Mossino and R. Temam [7]. It has been developed in a recent paper [6], and several applications in partial differential equations can be found in [6–9]. The usual rearrangement is also a relative rearrangement as it is proved in [6]. Some properties of the relative rearrangement have been given in [5, 6, 8]. In this paper, we prove some additional properties, which generalize well-known results for the usual rearrangement.

1. DEFINITIONS AND PRELIMINARY RESULTS

In this paper, we use only Lebesgue measure. Let Ω be a bounded measurable set of $\mathbb{R}^N$. For any measurable subset E of Ω, we denote by $|E|$ its measure. Let u be a real measurable function defined in Ω. We will say that u has a flat region of value t if meas$\{x \in \Omega, u(x) = t\} = |u = t|$ is strictly positive. There may exist a countable family of flat regions $P_i = \{u = t_i\}$. We denote $P = \bigcup_{i \in D} P_i$ the union of all flat regions of u.

DEFINITION 1. The *decreasing rearrangement* of u is defined on $\bar{\Omega}^* = [0, |\Omega|]$ by:

$$u_*(s) = \text{Inf}\{\theta \in \mathbb{R}, |u > \theta| \leqslant s\}.$$

We will also consider *the increasing rearrangement* of u: $u^*(s) = u_*(|\Omega| - s)$.

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Now, we recall the notion of relative rearrangement, as it appeared in [7]. Let $v \in L^1(\Omega)$, we define a function w in $\bar{\Omega}^*$ by:

$$w(s) = \begin{cases} \int_{u > u_*(s)} v(x)\, dx & \text{if} \quad |u = u_*(s)| = 0 \\ \int_{u > u_*(s)} v(x)\, dx + \int_0^{s - |u > u_*(s)|} (v|_{P(s)})_*(\sigma)\, d\sigma & \text{otherwise.} \end{cases}$$

(Here the last integrand is the decreasing rearrangement of the restriction of v to the set $P(s) = \{u = u_*(s)\}$ supposed to be of positive measure.) The following theorem was proved in [6].

THEOREM 1. *Let u be a measurable function defined in Ω, v in $L^p(\Omega)$ $(1 \leqslant p \leqslant +\infty)$ then:*

(i) $w \in W^{1,p}(\Omega^*)$

(ii) $\|dw/ds\|_{L^p(\Omega^*)} \leqslant \|v\|_{L^p(\Omega)}$,

where $\Omega^* =]0, |\Omega|[$.

DEFINITION 2 (*Relative rearrangement*). The function dw/ds is called the *rearrangment of v with respect to u* and is denoted v_{*u}.

We will need the following properties proved in [6] (see also [5]).

PROPOSITION 1. *If u is measurable function defined in Ω, v in $L^1(\Omega)$ then:*

(i.1) *for all constant c,* $v_{*c} = v_*$, $c_{*u} = c$

(i.2) $\int_{\Omega^*} v_{*u}(\sigma)\, d\sigma = \int_\Omega v(x)\, dx$

(i.3) *If we consider the relative rearrangement v_u^* (see [5, 6]) associated to the increasing rearrangement, we have*

$$v_u^* = -(-v)_{*-u}.$$

The following definitions concern the mean value operators introduced in [7].

DEFINITION 3. Let g be a measurable real function, almost everywhere defined in Ω^*. The the functions u and g, we can associate another function $M_u(g)\colon \Omega \to \mathbb{R}$ defined by:

$$\text{for a.e. } x, \quad M_u(g)(x) = \begin{cases} g(\boldsymbol{\beta}(u)(x)) & \text{if} \quad x \in \Omega \backslash P \\ \dfrac{1}{|P_i|} \displaystyle\int_{s_i'}^{s_i''} g(\sigma)\, d\sigma & \text{if} \quad x \in P_i, \end{cases}$$

where $s_i' = |u < t_i|$, $s_i'' = |u \leqslant t_i|$, $\boldsymbol{\beta}(u)(x) = |u < u(x)|$.

DEFINITION 4. Now, let us consider two measurable real functions defined in Ω. We denote by v_i the restriction of v to a flat region P_i of u. To the function g defined above, we can associate the function $M_{u,v}(g)\colon \Omega \to \mathbb{R}$ by

$$M_{u,v}(g)(x) = \begin{cases} M_u(g)(x) & \text{if } x \in \Omega \backslash P \\ M_{v_i}(h_i)(x) & \text{if } x \in P_i, \end{cases}$$

where $h_i(s) = g(s_i' + s)$ if $s \in [0, |P_i|]$; M_{v_i} is defined as M_u (with Ω replaced by P_i).

The proof of the following lemma is given in [5, 7].

LEMMA 1. *Let u, v be two measurable functions from Ω into $\mathbb{R}$, $v \in L^p(\Omega)$ $(1 < p \leqslant +\infty)$ and $g \in L^q(\Omega^*)$, $1/p + 1/q = 1$ then*

$$M_{u,v}(g) \in L^q(\Omega) \qquad \textit{and} \qquad \int_{\Omega^*} g v_u^* \, d\sigma = \int_\Omega M_{u,v}(g)\, v\, dx \tag{1}$$

Remark 1. If $v \in L^1(\Omega)$, $g \in \mathscr{C}^0(\bar{\Omega}^*)$ the relation (1) holds. In fact, if we consider first $g \in \mathscr{D}(\Omega^*)$, we can argue as in [7] to get relations (1). As $M_{u,v}$ belongs to $\mathscr{L}(L^\infty(\Omega^*), L^\infty(\Omega))$ (see [5]) and the mapping $g \in L^\infty(\Omega^*) \to \int_{\Omega^*} g v_u^* \, d\sigma$ is continuous, we can conclude by density of $\mathscr{D}(\Omega^*)$ in $\mathscr{C}^0(\bar{\Omega}^*)$.

Remark 2. One can give an explicit expression of v_{*u} when u is a regular function.

(R.1) Assume that Ω is a bounded open set of $\mathbb{R}^N$ and u is an element of $\mathscr{C}^\infty(\Omega)$ such that $1/|\nabla u| \in L^1(\Omega)$; then for any $v \in L^1(\Omega)$,

$$v_{*u}(s) = \frac{\int_{u = u_*(s)} (v(x)\, d\Gamma(x)/|\nabla u(x)|)}{\int_{u = u_*(s)} (d\Gamma(x)/|\nabla u(x)|)} \qquad \text{a.e. in } \Omega^*, \tag{2}$$

where $d\Gamma$ denote the $(N-1)$-dimensional Lebesgue measure.

Proof of Remark 2. We recall (see [1]) that a real t is called a regular value of u if $u^{-1}(t)$ is a compact $(N-1)$-dimensional manifold on which $\nabla u(x) \neq 0$.

A real t is said to be a critical value of u if it is not a regular value. The set of critical values is denoted by $\mathscr{C}$. According to Sard's theorem (see [1]), if $u \in \mathscr{C}^\infty(\Omega)$ then the Lebesgue measure of $\mathscr{C}$ is zero.

We denote by $\mu(t) = |u > t|$. We observe that u has no flat region because $1/|\nabla u|$ is in $L^1(\Omega)$ and on flat regions $\nabla u(x) = 0$ a.e. We then have

$$\text{for all } s \in \bar{\Omega}^*\colon \mu(u_*(s)) = s. \tag{3}$$

The function μ is absolutely continuous. In fact, let us take $(a, b) \in \mathbb{R}^2$ $(a < b)$. The function u is Lipschitz and $1/|\nabla u|$ is integrable. We can use Federer's theorem [2] to get

$$\mu(a) - \mu(b) = \int_{a \leqslant u \leqslant b} dx = \int_a^b dp \int_{u=p} \frac{d\Gamma}{|\nabla u(x)|}$$

and

$$\mu'(p) = \int_{u=p} \frac{d\Gamma}{|\nabla u(x)|}.$$

These last relations prove that μ is absolutely continuous. As absolutely continuous functions map null sets into null sets, we deduce

$$\text{meas } \mu(\mathscr{C}) = |\mu(\mathscr{C})| = 0.$$

Using relation (3), we get $\{s \in \bar{\Omega}^*, u_*(s) \in \mathscr{C}\}$ is included in $\mu(\mathscr{C})$. Thus, for almost every s in Ω^*, $u_*(s)$ is a regular value of u. In the following, we consider only such points s. The following computation is then true for almost every s of Ω^*.

Let $h > 0$; as u has no flat region, we get

$$w(s+h) - w(s) = \int_{u_*(s+h) \leqslant u \leqslant u_*(s)} v(x)\, dx.$$

One can check that for small h, $\nabla u(x) \neq 0$ for all x in the compact $K_h = \{u_*(s+h) \leqslant u \leqslant u_*(s)\}$, $1/|\nabla u| \in L^\infty(K_h)$ and $v/|\nabla u|$ is in $L^1(K_h)$. We use Federer's theorem [2] to get

$$\frac{w(s+h) - w(s)}{h} = -\frac{1}{h} \int_{u_*(s)}^{u_*(s+h)} dp \int_{u=p} \frac{v(x)\, d\Gamma}{|\nabla u(x)|}.$$

Let us write $u_*(s+h) = u_*(s) + R(s, h)$, where $R(s, h) = h \cdot (du_*/ds) + o(h)$ $(du_*/ds$ exist a.e. as u_* is decreasing and $R(s, h) \neq 0$, since u has no flat region). Then we get

$$\frac{w(s+h) - w(s)}{h}$$

$$= -\frac{R(s, h)}{h} \cdot \frac{1}{R(s, h)} \int_{u_*(s)}^{u_*(s) + R(s,h)} dp \int_{u=p} \frac{v(x)\, d\Gamma}{|\nabla u(x)|}.$$

And when h tends to zero,

$$v_{*u}(s) = \frac{dw}{ds} = -\frac{du_*}{ds} \int_{u=u_*(s)} \frac{v(x)\, d\Gamma}{u(x)} \qquad \text{a.e. in } \Omega^*.$$

This last relation is true for all $v \in L^1(\Omega)$. In particular, if $v = 1$ ($v_{*u} = 1$, see Proposition 1) and thus

$$\frac{du_*}{ds} = -1 \Big/ \int_{u = u_*(s)} \frac{d\Gamma}{|\nabla u(x)|} \qquad \text{a.e. in } \Omega^*.$$

These last formulas lead to (2).

2. General Properties of the Relative Rearrangement

The following is a generalization of the property of contraction for v_{*u}.

THEOREM 2. *Let ρ be a convex function defined in $\mathbb{R}$, $(v_1, v_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$, u a measurable function defined in Ω. Then*

$$\int_{\Omega^*} \rho(v_{1*u} - v_{2*u})\, d\sigma \leqslant \int_\Omega \rho(v_1 - v_2)\, dx.$$

This last formula is also valid for $(v_1, v_2) \in L^p(\Omega) \times L^p(\Omega)$ ($1 \leqslant p < +\infty$) if ρ satisfies

$$\exists \alpha \geqslant 0,\ \exists \beta \in \mathbb{R},\ \forall t \in \mathbb{R}, \qquad |\rho(t)| \leqslant \alpha |t|^p + \beta.$$

Remark 3. According to Kranoselskii [4], the last condition for ρ is necessary and sufficient to ensure that the mapping $v \to \rho(v)$ is continuous from $L^p(\Omega)$ (resp. $L^p(\Omega^*)$) into $L^1(\Omega)$ (resp. $L^1(\Omega^*)$).

Remark 4. Under the assumptions of Remark 2, if, moreover, ρ satisfies $|\rho(t)| \leqslant \alpha |t| + \beta$, we have the ponctual inequality,

$$\text{a.e. in } \Omega^*,\ \rho(v_{1*u}(s) - v_{2*u}(s)) \leqslant [\rho(v_1 - v_2)]_{*u}(s) \tag{5}$$

for any $(v_1, v_2) \in L^1(\Omega) \times L^1(\Omega)$. In fact, by relation (2),

$$v_{1*u}(s) - v_{2*u}(s) = \frac{\int_{u = u_*(s)} (v_1 - v_2)(x)(d\Gamma(x)/|\nabla u(x)|)}{\int_{u = u_*(s)} (d\Gamma(x)/|\nabla u(x)|)}.$$

Setting

$$dv = \frac{d\Gamma(x)}{|\nabla u(x)|} \cdot \left(1 \Big/ \int_{u = u_*(s)} \frac{d\Gamma}{|\nabla u(x)|}\right),$$

one can use Jensen inequality to get

$$\rho(v_{1*u}(s) - v_{2*u}(s)) = \rho\left(\int_{u = u_*(s)} (v_1 - v_2)(x)\, dv(x)\right) \leqslant \int_{u = u_*(s)} \rho(v_1 - v_2)\, dv$$

and

$$\int_{u=u_*(s)} \rho(v_1 - v_2)\, dv = [\rho(v_1 - v_2)]_{*u}(s) \qquad \text{(by relation (2))}.$$

This ponctuel relation leads to (4), since by integration,

$$\int_{\Omega^*} \rho(v_{1*u} - v_{2*u})\, d\sigma \leqslant \int_{\Omega^*} [\rho(v_1 - v_2)]_{*u}(\sigma)\, d\sigma = \int_{\Omega} \rho(v_1 - v_2)\, dx$$

(by Proposition 1(i.2)).

Remark 5. The ponctual relation (5(is not valid for any u. To see this, let us take, $u = \text{constant} = 1$, and $v_1 = v \in L^\infty(\Omega)$, $v_2 = 0$. Then, if it is true, we will get

$$\rho(v_{*1}) \leqslant [\rho(v)]_{*1}.$$

By Proposition 1(i.1), $v_{*1} = v_*$, $[\rho(v)]_{*1} = [\rho(v)]_*$; thus $\rho(v_*) \leqslant [\rho(v)]_*$. By equimeasurability, we deduce $\rho(v_*) = [\rho(v)]_*$ for any $v \in L^\infty(\Omega)$ and any convex function ρ; it is not difficult to see that this is impossible (for example, take ρ decreasing).

The proof of Theorem 2 needs the following lemma whose proof can be easily deduced from G. Chiti's result [3].

LEMMA 2. *Let ρ be a convex function defined in $\mathbb{R}$, u and v two measurable functions defined in Ω, $v \in L^\infty(\Omega)$; then we have*

$$\int_{\Omega^*} \rho((u+v)_* - u_*)\, d\sigma \leqslant \int_{\Omega} \rho(v)\, dx.$$

Proof of Theorem 2. Since ρ is a convex function, the mapping $v \in L^\infty(\Omega) \to \int_\Omega \rho(v)\, dx$ is L.S.C. for the weak star topology. Hence, if (v_1, v_2) are two elements of $L^\infty(\Omega)$, we know (see [5]) that for all $\lambda > 0$ and u measurable defined in Ω,

$$\left\| \frac{(u+\lambda v_1)_* - (u+\lambda v_2)_*}{\lambda} \right\|_\infty \leqslant \|v_1 - v_2\|_\infty$$

and

$$\frac{(u+\lambda v_1)_* - (u+\lambda v_2)_*}{\lambda} \xrightarrow[\lambda \to 0]{} v_{1*} - v_{2*u} \qquad \text{in } L^\infty(\Omega^*),$$

weak star.

We deduce from Lemma 2 and the remark above:

$$\int_{\Omega} \rho(v_1 - v_2)\, dx \geqslant \underline{\lim}_{\lambda} \int_{\Omega^*} \rho \left[\frac{(u+\lambda v_1)_* - (u+\lambda v_2)_*}{\lambda} \right] d\sigma$$

$$\geqslant \int_{\Omega^*} \rho(v_{1*u} - v_{2*u})\, d\sigma.$$

Assume that ρ satisfies the growth condition in Theorem 2. Since the mapping $v \in L^p(\Omega) \to v_{*u} \in L^p(\Omega^*)$ is continuous (see [6]), we deduce by Remark 3 that the mapping $v \in L^p(\Omega) \to (\int_\Omega \rho(v)\, dx, \int_{\Omega^*} \rho(v_{*u})\, d\sigma)$ is continuous. We can conclude using the density of $L^\infty(\Omega)$ into $L^p(\Omega)$.

COROLLARY 1. *Let $\rho(t)$, $t \geqslant 0$, be convex, non-negative, non-decreasing, (v_1, v_2) in $L^1(\Omega) \times L^1(\Omega)$, and u a measurable function. Then,*

$$\int_{\Omega^*} \rho(|v_{1*u} - v_{2*u}|)\, d\sigma \leqslant \int_{\Omega} \rho(|v_1 - v_2|)\, dx.$$

Proof. We argue as in [3]. We consider the real Lipschitz functions T_n defined by

$$T_n(z) = \begin{cases} n & \text{if } z \geqslant n \\ z & \text{if } |z| \leqslant n \\ -n & \text{if } z \leqslant -n. \end{cases}$$

Then the functions $v_{in} = T_n(v_i)$ $i = 1, 2$ are in $L^\infty(\Omega)$ and satisfy $|v_{1n} - v_{2n}| \leqslant |v_1 - v_2|$ a.e. Since the function ρ is non-decreasing, we deduce that

$$\int_{\Omega} \rho(|v_{1n} - v_{2n}|)\, dx \leqslant \int_{\Omega} \rho(|v_1 - v_2|)\, dx$$

and that the function $\rho(|t|)$ is convex; we apply Theorem 2 to get

$$\int_{\Omega^*} \rho(|v_{1n*u} - v_{2n*u}|)\, d\sigma \leqslant \int_{\Omega} \rho(|v_1 - v_2|)\, dx.$$

Since the sequence v_{in} tends to v_i in $L^1(\Omega)$ $i = 1, 2$ and the mapping $v \in L^1(\Omega) \to v_{*u} \in L^1(\Omega^*)$ is continuous, we can substract a sequence[1] denoted also v_{in*u} which converges almost everywhere in Ω^*. We apply Fatou's lemma to get that

$$\int_{\Omega^*} \rho(|v_{1*u} - v_{2*u}|)\, d\sigma \leqslant \underline{\lim}_{n} \int_{\Omega^*} \rho(|v_{1n*u} - v_{2n*u}|)$$

$$\leqslant \int_{\Omega} \rho(|v_1 - v_2|)\, dx. \quad \blacksquare$$

[1] That is, from the sequence (v_{1n*u}, v_{2n*u}).

These last results illustrate the convergence of the relative rearrangement in Orlicz spaces if the original functions belong to $L^1(\Omega)$ and converge in Orlicz spaces.

3. A Generalization of the Hardy–Littlewood Inequality

Before proving the result of generalization, we will need some lemmas:

LEMMA 3. *Let $v \in L^p(\Omega)$ $(1 \leqslant p \leqslant +\infty)$ then there exists a sequence $v_n \in L^p(\Omega)$ such that v_n has no flat region and v_n tends to v in $L^p(\Omega)$.*

Proof. Let $P = \bigcup_{i \in D} P_i$, where $P_i = \{v = \theta_i\}$, $|P_i| \neq 0$, and $\theta_i \neq 0$. We denote by χ_A the characteristic function of a measurable set A. We put $\lambda_n(x) = (1/n) \cdot (1/(1+|x|))$ for any $x \in \Omega$. We observe that

$$|\{x \in \Omega, \lambda_n(x) = \alpha\}| = |\{x \in \Omega, e^{-\lambda_n(x)} = \beta\}| = 0 \qquad \forall (\alpha, \beta) \in \mathbb{R}^2.$$

We define

$$v_n(x) = e^{-\lambda_n(x)\chi_P(x)}(v(x) + \lambda_n(x)\, \chi_{\{v=0\}}(x)).$$

One can check that

$$|v_n(x) - v(x)| \leqslant \frac{1}{n}(|v(x)| + 1).$$

So, v_n tends to v in $L^p(\Omega)$ $(1 \leqslant p \leqslant +\infty)$.

Let us prove that for all $t \in \mathbb{R}$, $|v_n = t| = 0$. We remark that

$$\begin{aligned}\{v_n = t\} = {} & \{x \in \Omega \backslash P, v(x) \neq 0, v_n(x) = t\} \\ & \cup \left(\bigcup_{i \in D} \{x \in P_i, v_n(x) = t\} \right) \\ & \cup \{x \in \Omega, v(x) = 0, v_n(x) = t\}.\end{aligned}$$

We deduce then:

$$\begin{aligned}|v_n = t| = {} & |\{x \in \Omega \backslash P, v(x) \neq 0, v(x) = t\}| \\ & + \sum_{i \in D} \left| \left\{ x \in P_i, e^{-\lambda_n(x)} = \frac{t}{\theta_i} \right\} \right| \\ & + |\{x \in \Omega, v(x) = 0, \lambda_n(x) = t\}|.\end{aligned}$$

By the remark above, each term of the summation vanishes. ∎

LEMMA 4. *Let $v \in L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$, such that v has no flat region then*

(i) *for all $a \in \bar{\Omega}^*$, there exists a measurable set $E(a)$ such that*

$$w(a) = \int_{E(a)} v(x)\, dx \qquad \text{and} \qquad |E(a)| = a.$$

In addition,

(ii) *if $a < b$ then $E(a) \subset E(b)$.*

Proof. (i) Let $a \in \bar{\Omega}^*$. If $|u = u_*(a)| = 0$, we set $E(a) = \{u > u_*(a)\}$. Then $|u > u_*(a)| = |u \geqslant u_*(a)| = a$.

If $|u = u_*(a)| \neq 0$, we denote v_a the restriction of v to the set $\{u = u_*(a)\}$; then by equimeasurability,

$$\int_0^{a - |u > u_*(a)|} (v_a)_* (\sigma)\, d\sigma = \int_{v_a > (v_a)_*(a - |u > u_*(a)|)} v(x)\, dx.$$

The set $\{v_a > (v_a)_*(a - |u > u_*(a)|)\}$ and $\{u > u_*(a)\}$ are disjoint. Hence, we have

$$|E(a)| = |u > u_*(a)| + |v_a > (v_a)_*(a - |u > u_*(a)|)| = a,$$

if we set $E(a) = \{u > u_*(a)\} \cup \{v_a > (v_a)_*(a - |u > u_*(a)|)\}$.

(ii) Let $a < b$. If $u_*(a) = u_*(b)$, then $v_a = v_b = k$; we deduce

$$k_*(a - |u > u_*(a)|) \geqslant k_*(b - |u > u_*(b)|)$$

so $E(a) \subset E(b)$.

If $u_*(a) > u_*(b)$: $E(a) \subset \{u \geqslant u_*(a)\} \subset \{u > u_*(b)\} \subset E(b)$.

Remark 6. If $]a, b[\, \cap \,]c, d[\, = \varnothing$ and if we set

$$E(a, b) = E(b) \backslash E(a)$$
$$E(c, d) = E(d) \backslash E(c),$$

then

$$E(a, b) \cap E(c, d) = \varnothing.$$

The following lemma is crucial to prove the result of generalization.

LEMMA 5. *Let u be a real measurable function defined in Ω and $v \in L^1(\Omega)$ then for all measurable sets E in Ω^*; we have*

$$\int_E v_{*u}(\sigma)\, d\sigma \geqslant \int_0^{|E|} v^*(\sigma)\, d\sigma \left(= \int_{|\Omega|-|E|}^{|\Omega|} v_*(\sigma)\, d\sigma \right),$$

where v^* denote the increasing rearrangement of v.

Proof Lemma 5. As the mappings $v \in L^1(\Omega) \to v_1^*$ or $v_{*u} \in L^1(\Omega^*)$ are continuous, thanks to Lemma 3, we can restrict to the case when v has no flat region.

Let $\mathcal{O}$ be an open set of Ω^* then $\mathcal{O}$ is the union (at most countable) of his disjoint connected components: $\mathcal{O} = \bigcup_{i \in D}]a_i, b_i[$,

$$\int_{\mathcal{O}} \frac{dw}{ds}\, d\sigma = \sum_{i \in D} \int_{a_i}^{b_i} \frac{dw}{ds}\, d\sigma.$$

According to Lemma 4,

$$\sum_{i \in D} \int_{a_i}^{b_i} \frac{dw}{ds}\, d\sigma = \sum_{i \in D} \int_{E(a_i, b_i)} v(x)\, dx.$$

Since $E(a_i, b_i) \cap E(a_j, b_j) = \varnothing$ for $i \neq j$ (see Remark 5), we deduce, via the Hardy–Littlewood inequality,

$$\int_{\mathcal{O}} \frac{dw}{ds}\, d\sigma = \int_{\bigcup_{i \in D} E(a_i, b_i)} v(x)\, dx \geqslant \int_0^{|\bigcup_{i \in D} E(a_i, b_i)|} v^*(\sigma)\, d\sigma,$$

$$\left| \bigcup_{i \in D} E(a_i, b_i) \right| = \sum_{i \in D} (b_i - a_i) = |\mathcal{O}|,$$

$$\int_{\mathcal{O}} \frac{dw}{ds}\, d\sigma \geqslant \int_0^{|\mathcal{O}|} v^*(\sigma)\, d\sigma. \tag{6}$$

If E is a measurable set of Ω^*, then there exists a sequence $\mathcal{O}_p$ of open sets such that $E \subset \mathcal{O}_{p+1} \subset \mathcal{O}_p$ and $|\mathcal{O}_p| \to_{p \to +\infty} |E|$. Then by (6)

$$\int_{\mathcal{O}_p} \frac{dw}{ds}\, d\sigma \geqslant \int_0^{|\mathcal{O}_p|} v^*(\sigma)\, d\sigma.$$

When we pass to the limit,

$$\int_E v_{*u}(\sigma)\, d\sigma = \int_E \frac{dw}{ds}\, d\sigma \geqslant \int_0^{|E|} v^*(\sigma)\, d\sigma.$$

The following theorem is the generalization of the Hardy–Littlewood inequality.

THEOREM 3. *Let u be a real measurable function on Ω, v_1 in $L^p(\Omega)$, and $v_2 \in L^q(\Omega^*)$, $1/p+1/q=1$, then*

$$\int_{\Omega^*} v_{1*u} v_2 \, d\sigma \leqslant \int_{\Omega^*} v_1^* v_2^* \, d\sigma \left(= \int_{\Omega^*} v_{1*} v_{2*} \, d\sigma \right). \tag{7}$$

Remark 7. If we change v_1 into $-v_1$, v_2 into $-v_2$, and u into $-u$, using Proposition 1(i.3), we get

$$\int_{\Omega^*} v_{1u}^* v_2 \, d\sigma \leqslant \int_{\Omega^*} v_1^* v_2^* \, d\sigma. \tag{8}$$

The proof of this theorem needs the following lemma whose proof is in [5].

LEMMA 6. *Let $f \in L^\infty(\Omega^*)$, $a \leqslant f \leqslant b$, $g \in L^1(\Omega^*)$; then*

$$\int_{\Omega^*} fg \, d\sigma = b \int_{\Omega^*} g \, d\sigma - \int_a^b dt \int_{f<t} g \, d\sigma.$$

Proof of Theorem 3. We begin with the case $v_1 \in L^p(\Omega)$ and $v_2 \in L^\infty(\Omega^*)$; then $v_{1*u} \in L^1(\Omega^*)$ and, by Lemma 6,

$$\int_{\Omega^*} v_{1*u} v_2 \, d\sigma = b \int_{\Omega^*} v_{1*u} \, d\delta - \int_a^b dt \int_{v_2<t} v_{1*u} \, d\sigma.$$

By Proposition 1(i.2) and equimeasurability,

$$\int_{\Omega^*} v_{1*u} \, d\sigma = \int_{\Omega^*} v_1^* \, d\sigma.$$

By Lemma 4,

$$\int_{v_2<t} v_{1*u} \, d\sigma \geqslant \int_0^{|v_2^*<t|} v_1^*(\sigma) \, d\sigma = \int_{v_2^*<t} v_1^*(\sigma) \, d\sigma.$$

Hence

$$\int_{\Omega^*} v_{1*u} v_2 \, d\sigma \leqslant b \int_{\Omega^*} v_1^* \, d\sigma - \int_a^b dt \int_{v_2^*<t} v_1^*(\sigma) \, d\sigma$$

$$= \int_{\Omega^*} v_1^* v_2^* \, d\sigma.$$

The result for $q < +\infty$ easily follows by density.

Remark 8. If $v_2 \geqslant 0$ is only measurable, $v_1 \geqslant 0$, and $v_1 \in L^1(\Omega)$, the relation (7) (or (8)) remains valid. In fact, there exists an increasing sequence $v_{2n} \in L^\infty(\Omega^*)$ such that

$$\lim_n v_{2n}(\sigma) = v_2(\sigma) \qquad \text{and} \qquad 0 \leqslant v_{2n}(\sigma) \leqslant v_2(\sigma) \text{ a.e.}$$

Then

$$\int_{\Omega^*} v_{2n}(\sigma)\, v_{1*u}(\sigma)\, d\sigma \leqslant \int_{\Omega^*} v_{2n}^* v_1^*\, d\sigma \leqslant \int_{\Omega^*} v_2^*(\sigma)\, v_1^*(\sigma)\, d\sigma.$$

As $v_1 \geqslant 0$ implies $v_{1*u} \geqslant 0$ (see [6]), then by Fatou's lemma,

$$\int_{\Omega^*} v_{1*u} v_2\, d\sigma \leqslant \int_{\Omega^*} v_1^* v_2^*\, d\sigma.$$

Remark 9. As a corollary, we recover the well-known Hardy–Littlewood theorem: For all $v \in L^p(\Omega)$, $h \in L^q(\Omega)$, $1/p + 1/q = 1$, we have

$$\int_\Omega hv\, dx \leqslant \int_{\Omega^*} h^* v^*\, d\sigma.$$

Proof. We begin with the case $h \in \mathscr{C}^0(\bar{\Omega})$ and $v \in L^1(\Omega)$. Using Remark 1 and Theorem 3,

$$\int_\Omega M_{h,v}(h^*)\, v\, dx \leqslant \int_{\Omega^*} h^* v^*\, d\sigma. \tag{9}$$

By Definition 4,

$$M_{h,v}(h^*)(x) = \begin{cases} h^*(\boldsymbol{\beta}(h)(x)) & \text{if} \quad x \in \Omega \backslash P \\ M_{v_i}(g)(x) & \text{if} \quad x \in P_i = \{h = t_i\} \end{cases}$$

$$g(s) = h^*(s_i' + s) \qquad \text{for} \quad s \in [0, s_i'' - s_i']$$

$$s_i' = |h < t_i|, \qquad s_i'' = |h \leqslant t_i|.$$

Then, we have $g(s) = h^*(s_i' + s) = h^*(s_i') = h^*(\boldsymbol{\beta}(h)(x))$. In any case $M_{h,v}(h^*)(x) = h^*(\boldsymbol{\beta}(h)(x)) = h(x)$ (since h is continuous). By (9), we get the Hardy–Littlewood inequality.

By density, the inequality remains valid for $h \in L^q(\Omega)$ and $v \in L^p(\Omega)$, $1/p + 1/q = 1$, $q > 1$, and then for $q = 1$. ∎

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