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A Further Generalization of Yannelis–Prabhakar's Continuous Selection Theorem and Its Applications

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In this article, we first prove one improved version of the Yannelis–Prabhakar continuous selection theorem and next, as its applications, a fixed point theorem in noncompact product spaces, a nonempty intersection theorem, some existence theorems of solutions for the generalized quasi-variational inequalities, and some equilibrium existence theorems for the abstract economies are given. © 1996 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

In 1983, Yannelis and Prabhakar [18] proved a continuous selection theorem as follows:

THEOREM A. *Let X be a nonempty paracompact Hausdorff topological space and Y be a Hausdorff topological vector space. Let $T: X \to 2^Y$ be a correspondence such that each $T(x)$ is open convex and for each $y \in Y$, $T^{-1}(y)$ is open in X. Then T has a continuous selection $f: X \to Y$, such that $f(x) \in T(x)$ for each $x \in X$.*

In 1992, Ding *et al.* [6] improved Theorem A by establishing the following:

THEOREM B. *Let X be a nonempty paracompact Hausdorff topological space and Y be a nonempty convex subset of a topological vector space.*

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Suppose $S, T: X \to 2^y$ *are correspondences such that*

(1) *for each* $x \in X$, co $S(x) \subset T(x)$ *and* $S(x) \neq \emptyset$,

(2) *for each* $y \in Y$, $S^{-1}(y)$ *is open in* X.

Then T *has a continuous selection.*

Using Theorem A, Yannelis and Prabhakar [18] obtained a fixed point theorem and an equilibrium existence theorem for an abstract economy. In [6], using Theorem B, Ding *et al.* obtained a fixed point theorem in product spaces, a nonempty intersection theorem, and two equilibrium existence theorems for an abstract economy.

In this paper, we first give one improved version of Theorem B and next by applying the result, we prove a fixed point theorem, a nonempty intersection theorem, some existence theorems of solutions for generalized quasi-variational inequalities, and some equilibrium existence theorems.

We need the following definitions.

Let X and Y be two topological spaces, $T: X \to 2^Y$ a multivalued mapping.

(1) T is said to be almost upper semicontinuous if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset \overline{V}$ for each $y \in U$.

(2) For each $y \in Y$, $T^{-1}(y) := \{x \in X: y \in T(x)\}$ is said to be the lower section of T.

(3) T is said to have local intersection property if for each $x \in X$ with $T(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of x such that $\bigcap_{z \in N(x)} T(z) \neq \emptyset$.

2. CONTINUOUS SELECTION THEOREM AND FIXED POINT THEOREM

In the paper, a subset of topological space is considered to have relative topology.

THEOREM 1. *Let* X *be a nonempty paracompact subset of a Hausdorff topological space* E *and* Y *be a nonempty subset of a Hausdorff topological vector space* F. *Suppose that* $S, T: X \to 2^Y$ *are two multivalued mappings with the following conditions*:

(i) *For each* $x \in X$, $S(x)$ *is nonempty and* co $S(x) \subset T(x)$.

(ii) S *has local intersection property.*

Then T *has a continuous selection*; *i.e., there is a continuous mapping* $f: X \to Y$ *such that* $f(x) \in T(x)$ *for each* $x \in X$.

Proof. For each x ∈ X, by condition (i) we know that $S(x) \neq \emptyset$; consequently, by condition (ii), there exists an open neighborhood $N(x)$ of x such that $M(x) := \bigcap_{z \in N(x)} S(z) \neq \emptyset$. Since X is paracompact, there is a locally finite open refinement $\mathscr{F} := \{U_\alpha : \alpha \in D\}$ of the $\{N(x): x \in X\}$ and a partition of unity subordinated to $\mathscr{F}$ $\{g_\alpha : \alpha \in D\}$ such that

(1) for each $\alpha \in D$, $g_\alpha : X \to [0,1]$ is continuous,

(2) $\overline{\{x \in X: g_\alpha(x) > 0\}} \subset U_\alpha$ for each $\alpha \in D$,

(3) $\sum_{\alpha \in D} g_\alpha(x) = 1$ for each $x \in X$.

Since $\mathscr{F}$ is a refinement of $\{N(x): x \in X\}$, for each $\alpha \in D$, there exists a $x_\alpha \in X$ such that $U_\alpha \subset N(x_\alpha)$. But with $M(x_\alpha) \neq \emptyset$, we may take a $y_\alpha \in M(x_\alpha)$.

Now, we define a mapping $f: X \to \operatorname{co} Y$ by

$$f(x) = \sum_{\alpha \in D} g_\alpha(x) y_\alpha, \qquad \forall x \in X.$$

Since $\mathscr{F}$ is locally finite, there are at most finitely many $g_\alpha(x) \neq 0$; hence f is continuous. For each $x \in X$ and each $\alpha \in D$, if $g_\alpha(x) \neq 0$, then $x \in U_\alpha \subset N(x_\alpha)$; consequently $y_\alpha \in S(x)$ and so $f(x) \in \operatorname{co} S(x) \subset T(x)$ by the condition (i). Therefore, $f: X \to Y$ is a continuous selection of T. ∎

Remark 1. Theorem 1 contains Theorems A and B. In fact, if for each $y \in Y$, $S^{-1}(y)$ is open, then for each $x \in X$ with $S(x) \neq \emptyset$, we take a fixed $y \in S(x)$ and let $N(x) = S^{-1}(y)$. Consequently $N(x)$ is an open neighborhood of x and $y \in \bigcap_{z \in N(x)} S(z)$. Hence S has local intersection property.

Remark 2. The following example shows that Theorem 1 is a true generalization of Theorems A and B.

EXAMPLE. Let $E = F = R$, $X = Y = [0,2)$, and $T(x) = S(x) = [x, 2)$ for each $x \in X$; then t satisfies all the conditions of Theorem 1. But for each $y \in Y$, $T^{-1}(y) = [0, y]$ is not open in X; hence T does not satisfy all the conditions of Theorems A and B.

THEOREM 2. *Let I be an index set. For each $i \in I$, let X_i be a convex subset of a locally convex Hausdorff topological vector space and D_i be a nonempty compact subset of X_i. Suppose that $X := \prod_{i \in I} X_i$ and $S_i, T_i: X \to 2^{D_i}$ are multivalued mappings with the following conditions*:

(i) *For each $x \in X$, $\operatorname{co} S_i(x) \subset T_i(x)$ and $S_i(x) \neq \emptyset$.*

(ii) *S_i has local intersection property.*

Then there exists a point $\hat{x} = \prod_{i \in I} \bar{x}_i \in D := \prod_{i \in I} D_i$ *such that* $\bar{x}_i \in T_i(\bar{x})$ *for each* $i \in I$.

Proof. Since D is compact in X, $\operatorname{co} D$ is paracompact in X by the Lemma 1 in [6]. By virtue of Theorem 1, there exists a continuous selection $f_i\colon \operatorname{co} D \to D_i$ of $T_i|_{\operatorname{co} D}$ for each $i \in I$. For each $x \in \operatorname{co} D$, let

$$f(x) = \prod_{i \in I} \{f_i(x)\};$$

then $f\colon \operatorname{co} D \to 2^D$ is upper semicontinuous by the Lemma 3 in [8]. Therefore, by Himmelberg's fixed point theorem [9], there exists $\bar{x} = \prod_{i \in I} \bar{x}_i \in D$ such that $\bar{x} \in f(\bar{x})$, i.e., $\bar{x}_i = f_i(\bar{x})$ for all $i \in I$. Hence

$$\bar{x} \in \prod_{i \in I} T_i(\bar{x}).$$

This completes the proof. ∎

Remark. Theorem 2 contains Theorem 3.2 of Yannelis and Prabhakar [18] and Theorem 2 of Ding *et al.* [6].

COROLLARY 3. *Let X be a convex subset of a locally convex Hausdorff topological vector space and D be a nonempty compact subset of X. Suppose that $T\colon X \to 2^D$ is a multivalued mapping with the following conditions*:

(i) *For each $x \in X$, $T(x)$ is nonempty convex.*

(ii) *T has local intersection property.*

Then there exists a point $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$.

Remark. Corollary 3 improves Browder's fixed point theorem [2].

3. GENERALIZED QUASI-VARIATIONAL INEQUALITIES

THEOREM 4. *Let X be a nonempty paracompact convex subset of a locally convex Hausdorff topological vector space E, let D be a nonempty compact subset of X, and let Y be a nonempty subset of a Hausdorff topological vector space F. Let $S\colon X \to 2^D$ be a continuous multivalued mapping with nonempty closed convex values and $T\colon X \to 2^Y$ be a multivalued mapping with nonempty convex values and local intersection property; $\varphi\colon X \times Y \times X \to R$ is a continuous functional. If the following conditions are fulfilled*:

(i) *$\varphi(x, y, z)$ is quasi-convex in z.*

(ii) *For each $x \in X$ and each $y \in T(x)$, $\varphi(x, y, x) \geq 0$.*

Then there exist $\bar{x} \in S(\bar{x})$ *and* $\bar{y} \in T(\bar{x})$ *such that*

$$\varphi(\bar{x}, \bar{y}, x) \geq 0 \qquad \text{for all} \quad x \in S(\bar{x}).$$

Proof. Since X is paracompact and $T\colon X \to 2^Y$ has nonempty convex values and local intersection property, there is a continuous selection $f\colon X \to Y$ of T by Theorem 1.

For each $x \in X$, let

$$H(x) = \left\{ z \in S(x) \colon \varphi(x, f(x), z) = \min_{u \in S(x)} \varphi(x, f(x), u) \right\}.$$

Since S has nonempty compact convex values, φ is continuous and $\varphi(x, y, z)$ is quasi-convex in z; hence $H\colon X \to 2^D$ has nonempty convex values.

For each $(x, u) \in X \times X$, let

$$\psi(x, u) = -\varphi(x, f(x), u).$$

By the continuity of φ and f we know that $\psi\colon X \times X \to R$ is continuous. Again, $S\colon X \to 2^D$ is continuous multivalued mapping with nonempty compact values and

$$H(x) = \left\{ z \in S(x) \colon \psi(x, z) = \max_{u \in S(x)} \psi(x, u) \right\};$$

hence $H\colon X \to 2^D$ is upper semicontinuous by Proposition 23 in [1, p. 120] and obviously, $H(x)$ is compact for each $x \in X$. Consequently, by virtue of Himmelberg's fixed point theorem [9] we know that there exists a point $\bar{x} \in D$ such that $\bar{x} \in H(\bar{x})$, i.e., $\bar{x} \in S(\bar{x})$ and $\varphi(\bar{x}, f(\bar{x}), \bar{x}) = \min_{u \in S(\bar{x})} \varphi(\bar{x}, f(\bar{x}), u)$. Now taking $\bar{y} = f(\bar{x})$, then $\bar{y} \in T(\bar{x})$ and $\varphi(\bar{x}, \bar{y}, \bar{x}) = \min_{u \in S(\bar{x})} \varphi(\bar{x}, \bar{y}, u)$. Therefore under condition (ii), for each $x \in S(\bar{x})$ we have

$$\varphi(\bar{x}, \bar{y}, x) \geq \min_{u \in S(\bar{x})} \varphi(\bar{x}, \bar{y}, u) = \varphi(\bar{x}, \bar{y}, \bar{x}) \geq 0.$$

This completes the proof. ∎

Remark. In [3, p. 209], under the conditions "F is quasi-complete, T is upper semicontinuous, $X = D$ and Y is closed convex," Chang proved the same conclusion; hence Theorem 4 improves Theorem 6.1.1 in [3].

COROLLARY 5. *Let X be a nonempty paracompact convex subset of a locally convex Hausdorff topological vector space E, D be a nonempty compact subset of X, and Y be a nonempty subset of E^* (E^* is dual space of E, with strong topology). Let $S\colon X \to 2^D$ be a continuous multivalued map-*

ping with nonempty closed convex values and $T\colon X \to 2^Y$ *be a multivalued mapping with nonempty convex values and local intersection property. Then there exists* $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ *such that* $\operatorname{Re}\langle \bar{y}, \bar{x} - z\rangle \leq 0$ *for all* $z \in S(\bar{x})$.

Proof. We take $\varphi(x, y, z) = \operatorname{Re}\langle y, z - x\rangle$; then by virtue of Theorem 4 we know that there exist $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$\operatorname{Re}\langle \bar{y}, z - \bar{x}\rangle \geq 0, \qquad \forall z \in S(\bar{x}).$$

i.e.,

$$\operatorname{Re}\langle \bar{y}, \bar{x} - z\rangle \leq 0, \qquad \forall z \in S(\bar{x}). \quad ∎$$

Remark. Corollary 5 improves the main result in [11].

THEOREM 6. *Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E and Y be a nonempty subset of a Hausdorff topological vector space F. Let* $S\colon X \to 2^X$ *be a continuous multivalued mapping with nonempty closed convex values,* $T\colon X \to 2^Y$ *be a multivalued mapping with nonempty convex values and local intersection property, and* $\varphi\colon X \times Y \times X \to R \cup \{\pm\infty\}$ *be upper semicontinuous. If the following conditions are fulfilled*:

(i) $\varphi(x, y, u)$ *is convex in* u,

(ii) *for each* $x \in X$ *and each* $y \in T(x)$, $\varphi(x, y, x) \geq 0$,

then there exist $\bar{x} \in S(\bar{x})$ *and* $\bar{y} \in T(\bar{x})$ *such that*

$$\varphi(\bar{x}, \bar{y}, x) \geq 0, \qquad \forall x \in S(\bar{x}).$$

Proof. By virtue of Theorem 1, there exists a continuous selection $f\colon X \to Y$ of T. For each $(x, u) \in X \times X$, let $\psi(x, u) = -\varphi(x, f(x), u)$; then by the upper semicontinuity of φ and continuity of f we know that $\psi\colon X \times X \to R \cup \{\pm\infty\}$ is lower semicontinuous and the condition (i) implies that $\psi(x, u)$ is concave in u. Consequently for each finite subset $\{u_1, u_2, \dots, u_n\} \subset X$ and each $u_0 \in \operatorname{co}\{u_1, \dots, u_n\}$,

$$u_0 = \sum_{i=1}^{n} \lambda_i u_i \qquad \left(\lambda \geq 0, \sum_{i=1}^{n} \lambda_i = 1\right),$$

we have

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i \psi(u_0, u_i) &\leq \psi\left(u_0, \sum_{i=1}^{n} \lambda_i u_i\right) \\
&= \psi(u_0, u_0) \\
&= -\varphi(u_0, f(u_o), u_0) \\
&\leq 0.
\end{aligned}$$

by condition (ii). Hence $\psi(x, u)$ is o-diagonally concave in u. Again,

$S: X \to 2^X$ is a continuous multivalued mapping with nonempty closed convex values; hence by virtue of Theorem 1 in [16], there exists $\bar{x} \in S(\bar{x})$ such that

$$\sup_{u \in S(\bar{x})} \psi(\bar{x}, u) \leq 0$$

i.e.,

$$\sup_{u \in S(\bar{x})} -\varphi(\bar{x}, f(\bar{x}), u) \leq 0.$$

Therefore taking $\bar{y} = f(\bar{x})$ we have $\bar{y} \in T(\bar{x})$ and

$$\varphi(\bar{x}, \bar{y}, u) \geq 0$$

for all $u \in S(\bar{x})$. This completes the proof. ∎

THEOREM 7. *Let X be a nonempty convex, perfectly normal, paracompact subset of a locally convex Hausdorff topological vector space E, Y be a nonempty subset of a Hausdorff topological vector space F, and D be a nonempty compact subset of X. Let $S: X \to 2^D$ be an almost upper semicontinuous multivalued mapping with nonempty convex values and open lower sections, $T: X \to 2^Y$ be a multivalued mapping with nonempty convex values and local intersection property, $\varphi: X \times Y \times X \to R \cup \{\pm\infty\}$. If the following conditions are fulfilled*:

(i) *$\varphi(x, y, u)$ is upper semicontinuous in (x, y) and is quasi-convex in u,*

(ii) *for each $x \in X$ and each $y \in T(x)$, $\varphi(x, y, x) \geq 0$,*

then there exist $\bar{x} \in \overline{S(\bar{x})}$ and $\bar{y} \in T(\bar{x})$ such that

$$\varphi(\bar{x}, \bar{y}, x) \geq 0, \qquad \forall x \in S(\bar{x}).$$

Proof. By virtue of Theorem 1, there exists a continuous selection $f: X \to Y$ of T. For each $x \in X$, let

$$G(x) = \{u \in S(x): \varphi(x, f(x), u) < 0\};$$

then by the second part of condition (i) and S having convex values we know that $G: X \to 2^D$ has convex values.

By the first part of condition (i) and the continuity of f, $\varphi(x, f(x), u)$ is upper semicontinuous in x; hence the set $\{x \in X: \varphi(x, f(x), u) < 0\}$ is open in X. Again, S has open lower sections; thus for each $u \in X$,

$$G^{-1}(u) = \{x \in X : u \in G(x)\}$$
$$= S^{-1}(u) \cap \{x \in X : \varphi(x, f(x), u) < 0\}$$

is an open subset of X. It implies that G has local intersection property and set

$$W = \{x \in X : G(x) \neq \emptyset\},$$
$$= \bigcup_{u \in X} G^{-1}(u)$$

is an open subset of X.

1. If $W = \emptyset$, then $G(x) = \emptyset$ for all $x \in X$; consequently for each $x \in X$ and each $u \in S(x)$, $\varphi(x, f(x), u) \geq 0$. But S has open lower sections and nonempty convex values; by virtue of Theorem 2, there is a point $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$. Now taking $\bar{y} = f(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$\varphi(\bar{x}, \bar{y}, u) \geq 0, \qquad \forall u \in S(\bar{x}).$$

2. If $W \neq \emptyset$, since X is perfectly normal and paracompact, W is an F_σ-set and hence W is paracompact by the Theorem 5.1.28 in [7, p. 383]. Consequently by Theorem 1, $G|_W : W \to 2^D$ has a continuous selection $g: W \to D$.

We define a multivalued mapping $H: X \to 2^D$ by

$$H(x) = \begin{cases} \{g(x)\}, & \text{if } x \in W. \\ \overline{S(x)}, & \text{if } x \in X \setminus W. \end{cases}$$

Then obviously, H has nonempty closed convex values. Otherwise, it follows from the almost upper semicontinuity of S that the mapping $\bar{S}: X \to 2^D$, defined by $\bar{s}(x) = \overline{S(x)}$ for each $x \in X$, is almost upper semicontinuous and hence $\bar{S}$ is upper semicontinuous by Lemma 1 in [13]. Again, since $g: W \to D$ is continuous and W is open, $H: X \to 2^D$ is upper semicontinuous. Consequently by virtue of Himmelberg's fixed point theorem there exists a point $\bar{x} \in D$ such that $\bar{x} \in H(\bar{x})$.

If $\bar{x} \in W$, then $\bar{x} = g(\bar{x}) \in G(\bar{x})$. Hence $\varphi(\bar{x}, f(\bar{x}), \bar{x}) < 0$ contradicts condition (ii). Hence $\bar{x} \in X \setminus W$ implies that $\bar{x} \in \overline{S(\bar{x})}$ and $G(\bar{x}) = \emptyset$; i.e., $\bar{x} \in \overline{S(\bar{x})}$ and $\varphi(\bar{x}, f(\bar{x}), x) \geq 0$ for all $x \in S(\bar{x})$. Now, taking $\bar{y} = f(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$\varphi(\bar{x}, \bar{y}, x) \geq 0, \qquad \forall x \in S(\bar{x}).$$

This completes the proof. ∎

COROLLARY 8. *Let X be a nonempty convex, perfectly normal, paracompact subset of a locally convex Hausdorff topological vector space E, E^* be dual space of E with strong topology, and D be a nonempty compact subset of X. $\emptyset \neq Y \subset E^*$. Let $S: X \to 2^D$ be an almost upper semicontinuous multivalued mapping with nonempty convex values and open lower sections and $T: X \to 2^Y$ be a multivalued mapping with nonempty convex values and local intersection property. Then there exist $\bar{x} \in \overline{S(\bar{x})}$ and $\bar{y} \in T(\bar{x})$ such that*

$$\operatorname{Re}\langle \bar{y}, \bar{x} - z \rangle \leq 0, \qquad \forall z \in S(\bar{x}).$$

Proof. Using Theorem 7, the proof is similar to that of Corollary 5 and thus omitted here. ∎

Remark. Corollary 8 improves Theorem 3 in [14].

4. INTERSECTION THEOREM AND EQUILIBRIUM EXISTENCE THEOREMS

We need the following definitions and notations.

Let Y be a topological space. The family of subsets in $Y\{A_\alpha: \alpha \in J\}$ is said to be open transfer complete; if $y \in A_\alpha$, then there exists $\alpha' \in J$ such that $y \in \operatorname{int} A_{\alpha'}$. Let $\{X_i: i \in I\}$ be a family of topological spaces,

$$X = \prod_{i \in I} X_i, \qquad \hat{X}_i = \prod_{\substack{j \in I \\ j \neq i}} X_j.$$

Let $\pi_i: X \to X_i$ and $\pi_i: X \to \hat{X}_i$ be the projections. If $x \in X$, we can write $\pi_i(x) = x_i$ and $\hat{\pi}_i(x) = \hat{x}_i$. Let $A \subset X$, $x_i \in X_i$, and $\hat{x}_i \in \hat{X}_i$; then $(x_i, \hat{x}_i)$ denotes the point $x \in X$ such that $\pi_i(x) = x_i$ and $\hat{\pi}_i(x) = \hat{x}_i$ and we define $A[x_i] = \{\hat{y}_i \in \hat{X}_i: (x_i, \hat{y}_i) \in A\}$ and $A[\hat{x}_i] = \{y_i \in X_i: (y_i, \hat{x}_i) \in A\}$. If $A_i \subset X_i$ and $\hat{A}_i \subset \hat{X}_i$, then $A_i \otimes \hat{A}_i$ denotes the set

$$\left\{(y_i, \hat{y}_i) \in X: y_i \in A_i \text{ and } \hat{y}_i \in \hat{A}_i\right\}.$$

THEOREM 9. *Let $\{X_i: i \in I\}$ be a family of nonempty convex sets, each in a locally convex Hausdorff topological vector space. For each $i \in I$, let D_i be a nonempty compact subset of X_i. Suppose that $\{A_i\}_{i \in I}$, $\{B_i\}_{i \in I}$ are two families of subsets of $X = \prod_{i \in I} X_i$ with the following conditions:*

(i) *For each $i \in I$, the family $\{B_i[x_i]: x_i \in D_i\}$ is open transfer complete in $\hat{X}_i$.*

(ii) *For each $i \in I$ and each $\hat{y}_i \in \hat{X}_i$, the set $B_i[\hat{y}_i] \cap D_i \neq \emptyset$ and $co\,(B_i[\hat{y}_i] \cap D_i) \subset A_i[\hat{y}_i] \cap D_i$.*

Then the set $\bigcap_{i \in I} A_i \neq \emptyset$.

Proof. For each $i \in I$ and each $x \in X$, let

$$S_i(x) = B_i[\hat{x}_i] \cap D_i,$$
$$T_i(x) = A_i[\hat{x}_i] \cap D_i;$$

then S_i, T_i: $X \to 2^{D_i}$ are two multivalued mappings with $\text{co}\, S_i(x) \subset T_i(x)$ and $S_i(x) \neq \emptyset$ for all $x \in X$.

Now we prove that S_i has local intersection property. For each $y \in X$, since $S_i(y) = B_i[\hat{y}_i] \cap D_i \neq \emptyset$, there exists a point $x_i \in B_i[\hat{y}_i] \cap D_i$ and hence $\hat{y}_i \in B_i[x_i]$ and $x_i \in D_i$, consequently, there exists $z_i \in D_i$ such that $\hat{y}_i \in \text{int}\, B_i[z_i]$ by condition (i). Thus there is an open neighborhood $N(\hat{y}_i)$ of $\hat{y}_i$ such that $N(\hat{y}_i) \subset B_i[z_i]$. Let

$$U(y) = X_i \otimes N(\hat{y}_i);$$

then $U(y)$ is an open neighborhood of y in X. For each $b \in U(y)$, we have $\hat{b}_i \in N(\hat{y}_i) \subset B_i[z_i]$; hence $z_i \in B_i[\hat{b}_i] \cap D_i = S_i(b)$. Thus

$$z_i \in \bigcap_{b \in U(y)} S_i(b).$$

Therefore S_i has local intersection property.

Summing up the above arguments we know that S_i, T_i satisfy all conditions of Theorem 2. Consequently, by virtue of Theorem 2, there exists a point $\bar{x} \in D := \prod_{i \in I} D_i$ such that

$$\bar{x}_i \in T_i(\bar{x})$$

for each $i \in I$, and hence

$$\bar{x} \in \bigcap_{i \in I} A_i.$$

This completes the proof. ∎

Remark. If for each $x_i \in D_i$, the set $B_i[x_i]$ is open in X_i, then the family $\{B_i[x_i]: x_i \in D_i\}$ is open transfer complete. Hence Theorem 9 contains Theorem 3 of Ding *et al.* [6].

Next we give some equilibrium existence theorems for an abstract economy. We first give some definitions in equilibrium theory. Let I be a set of agents. An abstract economy $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ is defined as a family of ordered quadruples (X_i, A_i, B_i, P_i), where A_i, B_i: $X := \prod_{j \in I} X_j \to 2^{X_i}$ are constraint correspondences and P_i: $X \to 2^{X_i}$ is a preference correspondence. An equilibrium for Γ is a point $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in \overline{B_i(\bar{x})}$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$.

THEOREM 10. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy, where I is a set of agents such that for each $i \in I$,*

(i) *X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space and D_i is a nonempty compact subset of X_i,*

(ii) *for each $x \in X := \prod_{i \in I} X_i$, $B_i(x)$ is nonempty convex and $A_i(x) \subset B_i(x) \subset D_i$,*

(iii) *the mapping $\overline{B}_i: X \to 2^{X_i}$, defined by $\overline{B}_i(x) = \overline{B_i(x)}$ for each $x \in X$, is upper semicontinuous,*

(iv) *the mapping $T_i: X \to 2^{D_i}$ defined by*

$$T_i(x) = A_i(x) \cap P_i(x)$$

has local intersection property,

(v) *for each $x \in X$, $x_i \notin co[A_i(x) \cap P_i(x)]$, and*

(vi) *the set $W_i := \{x \in X: A_i(x) \cap P_i(x) \neq \emptyset\}$ is a paracompact subset of X.*

Then Γ has an equilibrium choice $\bar{x} \in X$; i.e., for each $i \in I$, $\bar{x}_i \in \overline{B_i(\bar{x})}$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$.

Proof. For each $i \in I$, first, we prove that the mapping $S_i: X \to 2^{D_i}$ defined by

$$S_i(x) = \operatorname{co} T_i(x)$$

has local intersection property.

For each $x \in X$, if $S_i(x) \neq \emptyset$, then $T_i(x) \neq \emptyset$. Consequently, by condition (iv), there exists an open neighborhood U of x such that $\bigcap_{z \in U} T_i(z) \neq \emptyset$; hence $\bigcap_{z \in U} S_i(z) \supset \bigcap_{z \in U} T_i(z) \neq \emptyset$. Thus S_i has local intersection property.

By virtue of Theorem 1, $S_i|_{W_i}$ has a continuous selection $f_i: W_i \to D_i$. Define a mapping $G_i: X \to 2^{D_i}$ by

$$G_i(x) = \begin{cases} \{f_i(x)\}, & \text{if } x \in W_i, \\ \overline{B_i(x)}, & \text{if, } x \in X \setminus W_i. \end{cases}$$

Then $G_i(x)$ is nonempty convex for each $x \in X$.

For each $x \in W_i$, $T_i(x) \neq \emptyset$. Since T_i has local intersection property, there exists an open neighborhood $N(x)$ of x such that $\bigcap_{z \in N(x)} T_i(z) \neq \emptyset$ and hence $N(x) \subset W_i$. Thus W_i is open. Therefore, $G_i: X \to 2^{D_i}$ is upper semicontinuous by the continuity of f_i and condition (iii).

By virtue of Theorem 2, there exists $\bar{x} = \prod_{i \in I} \bar{x}_i \in D := \prod_{i \in I} D_i$ such that

$$\bar{x}_i \in G_i(\bar{x}), \qquad \forall i \in I.$$

Consequently, by condition (v) we know that

$$\bar{x}_i \in \overline{B_i(\bar{x})} \qquad \text{and} \qquad A_i(\bar{x}) \cap P_i(\bar{x}) = \varnothing$$

for all $i \in I$; i.e., $\bar{x}$ is an equilibrium choice of Γ. ∎

Remark. Theorem 10 contains Theorem 1 in [10], Theorem 6.1 in [18], Theorem 3.1 in [5], and correspondence result in [17].

THEOREM 11. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy, where I is a set of agents such that for each $i \in I$,*

(i) *X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space and D_i is a nonempty compact subset of X_i,*

(ii) *for each $x \in X$, $\varnothing \neq A_i(x) \subset B_i(x) \subset D_i$, and $B_i(x)$ is convex,*

(iii) *the mapping $\overline{B}_i \colon X \to 2^{X_i}$ defined by $\overline{B}_i(x) = \overline{B_i(x)}$ for each $x \in X$, is upper semicontinuous,*

(iv) *the mapping $T_i \colon X \to 2^{D_i}$ defined by $T_i(x) = co\, A_i(x) \cap co\, P_i(x)$ for each $x \in X$, has local intersection property,*

(v) *for each $x \in X$ $x_i \notin co\, P_i(x)$, and*

(vi) *the set $M_i := \{x \in X : co\, A_i(x) \cap co\, P_i(x) \neq \varnothing\}$ is paracompact.*

Then Γ has an equilibrium $\bar{x} \in D := \prod_{i \in I} D_i$; i.e., for each $i \in I$,

$$\bar{x}_i \in \overline{B_i(\bar{x})} \qquad \textit{and} \qquad A_i(\bar{x}) \cap P_i(\bar{x}) = \varnothing.$$

Proof. Since M_i is paracompact and $T_i|_{M_i} \colon M_i \to 2^{D_i}$ has local intersection property, $T_i|_{M_i}$ has a continuous selection $f_i \colon M_i \to D_i$. Define a mapping $G_i \colon X \to 2^{D_i}$ by

$$G_i(x) = \begin{cases} \{f_i(x)\}, & \text{if } x \in M_i, \\ \overline{B_i(x)}, & \text{if } x \in X \setminus M_i. \end{cases}$$

The following proofs are the same as the corresponding parts of Theorem 10 and hence omitted. ∎

Remark. Theorem 11 contains Theorem 4 in [6].

THEOREM 12. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy, where I is a set of agents such that for each $i \in I$,*

(i) *X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space and D_i is a nonempty compact subset of X_i,*

(ii) *for each* $x \in X := \prod_{i \in I} X_i$, $A_i(x) \subset B_i(x) \subset D_i$ *and* $B_i(x)$ *is nonempty convex,*

(iii) B_i *has local intersection property,*

(iv) *the mapping* $T_i: X \to 2^{D_i}$, *defined by* $T_i(x) = co\, P_i(x) \cap A_i(x)$ *for each* $x \in X$, *has local intersection property,*

(v) *for each* $x \in X$, $x_i \notin co\, P_i(x)$, *and*

(vi) *the set* $W_i = \{x \in X: A_i(x) \cap P_i(x) \neq \emptyset\}$ *is closed in* X.

Then there exists a point $\bar{x} \in D = \prod_{i \in I} D_i$ *such that* $\bar{x}_i \in B_i(\bar{x})$ *and* $A_i(\bar{x}) \cap P_i(\bar{x}) = \phi$ *for all* $i \in I$.

Proof. For each i $\in I$ define a mapping $G_i: X \to 2^{D_i}$ by

$$G_i(x) = \begin{cases} co(co\, P_i(x) \cap A_i(x)), & \text{if } x \in W_i, \\ B_i(x), & \text{if } x \in X \setminus W_i. \end{cases}$$

Then $G_i(x)$ is nonempty convex for each $x \in X$. For each $x \in X$, if $x \in W_i$, then $A_i(x) \cap P_i(x) \neq \emptyset$ and hence $\text{co}\, P_i(x) \cap A_i(x) \neq \emptyset$. By condition (iv), there exists an open neighborhood $N(x)$ of x such that

$$\bigcap_{z \in N(x)} \text{co}\, P_i(z) \cap A_i(z) \neq \emptyset.$$

Consequently $\bigcap_{z \in N(x)} G_i(z) \supset \bigcap_{z \in N(x)} \text{co}\, P_i(z) \cap A_i(z) \neq \emptyset$. If $x \in X \setminus W_i$, then by condition (iii), there is an open neighborhood $N_1(x)$ of x such that $\bigcap_{z \in N_1(x)} B_i(z) \neq \emptyset$. But by condition (vi), there exists an open neighborhood $N_2(x)$ of x such that $N_2(x) \subset X \setminus W_i$. Let $N(x) = N_1(x) \cap N_2(x)$; then $N(x)$ is an open neighborhood of x and

$$\bigcap_{z \in N(x)} G_i(z) = \bigcap_{z \in N(x)} B_i(z) \supset \bigcap_{z \in N_1(x)} B_i(z) \neq \emptyset.$$

This proves that G_i has local intersection property. ∎

By virtue of Theorem 2, there exists $\bar{x} \in D = \prod_{i \in I} D_i$ such that $\bar{x}_i \in G_i(\bar{x})$ for all $i \in I$. Again by condition (v) we know that

$$\bar{x}_i \in B_i(\bar{x}) \quad \text{and} \quad A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$$

for all $i \in I$.

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