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Gap functions and global error bounds for set-valued variational inequalities☆

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ABSTRACT

The set-valued variational inequality problem is very useful in economics theory and nonsmooth optimization. In this paper, we introduce some gap functions for set-valued variational inequality problems under suitable assumptions. By using these gap functions we derive global error bounds for the solution of the set-valued variational inequality problems. Our results not only generalize the previously known results for classical variational inequalities from single-valued case to set-valued, but also present a way to construct gap functions and derive global error bounds for set-valued variational inequality problems.

1. Introduction

Throughout this paper, unless otherwise mentioned, let K be a closed convex set in R^n, $G : R^n \to 2^{R^n}$ be an upper semicontinuous set-valued mapping with nonempty compact convex values.

We consider the following set-valued variational inequality problem.

$$\text{Find } x^* \in K \text{ and } u^* \in G(x^*), \quad \text{such that } \langle u^*, y - x^* \rangle \geq 0, \ \forall y \in K. \tag{1.1}$$

If G is a single-valued mapping, problem (1.1) reduces to the classical variational inequality problem.

$$\text{Find } x^* \in K, \quad \text{such that } \langle G(x^*), y - x^* \rangle \geq 0, \ \forall y \in K. \tag{1.2}$$

Variational inequality problems have many applications in different fields such as mathematical programming, game theory and economics. In recent years, much attention has been given to reformulate the classical variational inequality problem (1.2) as an equivalent optimization problem through a gap (merit) function. Gap function has turned out to be very useful in designing new globally convergent algorithms, in analyzing the rate of convergence of some iterative methods and in deriving the error bounds, which provide a measure of the distance between a solution set and an arbitrary point, for more details, see [1–5]. Error bounds have played an important role not only in theoretical analysis but also in convergence analysis of iterative algorithms for solving variational inequalities, see Pang [6] for an excellent survey of the theory and application. A few error bounds have been presented for the classical variational inequality problem (1.2), for example, see [1,3–5,7–12,22–25].

To the best of the authors' knowledge, very few gap functions and no error bounds results have been established for set-valued variational inequality problems. The set-valued variational inequality problems are useful in some practical applications. These problems involve economic theory, set-valued variational inclusions, complementarity and fixed point problems and nonsmooth optimization problems.

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In this paper, we establish gap functions which can be reformulated as unconstrained optimization problems equivalent to the original set-valued variational inequality problems. We also derive error bounds based on these gap functions under certain assumptions. Our results not only generalize the corresponding previously known results for variational inequality problem in [10] from single-valued case to set-valued case, but also provide a way to construct gap functions and derive error bounds for set-valued variational inequality problem.

This paper is organized as follows. We give some preliminaries which will be used in the rest of this paper in Section 2. Some new gap functions for the set-valued variational inequality problems are constructed in Section 3. Finally, in Section 4, we derive error bounds for set-valued variational inequality problem.

2. Preliminaries

A function $M : K \to R$ is called a gap function for the set-valued variational inequality problem (1.1) if and only if
(i) $M(x) \geq 0$, $\forall x \in K$;
(ii) $M(x^*) = 0$ if and only if $x^* \in K$ solves the set-valued variational inequality problem (1.1).

One of the many useful applications of gap functions is in deriving the so-called error bounds, i.e., upper estimation on the distance to the solution set S of the set-valued variational inequality problem (1.1):

$$d(x, S) \leq \gamma M(x)^{\lambda}, \quad \forall x \in K,$$

where $\gamma, \lambda > 0$ are independent of x.

Let us recall some gap functions proposed for the classical variational inequality problem (1.2) through optimization methods. Auslender [13] and Marcotte [14] considered it by minimizing a gap function f defined by

$$f(x) = \max\{\langle G(x), x - y\rangle | y \in K\}. \tag{2.1}$$

In order that the function f is well defined, the constrained set K has to be assumed to be compact in [14,15]. The regularized gap function for the classical variational inequality problem (1.2), introduced independently in [16,1], was defined as:

$$\begin{aligned} f_{\alpha,G,K}(x) &:= \max_{y\in K}\langle G(x), x - y\rangle - \frac{1}{2\alpha}\|x - y\|^2 \\ &= \langle G(x), x - P_K(x - \alpha G(x))\rangle - \frac{1}{2\alpha}\|x - P_K(x - \alpha G(x))\|^2. \end{aligned}$$

The above-mentioned authors reformulated the classical variational inequality (1.2) as a constrained optimization problem. Peng [3], Yamashita and Fukushima [10] reformulated it as an unconstrained optimization problem through different approaches, respectively. The former adopted the regularized gap function [1] and the implicit Lagrangian [17], while the latter utilized the Moreau–Yosida regularization. We are interested in the regularized gap functions used in [10], which are $f(\cdot; \alpha) : R^n \times (0, \infty) \to R \cup \{+\infty\}$ defined by

$$f(x; \alpha) = \sup_{y\in K}\{\langle G(x), x - y\rangle - \alpha\|x - y\|^2\}, \tag{2.2}$$

and $h(\cdot; \beta) : R^n \times [0, \infty) \to R \cup \{+\infty\}$ defined by

$$h(x; \beta) = \sup_{y\in K}\{\langle G(x), x - y\rangle + \beta\|x - y\|^2\}, \tag{2.3}$$

where α, β are nonnegative constants.

For the set-valued mapping G, we require the following concepts.

Definition 2.1. The set-valued mapping $G : R^n \to 2^{R^n}$ is said to be

(i) strongly monotone with modulus $\mu > 0$, if for each pair of points $x, y \in R^n$ and for all $u \in G(x)$, $v \in G(y)$, we have

$$\langle u - v, x - y\rangle \geq \mu\|x - y\|^2;$$

(ii) monotone, if for each pair of points $x, y \in R^n$ and for all $u \in G(x)$, $v \in G(y)$, we have

$$\langle u - v, x - y\rangle \geq 0;$$

(iii) pseudomonotone, if for each pair of points $x, y \in R^n$ and for all $u \in G(x)$, $v \in G(y)$, we have

$$\langle v, x - y\rangle \geq 0 \Rightarrow \langle u, x - y\rangle \geq 0;$$

(iv) strongly monotone with respect to $\bar{x}$ with modulus $\gamma > 0$ if for any point $x \in K$ and for all $u \in G(x)$, we have

$$\langle u, x - \bar{x}\rangle \geq \gamma\|x - \bar{x}\|^2,$$

where $\bar{x}$ solves set-valued variational inequality problem (1.1).

Remark 2.1. (i) It is well known that, if G is strongly monotone, then G is monotone; if G is monotone, then G is pseudomonotone. However, the converse are not true in general.

(ii) If $\bar{x}$ solves the set-valued variational inequality problem (1.1), G is strongly monotone with modulus $\mu > 0$, then G is strongly monotone with respect to $\bar{x}$ with modulus μ, the converse is not true in general. In fact, if there exists $\bar{v} \in G(\bar{x})$, such that

$$\langle \bar{v}, x - \bar{x} \rangle \geq 0, \quad \forall x \in K,$$

this implies that

$$\langle u, x - \bar{x} \rangle \geq \mu \|x - \bar{x}\|^2 + \langle \bar{v}, x - \bar{x} \rangle \geq \mu \|x - \bar{x}\|^2, \quad \forall x \in K, \ \forall u \in G(x).$$

(iii) If $\bar{x}$ solves the set-valued variational inequality problem (1.1), G is strongly monotone with respect to $\bar{x}$ with modulus $\gamma > 0$, similar discussion as in [4], we can obtain that the solution of the set-valued variational inequality problem (1.1) is unique.

Definition 2.2. The set-valued mapping $G : R^n \to 2^{R^n}$ is said to be upper semicontinuous, if for each $x \in R^n$ and each open set $V \subset R^n$ with $G(x) \subset V$, there exists an open neighborhood U of x such that $G(z) \subset V$ for each $z \in U$.

The following lemmas are important for the later use.

Lemma 2.1 (*[18]*). *Let X, Y be metric spaces, a set-valued map $F : X \to 2^Y$ and a function $f : Graph(F) \to R$ be given. If f and F are upper semicontinuous and if the values of F are compact, then the function $g : X \to R \cup \{+\infty\}$ defined by*

$$g(x) = \sup_{y \in F(x)} f(x, y),$$

is upper semicontinuous.

From the proof of Theorem 4.3 in [10], we have the following lemma:

Lemma 2.2. *For any $x \in R^n$, $x^* \in K$, then*

$$\inf_{z \in K} \{\|z - x^*\|^2 + \|z - x\|^2\} \geq \frac{1}{2} \|x - x^*\|^2.$$

3. Gap functions

Inspired by Yamashita and Fukushima [10], we define $g : R^n \times R^n \times (0, +\infty) \to R$ by

$$g(x; u; \alpha) = \sup_{y \in K} \{\langle u, x - y \rangle - \alpha \|x - y\|^2\}.$$

It is evident that $g(x; u; \alpha)$ can be rewritten as

$$g(x; u; \alpha) = \left\langle u, x - P_K \left(x - \frac{u}{2\alpha}\right)\right\rangle - \alpha \left\|x - P_K \left(x - \frac{u}{2\alpha}\right)\right\|^2,$$

thus $g(x; u; \alpha)$ is continuous in x and u.

Now we define the function $f : R^n \times (0, +\infty) \to R$ by

$$f(x; \alpha) = \inf_{u \in G(x)} g(x; u; \alpha). \tag{3.1}$$

Since $G(x)$ is compact and $g(x; u; \alpha)$ is continuous at u, thus $f(x; \alpha)$ is well defined.

Lemma 3.1. *The following conclusions hold:*

(i) *For any $\alpha > 0$, $f(\cdot; \alpha)$ is nonnegative on K.*

(ii) *For any $x \in R^n$, there exists some $u \in G(x)$ such that $f(x; \alpha) = g(x; u; \alpha)$.*

(iii) *For any $\alpha > 0$, the function $f(\cdot; \alpha)$ is lower semicontinuous.*

Proof. (i), (ii) are obvious. For (iii), since $g(x; u; \alpha)$ is continuous in x and u, G is upper semicontinuous and the values of G are compact, from Lemma 2.1, we obtain that the function $f(\cdot; \alpha)$ defined by

$$f(x; \alpha) = \inf_{u \in G(x)} g(x; u; \alpha) = - \sup_{u \in G(x)} (-g(x; u; \alpha))$$

is lower semicontinuous. This completes the proof. $\square$

We define another function $h(\cdot;\beta):R^n\times[0,+\infty)\to R\cup\{\infty\}$ by

$$h(x;\beta)=\sup_{y\in K,v\in G(y)}\{\langle v,x-y\rangle+\beta\|x-y\|^2\},\tag{3.2}$$

where $\beta\geq 0$ is a constant. For any $\beta\geq 0$, $h(\cdot;\beta)$ is nonnegative.

Lemma 3.2. *For any $\beta\geq 0$, the function $h(\cdot;\beta)$ is a lower semicontinuous convex function.*

Proof. The convexity of $h(\cdot;\beta)$ follows from the definition (3.2) directly, since $\langle v,\cdot-y\rangle+\beta\|\cdot-y\|^2$ is convex for every y and $v\in G(y)$. The lower semicontinuity of $h(\cdot;\beta)$ follows from the fact that a pointwise supremum of continuous functions yields a lower semicontinuous function. This completes the proof. □

It is noted that if G is a single-valued mapping, $f(\cdot;\alpha)$ reduces to the regularized gap function considered in [16,1,10] for $\alpha>0$; $h(\cdot;\beta)$ reduces to the gap function considered in [19] for $\beta=0$ and in [10] for $\beta>0$.

We now prove that $f(\cdot;\alpha)$ and $h(\cdot;\beta)$ can serve as gap functions for the set-valued variational inequality problem (1.1).

Lemma 3.3. *If $\alpha>0$, then $f(x^*;\alpha)=0$ if and only if x^* solves the set-valued variational inequality problem (1.1).*

Proof. If $f(x^*;\alpha)=0$. By Lemma 3.1(ii), there exists some $u^*\in G(x^*)$ such that

$$\begin{aligned}f(x^*;\alpha)&=g(x^*;u^*;\alpha)\\&=\sup_{y\in K}\{\langle u^*,x^*-y\rangle-\alpha\|x^*-y\|^2\}\\&=0.\end{aligned}$$

We claim that $\langle u^*,y-x^*\rangle\geq 0$, for any $y\in K$.

Indeed, we assume that there exists some $y_0\in K$ such that $\langle u^*,y_0-x^*\rangle<0$, or equivalently, $\langle u^*,x^*-y_0\rangle>0$. Letting $y_t=(1-t)x^*+ty_0$, where $t\in(0,1)$, then we have

$$\langle u^*,x^*-y_t\rangle=\langle u^*,t(x^*-y_0)\rangle=t\langle u^*,x^*-y_0\rangle.$$

Denote by $g_{y_t}(x^*;u^*;\alpha)=\langle u^*,x^*-y_t\rangle-\alpha\|x^*-y_t\|^2$, thus we have

$$g_{y_t}(x^*;u^*;\alpha)=t\langle u^*,x^*-y_0\rangle-t^2\alpha\|x^*-y_0\|.$$

Choose some $t_0\in(0,1)$ such that

$$g_{y_{t_0}}(x^*;u^*;\alpha)=t_0\langle u^*,x^*-y_0\rangle-{t_0}^2\alpha\|x^*-y_0\|>0.$$

By the definition of g, we have $g(x^*;u^*;\alpha)\geq g_{y_{t_0}}(x^*;u^*;\alpha)$. Thus we obtain that

$$g(x^*;u^*;\alpha)\geq g_{y_{t_0}}(x^*;u^*;\alpha)>0,$$

which contradicts the fact that

$$f(x^*;\alpha)=g(x^*;u^*;\alpha)=0.$$

Conversely, if x^* solves the set-valued variational inequality problem (1.1), then there exists some $u^*\in G(x^*)$ such that

$$\langle u^*,y-x^*\rangle\geq 0,\quad\forall y\in K.$$

It is clear that

$$\langle u^*,x^*-y\rangle-\alpha\|x^*-y\|^2\leq 0,\quad\forall y\in K,$$

which yields that

$$f(x^*;\alpha)=g(x^*;u^*;\alpha)\leq 0.$$

Combining with the nonnegativity of $f(\cdot;\alpha)$, we have that $f(x^*;\alpha)=0$. We completes the proof. □

Lemma 3.4. (i) *If G is pseudomonotone, then x^* solves the set-valued variational inequality problem (1.1) if and only if $h(x^*;0)=0$.*

(ii) *If G is pseudomonotone, if there exists $\beta>0$ with $h(x^*;\beta)=0$, then x^* solves the set-valued variational inequality problem (1.1).*

(iii) *If x^* solves the set-valued variational inequality problem (1.1), G is strongly monotone with respect to x^* with modulus $\mu>0$, and β is chosen to satisfy $0\leq\beta\leq\mu$, then $h(x^*;\beta)=0$.*

(iv) *If G is strongly monotone with modulus $\mu>0$, and β is chosen to satisfy $0\leq\beta\leq\mu$, then x^* solves the set-valued variational inequality problem (1.1) if and only if $h(x^*;\beta)=0$.*

Proof. (i) If x^* solves the set-valued variational inequality problem (1.1), then there exists some $u^* \in G(x^*)$ such that

$$\langle u^*, y - x^* \rangle \geq 0, \quad \forall y \in K.$$

Since G is pseudomonotone, then we have

$$\langle v, y - x^* \rangle \geq 0, \quad \forall y \in K, v \in G(y),$$

which yields that

$$h(x^*; 0) = \sup_{y \in K, v \in G(y)} \{\langle v, x^* - y \rangle\} \leq 0.$$

Combining with the nonnegativity of $h(\cdot; \beta)$, we have that $h(x^*; 0) = 0$.

Conversely, if $h(x^*; 0) = 0$, i.e.,

$$\langle v, x^* - y \rangle \leq 0, \quad \forall y \in K, v \in G(y). \tag{3.3}$$

Since G is pseudomonotone and upper semicontinuous with nonempty compact convex values, by Proposition 1 in [20], variational inequality (3.3) implies that there exists some $u^* \in G(x^*)$ such that

$$\langle u^*, y - x^* \rangle \geq 0, \quad \forall y \in K,$$

which means that x^* solves the set-valued variational inequality problem (1.1).

(ii) If $\beta > 0$ and $h(x^*; \beta) = 0$, it is easy to see that

$$\langle v, x^* - y \rangle \leq 0, \quad \forall y \in K, v \in G(y). \tag{3.4}$$

From the proof of (i), we know that x^* solves the set-valued variational inequality problem (1.1).

(iii) Since G is strongly monotone with respect to x^* with modulus $\mu > 0$, for any $y \in K$, $v \in G(y)$, we have

$$\langle v, y - x^* \rangle \geq \mu \|y - x^*\|^2.$$

This implies

$$\langle v, x^* - y \rangle + \beta \|y - x^*\|^2 \leq (\beta - \mu) \|y - x^*\|^2,$$

which yields that

$$h(x^*, \beta) \leq 0,$$

Combining with the nonnegativity of $h(\cdot; \beta)$, we have that $h(x^*; \beta) = 0$.

(iv) Since G is strongly monotone with modulus $\mu > 0$, G is pseudomonotone. The conclusion follows immediately from (ii) and (iii). □

The gap functions mentioned above can be reformulated as constrained minimization problems equivalent to the set-valued variational inequality problem (1.1). Next, we consider the following functions defined by

$$\phi_f(x; \alpha, \lambda) = \inf_{z \in K} \{f(z; \alpha) + \lambda \|x - z\|^2\} \tag{3.5}$$

and

$$\phi_h(x; \beta, \lambda) = \inf_{z \in K} \left\{h(z; \beta) + \lambda \|x - z\|^2\right\}, \tag{3.6}$$

where λ is a positive constant, $f(\cdot; \alpha)$ and $h(\cdot; \beta)$ are defined by (3.1) and (3.2), respectively. In fact, combining with the definitions of $f(\cdot; \alpha)$ and $h(\cdot; \beta)$, $\phi_f(\cdot; \alpha, \lambda)$ and $\phi_h(\cdot; \beta, \lambda)$ can be rewritten as

$$\phi_f(x; \alpha, \lambda) = \inf_{z \in K, u \in G(z)} \left\{ \sup_{y \in K} \{\langle u, z - y \rangle - \alpha \|z - y\|^2\} + \lambda \|x - z\|^2 \right\}$$

and

$$\phi_h(x; \beta, \lambda) = \inf_{z \in K} \left\{ \sup_{y \in K, v \in G(y)} \{\langle v, z - y \rangle + \beta \|z - y\|^2\} + \lambda \|x - z\|^2 \right\},$$

respectively.

Consequently, by using the lemmas above, we prove that the unconstrained minimization of ϕ_f and Φ_h are equivalent to the set-valued variational inequality problem (1.1) under certain assumptions of G.

Theorem 3.1. (i) *For any* $\alpha > 0$, $\beta \geq 0$, $\lambda > 0$, *then* $\phi_f(\cdot; \alpha, \lambda)$ *and* $\phi_h(\cdot; \beta, \lambda)$ *are nonnegative on* R^n.

(ii) *For any* $\alpha > 0$, $\lambda > 0$, *then* x^* *solves the set-valued variational inequality problem* (1.1) *if and only if* $\phi_f(x^*; \alpha, \lambda) = 0$.

(iii) *If* G *is pseudomonotone, then for any* $\lambda > 0$, x^* *solves the set-valued variational inequality problem* (1.1) *if and only if* $\phi_h(x^*; 0, \lambda) = 0$.

(iv) *Let* x^* *solves the set-valued variational inequality problem* (1.1), *if* G *is strongly monotone with respect to* x^* *with modulus* $\mu > 0$, β *is chosen to satisfy* $0 \leq \beta \leq \mu$, *then for any* $\lambda > 0$, $\phi_h(x^*; \beta, \lambda) = 0$.

(v) *If* G *is strongly monotone with modulus* $\mu > 0$, *for any* β, λ *satisfying* $0 \leq \beta \leq \mu$ *and* $\lambda > 0$, *then* x^* *solves the set-valued variational inequality problem* (1.1) *if and only if* $\phi_h(x^*; \beta, \lambda) = 0$.

Proof. (i) For any $\alpha > 0, \beta \geq 0, f(\cdot; \alpha)$ and $h(\cdot; \beta)$ are nonnegative on R^n, thus we have $\phi_f(\cdot; \alpha, \lambda)$ and $\phi_h(\cdot; \beta, \lambda)$ are nonnegative on R^n.

(ii) If x^* solves the set-valued variational inequality problem (1.1), from Lemma 3.3, it holds that $f(x^*; \alpha) = 0$. Thus we have

$$\begin{aligned}\phi_f(x^*; \alpha, \lambda) &= \inf_{z\in K}\{f(z; \alpha) + \lambda\|x^* - z\|^2\}\\ &\leq f(x^*; \alpha) + \lambda\|x^* - x^*\|^2\\ &= 0.\end{aligned}$$

Combining with the nonnegativity of $\Phi_f(\cdot; \alpha, \lambda)$, we have that $\Phi_f(x^*; \alpha, \lambda) = 0$.

Conversely, suppose that $\Phi_f(x^*; \alpha, \lambda) = 0$. From the definition of $\phi_f(\cdot; \alpha, \lambda)$, there exists a minimizing sequence $\{z_n\}$ in K such that, for any positive integer n, we have

$$f(z_n; \alpha) + \lambda\|z_n - x^*\|^2 < \frac{1}{n},$$

i.e., there exists a sequence $\{z_n\}$ in K such that $f(z_n; \alpha) \to 0$ and $\|z_n - x^*\| \to 0$. Since the set K is closed, $z_n \to x^*$ and $z_n \in K$ imply that $x^* \in K$. Since $f(\cdot; \alpha)$ is lower semicontinuous and nonnegative, we have

$$0 \leq f(x^*; \alpha) \leq \liminf_{n\to\infty} f(z_n; \alpha) = 0,$$

which yields that

$$f(x^*; \alpha) = 0.$$

Therefore from Lemma 3.1, we obtain that x^* is a solution of the set-valued variational inequality problem (1.1).

(iii) If G is pseudomonotone and x^* solves the set-valued variational inequality problem (1.1), it follows from Lemma 3.4(i) that $h(x^*; 0) = 0$. Thus we have

$$\begin{aligned}\phi_h(x^*; 0, \lambda) &= \inf_{z\in K}\{h(z; 0) + \lambda\|x^* - z\|^2\}\\ &\leq h(x^*; 0) + \lambda\|x^* - x^*\|^2\\ &= 0.\end{aligned}$$

Combining with the nonnegativity of $\phi_h(\cdot; 0, \lambda)$, we obtain that

$$\phi_h(x^*; 0, \lambda) = 0.$$

Conversely, suppose $\phi_h(x^*; 0, \lambda) = 0$, the proof is similar to that of (ii).

(iv) If G is strongly monotone with respect to x^* with modulus $\mu > 0$, β is chosen to satisfy $0 \leq \beta \leq \mu$ and x^* solves the set-valued variational inequality problem (1.1), it follows from Lemma 3.4(iii) that $h(x^*; \beta) = 0$. Thus we have

$$\begin{aligned}\phi_h(x^*; \beta, \lambda) &= \inf_{z\in K}\{h(z; \beta) + \lambda\|x^* - z\|^2\}\\ &\leq h(x^*; \beta) + \lambda\|x^* - x^*\|^2\\ &= 0.\end{aligned}$$

Combining with the nonnegativity of $\phi_h(\cdot; \beta, \lambda)$, we obtain that

$$\phi_h(x^*; \beta, \lambda) = 0.$$

Conversely, suppose $\phi_h(x^*; \beta, \lambda) = 0$, the proof is similar to that of (ii). □

Theorem 3.1 shows us that the unconstrained minimization problems

$$\min_{x\in R^n} \phi_f(x; \alpha, \lambda) \quad \text{and} \quad \min_{x\in R^n} \phi_h(x; \beta, \lambda)$$

are equivalent to the set-valued variational inequality problem (1.1) under certain assumptions of G and the associated parameters. Thus it is convenient to use unconstrained minimization methods to solve the set-valued variational inequality problem (1.1) which satisfies the conditions in Theorem 3.1.

From the application point of view, it is desirable that the gap functions $\phi_f(x;\alpha,\lambda)$ and $\phi_h(x;\beta,\lambda)$ are differentiable everywhere. Thus, we define the functions $\Phi_f(\cdot,\cdot;\alpha,\lambda) : R^n\times R^n\times(0,+\infty)\times(0,+\infty)\to R$ and $\Phi_h(\cdot,\cdot;\beta,\lambda) : R^n\times R^n\times(0,+\infty)\times(0,+\infty)\to R\cup(+\infty)$ by

$$\Phi_f(x,z;\alpha,\lambda)=f(z;\alpha)+\lambda\|x-z\|^2$$

and

$$\Phi_h(x,z;\beta,\lambda)=h(z;\beta)+\lambda\|x-z\|^2,$$

respectively. By the definitions of $\phi_f(\cdot;\alpha,\lambda)$ in (3.5) and $\phi_h(\cdot;\beta,\lambda)$ in (3.6), we know that

$$\phi_f(x;\alpha,\lambda)=\inf_{z\in K}\Phi_f(x,z;\alpha,\lambda)$$

and

$$\phi_h(x;\beta,\lambda)=\inf_{z\in K}\Phi_h(x,z;\beta,\lambda).$$

Proposition 3.1. *Let $\alpha>0$ and $\lambda>0$. Suppose the function $\Phi_f(x,\cdot;\alpha,\lambda)$ attains its unique minimum $z_f(x;\alpha,\lambda)$ on K for each $x\in R^n$, then $\phi_f(\cdot;\alpha,\lambda)$ is differentiable on R^n and*

$$\nabla\phi_f(x;\alpha,\lambda)=2\lambda(x-z_f(x;\alpha,\lambda)).$$

Proof. The conclusion follows from Theorem 1.7 in Chapter 4 in [13]. □

Proposition 3.2. *If $\beta\geq 0$ and $\lambda>0$, then the function $\Phi_h(x,\cdot;\beta,\lambda)$ attains its unique minimum $z_h(x;\beta,\lambda)$ uniquely. Moreover, $\phi_h(\cdot;\beta,\lambda)$ is differentiable convex function on R^n and*

$$\nabla\phi_h(x;\beta,\lambda)=2\lambda(x-z_h(x;\beta,\lambda)).$$

Proof. From Lemma 3.2, $h(\cdot;\beta)$ is a closed convex function. By the strictly convexity of the function $\|\cdot-x\|^2$, we know that $\Phi_h(x,\cdot;\beta,\lambda)$ is strictly convex, thus $\Phi_h(x,\cdot;\beta,\lambda)$ attains its minimum on K uniquely. Therefore, by Theorem 1.7 in Chapter 4 in [13], $\phi_h(\cdot;\beta,\lambda)$ is differentiable and its gradient is exhibited as stated in the proposition. The convexity of $\phi_h(\cdot;\beta,\lambda)$ follows from the convexity of $h(\cdot;\beta)$ (see the proof of Proposition 4.1 in [21]). This completes the proof. □

Theorem 3.2. (i) *If G is pseudomonotone, then for any $\lambda>0$, then any stationary point of $\phi_h(\cdot;0,\lambda)$ solves the set-valued variational inequality problem (1.1).*

(ii) *If G is strongly monotone with modulus $\mu>0$, for any β,λ satisfying $0\leq\beta\leq\mu$ and $\lambda>0$, then any stationary point of $\phi_h(\cdot;\beta,\lambda)$ solve set-valued variational inequality problem (1.1).*

Proof. By Proposition 3.2, the function $\phi_h(\cdot;\beta,\lambda)$ is a differentiable convex function on R^n, for any $\beta\geq 0$ and $\lambda>0$. The desired result then follows from Theorem 3.1(iii) and (v). □

4. Global error bounds

In this section, we present error bounds based on the gap functions $f(\cdot;\alpha)$, $h(\cdot;\beta)$, $\phi_f(\cdot;\alpha,\lambda)$ and $\phi_h(\cdot;\beta,\lambda)$ for set-valued variational inequality problem (1.1). We assume that x^* is the unique solution of set-valued variational inequality problem (1.1) and G is strongly monotone with respect to x^* with modulus $\mu>0$. We do not need that G satisfies some Lipschitz continuity as usual.

First, we discuss how the gap functions $f(x;\alpha)$ and $h(x^*;\beta)$ provide error bounds for the set-valued variational inequality problem (1.1) on K.

Lemma 4.1. *Suppose that $x^*\in K$ is the unique solution of the set-valued variational inequality problem (1.1) and G is strongly monotone with respect to x^* with modulus $\mu>0$. If α is chosen to satisfy $0<\alpha<\mu$, then we have*

$$f(x;\alpha)\geq(\mu-\alpha)\|x-x^*\|^2,\quad\forall x\in K.$$

Proof. By Lemma 3.1(ii), for any $x \in K$, there exists some $v_x \in G(x)$ such that $f(x;\alpha) = g(x; v_x; \alpha)$. Since G is strongly monotone with respect to x^* with modulus $\mu > 0$, it holds that $\langle v_x, x - x^* \rangle \geq \mu \|x - x^*\|^2$. Thus we have

$$\begin{aligned} f(x;\alpha) &= g(x; v_x; \alpha) \\ &= \sup_{y\in K}\{\langle v_x, x-y\rangle - \alpha\|x-y\|^2\} \\ &\geq \langle v_x, x-x^*\rangle - \alpha\|x-x^*\|^2 \\ &\geq \mu\|x-x^*\|^2 - \alpha\|x-x^*\|^2 \\ &\geq (\mu-\alpha)\|x-x^*\|^2, \end{aligned}$$

this completes the proof. □

Lemma 4.2. *If $x^* \in K$ solves the set-valued variational inequality problem (1.1), G is strongly monotone with respect to x^* with modulus $\mu > 0$, and β is chosen to satisfy $0 < \beta \leq \mu$, then we have $h(x^*;\beta) = 0$ and*

$$h(x;\beta) \geq \beta\|x-x^*\|^2, \quad \forall x \in K.$$

Proof. From Lemma 3.4(iii), we have

$$h(x^*;\beta) = 0.$$

From the definition of $h(\cdot;\beta)$ and $x^* \in K$ solves the set-valued variational inequality problem (1.1), we have

$$\begin{aligned} h(z;\beta) &= \sup_{y\in K, v\in G(y)}\{\langle v, z-y\rangle + \beta\|z-y\|^2\} \\ &\geq \langle u^*, z-x^*\rangle + \beta\|z-x^*\|^2 \\ &\geq \beta\|z-x^*\|^2. \end{aligned}$$

This completes the proof. □

By using the results obtained above, we now construct global error bounds for the set-valued variational inequality problem (1.1).

Theorem 4.1. *Suppose that $x^* \in K$ is the unique solution of the set-valued variational inequality problem (1.1) and G is strongly monotone with respect to x^* with modulus $\mu > 0$, if α is chosen to satisfy $0 < \alpha < \mu$. Then for any $\lambda > 0$, we have*

$$\frac{1}{2}\min\{\mu-\alpha, \lambda\}\|x-x^*\|^2 \leq \phi_f(x;\alpha,\lambda) \leq \lambda\|x-x^*\|^2, \quad \forall x \in R^n. \tag{4.1}$$

Proof. From Lemma 3.3, we have

$$f(x^*;\alpha) = 0.$$

Thus we obtain

$$\begin{aligned} \phi_f(x;\alpha,\lambda) &= \inf_{z\in K}\{f(z;\alpha) + \lambda\|x-z\|^2\} \\ &\leq f(x^*;\alpha) + \lambda\|x-x^*\|^2 \\ &= \lambda\|x-x^*\|^2, \end{aligned}$$

which implies the right-most inequality in (4.1).

On the other hand, it is easy to see that the function $f(\cdot;\alpha) + \lambda\|\cdot - x\|^2$ is coercive, then there exist some $z_0 \in K$ such that

$$\phi_f(x;\alpha,\lambda) = \inf_{z\in K}\{f(z;\alpha) + \lambda\|x-z\|^2\} = f(z_0;\alpha) + \lambda\|x-z_0\|^2. \tag{4.2}$$

By Lemma 4.1, we have

$$f(z_0;\alpha) \geq (\mu-\alpha)\|z_0-x^*\|^2. \tag{4.3}$$

Replacing $f(z;\alpha)$ with (4.3) in the definition of $\phi_f(x;\alpha,\lambda)$ in (3.5) and combining with (4.2), we obtain

$$\begin{aligned}\phi_f(x;\alpha,\lambda) &= \inf_{z\in K}\{f(z;\alpha)+\lambda\|x-z\|^2\}\\ &= f(z_0;\alpha)+\lambda\|x-z_0\|^2\\ &\geq (\mu-\alpha)\|z_0-x^*\|^2+\lambda\|x-z_0\|^2\\ &\geq \inf_{z\in K}\{(\mu-\alpha)\|z-x^*\|^2+\lambda\|x-z\|^2\}\\ &\geq \min\{\mu-\alpha,\lambda\}\inf_{z\in K}\{\|z-x^*\|^2+\|x-z\|^2\}\\ &\geq \frac{1}{2}\min\{\mu-\alpha,\lambda\}\|x-x^*\|^2,\end{aligned}$$

where the last inequality follows from Lemma 2.2. □

Theorem 4.2. *If $x^*\in K$ solves the set-valued variational inequality problem (1.1), G is strongly monotone with respect to x^* with modulus $\mu>0$, and β is chosen to satisfy $0<\beta\leq\mu$. Then for any $\lambda>0$, we have*

$$\frac{1}{2}\min\{\beta,\lambda\}\|x-x^*\|^2\leq\phi_h(x;\beta,\lambda)\leq\lambda\|x-x^*\|^2,\quad\forall x\in R^n. \tag{4.4}$$

Proof. Since $x^*\in K$ solves the set-valued variational inequality problem (1.1), then there exists some $u^*\in G(x^*)$ such that

$$\langle u^*,y-x^*\rangle\geq 0,\quad\forall y\in K.$$

From Lemma 3.4(iii), we have that

$$h(x^*;\beta)=0.$$

Thus we obtain

$$\begin{aligned}\phi_h(x;\beta,\lambda) &= \inf_{z\in K}\{h(z;\beta)+\lambda\|x-z\|^2\}\\ &\leq h(x^*;\beta)+\lambda\|x-x^*\|^2\\ &= \lambda\|x-x^*\|^2,\end{aligned}$$

which implies the right-most inequality in (4.4).

On the other hand, from Lemma 4.2, we have

$$h(z;\beta)\geq\beta\|z-x^*\|^2,\quad\forall z\in K. \tag{4.5}$$

Replacing $h(z;\beta)$ with (4.5) in the definition of $\phi_h(x;\beta,\lambda)$ in (3.6), we obtain

$$\begin{aligned}\phi_h(x;\beta,\lambda) &= \inf_{z\in K}\{h(z;\beta)+\lambda\|x-z\|^2\}\\ &\geq \inf_{z\in K}\{\beta\|z-x^*\|^2+\lambda\|x-z\|^2\}\\ &\geq \min\{\beta,\lambda\}\inf_{z\in K}\{\|z-x^*\|^2+\|x-z\|^2\}\\ &\geq \frac{1}{2}\min\{\beta,\lambda\}\|x-x^*\|^2,\end{aligned}$$

where the last inequality follows from Lemma 2.2. This completes the proof. □

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