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On $L(2,1)$-coloring split, chordal bipartite, and weakly chordal graphs[☆]

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ABSTRACT

An $L(2,1)$-coloring, or λ-coloring, of a graph is an assignment of non-negative integers to its vertices such that adjacent vertices get numbers at least two apart, and vertices at distance two get distinct numbers. Given a graph G, λ is the minimum range of colors for which there exists a λ-coloring of G. A conjecture by Griggs and Yeh [J.R. Griggs, R.K. Yeh, Labelling graphs with a condition at distance 2, SIAM Journal on Discrete Mathematics 5 (1992) 586–595] states that λ is at most Δ^2, where Δ is the maximum degree of a vertex in G. We prove that this conjecture holds for weakly chordal graphs. Furthermore, we improve the known upper bounds for chordal bipartite graphs, and for split graphs.

1. Introduction

Radio frequency assignment is the task of assigning radio frequencies to transmitters at different locations without causing interference. This problem is closely related to coloring the vertices of a graph where the vertices represent the transmitters and adjacencies indicate possible interferences. To satisfy the λ-coloring constraint, close transmitters (distance 2 apart) must receive different frequencies and very close transmitters (distance 1 apart) must receive frequencies at least 2 apart, that is a more realistic scenario to avoid the interferences.

Fig. 1 shows that if a graph coloring approach is used, then may exist several transmitters sending information to a central one within the same frequency, and with the λ-coloring approach, all the transmitters with a common neighbor send information with different frequencies, avoiding this problem.

Let $\mathrm{dist}(u,v)$ be the number of edges in a shortest path between the vertices u and v in a graph $G=(V,E)$. A k-λ-coloring of G is a function $c : V \to \{0,\dots,k\}$ such that if $uv \in E$, then $|c(u)-c(v)| \geq 2$ and if $\mathrm{dist}(u,v)=2$, then $c(u) \neq c(v)$. The λ-coloring problem for a graph G is to determine λ, the minimum k for which there exists a k-λ-coloring of G.

Fig. 2 shows examples of *optimum* λ-coloring of graphs.

As with the chromatic number of a graph G, it is natural to seek a bound on λ in terms of its maximum degree Δ. Griggs and Yeh [6] proved the general upper bound $\lambda \leq n+\Delta-2$ for a graph with n vertices and conjectured that $\lambda \leq \Delta^2$ for every graph with $\Delta \geq 2$. This has been a motivating problem for research in this field, and some results are known. Optimum 4-λ-coloring of graphs with $\Delta = 2$ are given by Griggs and Yeh [6], so, from now on, we only consider graphs with $\Delta \geq 3$. Moreover, for the purposes of the λ-coloring problem it suffices to consider only connected graphs.

The Griggs and Yeh conjecture was verified for some classes of graphs such as chordal graphs and planar graphs [2,3] but it is still open for general graphs. An up to date survey on the subject is [3]. In this paper we prove this conjecture for weakly chordal graphs. Moreover, we show that for chordal bipartite graphs, $\lambda \leq \Delta^2-\Delta+2$, improving the Δ^2 upper bound

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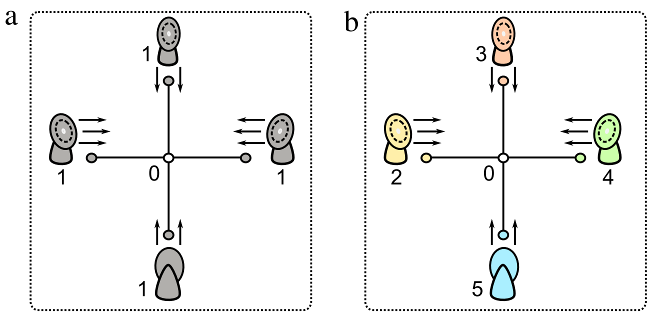

Fig. 1. (a) Coloring; and (b) λ-coloring approaches.

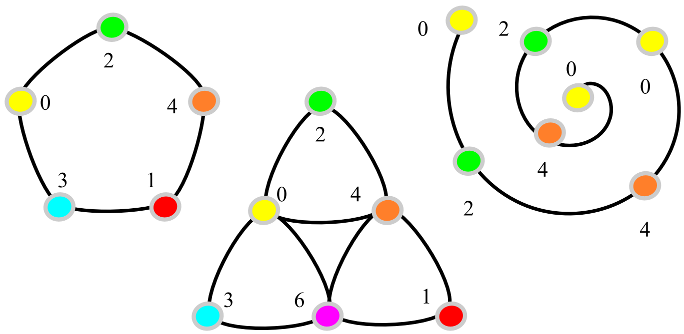

Fig. 2. Optimum λ-colorings of graphs.

from [4]. Besides, we also improve the upper bound for split graphs given by Bodlaender et al. [2] and Kral [7], proving that for split graphs $\lambda \leq \frac{2\sqrt{3}}{9}\Delta^{1.5} + O(\Delta)$.

A graph is *split* if its vertex set can be partitioned into a stable set and a complete set. A graph is *chordal* if it has no induced cycle of length at least four. A graph is *weakly chordal* if neither the graph nor its complement has an induced cycle of length at least five. A bipartite graph is *chordal bipartite* if it is a weakly chordal graph. For more information on these graph classes we refer to [5].

2. Weakly chordal graphs and chordal bipartite graphs

A *breadth-first search* in a graph G defines a rooted spanning tree T, called a BFST (*breadth-first search tree*), of G. If T is rooted in r and v is a vertex of G, let $\mathrm{depth}(v) = \mathrm{dist}(v, r)$ in T. A BFST has the property that for every vertex v, $\mathrm{dist}(v, r)$ in G is exactly $\mathrm{depth}(v)$. Moreover, for every edge uv of G, $|\mathrm{depth}(v) - \mathrm{depth}(u)| \leq 1$. A breadth-first search in a graph G also produces an ordering of V given by the order in which the vertices are visited by the search. This ordering satisfies: if u precedes v, then $\mathrm{depth}(u) \leq \mathrm{depth}(v)$.

Lemma 1. *Let T be a BFST of a weakly chordal graph $G = (V, E)$. For vertices u and v, if $\mathrm{dist}(u, v) = 2$ in G and $\mathrm{depth}(u) \leq \mathrm{depth}(v)$, then there exists a vertex z such that $uz \in E$, $vz \in E$, and $\mathrm{depth}(z) \leq \mathrm{depth}(v)$.*

Proof. Assume there is no such vertex z. Hence, since $\mathrm{dist}(u, v) = 2$ in G, there must be a vertex z' adjacent to both u and v such that $\mathrm{depth}(z') > \mathrm{depth}(v)$. As T is a BFST of G, and $uz' \in E$, $\mathrm{depth}(u) = \mathrm{depth}(v)$. Let u' be the vertex adjacent to u in T with $\mathrm{depth}(u') = \mathrm{depth}(u) - 1$, that is, u' is the father of u. Analogously, take v' the father of v.

By supposition $u'v \notin E$ and $uv' \notin E$. Moreover, since $\mathrm{dist}(u, v) = 2$, $uv \notin E$. Finally, $z'v'$ and $z'u'$ are not edges of G because T is a BFST of G. Then the vertices u', u, z', v and v' belong to an induced cycle of length at least five in G, a contradiction. (See Fig. 3.) □

Let $G = (V, E)$ be a graph and T a BFST of G. Throughout this section, for a vertex $v \in V$, P_v denotes the set of vertices u adjacent to v in G satisfying $\mathrm{depth}(u) \leq \mathrm{depth}(v)$. Moreover, Q denotes the set of vertices of P_v with the same depth of v, A denotes the ones with the depth one unit lower than the depth of v, and A' is a maximal stable set of A. Note that $P_v = Q \cup A$, the father of v in T always belongs to P_v, $v \notin P_v$, and the vertex z in the statement of Lemma 1 is in A. As usual, $N(v)$ denotes the set of all neighbors of v.

Lemma 2. *Let T be a BFST of a weakly chordal graph $G = (V, E)$. If $v \in V$ with $\mathrm{depth}(v) \geq 2$, then there is a vertex z with $\mathrm{depth}(z) = \mathrm{depth}(v) - 2$ such that $S \subseteq N(z)$, where S is any stable set of A.*

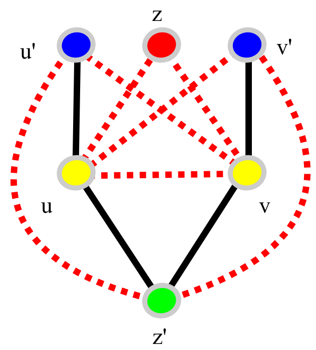

Fig. 3. Edges in a BFST of a weakly chordal graph.

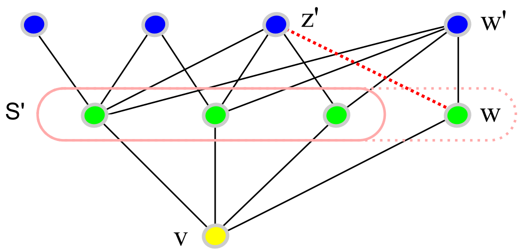

Fig. 4. There is a vertex w' with $S \subseteq N(w')$.

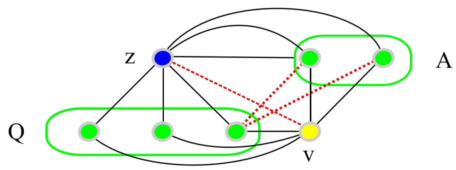

Fig. 5. There is no vertex z such that $A \subseteq N(z)$.

Proof. The proof goes by induction on $|S|$. If $|S| = 1$, then take z as the father in T of the vertex in S. Now, let $S = S' \setminus \{w\}$ with $|S'| \geq 1$. By induction, there is a vertex z' such that $S' \subseteq N(z')$.

As $\text{depth}(v) \geq 2$, there is at least one vertex w' such that $\text{depth}(w') = \text{depth}(w) - 1$ and $ww' \in E(T)$. If $z'w \in E$ we are done. Otherwise, we have that $w'y \in E$ for every $y \in S'$. In fact, if it is not the case, the vertices z', y, v, w and w' would belong to an induced cycle of length at least 5. Therefore, $S \subseteq N(w')$. (See Fig. 4.) $\square$

Lemma 3. *Let T be a BFST of a weakly chordal graph G. For a vertex v such that $\text{depth}(v) \geq 2$, if P_v is a stable set, $Q = \emptyset$, and $|A| = \Delta$, then there is only one vertex z such that $\text{depth}(z) = \text{depth}(v) - 2$. Moreover, z is the root of T.*

Proof. By Lemma 2, there is a vertex z with $\text{depth}(z) = \text{depth}(v) - 2$ such that $P_v \subseteq N(z)$. As $|A| = \Delta$, we have $N(z) = A$. Thus, z is the root of T. $\square$

Lemma 4. *Let T be a BFST of a weakly chordal graph G. There is no vertex v such that $\text{depth}(v) \geq 2$, P_v is a stable set, $Q \neq \emptyset$, and $|P_v| = \Delta$.*

Proof. Suppose that there is such a vertex v. Using an argument similar to that used in the proof of Lemma 2, there is a vertex $z \notin P_v$ (if $z \in P_v$ there would be an edge connecting vertices in P_v) such that $\text{depth}(z) = \text{depth}(v) - 1$ and $Q \subseteq N(z)$.

Moreover, by supposition, $zv \notin E$, and for every $w \in A$, $zw \in E$, otherwise the vertices $z, y \in Q$, v and w would belong to an induced cycle of length at least 5 (as seen in Fig. 5).

As $|P_v| = \Delta$, there is no vertex $u \in N(z)$ such that $\text{depth}(u) < \text{depth}(z)$, hence z is the root of the tree. However, there is at least one vertex $x \in A$ for which $\text{depth}(x) = \text{depth}(z)$, a contradiction. $\square$

Lemma 5. *Let T be a BFST of a weakly chordal graph $G = (V, E)$. For any vertex v the number of vertices u such that $\text{dist}(u, v) = 2$ and $\text{depth}(u) \leq \text{depth}(v)$ is at most $|P_v|(\Delta - 2) + 1$.*

Proof. Let v be a vertex in V and U be the set of vertices u such that $\text{dist}(u, v) = 2$ and $\text{depth}(u) \leq \text{depth}(v)$. By Lemma 1, every $u \in U$ has at least one neighbor in P_v. Let R_1 be the set of edges having one end in U and the other in P_v and R_2 be the set of edges having both ends in P_v.

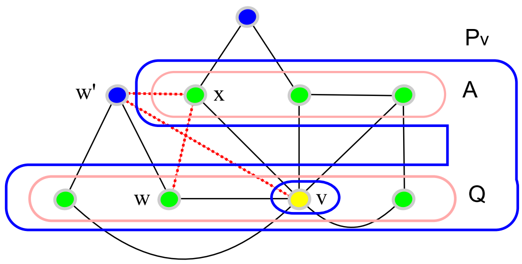

Fig. 6. Contradiction in a BFST of a weakly chordal graph.

An upper bound for the size of set U could be given by the number of paths of length 2 starting in v. Since this is a very rough estimate it can be decreased by considering the paths that do not contribute by ending in a vertex not in U, and the pair of paths ending in the same vertex in U. First, observe that an edge $xy \in R_2$ implies the existence of two paths of length two starting in v, one from v to y passing through x and another from v to x passing through y. However, as both x and y are in the neighborhood of v, these vertices do not have distance two to v. Obviously, $|R_1| \geq |U|$. Moreover, for every vertex $u \in U$ with more than one edge in R_1, only the first edge of R_1 where u belongs will increase the number of vertices with distance two to v.

So, one can give an upper bound for the size of U as follows: $|U| \leq |P_v|(\Delta - 1) - (|R_1| - |U|) - 2|R_2|$. Now, we analyze the values of $|R_1|$ and $|R_2|$ for the worst case scenario:

(1) vertices in A:

(1.1) vertices in A' (a maximal stable set in A).
By Lemma 2, there is a vertex z, with $\text{depth}(z) = \text{depth}(v) - 2$, such that $A' \subseteq N(z)$. In this case, as $z \in U$, $|R_1| \geq |U| + (|A'| - 1)$.

(1.2) vertices in $A \setminus A'$.

Each vertex in $A \setminus A'$ is adjacent to one in A' (or A' would not be maximal). Moreover, each edge connecting vertices in A increases the value of $|R_2|$ by one unit.

Hence, in the worst case, all the vertices of A are in Case (1.1), because each vertex in A' reduces the upper bound for $|U|$ by one unit and each vertex in (1.2) reduces it by two, i.e., $|R_1| \geq |U| + (|A| - 1)$.

(2) vertices in Q:

If v is the root of T the theorem holds (as there is no vertex u such that $\text{dist}(u, v) = 2$ and $\text{depth}(u) \leq \text{depth}(v)$). Let v be a vertex different from the root, so $A \neq \emptyset$ and, for every vertex $w \in Q$:

(2.1) If $wx \in E$ (for some $x \in A$), then $|R_2|$ is increased by one unit. In this case, the edge cannot be the same edge counted in Case (1.2), because they are connecting vertices in A and Q, and the ones in Case (1.2) are connecting two vertices in A.

(2.2) Otherwise, $wx \notin E$. If there is a vertex z' such that $z'w \in E, z'x \in E$ (for some $x \in A$), $\text{depth}(z') = \text{depth}(x)$, and $z' \notin P_v$ (otherwise we would be in Case (2.1)), then $|R_1|$ is increased by one unit, because there are two paths of size two from v to z'. Moreover, the edge that increases the value of $|R_1|$ is not the same as in Case (1.1), because, in this case, the edges connect the vertex z' (with $\text{depth}(z') = \text{depth}(v) - 1$) and a vertex in A or Q and the edges in Case (1.1) connect a vertex z (with $\text{depth}(z) = \text{depth}(v) - 2$) to vertices in A.

(2.3) Otherwise, $wx \notin E$, and there is no vertex z' such that $\text{depth}(z') = \text{depth}(x)$ where $z'w \in E$ and $z'x \in E$. Let w' be the father of w in T. So, $w'v \notin E$ (otherwise we would be in Case (2.1)), $xw \notin E$ (otherwise we would be in Case (2.1)), and $w'x \notin E$ (otherwise we would be in Case (2.2)), and the vertices w', w, x, and v would belong to an induced cycle of length at least five, a contradiction. (See Fig. 6.)

So, in the worst scenario, $|R_1|$ is increased by at least $|Q|$ units, i.e., $|R_1| \geq |U| + |A| - 1 + |Q| = |U| + |P_v| - 1$, and $|R_2| = 0$.

Hence, $|U| \leq |P_v|(\Delta - 1) - (|R_1| - |U|) - 2|R_2| \leq |P_v|(\Delta - 1) - |P_v| + 1 = |P_v|(\Delta - 2) + 1$. $\square$

Theorem 6 *([8]). For any ordering of the vertices of a graph $G = (V, E)$, there exists a λ-coloring such that v receives a color in $\{0, \ldots, 2a + b + c\}$, where a is the number of vertices that are adjacent to v and precede v in the ordering, b is the number of vertices that are adjacent to v and succeed v in the ordering, and c is the number of vertices with distance two to v and precede v in the ordering. Hence, $\lambda \leq \max_{v \in V}\{2a + b + c\}$.*

Theorem 7. *If G is a weakly chordal graph with $\Delta \geq 2$, then $\lambda \leq \Delta^2$.*

Proof. When $\Delta = 2$ we know that $\lambda \leq 4 = \Delta^2$ [6]. Let G be a weakly chordal graph with $\Delta \geq 3$, let T be a BFST of G, and let σ be the ordering of the vertices produced by it. By Lemma 5, for any vertex v of G, the maximum number of vertices at distance two to v that precede it in σ will be at most $|P_v|(\Delta - 2) + 1$.

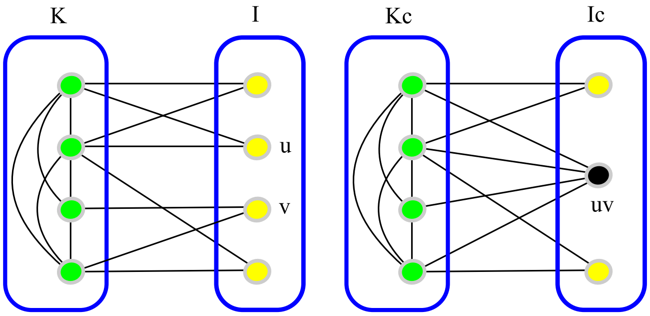

Fig. 7. A condensed split graph.

By Theorem 6, if $|P_v| \leq \Delta - 1$, v receives a color at most $2|P_v| + (\Delta - |P_v|) + |P_v|(\Delta - 2) + 1 = |P_v|(\Delta - 1) + \Delta + 1 \leq (\Delta - 1)(\Delta - 1) + \Delta + 1 = \Delta^2 - \Delta + 2$. Otherwise, $|P_v| = \Delta$ and a similar calculation provides an upper bound of $|P_v|\Delta + 1 \leq \Delta^2 + 1$.

Now the proof of the theorem will be divided into two cases according to P_v being or not a stable set. In each case we show that the estimation of the maximum number of vertices at distance two to v when $|P_v| = \Delta$ can be reduced by one.

If there exists an edge connecting vertices in P_v, it is straightforward to check that this number can be reduced by one, applying a reasoning similar to the one used in the proof of Lemma 5.

Otherwise, P_v is a stable set, and by Lemma 4, $Q = \emptyset$. By Lemma 3, there is a vertex z such that $\text{depth}(z) = \text{depth}(v) - 2$, adjacent to all the vertices in P_v. Additionally, this vertex must be the root of the tree.

Now, let U be the set of vertices $u \in U$ such that $\text{depth}(u) = 2$ and $|P_u| = \Delta$. Since there are at most Δ vertices in U, rearranging them in σ in a way that they are the first ones in the subordering of vertices x such that $\text{depth}(x) = 2$, by Theorem 6, they receive a color at most $2\Delta + \Delta - 1 = 3\Delta - 1 \leq \Delta^2$. □

If the method described above is applied to color the more specific class of chordal bipartite graphs, a better upper bound for λ can be obtained.

Corollary 8. *If G is a chordal bipartite graph, then $\lambda \leq \Delta^2 - \Delta + 2$.*

Proof. As in the proof of Theorem 7, the coloring procedure is done in an ordering of vertices produced by a BFST of G. Since a chordal bipartite graph is a weakly chordal graph, as seen in the proof of Theorem 7, if $|P_v| \leq \Delta - 1$ for some vertex v, then we can use a color of number at most $\Delta^2 - \Delta + 2$ to v. Otherwise, these vertices must be at distance 2 from the root of the BFST and changing the ordering of the vertices obtained by BFST by moving all these vertices with $|P_v| = \Delta$ to the first positions, it will be possible to use a color of number at most $3\Delta - 1 \leq \Delta^2 - \Delta + 2$, for $\Delta \geq 3$, producing a λ-coloring of G as required. □

Conjecture 9. *If G is a weakly chordal graph, then $\lambda \leq \Delta^2 - \Delta + 2$.*

By Theorem 7, if for every vertex v we have $|P_v| \leq \Delta - 1$ or $|P_v| = \Delta$ and P_v is a stable set, then the upper bound of the Conjecture 9 holds. The only case that was not verified occurs when $|P_v| = \Delta$ and there is an edge connecting vertices in P_v.

3. Split graphs

A graph G_c is *condensed split* if it can be obtained from a (connected) split graph G by recursively removing two vertices u and v with $\text{dist}(u, v) = 3$ in G and adding one new vertex whose neighborhood is the union of the neighborhood of the removed vertices, until there are no more such pairs of vertices. Fig. 7 shows a condensed split graph. Note that G_c is always a connected graph.

Lemma 10. *If G is a split graph, then G_c is split. Moreover, $\Delta(G_c) = \Delta(G)$.*

Proof. The class of split graphs is hereditary, so the graph obtained removing a vertex from a split graph is also a split graph. Besides, if we add a vertex to the stable set I without adding edges to vertices in I, the graph obtained still is a split graph. Thus, if G is a split graph, then G_c is too.

If G is a complete graph, then $G_c = G$. Otherwise, there is a vertex in the complete set K of G with degree at least $|K|$ and each new vertex added to G_c has as neighbors only vertices in K, so its degree is at most $|K|$. Hence, $\Delta(G) = \Delta(G_c)$. □

Lemma 11. *If $G_c = (I_c, K_c)$ is a condensed split graph, $i = |I_c|$ and $w = |K_c|$, then $\binom{i}{2} \leq w\binom{\Delta - w + 1}{2}$.*

Proof. First observe that each of the $\binom{i}{2}$ pairs of vertices in I_c must have at least a common neighbor in K_c. Besides, a vertex v in K_c can be such a common neighbor only for the pairs whose vertices belong to the neighborhood of v in I_c. Finally, observe that this set has size at most $\Delta - (w - 1)$. □

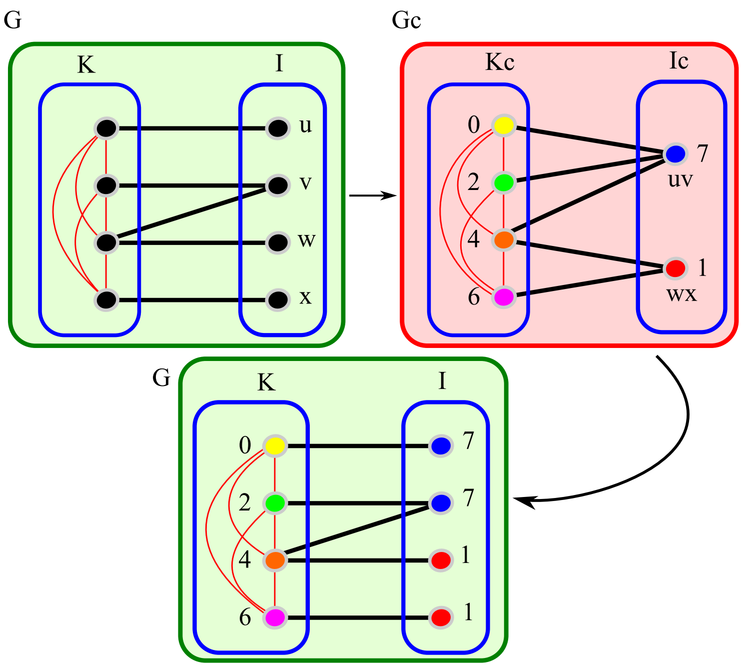

Fig. 8. Extending a λ-coloring of a condensed split graph to the original split graph.

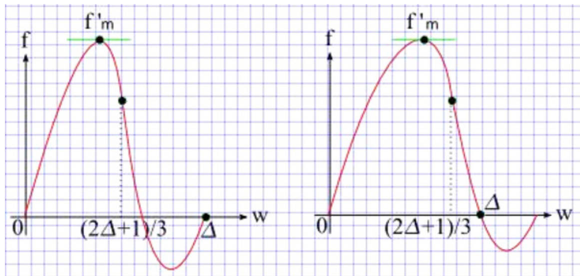

Fig. 9. Function $f(w)$.

Theorem 12. *If G is a split graph, then* $\lambda \leq \frac{2\sqrt{3}}{9}\Delta^{1.5} + 2\Delta + \Delta^{0.5} - 2$.

Proof. Let $G_c = (I_c, K_c)$ with $|I_c| = i$ and $|K_c| = w$ be a condensed split graph obtained from G. Since any λ-coloring of G_c can be used to obtain a λ-coloring of G, by definition of condensed split graph, we have $\lambda(G) \leq \lambda(G_c)$. (See Fig. 8.) Moreover, by Lemma 10, $\Delta = \Delta(G_c) = \Delta(G)$.

By Lemma 11 we have that $\binom{i}{2} \leq w\binom{\Delta-w+1}{2}$, implying that $i^2 - i \leq w(\Delta - w + 1)(\Delta - w)$. Now, take the mapping $f(w) = w(\Delta-w+1)(\Delta-w)$. The maximum value of $f(w)$ for integer w can be obtained with an analysis of the derivatives of f.

As $f(w) = w(\Delta-w+1)(\Delta-w) = w^3+w^2(-2\Delta-1)+w(\Delta^2+\Delta)$, we have that $f(0) = 0$ and $f(\Delta) = 0$. Now we can find the first and the second derivatives of $f(w)$, given as $f'(w) = 3w^2+w(-4\Delta-2)+(\Delta^2+\Delta)$ and $f''(w) = 6w-4\Delta-2$. When $f''(w) = 0$ we have an inflection point of the function $f(w)$, this happens when $w = \frac{2\Delta+1}{3}$, and the function $f(w)$ must looks like one of the two types of Fig. 9.

In order to verify the maximum value of the function $f(w)$, such that w is an integer, let us consider the first derivative $f'(w) = 0$. This happens when $w = \frac{(2\Delta+1)-\sqrt{\Delta^2+\Delta+1}}{3}$. So, the possible values for w being an integer are: $\frac{\Delta-2}{3}$, $\frac{\Delta-1}{3}$, $\frac{\Delta}{3}$, $\frac{\Delta+1}{3}$, and $\frac{\Delta+2}{3}$. The values of $f(w)$ for those w are given in Table 1. The maximum is attained when $w = \frac{\Delta}{3}$ with $f(w) = \frac{4\Delta^3+6\Delta^2}{27}$.

Assuming $i > \frac{2\sqrt{3}}{9}\Delta^{1.5} + \Delta^{0.5}$ we would have $\binom{i}{2} > w\binom{\Delta-w+1}{2}$, contradicting Lemma 11. So, $i \leq \frac{2\sqrt{3}}{9}\Delta^{1.5} + \Delta^{0.5}$.

Griggs and Yeh [6] proved the $\lambda \leq n + \Delta - 2$ for a graph with n vertices. Since the number of vertices in G_c is $i + \omega$, we have $n = i + \omega \leq \frac{2\sqrt{3}}{9}\Delta^{1.5} + \Delta^{0.5} + \Delta$, implying $\lambda(G_c) \leq \frac{2\sqrt{3}}{9}\Delta^{1.5} + 2\Delta + \Delta^{0.5} - 2$. □

Table 1
Function $f(w)$.

w	$f(w)$
$\frac{\Delta-2}{3}$	$\frac{4\Delta^3+6\Delta^2-18\Delta-20}{27}$
$\frac{\Delta-1}{3}$	$\frac{4\Delta^3+6\Delta^2-6\Delta-4}{27}$
$\frac{\Delta}{3}$	$\frac{4\Delta^3+6\Delta^2}{27}$
$\frac{\Delta+1}{3}$	$\frac{4\Delta^3+6\Delta^2-2}{27}$
$\frac{\Delta+2}{3}$	$\frac{4\Delta^3+6\Delta^2-6\Delta-4}{27}$

Theorem 12 establishes a special case of the following conjecture by Kral:

Conjecture 13 *([7]). If G is chordal, then* $\lambda \leq \frac{2\sqrt{3}}{9}\Delta^{1.5} + O(\Delta)$.

For any *simplicial* vertex v (a simplicial vertex is one whose neighborhood is a complete set) of a chordal graph G we can consider the subgraph induced by v, the vertices adjacent to v, and those which are at distance 2 away from v. Since it is a split graph, it can be λ-colored according to Theorem 12. As any chordal graph has a perfect elimination scheme, if we could combine the λ-colorings of these split subgraphs into a λ-coloring of G, we could set Kral's conjecture by giving that specific upper bound for λ.

4. Perspectives

It should be interesting to extend the upper bounds for λ from chordal bipartite graphs and split graphs here presented to weakly chordal graphs and chordal graphs, respectively. Furthermore, to find examples of graphs such that these upper bounds are sharp is also a matter for further study. We note that a split graph G with $\lambda > \frac{1}{3}\sqrt{\frac{2}{3}}\Delta^{1.5}$ can be found in [2], and a chordal graph G with $\lambda > \frac{2\sqrt{3}}{9}\Delta^{1.5}$ appears in [7].

The λ-coloring problem is $\mathcal{NP}$-complete even when restricted to split (and so to chordal and weakly chordal) graphs [2]. Another interesting question is to settle this problem for chordal bipartite graphs. As a matter of fact the computational complexity of the problem is open even for bipartite permutation graphs [1], a subclass of chordal bipartite graphs.

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