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Some nonexistence results for nonconstant stationary solutions to the Gray–Scott model in a bounded domain

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ABSTRACT

In the present paper, we are concerned with a reaction–diffusion system well-known as the Gray–Scott model in a bounded domain and study the corresponding steady-state problem. We establish some results for the nonexistence of nonconstant positive stationary solutions.

1. Introduction

In chemistry, there is a reaction–diffusion system well-known as the Gray–Scott model which is also called the cubic autocatalysis. The steady-state problem of this chemical reaction satisfies the following elliptic system:

$$\begin{cases} -d_1\Delta u = F(1-u) - uv^2 & \text{in } \Omega, \\ -d_2\Delta v = uv^2 - (F+k)v & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$ and $\partial_\nu = \partial/\partial\nu$. The unknown functions u and v represent the concentrations of two reactants and are considered to be nonnegative, and the parameters d_1, d_2, F and k are always assumed to be positive constants. It is clear that only nonnegative solutions of (1.1) are of real interest. For the more detailed background of this model, please refer to [2,3,8–11,16] and the references therein.

Problem (1.1) has received extensive concerns in both numerical and analytical studies whether the domain of the reactor is bounded or unbounded.

For example, if $n = 1$ and Ω is a finite open interval, in [1,5–7,9,11], etc., many interesting phenomena of patterns have been observed through the Gray–Scott model. In particular, Pearson in [11] carried out some thorough numerical analysis for the patterns of (1.1) and observed a complex structure for the positive nonconstant solutions (namely, patterns). In a more recent paper [8], McGough and Riley used more robust numerical schemes to confirm the results of [11]. Moreover, if Ω is a bounded smooth domain in $\mathbf{R}^n$ ($n \geq 1$), the authors of [8] discussed the stability of nonnegative constant solutions, obtained *a priori* bounds for the positive solutions of (1.1), and gave a bifurcation analysis showing the (local) existence of patterns of (1.1). Much more recently, we [13] investigated this Gray–Scott model in the case where Ω is bounded and smooth, and presented some further results for positive stationary solutions. More precisely, we derived a refined *a*

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priori estimates of positive stationary solutions, and improved some previous results for the nonexistence and existence of positive nonconstant stationary solutions as the parameters are varied, which imply certain conditions where the pattern has occurred or not.

When the domain Ω is unbounded, in [3], Hale et al. dealt with the model in one dimensional entire space and proved that a family of explicit stationary homoclinic orbits is unstable and that an explicit heteroclinic orbit is asymptotically and exponentially stable. In the work of [4], they obtained the existence of positive stationary solutions. Recently, Sato [15] established some sufficient conditions about the existence of nonconstant positive stationary solutions in this case of unbounded domain.

Simple analysis shows that if $F > 4(F+k)^2$, then (1.1) possesses two different constant positive solutions while (1.1) has a unique constant positive solution for $F = 4(F+k)^2$. In addition, $(1, 0)$ is the third nonnegative constant solution of (1.1).

In the present paper, we shall continue to deal with (1.1) and derive some further results for the nonexistence of positive nonconstant solutions. In particular, we observe an interesting phenomenon of pattern formation. That is, we show that there is no nonconstant positive stationary solution to the Gray–Scott model provided that $F < 4(F+k)^2$ and the diffusion coefficient d_1 is sufficiently small; and this result is in sharp contrast with the case of $F > 4(F+k)^2$ and small d_1, where we in [13] confirmed that nonconstant positive stationary solutions may occur. Thus, these conclusions indicate that, when d_1 is small enough, the condition of $F > 4(F+k)^2$ plays a decisive role in leading to spatially nonhomogeneous distribution of the two reactants involved in (1.1).

From now on, without special statement, the solutions of (1.1) we consider below always refer to the classical nonnegative ones. The positive solution means that the two components u and v are positive on $\overline{\Omega}$. The organization of this paper is as follows. In Section 2, we first state some preliminary results, and in Section 3, we shall discuss the nonexistence of positive nonconstant solutions to (1.1).

2. Some preliminary results

In this section, we collect some preliminary results which will be used in the forthcoming section. First of all, we recall the *a priori* estimates for the positive solutions of (1.1) obtained in [13], these conclusions have improved the ones of [8].

For simplicity, we denote

$$a = \max\{d_1/d_2, F/(F+k)\}.$$

Theorem 2.1 (*Theorem 2.1, [13]*). *For any positive solution (u, v) of (1.1), the following is satisfied:*

$$\max_{\overline{\Omega}} u(x) < 1 \quad \text{and} \quad \max_{\overline{\Omega}} v(x) < a.$$

Moreover,

$$\min_{\overline{\Omega}} u(x) > F/(F+a^2) \quad \text{and} \quad \max_{\overline{\Omega}} v(x) > F+k.$$

With the help of Theorem 2.1 and the Harnack inequality, we also established the positive upper and lower bounds for positive solutions to (1.1). More precisely, we can state

Theorem 2.2 (*Theorem 2.2, [13]*). *Let d and D be two given positive numbers. Then there exist positive constants $\underline{C}$ and $\overline{C}$, which depend only on $d, D, F,\ k$ and Ω such that if $d_2 \geq d$ and $d_1/d_2 \leq D$, any positive solution (u, v) of (1.1) satisfies*

$$\underline{C} < \min_{\overline{\Omega}}\{u(x), v(x)\} \leq \max_{\overline{\Omega}}\{u(x), v(x)\} < \overline{C}.$$

In what follows, we will prove a useful identity for any nonnegative solution to (1.1). This result can provide the relationship of the gradients of u and v. For our purpose, we need to set

$$\overline{u} = \frac{1}{|\Omega|}\int_\Omega u\mathrm{d}x \quad \text{and} \quad \overline{v} = \frac{1}{|\Omega|}\int_\Omega v\mathrm{d}x,$$

and

$$\phi = u - \overline{u} \quad \text{and} \quad \psi = v - \overline{v}.$$

As a consequence, we are able to claim that

Theorem 2.3. *For any nonnegative solution (u, v) of (1.1), the following holds:*

$$d_1^2\int_\Omega |\nabla\phi|^2\mathrm{d}x + Fd_1\int_\Omega \phi^2\mathrm{d}x + [(F+k)d_1 - Fd_2]\int_\Omega \phi\psi\,\mathrm{d}x = d_2^2\int_\Omega |\nabla\psi|^2\mathrm{d}x + (F+k)d_2\int_\Omega \psi^2\mathrm{d}x. \tag{2.1}$$

Proof. As in [8], we integrate the two equations in (1.1) to deduce that

$$F\int_\Omega u\mathrm{d}x + (F+k)\int_\Omega v\mathrm{d}x = F|\Omega|. \tag{2.2}$$

Now, let $w(x) = d_1u(x) + d_2v(x)$. Thus, by use of (2.2), w satisfies

$$-\Delta w = F(1-u) - (F+k)v = -F\phi - (F+k)\psi \quad \text{in } \Omega, \qquad \partial_\nu w = 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

Since

$$\int_\Omega \phi \mathrm{d}x = \int_\Omega \psi \mathrm{d}x = 0,$$

it follows from (2.3) that

$$\begin{aligned}\int_\Omega |\nabla w|^2\mathrm{d}x &= -\int_\Omega [F\phi + (F+k)\psi](d_1u + d_2v)\mathrm{d}x \\ &= -Fd_1\int_\Omega \phi^2\mathrm{d}x - [(F+k)d_1 + Fd_2]\int_\Omega \phi\psi\mathrm{d}x - (F+k)d_2\int_\Omega \psi^2\mathrm{d}x,\end{aligned} \tag{2.4}$$

which clearly implies

$$\int_\Omega \phi\psi\mathrm{d}x \le 0. \tag{2.5}$$

On the other hand, owing to the definition of w we have

$$\int_\Omega |\nabla w|^2\mathrm{d}x = \int_\Omega |\nabla(d_1u + d_2v)|^2\mathrm{d}x = d_1^2\int_\Omega |\nabla\phi|^2\mathrm{d}x + 2d_1d_2\int_\Omega \nabla\phi\cdot\nabla\psi\mathrm{d}x + d_2^2\int_\Omega |\nabla\psi|^2\mathrm{d}x. \tag{2.6}$$

In order to get rid of the term $\int_\Omega \nabla\phi\cdot\nabla\psi\mathrm{d}x$ in (2.6), we multiply (2.3) by ϕ and integrate over Ω to see

$$-F\int_\Omega \phi^2\mathrm{d}x - (F+k)\int_\Omega \phi\psi\mathrm{d}x = \int_\Omega \nabla w\cdot\nabla\phi\mathrm{d}x = d_1\int_\Omega |\nabla\phi|^2\mathrm{d}x + d_2\int_\Omega \nabla\phi\cdot\nabla\psi\mathrm{d}x. \tag{2.7}$$

Hence, we can combine (2.4), (2.6) and (2.7) to yield (2.1). The proof is complete. □

3. Nonexistence of positive nonconstant solutions

This section is devoted to the analysis of the nonexistence for nonconstant positive solutions of (1.1). From now on, let us denote μ_1 to be the first positive eigenvalue of the operator $-\Delta$ in Ω with the homogeneous Neumann boundary condition.

Theorem 3.1. *If one of the following cases holds, then (1.1) has no nonconstant positive solution:*

(i) $\frac{F}{F+k} > \frac{d_1}{d_2}$ *and* $d_2 > \frac{F}{(F+k)^2}\cdot\frac{d_1+\mu_1^{-1}F}{\mu_1d_2+F+k}$;

(ii) $\frac{d_1}{d_2} \ge \max\{\frac{F}{F+k}, \frac{1}{2}(F+k)\}$ *and* $d_2 > \frac{d_1[d_2+\mu_1^{-1}(F+k)]}{2d_2(\mu_1d_1+F)} + \frac{1}{\mu_1}\left[\frac{1}{2}\left(\frac{d_1}{d_2}\right)^2 + 2\left(\frac{d_1}{d_2}\right) - (F+k)\right]$.

Proof. We first verify case (i). Let us define $\overline{u}$, $\overline{v}$, ϕ and ψ as before and assume that (u, v) is a positive solution of (1.1). If $\frac{F}{F+k} > \frac{d_1}{d_2}$, then $a = \frac{F}{F+k}$. Thus, (2.1) and (2.5) enable us to assert

$$d_2^2\int_\Omega |\nabla\psi|^2\mathrm{d}x + (F+k)d_2\int_\Omega \psi^2\mathrm{d}x \le d_1^2\int_\Omega |\nabla\phi|^2\mathrm{d}x + Fd_1\int_\Omega \phi^2\mathrm{d}x. \tag{3.1}$$

Consequently, thanks to the well-known Poincaré inequality

$$\mu_1\int_\Omega \phi^2\mathrm{d}x \le \int_\Omega |\nabla\phi|^2\mathrm{d}x,$$

it follows from (3.1) that

$$\int_\Omega \psi^2\mathrm{d}x \le \frac{d_1+\mu_1^{-1}F}{\mu_1d_2+F+k}\cdot\frac{d_1}{d_2}\int_\Omega |\nabla\phi|^2\mathrm{d}x. \tag{3.2}$$

On the other hand, multiplying the first equation of (1.1) by ϕ, integrating over Ω, and applying Theorem 2.1 and the Cauchy inequality, we obtain

$$\begin{aligned} d_1 \int_\Omega |\nabla\phi|^2 \mathrm{d}x &= -F\int_\Omega \phi^2 \mathrm{d}x - \int_\Omega (uv^2 - u\overline{v}^2 + u\overline{v}^2 - \overline{u}\overline{v}^2)\phi \mathrm{d}x \\ &= -(F+\overline{v}^2)\int_\Omega \phi^2 \mathrm{d}x - \int_\Omega u(v+\overline{v})\phi\psi \mathrm{d}x \\ &\le -F\int_\Omega \phi^2 \mathrm{d}x + \frac{2F}{F+k}\int_\Omega |\phi|\cdot|\psi| \mathrm{d}x \\ &\le \frac{F}{(F+k)^2}\int_\Omega \psi^2 \mathrm{d}x. \end{aligned} \tag{3.3}$$

Hence, due to (3.2) and (3.3), we have

$$d_1 \int_\Omega |\nabla\phi|^2 \mathrm{d}x \le \frac{F}{(F+k)^2}\cdot\frac{(d_1+\mu_1^{-1}F)d_1}{(\mu_1 d_2+F+k)d_2}\int_\Omega |\nabla\phi|^2 \mathrm{d}x,$$

which obviously leads to $\phi \equiv 0$, that is, $u \equiv \overline{u} =$ constant under condition (i). Thereby, in this case, (1.1) admits no nonconstant positive solution by (2.1).

If $\frac{F}{F+k} \le \frac{d_1}{d_2}$, then $a = \frac{d_1}{d_2}$. Similarly, from (2.1) and (2.5), applying the Poincaré inequality, we derive

$$\int_\Omega \phi^2 \mathrm{d}x \le \frac{d_2+\mu_1^{-1}(F+k)}{\mu_1 d_1+F}\cdot\frac{d_2}{d_1}\int_\Omega |\nabla\psi|^2 \mathrm{d}x. \tag{3.4}$$

Multiplying the equation for v in (1.1) by ψ, and then integrating over Ω, together with Theorem 2.1, we also find

$$d_2 \int_\Omega |\nabla\psi|^2 \mathrm{d}x \le \left(\frac{d_1}{d_2}\right)^2 \int_\Omega |\phi|\cdot|\psi| \mathrm{d}x + \left[2\left(\frac{d_1}{d_2}\right) - (F+k)\right]\int_\Omega \psi^2 \mathrm{d}x. \tag{3.5}$$

Suppose that $\frac{d_1}{d_2} \ge \frac{1}{2}(F+k)$, it is easy to know from (3.5) that

$$d_2 \int_\Omega |\nabla\psi|^2 \mathrm{d}x \le \frac{1}{2}\cdot\left(\frac{d_1}{d_2}\right)^2 \int_\Omega \phi^2 \mathrm{d}x + \left[\frac{1}{2}\cdot\left(\frac{d_1}{d_2}\right)^2 + 2\left(\frac{d_1}{d_2}\right) - (F+k)\right]\int_\Omega \psi^2 \mathrm{d}x. \tag{3.6}$$

Therefore, by virtue of (3.4) and (3.6), the same argument as in case (i) concludes that condition (ii) ensures the nonexistence of nonconstant positive solutions to (1.1).

In the case of $\frac{F}{F+k} \le \frac{d_1}{d_2} < \frac{1}{2}(F+k)$, (i) of Theorem 3.1 in [13] indicates that (1.1) will have no nonconstant positive solutions. Now, the proof is complete. □

Remark 3.1. In [13], we yielded some conditions for the nonexistence of nonconstant positive solutions to (1.1). Simple observation shows that, in some cases, the results here are the improvement of the ones in [13] (see Theorem 3.1 and Theorem 3.2 there). Actually, if we let $d_1, k \to 0$ and $F \to 1/4$, then (i) of Theorem 3.1 is true if $\mu_1 d_2 > (\sqrt{65}-1)/8$ while the conditions in (i) and (ii) of Theorem 3.2 in [13] become $\mu_1 d_2 > 43/4$, and $\epsilon > 9$ and $\mu_1 d_2 > 7/4+\epsilon$, respectively. Therefore, for such chosen parameters, (i) of our Theorem 3.1 here improves Theorem 3.2 in [13]. On the other hand, if we take d_1, d_2 such that both d_1/d_2 and d_2 are sufficiently large, it is clearly noted that the condition (ii) in Theorem 3.1 obtained here is superior to those given by Theorem 3.1 and Theorem 3.2 in [13]. However, we would also like to remark that, Theorem 3.1 and Theorem 3.2 in [13] are not strictly covered by the results in this paper. Roughly speaking, (1.1) has no nonconstant positive solutions if either d_2 or μ_1 is large enough. In the sense of [12], one sees that large μ_1 corresponds to small size of the reactor Ω. As a result, in terms of chemistry, our conclusions demonstrate that the fast diffusion of the reactant v or the small size of the reactor will contribute to the spatially uniform distribution of the two reactants of the Gray–Scott system.

Finally, we want to claim that (1.1) possesses no positive solution for $F < 4(F+k)^2$ and small d_1. To achieve this goal, we need to recall a useful lemma, whose proof was given in [14].

Lemma 3.1. *Assume that $\mu > 0$ is a constant and $b(x)$ is a continuous positive function on $\overline{\Omega}$. Then, the following problem*

$$-\Delta z = \mu(1-b^{-1}(x)z) \quad \text{in } \Omega, \qquad \partial_\nu z = 0 \quad \text{on } \partial\Omega. \tag{3.7}$$

has a unique positive solution z_μ, and $z_\mu \to b(x)$ uniformly on $\overline{\Omega}$ as $\mu \to \infty$.

The final result of this section is stated as follows.

Theorem 3.2. *Assume that $F < 4(F+k)^2$ and let d be an arbitrary positive number. Then there exists a small positive constant ε_1 depending only on d, F, k and Ω, such that (1.1) has no positive solution provided that $0 < d_1 < \varepsilon_1$ and $d < d_2$.*

Proof. Since (1.1) has no nonconstant positive solution if d_1 is small and d_2 is sufficiently large by (i) in Theorem 3.1, it suffices to consider the case of small d_1 and $d \leq d_2 \leq D$, where D is a fixed large constant. To this end, we adopt an indirect argument. Suppose that our conclusion is not true. Then, there exists a sequence $\{(d_{1,i},\ d_{2,i})\}_{i=1}^{\infty}$ with $d_{1,i} \to 0$ and $d \leq d_{2,i} \leq D$, such that (1.1) possesses a positive solution (u_i, v_i) corresponding to $(d_1, d_2) = (d_{1,i},\ d_{2,i})$. It may be assumed that $d_{2,i} \to d_2 \in [d, D]$.

By the standard L^p and Schauder theory for elliptic equations and the embedding theorem, together with Theorem 2.2, passing to a subsequence of $\{v_i\}$ if necessary, we see from the equation for v_i in (1.1) that $v_i \to v$ in $C^1(\overline{\Omega})$ as $i \to \infty$, and v is a positive function over $\overline{\Omega}$. As a result, due to Lemma 3.1, it can easily follow that there exists a subsequence of $\{u_i\}$ still labelled by itself, such that $u_i \to F/(F+v^2)$ uniformly on $\overline{\Omega}$. This analysis implies that v is a positive solution of the problem

$$-d_2 \Delta v = -\frac{v}{F+v^2}[(F+k)v^2 - Fv + F(F+k)] \quad \text{in } \Omega, \qquad \partial_\nu v = 0 \quad \text{on } \partial\Omega.$$

We notice that $F < 4(F+k)^2$ gives $(F+k)v^2 - Fv + F(F+k) > 0$. Obviously, the above equation has no positive solution in this case. Hence, our Theorem 3.2 holds and the proof is complete. $\square$

Remark 3.2. In (i) of Theorem 3.1 of [13], it was proved that (1.1) has no nonconstant positive solutions if $a \leq F+k$. Thus, Theorem 3.2 here is an improved result if d_1 is small. Moreover, (ii) of Remark 3.2 there showed that (1.1) may admit nonconstant positive solutions for small d_1 when $F > 4(F+k)^2$ holds. In other words, if the reactant u diffuses slowly, the existence of two constant positive solutions, which is guaranteed by $F > 4(F+k)^2$, helps to enhance the formation of patterns (i.e., nonhomogeneous stationary solutions) of (1.1) while the disappearance of these two constant positive solutions (equivalently, $F < 4(F+K)^2$) seems to tend to cause the spatially homogeneous distribution of the two reactants in this system.

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