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Toric complexes and Artin kernels

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Abstract

A simplicial complex L on n vertices determines a subcomplex T_L of the n-torus, with fundamental group the right-angled Artin group G_L. Given an epimorphism $\chi : G_L \to \mathbb{Z}$, let T_L^χ be the corresponding cover, with fundamental group the Artin kernel N_χ. We compute the cohomology jumping loci of the toric complex T_L, as well as the homology groups of T_L^χ with coefficients in a field $\Bbbk$, viewed as modules over the group algebra $\Bbbk\mathbb{Z}$. We give combinatorial conditions for $H_{\leqslant r}(T_L^\chi; \Bbbk)$ to have trivial $\mathbb{Z}$-action, allowing us to compute the truncated cohomology ring, $H^{\leqslant r}(T_L^\chi; \Bbbk)$. We also determine several Lie algebras associated to Artin kernels, under certain triviality assumptions on the monodromy $\mathbb{Z}$-action, and establish the 1-formality of these (not necessarily finitely presentable) groups.

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Keywords: Toric complex; Right-angled Artin group; Artin kernel; Bestvina–Brady group; Cohomology ring; Stanley–Reisner ring; Cohomology jumping loci; Monodromy action; Holonomy Lie algebra; Malcev Lie algebra; Formality

Contents

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1. Introduction

The underlying theme of this paper is the interplay between topology, discrete geometry, group theory, and commutative algebra, as revealed by the intricate connections tying up the combinatorics of a simplicial complex with the algebraic and geometric topology of certain spaces modeled on it.

Our first goal is to better understand the topology of a toric complex T_L, and how to compute some of its homotopy-type invariants, directly from the combinatorial data encoded in the input simplicial complex L.

Our second—and more ambitious—goal is to understand how the algebraic topology of the infinite cyclic Galois covers of T_L depends on the vertex labellings parametrizing those covers.

1.1. *Toric complexes and right-angled Artin groups*

Let L be a finite simplicial complex on vertex set V, and let T^n be the torus of dimension $n = |\mathsf{V}|$, with the standard CW-decomposition. The *toric complex* associated to L, denoted T_L, is the subcomplex of T^n obtained by deleting the cells corresponding to the non-faces of L. Much is known about the topology of these combinatorially defined spaces.

On one hand, the fundamental group $G_L = \pi_1(T_L)$ is the *right-angled Artin group* determined by the graph $\Gamma = L^{(1)}$, with presentation consisting of a generator v for each vertex v in V, and a commutator relation $vw = wv$ for each edge $\{v, w\}$ in Γ. As shown by Droms [14], $G_L \cong G_{L'}$ if and only if the corresponding graphs are isomorphic. The associated graded Lie algebras and the Chen Lie algebras of right-angled Artin groups were computed in [15,16,27] and [27], respectively. For a survey of the geometric properties of such groups, we refer to Charney [6].

On the other hand, the cohomology ring $H^*(T_L, \Bbbk)$ is the exterior Stanley–Reisner ring $\Bbbk\langle L\rangle$, with generators the duals v^*, and relations the monomials corresponding to the missing faces of L, see Kim and Roush [20] and Charney and Davis [7].

1.2. *Higher homotopy groups*

As mentioned above, one of our goals here is to better understand the homotopy type invariants attached to a toric complex. In particular, we seek to generalize known results, from the special case when L is a flag complex (that is, one for which every subset of pairwise adjacent vertices spans a simplex), to the general case. We start in Section 2 with a study of the higher homotopy groups of a toric complex.

It has long been known that T_L is aspherical, whenever L is a flag complex, see [7,24]. Recently, Leary and Saadetoğlu [21] have shown that the converse also holds. In Theorem 2.1, we make this result more precise, by giving a combinatorial description of the first non-vanishing higher homotopy group of a non-flag complex L, viewed as a module over $\mathbb{Z}G_L$.

1.3. Cohomology jumping loci

The characteristic varieties $\mathcal{V}^i_d(X,\Bbbk)$ and the resonance varieties $\mathcal{R}^i_d(X,\Bbbk)$ of a finite-type CW-complex X provide a unifying framework for the study of a host of questions, both quantitative and qualitative, concerning the space X and its fundamental group. For instance, counting certain torsion points on the character torus, according to their depth with respect to the stratification by the characteristic varieties, yields information about the homology of finite abelian covers of X. The subtle interplay between the geometry of these two sets of varieties leads to powerful formality and quasi-projectivity obstructions, see [13]. Finally, the cohomology jumping loci of a classifying space $K(G,1)$ provide computable upper bounds for the Bieri–Neumann–Strebel–Renz (BNSR) invariants of a group G, see [30].

The degree 1 jump loci of a toric complex have been computed in [13,27], leading to a complete solution of Serre's quasi-projectivity problem within the class of right-angled Artin groups. We determine here, in Section 3, the higher-degree jump loci of toric complexes. Applications and further discussion can be found in [30].

In Theorem 3.8, we compute the resonance varieties $\mathcal{R}^i_d(T_L,\Bbbk) \subseteq \operatorname{Hom}(G_L,\Bbbk)$, associated to the cohomology ring $\Bbbk\langle L\rangle$, while in Theorem 3.12, we compute the jumping loci for cohomology with coefficients in rank 1 local systems, $\mathcal{V}^i_d(T_L,\Bbbk) \subseteq \operatorname{Hom}(G_L,\Bbbk^\times)$, over an arbitrary field $\Bbbk$, and for all integers $i,d \geqslant 1$. Explicitly,

$$\mathcal{R}^i_d(T_L,\Bbbk) = \bigcup_{\mathsf{W}} \Bbbk^{\mathsf{W}}, \qquad \mathcal{V}^i_d(T_L,\Bbbk) = \bigcup_{\mathsf{W}} \bigl(\Bbbk^\times\bigr)^{\mathsf{W}}, \tag{1}$$

where, in both cases, the union is taken over all subsets $\mathsf{W} \subset \mathsf{V}$ for which the ith "Aomoto–Betti" number, $\beta_i(\Bbbk\langle L\rangle,\mathsf{W})$, is at least d. These numbers can be computed directly from L, using the following formula of Aramova, Avramov, and Herzog [1]:

$$\beta_i\bigl(\Bbbk\langle L\rangle,\mathsf{W}\bigr) = \sum_{\sigma\in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}\bigl(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma),\Bbbk\bigr), \tag{2}$$

where L_{W} is the subcomplex induced by L on W, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$. In the particular case when $i=d=1$, the formulas from (1) recover results from [27] and [13], respectively.

1.4. Infinite cyclic covers and Artin kernels

Consider a homomorphism $\chi : G_L \to \mathbb{Z}$, specified by assigning an integer weight, $m_v = \chi(v)$, to each vertex v in V. Assume χ is onto, and let $\pi : T_L^\chi \to T_L$ be the corresponding Galois cover. The fundamental group

$$N_\chi := \pi_1\bigl(T_L^\chi\bigr) = \ker(\chi : G_L \to \mathbb{Z}) \tag{3}$$

is called the *Artin kernel* associated to χ. A classifying space for this group is the space $T^\chi_{\Delta_\Gamma}$, where Δ_Γ is the flag complex of $\Gamma = L^{(1)}$. As mentioned above, a major goal of this paper is to understand how the algebraic topology of the spaces T_L^χ, and some of the properties of the groups N_χ, depend on the epimorphism χ.

Noteworthy is the case when χ is the "diagonal" homomorphism $\nu : G_L \twoheadrightarrow \mathbb{Z}$, which assigns to each vertex the weight 1. The corresponding Artin kernel, $N_\Gamma = N_\nu$, is called the *Bestvina–Brady group* associated to Γ. As hinted at by Stallings [34] and Bieri [3] in their pioneering work, and as proved in full generality by Bestvina and Brady in their landmark paper [2], the geometric and homological finiteness properties of the group N_Γ are intimately connected to the topology of the flag complex Δ_Γ. For example, N_Γ is finitely generated if and only if Γ is connected, and N_Γ is finitely presented if and only if Δ_Γ is simply-connected. When $\pi_1(\Delta_\Gamma) = 0$, an explicit finite presentation for N_Γ was given by Dicks and Leary [11], and the presentation was further simplified in [28].

1.5. Homology of $\mathbb{Z}$-covers

To get a handle on the coverings $T_L^\chi \to T_L$, we focus in Sections 4–6 on the homology groups $H_*(T_L^\chi, \Bbbk)$, viewed as modules over the group algebra $\Bbbk\mathbb{Z}$, with coefficients in an arbitrary field $\Bbbk$. After some preparatory material in Section 4, we compute those homology groups, in two steps. First, we give in Theorem 5.1 a combinatorial formula for their $\Bbbk\mathbb{Z}$-ranks:

$$\operatorname{rank}_{\Bbbk\mathbb{Z}} H_i\bigl(T_L^\chi, \Bbbk\bigr) = \beta_i\bigl(\Bbbk\langle L\rangle, \mathsf{V}_0(\chi)\bigr), \tag{4}$$

where $\mathsf{V}_0(\chi) = \{v \mid \chi(v) \neq 0\}$ is the support of χ.

The computation of the $\Bbbk\mathbb{Z}$-torsion part is more complicated. The algorithm, which we summarize in Theorem 5.9, involves two steps. The first, arithmetic in nature, requires factoring certain cyclotomic polynomials in $\Bbbk[t]$. The second, algebro-combinatorial in nature, requires diagonalizing certain monomial matrices over $\Bbbk[[t]]$. In Section 5.6, we note that the f-primary part of $H_i(T_L^\chi, \Bbbk)$ is non-trivial only when the irreducible polynomial $f \in \Bbbk[t]$ divides some cyclotomic polynomial, and we show that this is the only restriction on non-trivial torsion, for $i > 0$.

As an application of our method, we give in Corollary 5.2 and Theorem 6.2 combinatorial tests for deciding whether the following conditions are satisfied (for fixed $r > 0$):

(a) For each $i \leqslant r$, the $\Bbbk$-vector space $H_i(T_L^\chi, \Bbbk)$ is finite-dimensional.
(b) For each $i \leqslant r$, the $\Bbbk\mathbb{Z}$-module $H_i(T_L^\chi, \Bbbk)$ is trivial.

The first test amounts to the vanishing of the ranks from (4), for all $i \leqslant r$. The second test amounts to the vanishing of $\beta_i(\Bbbk\langle L\rangle, \mathsf{V}_q(\chi))$, for all $i \leqslant r$, where q runs through a finite list of primes (and 0), depending on χ and $p = \operatorname{char}\Bbbk$, and $\mathsf{V}_q(\chi)$ is the support of χ in characteristic q. Clearly (b) $\Rightarrow$ (a), but not the other way, cf. Remark 6.3.

For the Bestvina–Brady covers T_L^ν, we show in Corollary 7.4 that (a) and (b) hold simultaneously, and this happens precisely when $\nu_\Bbbk \notin \bigcup_{i \leqslant r} \mathcal{R}_1^i(T_L, \Bbbk)$, where $\nu_\Bbbk$ is the cohomology class in degree one naturally associated to ν.

1.6. Cohomology ring and finiteness properties

In Section 7, we illustrate our techniques by determining the cohomology ring of T_L^χ (truncated in a certain degree), with coefficients in a field $\Bbbk$. The cohomology ring of a Bestvina–Brady cover T_L^ν was computed in [21], up to any degree r for which condition (a) holds. In

Theorem 7.1, we extend this computation to arbitrary χ, up to any degree r for which condition (b) holds:

$$H^{\leqslant r}\big(T_L^\chi, \Bbbk\big) \cong H^{\leqslant r}(T_L, \Bbbk)/(\chi_\Bbbk). \tag{5}$$

Next, we consider the finiteness properties of Artin kernels. A group G is said to be of type FP_r $(1 \leqslant r \leqslant \infty)$ if the trivial G-module $\mathbb{Z}$ has a projective $\mathbb{Z}G$-resolution, finitely generated in degrees up to r, see Serre [32]. Using a result of Meier, Meinert, and VanWyk [23], as reinterpreted by Bux and Gonzalez [5], we note in Theorem 7.3 that

$$N_\chi \text{ is of type } \mathrm{FP}_r \quad \Longleftrightarrow \quad \dim_\Bbbk H_i(N_\chi, \Bbbk) < \infty, \quad \text{for all } i \leqslant r, \text{ and all } \Bbbk. \tag{6}$$

This is to be compared with a result of the same flavor from [21], where it is shown that the cohomological dimension of a Bestvina–Brady group N_ν equals its trivial cohomological dimension.

The finiteness properties of the groups N_χ are controlled by the Bieri–Neumann–Strebel–Renz invariants of G_L, and those invariants are also computed in [5,23]. A close relationship between the first resonance variety $\mathcal{R}_1^1(G_L, \mathbb{R})$ and the BNS invariant $\Sigma^1(G_L)$ was first noted in [27]. A generalization to the higher resonance varieties and the higher BNSR invariants, based on formula (1), is given in [30].

1.7. Graded Lie algebras

In Sections 8 and 9 we study several graded Lie algebras attached to groups of the form N_χ, generalizing results from [28], valid only in the case where $\chi = \nu$, the diagonal character.

We start with some general properties of holonomy Lie algebras. Given a graded algebra A satisfying some mild assumptions, its *holonomy Lie algebra*, $\mathfrak{h}(A) = \bigoplus_{s\geqslant 1} \mathfrak{h}_s(A)$, is defined as the free Lie algebra on the dual of A^1, modulo the (homogeneous) ideal generated by the image of the comultiplication map. The main result here is Theorem 8.1, where we show that, if an element $a \in A^1$ is non-resonant up to degree r, then

$$\big(\mathfrak{h}(A)/\mathfrak{h}''(A)\big)_s \cong \big(\mathfrak{h}(A/aA)/\mathfrak{h}''(A/aA)\big)_s, \quad \text{for } 2 \leqslant s \leqslant r+1. \tag{7}$$

Next, we consider the associated graded Lie algebra, $\mathrm{gr}(N_\chi)$, arising from the lower central series of an Artin kernel N_χ, and the rational holonomy Lie algebra, $\mathfrak{h}(N_\chi)$, arising from the cohomology ring $A = H^*(N_\chi, \mathbb{Q})$. In Proposition 9.2 and Corollary 9.3, we assume that $H_1(N_\chi, \mathbb{Q})$ is $\mathbb{QZ}$-trivial, and determine the associated graded Lie algebra, $\mathrm{gr}(N_\chi)$, and the Chen Lie algebra, $\mathrm{gr}(N_\chi/N_\chi'')$. Using computations from [27], we show that their graded ranks, ϕ_k and θ_k, are given by

$$\prod_{k=1}^{\infty}\big(1-t^k\big)^{\phi_k} = \frac{P(-t)}{1-t} \quad \text{and} \quad \sum_{k=2}^{\infty}\theta_k t^k = Q\left(\frac{t}{1-t}\right), \tag{8}$$

where $P(t)$ and $Q(t)$ are the clique and cut polynomials of the graph Γ. In Theorem 9.5, we assume additionally that $H_2(N_\chi, \mathbb{Q})$ is $\mathbb{QZ}$-trivial, and determine the graded Lie algebra $\mathfrak{h}(N_\chi)$ in this instance; in particular, we find that $\mathfrak{h}'(N_\chi) = \mathfrak{h}'(G_\Gamma)$.

1.8. The 1-formality property

A finitely generated group G is said to be 1-*formal*, in the sense of Sullivan [35], if there is a filtration-preserving Lie algebra isomorphism between the Malcev Lie algebra of G, as defined by Quillen [31], and the degree completion of the rational holonomy Lie algebra of G, as defined by Chen [9]. Put another way, G is 1-formal if its Malcev Lie algebra, $\mathfrak{m}(G)$, is quadratically presented. Examples include fundamental groups of compact Kähler manifolds and complements of algebraic hypersurfaces in $\mathbb{CP}^n$, certain pure braid groups of Riemann surfaces, and certain Torelli groups. The 1-formality property of a group has many remarkable consequences. We refer to [13,26,35] for more details and references.

In [19], Kapovich and Millson showed that all finitely generated Artin groups (in particular, all groups of the form G_Γ) are 1-formal. In [28], we showed that all finitely presented Bestvina–Brady groups, i.e., all groups N_Γ for which $\pi_1(\Delta_\Gamma)=0$, are 1-formal.

We generalize this last result here, in Theorem 10.1: Suppose $H_i(N_\chi,\mathbb{Q})$ is $\mathbb{QZ}$-trivial, for $i=1,2$ (when $\chi=\nu$, this boils down to $\widetilde{H}_i(\Delta_\Gamma,\mathbb{Q})=0$, for $i=0,1$); then the Artin kernel N_χ is a finitely generated, 1-formal group. Using this theorem, we are able to construct 1-formal groups which are not finitely presentable. To the best of our knowledge, these examples are the first of their kind.

2. Toric complexes and right-angled Artin groups

In this section, we discuss in more detail some of our main characters—the spaces T_L and their fundamental groups, G_L—and compute certain homotopy groups of T_L.

2.1. Toric complexes

Let L be a finite simplicial complex, on vertex set V. For each simplex $\sigma=\{v_{i_1},\dots,v_{i_k}\}$ of L, let T_σ be the torus formed by identifying parallel faces of a k-cube ($T_\emptyset$ is a point). The *toric complex* associated to L is the identification space

$$T_L=\coprod_{\sigma\in L} T_\sigma/\{T_\sigma\cap T_{\sigma'}=T_{\sigma\cap\sigma'}\}. \tag{9}$$

An equivalent description is as follows. Let T^n be the torus of dimension $n=|\mathsf{V}|$, with the standard CW-decomposition. Then T_L is the subcomplex of T^n obtained by deleting the cells corresponding to the non-faces of L. Note that the k-cells c_σ in T_L are in one-to-one correspondence with the $(k-1)$-simplices σ in L.

In the terminology from [10], $T_L=\mathcal{Z}_L(S^1)$ is an example of a "generalized moment–angle complex." As such, the construction enjoys many natural properties. For instance, $T_{K*L}=T_K\times T_L$, where $*$ denotes simplicial join; moreover, if $K\subset L$ is a simplicial subcomplex, then $T_K\subset T_L$ is a CW-subcomplex.

2.2. The cohomology ring

Fix a coefficient ring $\Bbbk$ (we will mainly be interested in the case when $\Bbbk=\mathbb{Z}$, or $\Bbbk$ is a field). Let $C_\bullet(L,\Bbbk)$ be the simplicial chain complex of L, and let $C_\bullet(T_L,\Bbbk)$ be the cellular chain complex of T_L. Since T_L is a subcomplex of T^n, all differentials in $C_\bullet(T_L, \Bbbk)$ vanish, i.e., T_L is a *minimal* CW-complex. It follows that

$$H_k(T_L, \Bbbk) = C_{k-1}(L, \Bbbk), \quad \text{for all } k > 0. \tag{10}$$

In other words, $H_k(T_L, \Bbbk)$ is a free $\Bbbk$-module of rank $d_k(L) = \#\{\sigma \in L \mid |\sigma| = k\}$, where $|\sigma| = \dim(\sigma) + 1$. In particular, the Betti numbers $b_k(T_L) = \dim_\Bbbk H_k(T_L, \Bbbk)$ depend only on L, and not on $\Bbbk$.

As shown in [7,20], the cohomology ring of T_L may be identified with the exterior Stanley–Reisner ring of L. More precisely, let V be the free $\Bbbk$-module on the set V, and $V^* = \mathrm{Hom}_\Bbbk(V, \Bbbk)$ its dual. Set $\Bbbk\langle L\rangle = \bigwedge V^*/J_L$, where $\bigwedge V^*$ is the exterior algebra on V^* and J_L is the ideal generated by all monomials $t_\sigma = v_{i_1}^* \cdots v_{i_k}^*$ corresponding to simplices $\sigma = \{v_{i_1}, \dots, v_{i_k}\}$ not belonging to L. Then

$$H^*(T_L, \Bbbk) = \Bbbk\langle L\rangle. \tag{11}$$

In the case when L is a flag complex, J_L is a quadratic monomial ideal, and so $\Bbbk\langle L\rangle$ is a Koszul algebra.

2.3. Right-angled Artin groups

The 1-skeleton $L^{(1)}$ can be viewed as a (finite) simple graph $\Gamma = (\mathsf{V}, \mathsf{E})$, with vertex set V consisting of the 0-cells, and edge set E consisting of the 1-cells. It is readily seen that the fundamental group $G_L = \pi_1(T_L)$ is isomorphic to

$$G_\Gamma := \bigl\langle v \in \mathsf{V} \bigm| uv = vu \text{ if } \{u, v\} \in \mathsf{E}\bigr\rangle, \tag{12}$$

the right-angled Artin group associated to Γ. These groups interpolate between free groups (if $\Gamma = \overline{K}_n$ is the discrete graph on n vertices, then $G_\Gamma = F_n$), and free abelian groups (if $\Gamma = K_n$ is the complete graph on n vertices, then $G_\Gamma = \mathbb{Z}^n$). Moreover, if $\Gamma = \Gamma' \sqcup \Gamma''$ is the disjoint union of two graphs, then $G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$, and if $\Gamma = \Gamma' * \Gamma''$ is their join, then $G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$.

Let $\Delta = \Delta_\Gamma$ be the flag complex of Γ, i.e., the maximal simplicial complex with 1-skeleton equal to Γ. Clearly, L is a subcomplex of Δ, sharing the same 1-skeleton. Moreover, the k-simplices of Δ correspond to the $(k+1)$-cliques of Γ. A classifying space for $G = G_\Gamma$ is the toric complex associated to Δ,

$$K(G, 1) = T_\Delta, \tag{13}$$

see Charney and Davis [7] and Meier and VanWyk [24]. For example, if $\Gamma = \overline{K}_n$, then $\Delta = \Gamma$, and $T_\Delta = \bigvee^n S^1$, whereas if $\Gamma = K_n$, then $\Delta = \Delta_{n-1}$, the $(n-1)$-simplex, and $T_\Delta = T^n$.

Let $\widetilde{C}_\bullet = (C_\bullet(\widetilde{T}_L, \Bbbk), \tilde{\partial}_\bullet)$ be the equivariant chain complex of the universal cover of T_L. Under the identification $\widetilde{C}_k = \Bbbk G \otimes_\Bbbk C_k$, where $C_k = C_k(T_L, \Bbbk)$, the boundary map $\tilde{\partial}_k : \widetilde{C}_k \to \widetilde{C}_{k-1}$ is given by

$$\tilde{\partial}_k(1 \otimes c_\sigma) = \sum_{r=1}^{k} (-1)^{r-1}(v_r - 1) \otimes c_{\sigma \setminus \{v_r\}} \tag{14}$$

where $\sigma = \{v_1, \dots, v_k\}$ is a $(k-1)$-simplex in L.

2.4. Higher homotopy groups of toric complexes

By the above-mentioned result, all the higher homotopy groups of the toric complex associated to a flag complex vanish. In general, though, T_L will have $\pi_r(T_L) \neq 0$, for some $r > 1$. The next result identifies the first integer r for which this happens, and computes the rank of the G_L-coinvariants for the corresponding homotopy group (viewed as a module over $\mathbb{Z}G_L$).

Theorem 2.1. *For a finite simplicial complex L, with associated flag complex Δ, set $p = p(L) := \sup\{k \mid d_{\leqslant k}(\Delta) = d_{\leqslant k}(L)\}$. Let T_L be the corresponding toric complex, and $G = \pi_1(T_L)$. Then, the following are equivalent:*

(i) *T_L is aspherical.*
(ii) *L is a flag complex.*
(iii) $p = \infty$.

Moreover, if $p < \infty$, then:

(iv) $\pi_2(T_L) = \cdots = \pi_{p-1}(T_L) = 0$.
(v) *The pth homotopy group of T_L, when viewed as a module over $\mathbb{Z}G$, has a finite presentation of the form $\pi_p(T_L) = \operatorname{coker}(\Pi_{p+1} \circ \tilde{\partial}_{p+2})$, where $\Pi_{p+1} : \mathbb{Z}G^{d_{p+1}(\Delta)} \to \mathbb{Z}G^{d_{p+1}(\Delta)-d_{p+1}(L)}$ is the canonical projection.*
(vi) *The group of coinvariants $(\pi_p(T_L))_G$ is free abelian, of rank $d_{p+1}(\Delta) - d_{p+1}(L) > 0$.*

Proof. First suppose $p < \infty$. Then L is a proper simplicial subcomplex of Δ, and both have the same 1-skeleton, Γ. Thus, T_L is a proper subcomplex of the aspherical CW-complex T_Δ, and both share the same 2-skeleton, T_Γ. Moreover, T_L and T_Δ are minimal CW-complexes, and their cohomology rings, $H^*(T_L;\mathbb{Z}) = \mathbb{Z}\langle T_L\rangle$ and $H^*(T_\Delta;\mathbb{Z}) = \mathbb{Z}\langle T_\Delta\rangle$, are generated in degree 1. Statements (iv)–(vi) now follow at once from [25], Theorem 2.10, Corollary 2.11, and Remark 2.13.

The implication (iii) $\Rightarrow$ (ii) is clear, whereas (ii) $\Rightarrow$ (i) follows from (13). To prove (i) $\Rightarrow$ (iii), assume p is finite. Then $\pi_p(T_L) \neq 0$, by (vi); thus, T_L is not aspherical. $\square$

Equivalence (i) $\Leftrightarrow$ (ii) in the above recovers a result of Leary and Saadetoğlu [21, Proposition 4], which they proved by different means.

3. Aomoto–Betti numbers and cohomology jumping loci

In this section, we determine the resonance varieties $\mathcal{R}^i_d(T_L, \Bbbk)$ and characteristic varieties $\mathcal{V}^i_d(T_L, \Bbbk)$ of a toric complex T_L, in all degrees $i, d \geqslant 1$, and for all fields $\Bbbk$.

3.1. Resonance varieties of algebras

Let A be a connected, locally finite, graded algebra over a field $\Bbbk$. Denote the Betti numbers of A by $b_i(A) = \dim_\Bbbk A^i$. For each element $z \in A^1$ with $z^2 = 0$, right-multiplication by z defines a cochain complex

$$(A, z):\quad A^0 \xrightarrow{z} A^1 \xrightarrow{z} \cdots \longrightarrow A^{i-1} \xrightarrow{z} A^i \xrightarrow{z} A^{i+1} \longrightarrow \cdots, \tag{15}$$

also known as the *Aomoto complex*. Denote the Betti numbers of this complex by

$$\beta_i(A,z) = \dim_{\Bbbk} H^i(A,z). \tag{16}$$

Remark 3.1. The following properties of the Aomoto Betti numbers are immediate:

(1) $\beta_i(A,z) \leqslant b_i(A)$, for all $z \in A^1$, and $\beta_i(A,0) = b_i(A)$.
(2) $\beta_0(A,z) = 0$, for all $z \neq 0$, and $\beta_0(A,0) = 1$.

Suppose $z^2 = 0$, for all $z \in A^1$. Then, for each $i \geqslant 1$ and each $0 \leqslant d \leqslant b_i = b_i(A)$, we may define the *resonance variety*

$$\mathcal{R}^i_d(A) = \bigl\{ z \in A^1 \bigm| \beta_i(A,z) \geqslant d \bigr\}. \tag{17}$$

It is readily seen that each set $\mathcal{R}^i_d = \mathcal{R}^i_d(A)$ is a homogeneous algebraic subvariety of the affine space $A^1 = \Bbbk^{b_1}$. For each degree $i \geqslant 1$, the resonance varieties provide a descending filtration $A^1 = \mathcal{R}^i_0 \supset \mathcal{R}^i_1 \supset \cdots \supset \mathcal{R}^i_{b_i} \supset \mathcal{R}^i_{b_{i+1}} = \emptyset$.

Remark 3.2. Given $z \in A^1$, and an integer $r \geqslant 1$, the following are equivalent:

(1) $\beta_i(A,z) = 0$, for all $i \leqslant r$.
(2) $z \notin \mathcal{R}^i_1(A)$, for all $i \leqslant r$.

A simple example is provided by the exterior algebra $E = \bigwedge \Bbbk^n$. It is readily checked that the Aomoto complex (15) is exact in this case. Hence, $\mathcal{R}^i_d(E) = \{0\}$, for all $1 \leqslant i \leqslant n$ and $1 \leqslant d \leqslant \binom{n}{i}$.

Now let X be a connected CW-complex with finitely many cells in each dimension. Fix a field $\Bbbk$, and let $A = H^*(X,\Bbbk)$ be the respective cohomology ring. (If $\operatorname{char}\Bbbk = 2$, we need to assume $H_1(X,\mathbb{Z})$ is torsion-free.) Then A is a locally finite, graded algebra with $z^2 = 0$, for all $z \in A^1$. Thus, for every $z \in A^1$, we may define the cochain complex $(A, \cdot z)$, with Aomoto Betti numbers $\beta^{\Bbbk}_i(X,z) := \beta_i(H^*(X,\Bbbk), z)$. The resonance varieties of X (over the field $\Bbbk$) are simply those of its cohomology ring: $\mathcal{R}^i_d(X,\Bbbk) := \mathcal{R}^i_d(H^*(X,\Bbbk))$. Similarly, if G is a group admitting a classifying space of finite type, $\mathcal{R}^i_d(G,\Bbbk) := \mathcal{R}^i_d(H^*(G,\Bbbk))$.

3.2. *Aomoto Betti numbers for* $\Bbbk\langle L\rangle$

We now specialize to the case when $X = T_L$ is the toric complex associated to a finite simplicial complex L, and $\Bbbk\langle L\rangle = H^*(T_L,\Bbbk)$ is the corresponding exterior face ring. Since $H_1(T_L,\mathbb{Z})$ is torsion-free, the resonance varieties $\mathcal{R}^i_d(T_L,\Bbbk) = \mathcal{R}^i_d(\Bbbk\langle L\rangle)$ are defined over any field $\Bbbk$.

From the definitions, $\Bbbk\langle L\rangle^1 = V^*$, where V is the $\Bbbk$-vector space with basis indexed by the vertex set V of L, and V^* its dual. Given an element $z \in V^*$, write $z = \sum_{v\in\mathsf{V}} z_v v^*$, with $z_v \in \Bbbk$, and define the *support* of z as

$$\operatorname{supp}(z) = \{v \in \mathsf{V} \mid z_v \neq 0\}. \tag{18}$$

Conversely, given a subset $\mathsf{W} \subset \mathsf{V}$ of the vertex set, we may define a “canonical” element $z_{\mathsf{W}} \in V^*$ by

$$z_{\mathsf{W}} = \sum_{v \in \mathsf{W}} v^*, \tag{19}$$

with the convention that $z_{\emptyset} = 0$. Obviously, $\operatorname{supp}(z_{\mathsf{W}}) = \mathsf{W}$. For simplicity, we will write

$$\beta_i\big(\Bbbk\langle L\rangle, \mathsf{W}\big) := \beta_i\big(\Bbbk\langle L\rangle, z_{\mathsf{W}}\big). \tag{20}$$

Remark 3.3. Unlike the Betti numbers $b_i(T_L) = \dim_{\Bbbk} H_i(T_L, \Bbbk)$, the Aomoto Betti numbers $\beta_i(\Bbbk\langle L\rangle, \mathsf{W})$ do depend on $\Bbbk$ (in fact, just on $p = \operatorname{char} \Bbbk$), and not only on L (and W); see Proposition 3.6 below. Note also that $\beta_0(\Bbbk\langle L\rangle, \mathsf{W}) = 0$, whenever $\mathsf{W} \neq \emptyset$, and $\beta_i(\Bbbk\langle L\rangle, \emptyset) = b_i(T_L)$.

The following result is due to Aramova, Avramov, and Herzog (Proposition 4.3 in [1]). For the sake of completeness, we reproduce the proof.

Lemma 3.4. *(See [1].) Let z, $z' \in \Bbbk\langle L\rangle^1$. If* $\operatorname{supp}(z) = \operatorname{supp}(z')$*, then* $\beta_i(\Bbbk\langle L\rangle, z) = \beta_i(\Bbbk\langle L\rangle, z')$*, for all $i \geqslant 1$.*

Proof. Write $z = \sum_{v \in \mathsf{W}} z_v v^*$ and $z' = \sum_{v \in \mathsf{W}} z'_v v^*$, where W is the common support. The linear map $\phi : V^* \to V^*$ given on basis elements by

$$\phi(v^*) = \begin{cases} \frac{z'_v}{z_v} v^* & \text{for } v \in \mathsf{W}, \\ v^* & \text{otherwise}, \end{cases} \tag{21}$$

extends to an algebra isomorphism $\phi : \Bbbk\langle L\rangle \to \Bbbk\langle L\rangle$, taking z to z'. The conclusion follows. $\square$

Lemma 3.5. *If* $\mathsf{W}' \subset \mathsf{W}$*, then* $\beta_i(\Bbbk\langle L\rangle, \mathsf{W}') \geqslant \beta_i(\Bbbk\langle L\rangle, \mathsf{W})$*, for all* $i \geqslant 1$.

Proof. For each $t \in \Bbbk$, define

$$z_t = \sum_{v \in \mathsf{W}'} v^* + t \sum_{v \in \mathsf{W} \setminus \mathsf{W}'} v^*.$$

Note that $z_0 = z_{\mathsf{W}'}$ and $z_1 = z_{\mathsf{W}}$. By lower semicontinuity of Betti numbers,

$$\beta_i\big(\Bbbk\langle L\rangle, z_t\big) \leqslant \beta_i\big(\Bbbk\langle L\rangle, z_0\big),$$

for t in a Zariski open subset of $\Bbbk$, containing 0. On the other hand, for $t \neq 0$, $\beta_i(\Bbbk\langle L\rangle, z_t) = \beta_i(\Bbbk\langle L\rangle, z_1)$, by Lemma 3.4. This finishes the proof. $\square$

3.3. The Aramova–Avramov–Herzog formula

The Aomoto Betti numbers of the exterior face ring $\Bbbk\langle L\rangle$ can be computed in purely combinatorial terms, by using the following Hochster-type formula from [1, Proposition 4.3], suitably interpreted and corrected.

Proposition 3.6. *(See [1].) Let L be a finite simplicial complex on vertex set V, and let $\mathsf{W} \subset \mathsf{V}$. Then:*

$$\beta_i\big(\Bbbk\langle L\rangle, \mathsf{W}\big) = \sum_{\sigma \in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}\big(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \Bbbk\big).$$

Here $L_{\mathsf{W}} = \{\tau \in L \mid \tau \subset \mathsf{W}\}$ is the simplicial complex obtained by restricting L to W, and $\mathrm{lk}_{L_{\mathsf{W}}}(\sigma) = \{\tau \in L_{\mathsf{W}} \mid \tau \cup \sigma \in L\}$ is the link of a simplex σ in L_{W}. The range of summation in the above formula includes the empty simplex, with the convention that $|\emptyset| = 0$ and $\widetilde{H}_{-1}(\emptyset, \Bbbk) = \Bbbk$. In particular,

$$\begin{aligned}\beta_1\big(\Bbbk\langle L\rangle, \mathsf{W}\big) &= \dim_{\Bbbk} \widetilde{H}_0\big(\mathrm{lk}_{L_{\mathsf{W}}}(\emptyset), \Bbbk\big) + \sum_{v \in \mathsf{V}\setminus\mathsf{W}} \dim_{\Bbbk} \widetilde{H}_{-1}\big(\mathrm{lk}_{L_{\mathsf{W}}}(v), \Bbbk\big) \\ &= \tilde{b}_0(L_{\mathsf{W}}) + \Big|\big\{v \in \mathsf{V}\setminus\mathsf{W} \;\big|\; \mathrm{lk}_{L_{\mathsf{W}}}(v) = \emptyset\big\}\Big|. \end{aligned} \tag{22}$$

From the proposition, we obtain the following immediate corollary.

Corollary 3.7. *Let L be a finite simplicial complex on vertex set V. For a subset $\mathsf{W} \subset \mathsf{V}$, and an integer $r > 0$, the following are equivalent:*

(i) $\beta_i(\Bbbk\langle L\rangle, \mathsf{W}) = 0$, *for* $1 \leqslant i \leqslant r$.
(ii) $\widetilde{H}_i(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \Bbbk) = 0$, *for all* $\sigma \in L_{\mathsf{V}\setminus\mathsf{W}}$ *and* $-1 \leqslant i \leqslant r - 1 - |\sigma|$.

3.4. Resonance of toric complexes

Denote the $\Bbbk$-vector space $V^* = \Bbbk\langle L\rangle^1$ by $\Bbbk^{\mathsf{V}}$. For a subset $\mathsf{W} \subset \mathsf{V}$, denote by $\Bbbk^{\mathsf{W}}$ the corresponding coordinate subspace.

Theorem 3.8. *Let L be a finite simplicial complex on vertex set V. Then, the resonance varieties of the toric complex T_L (over a field $\Bbbk$) are given by:*

$$\mathcal{R}^i_d(T_L, \Bbbk) = \bigcup_{\substack{\mathsf{W} \subset \mathsf{V} \\ \beta_i(\Bbbk\langle L\rangle, \mathsf{W}) \geqslant d}} \Bbbk^{\mathsf{W}}.$$

Proof. Suppose $z \in \mathcal{R}^i_d(T_L, \Bbbk)$. By definition, this means $\beta_i(\Bbbk\langle L\rangle, z) \geqslant d$. Set $\mathsf{W} = \mathrm{supp}(z)$; then z belongs to the subspace $\Bbbk^{\mathsf{W}}$. By Lemma 3.4, $\beta_i(\Bbbk\langle L\rangle, \mathsf{W}) \geqslant d$.

Now suppose $z \in \Bbbk^{\mathsf{W}}$, for some subset $\mathsf{W} \subset \mathsf{V}$ for which $\beta_i(\Bbbk\langle L\rangle, \mathsf{W}) \geqslant d$. Write $\mathsf{W}' = \operatorname{supp}(z)$; clearly, $\mathsf{W}' \subset \mathsf{W}$. By Lemmas 3.4 and 3.5,

$$\beta_i\big(\Bbbk\langle L\rangle, z\big) = \beta_i\big(\Bbbk\langle L\rangle, \mathsf{W}'\big) \geqslant \beta_i\big(\Bbbk\langle L\rangle, \mathsf{W}\big) \geqslant d,$$

and so $z \in \mathcal{R}^i_d(T_L, \Bbbk)$. □

As a corollary, we recover the description from [27, Theorem 5.5] of the first resonance variety of a right-angled Artin group.

Corollary 3.9. *(See [27].) Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a finite graph. Then $\mathcal{R}^1_1(G_\Gamma, \Bbbk) = \bigcup_{\mathsf{W}} \Bbbk^{\mathsf{W}}$, where the union is over all subsets $\mathsf{W} \subset \mathsf{V}$ such that the induced subgraph Γ_{W} is disconnected.*

Proof. By definition, the resonance varieties of G_Γ are those of $\Bbbk\langle L\rangle$, where $L = \Delta_\Gamma$. By (22), we have $\beta_1(\Bbbk\langle L\rangle, \mathsf{W}) = 0$ if and only if L_{W} is connected and dominating, i.e., for all $v \in \mathsf{V} \setminus \mathsf{W}$, there is a $w \in \mathsf{W}$ such that $\{v, w\} \in \mathsf{E}$. The conclusion easily follows. □

3.5. Non-propagation of resonance

One may wonder whether, for a graded algebra A as in Section 3.1, resonance "propagates," i.e., whether $z \in \mathcal{R}^i_1(A)$ implies $z \in \mathcal{R}^k_1(A)$, for all $k \geqslant i$ such that $A^j \neq 0$, for $i \leqslant j \leqslant k$. Such a phenomenon is believed to hold when A is the Orlik–Solomon algebra of a complex hyperplane arrangement. The following example shows that resonance in degree 1 does not propagate in higher degrees, even for the exterior Stanley–Reisner rings of flag complexes.

Example 3.10. Let $\Gamma = \Gamma_1 \sqcup \Gamma_2$, where $\Gamma_j = K_{n_j}$ are complete graphs on $n_j \geqslant 2$ vertices ($j = 1, 2$), and consider the toric complex T_{Δ_Γ}. The simplest example (with $n_1 = n_2 = 2$) is the graph $\Gamma = \Delta_\Gamma$ depicted below:

Using Theorem 3.8, it is readily seen that

$$\mathcal{R}^i_1(T_{\Delta_\Gamma}, \Bbbk) = \begin{cases} \Bbbk^{n_1+n_2}, & \text{if } i = 1, \\ \Bbbk^{n_1} \times \{0\} \cup \{0\} \times \Bbbk^{n_2}, & \text{if } 1 < i \leqslant \min(n_1, n_2). \end{cases}$$

3.6. Characteristic varieties

Let X be a connected CW-complex with finitely many cells in each dimension. Let $G = \pi_1(X)$ be the fundamental group, and $\operatorname{Hom}(G, \Bbbk^\times)$ its group of $\Bbbk$-valued characters. The *characteristic varieties* of X (over $\Bbbk$) are the jumping loci for homology with coefficients in rank 1 local systems:

$$\mathcal{V}^i_d(X, \Bbbk) = \big\{\rho \in \operatorname{Hom}\big(G, \Bbbk^\times\big) \bigm| \dim_{\Bbbk} H_i(X, \Bbbk_\rho) \geqslant d\big\}. \tag{23}$$

Here $\Bbbk_\rho$ is the 1-dimensional $\Bbbk$-vector space, viewed as a module over the group ring $\Bbbk G$ via $g \cdot a = \rho(g)a$, for $g \in G$ and $a \in \Bbbk$. By definition, $H_*(X, \Bbbk_\rho)$ is the homology of the chain complex $(\Bbbk_\rho \otimes_{\Bbbk G} C_\bullet(\widetilde{X}, \Bbbk), \partial^\rho)$, where $\widetilde{X}$ is the universal cover, $(C_\bullet(\widetilde{X}, \Bbbk), \tilde{\partial})$ is the (G-equivariant) cellular chain complex, and $\partial^\rho = \mathrm{id} \otimes_{\Bbbk G} \tilde{\partial}$.

Now let $X = T_L$ be the toric complex corresponding to a finite simplicial complex L on vertex set V. The character variety $\mathrm{Hom}(G_L, \Bbbk^\times)$ may be identified with the algebraic torus $(\Bbbk^\times)^{\mathsf{V}}$. For a subset $\mathsf{W} \subset \mathsf{V}$, denote by $(\Bbbk^\times)^{\mathsf{W}}$ the corresponding coordinate subtorus.

Lemma 3.11. *Let $\rho : G_L \to \Bbbk^\times$ be a character. Then*

$$\dim_\Bbbk H_i(T_L, \Bbbk_\rho) = \beta_i\big(\Bbbk\langle L\rangle, z_\rho\big),$$

where $z_\rho = \sum_{v \in \mathsf{V}} (\rho(v) - 1) v^*$.

Proof. The equivariant chain complex $(C_\bullet(\widetilde{T}_L, \Bbbk), \tilde{\partial}_\bullet)$ has boundary maps given by (14). Note that $\partial_k^\rho(1 \otimes c_\sigma) = \sum_{r=1}^k (-1)^{r-1} (\rho(v_r) - 1) \otimes c_{\sigma \setminus \{v_r\}}$. It is readily checked that the chain complex $(\Bbbk_\rho \otimes_{\Bbbk G} C_\bullet(\widetilde{X}, \Bbbk), \partial^\rho)$ is dual to the cochain complex $(\Bbbk\langle L\rangle, z_\rho)$. $\square$

Using Theorem 3.8 and Lemma 3.11, an argument similar to that in [12, Proposition 10.5] yields the following description of the characteristic varieties of a toric complex.

Theorem 3.12. *Let L be a finite simplicial complex on vertex set* V. *Then,*

$$\mathcal{V}_d^i(T_L, \Bbbk) = \bigcup_{\substack{\mathsf{W} \subset \mathsf{V} \\ \beta_i(\Bbbk\langle L\rangle, \mathsf{W}) \geqslant d}} \big(\Bbbk^\times\big)^{\mathsf{W}}.$$

As a corollary, we recover the description from [12, Proposition 10.5] of the first characteristic variety of a right-angled Artin group.

Corollary 3.13. *Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a finite graph. Then $\mathcal{V}_1^1(G_\Gamma, \Bbbk) = \bigcup_{\mathsf{W}} (\Bbbk^\times)^{\mathsf{W}}$, where the union is over all subsets $\mathsf{W} \subset \mathsf{V}$ such that the induced subgraph Γ_{W} is disconnected.*

4. Homology of infinite cyclic covers

Let $X^\nu \to X$ be a Galois $\mathbb{Z}$-cover. In this section, we study the homology groups $H_*(X^\nu, \Bbbk)$, viewed as modules over $\Bbbk\mathbb{Z}$, and the cohomology ring $H^*(X^\nu, \Bbbk)$. We start with some of the relevant algebraic background.

4.1. *Finitely generated modules over $\Bbbk\mathbb{Z}$*

Let $\Bbbk$ be a field. We will identify the group algebra $\Bbbk\mathbb{Z}$ with the ring of Laurent polynomials, $\Lambda = \Bbbk[t^{\pm 1}]$. The irreducible polynomials in $\Bbbk[t^{\pm 1}]$ coincide (up to units) with the irreducible polynomials, different from t, in the subring $\Bbbk[t]$. Since Λ is a principal ideal domain, every finitely generated Λ-module M decomposes as a finite direct sum

$$M = F(M) \oplus \bigoplus_{\substack{t \neq f \in \Bbbk[t] \\ f \text{ irreducible}}} T_f(M), \tag{24}$$

where $F(M) = \Lambda^{\operatorname{rank} M}$ denotes the free part, and $T_f(M) = \bigoplus_{j \geqslant 1} (\Lambda/f^j\Lambda)^{e_{j,f}(M)}$ denotes the f-primary part.

Particularly simple is the case when $f = t - 1$ and $j = 1$. Note that $\Lambda/(t-1)\Lambda = \Bbbk$, with module structure given by $t \cdot 1 = 1$. We say a finitely generated Λ-module M is *trivial* (or, has trivial $\mathbb{Z}$-action), if M decomposes as a direct sum of such modules, i.e., $M = (\Lambda/(t-1)\Lambda)^e$; equivalently, $t \cdot m = m$, for all $m \in M$.

Suppose now $\Bbbk$ is algebraically closed. Then all irreducible polynomials in Λ are of the form $t - a$, for some $a \in \Bbbk^\times$. For simplicity, write

$$T_a(M) := T_{t-a}(M) = \bigoplus_{j \geqslant 1} \bigl(\Lambda/(t-a)^j\Lambda\bigr)^{e_{j,a}(M)}. \tag{25}$$

The homomorphism $\mathrm{ev}_a : \mathbb{Z} \to \Bbbk^\times$, $\mathrm{ev}_a(n) = a^n$ defines a Λ-module $\Bbbk_a$, via $g \cdot x := g(a)x$, for $g \in \Lambda$ and $x \in \Bbbk$. Clearly, $\Bbbk_a = T_a(\Bbbk_a) = \Lambda/(t-a)\Lambda$.

4.2. The (f)-adic completion of $\Bbbk\mathbb{Z}$

Returning to the general case, fix an irreducible polynomial $f \in \Bbbk[t]$, $f \neq t$, and let $\widehat{\Lambda}$ be the completion of $\Lambda = \Bbbk[t^{\pm 1}]$ with respect to the filtration $(f^k)_{k \geqslant 0}$. It is easy to see that the (f)-adic filtration on Λ is separated. Hence, the canonical map $\iota : \Lambda \to \widehat{\Lambda}$ is injective.

Denote by $\widehat{(f^k)}_{k \geqslant 0}$ the corresponding filtration on $\widehat{\Lambda}$. By construction, $\iota : \Lambda \to \widehat{\Lambda}$ preserves filtrations, and induces an isomorphism $\mathrm{gr}(\iota) : \mathrm{gr}(\Lambda) \to \mathrm{gr}(\widehat{\Lambda})$ between the associated graded rings. Note that $\mathrm{gr}^0(\Lambda)$ is simply the residue field $\Bbbk_f := \Bbbk[t]/(f)$.

Remark 4.1. Let $u \in \Bbbk[t]$ be a polynomial, $\iota(u)$ its image in $\widehat{\Lambda}$, and $\bar{u}$ its image in $\Bbbk_f$. Then $\iota(u)$ is a unit in $\widehat{\Lambda} \Leftrightarrow \bar{u} \neq 0$ in $\Bbbk_f \Leftrightarrow f \nmid u$.

Denote by λ_f the homothety associated to f.

Lemma 4.2. *There is an isomorphism of graded $\Bbbk_f$-algebras,* $\mathrm{gr}(\widehat{\Lambda}) \cong \Bbbk_f[t]$.

Proof. We may replace $\mathrm{gr}(\widehat{\Lambda})$ by $\mathrm{gr}(\Lambda)$. It is straightforward to check that λ_f induces $\Bbbk_f$-linear isomorphisms, $\lambda_f : f^k\Lambda/f^{k+1}\Lambda \xrightarrow{\simeq} f^{k+1}\Lambda/f^{k+2}\Lambda$, for all $k \geqslant 0$. □

Corollary 4.3. *The map $\lambda_f : \widehat{\Lambda} \to \widehat{\Lambda}$ is injective (that is, f is not a zero-divisor in $\widehat{\Lambda}$).*

Proof. Due to the above lemma, $\widehat{\Lambda}$ is an integral domain; see [33, Proposition 7 on p. II-8]. Our claim follows then from the injectivity of ι. □

4.3. $\mathbb{Z}$-covers

Now let X be a connected CW-complex, and let $\nu : \pi_1(X) \to \mathbb{Z}$ be a non-trivial homomorphism. Since $\mathbb{Z}$ is abelian, ν factors through $H_1(X, \mathbb{Z})$, thus defining a 1-dimensional cohomology class, $\nu_{\mathbb{Z}} \in H^1(X, \mathbb{Z})$. Let $\omega \in H^1(S^1, \mathbb{Z}) = \mathbb{Z}$ be the standard generator. By obstruction theory, there is a map $h : X \to S^1$ such that $\nu_{\mathbb{Z}} = h^*(\omega)$. It is readily seen that the induced homomorphism, $h_\sharp : \pi_1(X) \to \pi_1(S^1)$, coincides with ν.

Denote by X^ν the Galois (connected) cover of X corresponding to $\ker(\nu)$. Without loss of generality, we may assume ν is surjective. Indeed, if the image of ν has index m in $\mathbb{Z}$, then ν is the composite $\pi_1(X) \xrightarrow{\nu'} \mathbb{Z} \xrightarrow{\times m} \mathbb{Z}$, where ν' is onto. Clearly, $\ker(\nu) = \ker(\nu')$, and so the corresponding covers are equivalent.

We may obtain the infinite cyclic Galois cover, $\pi : X^\nu \to X$, with $\operatorname{im}(\pi_\sharp) = \ker(\nu)$, by pulling back the universal cover $\exp : \mathbb{R} \to S^1$ along h. Note that X^ν is the homotopy fiber of h. The spaces and maps defined so far fit into the following diagram:

$$\begin{tikzcd} & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ X^\nu \arrow[r, equal] \arrow[d, equal] & X^\nu \arrow[r] \arrow[d, "\pi"] & \mathbb{R} \arrow[d, "\exp"] \\ X^\nu \arrow[r, "\pi"] & X \arrow[r, "h"] & S^1 \end{tikzcd} \tag{26}$$

4.4. Cohomology of $\mathbb{Z}$-covers

Let $\Bbbk$ be a coefficient field. The cell structure on X lifts to an equivariant cell structure on X^ν, so that the group $\mathbb{Z}$ acts by deck transformations on X^ν, permuting the cells. In this fashion, each homology group $H_i(X^\nu, \Bbbk)$ acquires the structure of a module over the group ring $\Bbbk\mathbb{Z}$. Since ν is surjective, Shapiro's lemma yields an isomorphism of $\Bbbk\mathbb{Z}$-modules,

$$H_i\big(X^\nu, \Bbbk\big) \cong H_i(X, \Bbbk\mathbb{Z}_\nu), \tag{27}$$

where $\Bbbk\mathbb{Z}_\nu$ denotes the ring $\Bbbk\mathbb{Z}$, viewed as a module over $\Bbbk\pi_1(X)$ via the linear extension $\tilde{\nu} : \Bbbk\pi_1(X) \to \Bbbk\mathbb{Z}$.

Let $\nu_* : H_1(X, \Bbbk) \to H_1(\mathbb{Z}, \Bbbk) = \Bbbk$ be the homomorphism induced by ν. The corresponding cohomology class, $\nu_\Bbbk \in H^1(X, \Bbbk)$, is the image of $\nu_{\mathbb{Z}}$ under the coefficient homomorphism $\mathbb{Z} \to \Bbbk$. Since $\nu_{\mathbb{Z}} \cup \nu_{\mathbb{Z}} = h^*(\omega \cup \omega) = 0$, we also have $\nu_\Bbbk \cup \nu_\Bbbk = 0$, by naturality of cup products with respect to coefficient homomorphisms. We will denote by $(\nu_\Bbbk)$ the ideal of $A = H^*(X, \Bbbk)$ generated by $\nu_\Bbbk$. Moreover, we will write $A^{\leqslant r}$ for the ring A, modulo the ideal $\bigoplus_{i>r} A^i$; of course, additively $A^{\leqslant r} = \bigoplus_{i=0}^r A^i$.

Proposition 4.4. *Let $\pi : X^\nu \to X$ be a $\mathbb{Z}$-cover as above. Then:*

(1) *The induced homomorphism in cohomology, $\pi^* : H^*(X, \Bbbk) \to H^*(X^\nu, \Bbbk)$, factors through a ring map, $\bar{\pi}^* : H^*(X, \Bbbk)/(\nu_\Bbbk) \to H^*(X^\nu, \Bbbk)$.*
(2) *Suppose $H_i(X^\nu, \Bbbk)$ has trivial $\mathbb{Z}$-action, for all $i \leqslant r$. Then $\bar{\pi}^*$ restricts to a ring isomorphism, $\bar{\pi}^* : H^{\leqslant r}(X, \Bbbk)/(\nu_\Bbbk) \xrightarrow{\simeq} H^{\leqslant r}(X^\nu, \Bbbk)$.*

Proof. Consider the cohomology spectral sequence with $\Bbbk$-coefficients associated to the homotopy fibration $X^\nu \xrightarrow{\pi} X \xrightarrow{h} S^1$. Clearly, $E_2 = E_\infty$. Note also that the $\mathbb{Z}$-action on the homology of X^ν given by (27) may be identified with the action of $\pi_1(S^1)$ on the homology of the fiber, and similarly for cohomology. The factorization in Part (1) follows from the fact that $\nu_{\Bbbk} = h^*(\omega_{\Bbbk})$. The claim in Part (2) now follows from [4]: the surjectivity of $\bar{\pi}^*$ from (13.5) on p. 39, and the injectivity of $\bar{\pi}^*$ from Theorem 14.2(b) on p. 42. □

4.5. Field of fractions

For the rest of this section, we shall assume X has finite k-skeleton, for some $k \geqslant 1$; in particular, $\pi_1(X)$ is finitely generated. We seek to compute the finitely generated $\Bbbk\mathbb{Z}$-modules $H_i(X^\nu, \Bbbk)$, for $i \leqslant k$. As a first step, we reduce the computation of the ranks of these $\Bbbk\mathbb{Z}$-modules to that of certain Betti numbers with coefficients in rank 1 local systems.

Let $\mathbb{K} = \Bbbk(t)$ be the field of fractions of the integral domain $\Bbbk\mathbb{Z} = \Bbbk[t^{\pm 1}]$. Set $G := \pi_1(X)$. Composing the natural inclusion $\Bbbk\mathbb{Z} \hookrightarrow \mathbb{K}$ with the linear extension of ν to group rings, $\Bbbk G \to \Bbbk\mathbb{Z}$, and restricting to G defines a homomorphism $\bar{\nu} : G \to \mathbb{K}^\times$. In this manner, we have a rank 1 local system, $\mathbb{K}_{\bar{\nu}}$, on X.

Lemma 4.5. *With notation as above,*

$$\operatorname{rank}_{\Bbbk\mathbb{Z}} H_i\big(X^\nu, \Bbbk\big) = \dim_{\mathbb{K}} H_i(X, \mathbb{K}_{\bar{\nu}}), \quad \text{for } i \leqslant k.$$

Proof. Note that the inclusion $\Bbbk\mathbb{Z} \hookrightarrow \mathbb{K}$ is a flat morphism; thus, $\operatorname{rank}_{\Bbbk\mathbb{Z}} H_i(X, \Bbbk\mathbb{Z}_\nu) = \dim_{\mathbb{K}} H_i(X, \Bbbk\mathbb{Z}_\nu \otimes_{\Bbbk\mathbb{Z}} \mathbb{K})$. Since $\Bbbk\mathbb{Z}_\nu \otimes_{\Bbbk\mathbb{Z}} \mathbb{K} = \mathbb{K}_{\bar{\nu}}$, we are done. □

4.6. Completion

In each degree $0 \leqslant i \leqslant k$, the finitely generated Λ-module $H_i(X^\nu, \Bbbk)$ decomposes as

$$H_i\big(X^\nu, \Bbbk\big) = \Lambda^{r_i} \oplus \bigoplus_{\substack{t \neq f \in \Bbbk[t] \\ f \text{ irreducible}}} \bigoplus_{j \geqslant 1} \big(\Lambda / f^j \Lambda\big)^{e^i_j(f)}, \tag{28}$$

where r_i is the Λ-rank, and $e^i_j(f)$ is the number of f-primary Jordan blocks of size j. Of course, since X^ν is path-connected, $H_0(X^\nu, \Bbbk) = \Lambda/(t-1)$, and so $r_0 = 0$, and $e^0_1(t-1) = 1$ is the only non-zero entry among the multiplicities $e^0_j(f)$.

Now fix an irreducible polynomial $f \in \Lambda$, and let $\widehat{\Lambda}$ be the completion of Λ with respect to the (f)-adic filtration. Recall we have an injective map $\iota : \Lambda \to \widehat{\Lambda}$. Since ι is a flat morphism, decomposition (28) yields, for each $i \leqslant k$,

$$\begin{aligned} H_i(X, \widehat{\Lambda}_\nu) &= H_i(X, \Lambda_\nu) \otimes_\Lambda \widehat{\Lambda} \\ &= \widehat{\Lambda}^{r_i} \oplus \bigoplus_{j \geqslant 1} \big(\widehat{\Lambda}/f^j \widehat{\Lambda}\big)^{e^i_j(f)}. \end{aligned} \tag{29}$$

Note that this gives the canonical decomposition of $H_i(X, \widehat{\Lambda}_\nu)$ over the principal ideal domain $\widehat{\Lambda}$, since f is (the unique) irreducible in $\widehat{\Lambda}$.

5. Homology of $\mathbb{Z}$-covers of toric complexes

We now specialize to the case when X is a toric complex, and compute the homology groups of an infinite cyclic cover, viewed as modules over the ring of Laurent polynomials.

5.1. $\mathbb{Z}$-covers and Artin kernels

As before, let L be a finite simplicial complex, on vertex set V, and let T_L be the corresponding toric complex. Recall that the fundamental group $G_L = \pi_1(T_L)$ depends only on the graph $\Gamma = L^{(1)}$, and that a classifying space for G_L is the toric complex T_Δ, where $\Delta = \Delta_\Gamma$ is the flag complex of Γ.

Consider now an epimorphism $\chi : G_L \twoheadrightarrow \mathbb{Z}$. The construction reviewed in Section 4.3 defines an infinite cyclic (regular) cover, $\pi : T_L^\chi \to T_L$. The fundamental group

$$N_\chi := \pi_1\bigl(T_L^\chi\bigr) = \ker(\chi : G_L \to \mathbb{Z}) \tag{30}$$

is called the *Artin kernel* associated to χ. Clearly, a classifying space for this group is the space T_Δ^χ.

The most basic example is provided by the "diagonal" homomorphism, $\nu : G_L \twoheadrightarrow \mathbb{Z}$, given by $\nu(v) = 1$, for all $v \in \mathsf{V}$. The corresponding Artin kernel, N_ν, is simply denoted by N_Γ, and is called the *Bestvina–Brady group* associated to Γ. This group need not be finitely generated. For example, if $\overline{K}_n$ is the discrete graph on $n > 1$ vertices, then $G_{\overline{K}_n} = F_n$ and $N_{\overline{K}_n}$ is a free group of countably infinite rank. More generally, as shown by Meier and VanWyk [24] and Bestvina and Brady [2] the group N_Γ is finitely generated if and only if the graph Γ is connected. Even then, N_Γ may not admit a finite presentation. For example, if $K_{2,2} = \overline{K}_2 * \overline{K}_2$ is a 4-cycle, then $G_{K_{2,2}} = F_2 \times F_2$, and, as noted by Stallings [34], $N_{K_{2,2}}$ is not finitely presentable. In fact, as shown in [2], N_Γ is finitely presented if and only if Δ_Γ is simply-connected.

More generally, we have the following characterization from Meier, Meinert and VanWyk [23] and Bux and Gonzalez [5]. Assume L is a flag complex. Let $\mathsf{W} = \{v \in \mathsf{V} \mid \chi(v) \neq 0\}$ be the support of χ. Then:

(a) N_χ is finitely generated if and only if L_W is connected, and dominant, i.e., for all $v \in \mathsf{V} \setminus \mathsf{W}$, there is a $w \in \mathsf{W}$ such that $\{v, w\} \in L$.
(b) N_χ is finitely presented if and only if L_W is 1-connected and, for every simplex σ in $L_{\mathsf{V}\setminus\mathsf{W}}$, the space $\mathrm{lk}_{L_\mathsf{W}}(\sigma) := \{\tau \in L_\mathsf{W} \mid \tau \cup \sigma \in L\}$ is $(1 - |\sigma|)$-acyclic.

5.2. $\Bbbk\mathbb{Z}$-ranks for $\mathbb{Z}$-covers of toric complexes

Let L be an arbitrary finite simplicial complex. Fix a cover $T_L^\chi \to T_L$, defined by a homomorphism $\chi : G_L \twoheadrightarrow \mathbb{Z}$. Our goal in this section is to compute the homology groups $H_i(T_L^\chi, \Bbbk)$, viewed as modules over the group ring $\Bbbk\mathbb{Z}$, for a fixed coefficient field $\Bbbk$. In view of formula (28), we need to compute the $\Bbbk\mathbb{Z}$-ranks, r_i, and the multiplicities of the Jordan blocks, $e_j^i(f)$, for all integers $i \geqslant 0$, $j \geqslant 1$, and all irreducible polynomials $f \in \Bbbk[t]$, $f \neq t$. We begin with a combinatorial formula for the $\Bbbk\mathbb{Z}$-ranks.

Define $\mathsf{V}_\Bbbk(\chi) = \mathrm{supp}(\chi_\Bbbk)$, where $\chi_\Bbbk \in H^1(T_L, \Bbbk) = \Bbbk\langle L\rangle^1$ is the corresponding cohomology class. Clearly, the subset $\mathsf{V}_\Bbbk(\chi) \subset \mathsf{V}$ depends only on χ and $p = \mathrm{char}\,\Bbbk$, so we simply write

$$\mathsf{V}_p(\chi) := \mathsf{V}_{\Bbbk}(\chi) = \operatorname{supp}(\chi_{\Bbbk}). \tag{31}$$

Theorem 5.1. *For all $i \geqslant 0$,*

$$\operatorname{rank}_{\Bbbk\mathbb{Z}} H_i\big(T_L^{\chi}, \Bbbk\big) = \beta_i\big(\Bbbk\langle L\rangle, \mathsf{V}_0(\chi)\big).$$

Proof. Set $\mathbb{K} = \Bbbk(t)$, and let $\bar{\chi} : G_L \to \mathbb{K}^{\times}$ be the 1-dimensional representation defined by χ. The cohomology class $z_{\bar{\chi}} = \sum_{v\in\mathsf{V}}(\bar{\chi}(v) - 1)v^* \in \mathbb{K}\langle L\rangle^1$ has support

$$\operatorname{supp}(z_{\bar{\chi}}) = \big\{v \in \mathsf{V} \;\big|\; \bar{\chi}(v) \neq 1\big\} = \big\{v \in \mathsf{V} \;\big|\; \chi(v) \neq 0\big\} = \mathsf{V}_0(\chi). \tag{32}$$

We have the chain of equalities

$$\begin{aligned}
\operatorname{rank}_{\Bbbk\mathbb{Z}} H_i\big(T_L^{\chi}, \Bbbk\big) &= \dim_{\mathbb{K}} H_i(T_L, \mathbb{K}_{\bar{\chi}}) && \text{by Lemma 4.5}\\
&= \beta_i\big(\mathbb{K}\langle L\rangle, z_{\bar{\chi}}\big) && \text{by Lemma 3.11}\\
&= \beta_i\big(\mathbb{K}\langle L\rangle, \operatorname{supp}(z_{\bar{\chi}})\big) && \text{by Lemma 3.4}\\
&= \beta_i\big(\mathbb{K}\langle L\rangle, \mathsf{V}_0(\chi)\big) && \text{by (32)}\\
&= \beta_i\big(\Bbbk\langle L\rangle, \mathsf{V}_0(\chi)\big) && \text{since } \operatorname{char}\mathbb{K} = \operatorname{char}\Bbbk.
\end{aligned}$$

This ends the proof. □

Here is an immediate consequence of Theorem 5.1 (together with Corollary 3.7 and Remark 3.2).

Corollary 5.2. *For each $r \geqslant 1$, the following are equivalent:*

(1) $\dim_{\Bbbk} H_i(T_L^{\chi}, \Bbbk) < \infty$*, for all $i \leqslant r$.*
(2) $\beta_i(\Bbbk\langle L\rangle, \mathsf{V}_0(\chi)) = 0$*, for all $i \leqslant r$.*
(3) $\widetilde{H}_i(\operatorname{lk}_{L_{\mathsf{V}_0(\chi)}}(\sigma), \Bbbk) = 0$*, for all $\sigma \in L_{\mathsf{V}\setminus\mathsf{V}_0(\chi)}$ and $-1 \leqslant i \leqslant r - 1 - |\sigma|$.*

If $\operatorname{char}\Bbbk = 0$*, the above conditions are also equivalent to*

(4) $\chi_{\Bbbk} \notin \mathcal{R}_1^i(T_L, \Bbbk)$*, for all $i \leqslant r$.*

5.3. A chain isomorphism

Let R be an arbitrary commutative ring, and suppose we are given a vector $\gamma = (\gamma_v)_{v\in\mathsf{V}} \in R^{\mathsf{V}}$. Let $C_\bullet(T_L, R_\gamma)$ be the R-chain complex defined as follows. Set $C_k(T_L, R_\gamma) = R \otimes C_k$, where $C_k = C_k(T_L, \mathbb{Z})$ are the usual cellular k-chains on T_L. The boundary maps $\partial^{\gamma} : R \otimes C_k \to R \otimes C_{k-1}$ are given by

$$\partial^{\gamma}(1 \otimes c_\sigma) = \sum_{r=1}^{k}(-1)^{r-1}\gamma_{v_r} \otimes c_{\sigma\setminus\{v_r\}}, \tag{33}$$

where $\sigma = \{v_1, \dots, v_k\}$ is a $(k-1)$-simplex in L.

Given two vectors $\gamma = (\gamma_v)_{v\in \mathsf{V}}$ and $\xi = (\xi_v)_{v\in \mathsf{V}}$ in R^{V}, let us write $\gamma \doteq \xi$ if there exist units $u_v \in R^\times$ such that $\gamma_v = u_v \xi_v$, for all $v \in \mathsf{V}$.

Lemma 5.3. *If $\gamma \doteq \xi$, then $H_*(T_L, R_\gamma) \cong H_*(T_L, R_\xi)$, as R-modules.*

Proof. Define a chain map $C_\bullet(T_L, R_\gamma) \to C_\bullet(T_L, R_\xi)$ by $1 \otimes c_\sigma \mapsto u_{v_1} \cdots u_{v_k} \otimes c_\sigma$, for $c_\sigma \in C_k$. Clearly, this is a chain isomorphism. □

The computation of the $\Bbbk\mathbb{Z}$-torsion part of $H_i(T_L^\chi, \Bbbk)$, encoded by the multiplicities $e^i_j(f)$, is in two steps: the first, arithmetic, and the second, combinatorial.

5.4. The arithmetic step

Fix an irreducible polynomial $f \in \Bbbk[t]$, different from t. Set $m_v = \chi(v) \in \mathbb{Z}$, for each $v \in \mathsf{V}$. Define the vector $b = (b_v)_{v\in\mathsf{V}}$, with components

$$b_v = b_v(\chi, f) = \begin{cases} \operatorname{ord}_f(t^{m_v} - 1), & \text{if } m_v \neq 0, \\ -\infty, & \text{if } m_v = 0, \end{cases} \tag{34}$$

where $\operatorname{ord}_f(g) = \max\{k \geqslant 0\colon\ f^k | g\}$, for $0 \neq g \in \Lambda$.

Example 5.4. To illustrate the computation, suppose $\Bbbk$ is an algebraically closed field, of characteristic p. Then any irreducible polynomial in Λ is of the form $t - a$, for some $a \in \Bbbk^\times$. Assume $m_v \neq 0$. If either $\operatorname{ord}(a) = \infty$, or $\operatorname{ord}(a) < \infty$ and $\operatorname{ord}(a) \nmid m_v$, then $b_v = 0$; otherwise,

$$b_v = \begin{cases} 1, & \text{if } p = 0, \\ p^s, & \text{if } m_v = p^s q, \text{ with } (q, p) = 1. \end{cases} \tag{35}$$

Returning now to the general case, denote by $\widehat{\Lambda}$ the (f)-adic completion of Λ. Define a new vector, $\gamma = \gamma(\chi, f) \in \widehat{\Lambda}^{\mathsf{V}}$, by setting $\gamma_v = f^{b_v}$, with the convention that $f^{-\infty} = 0$. The associated differential, $\partial^\gamma : \widehat{\Lambda} \otimes C_k \to \widehat{\Lambda} \otimes C_{k-1}$, is given by

$$\partial^\gamma(1 \otimes c_\sigma) = \sum_{r=1}^{k} (-1)^{r-1} f^{b_{v_r}} \otimes c_{\sigma \setminus \{v_r\}}. \tag{36}$$

Using formula (14) and Remark 4.1, we see that ∂^γ equals $\mathrm{id}_{\widehat{\Lambda}_\chi} \otimes_{\Bbbk G_L} \widetilde{\partial}$, modulo units in $\widehat{\Lambda}$. From Lemma 5.3, we derive the following corollary.

Corollary 5.5. *$H_*(T_L, \widehat{\Lambda}_\chi) = H_*(T_L, \widehat{\Lambda}_\gamma)$, as $\widehat{\Lambda}$-modules.*

5.5. Independence of f

The output of the preceding step is the vector $b = (b_v)_{v\in\mathsf{V}}$, constructed in (34) via the factorization properties of the ring Λ. We will use this vector as input in the second, combinatorial, step of our $\Bbbk\mathbb{Z}$-torsion computation, in the following way.

Let $S=\Bbbk[[t]]$ be the power-series ring in variable t. Define a vector $\xi=(\xi_v)_{v\in\mathsf{V}}\in S^{\mathsf{V}}$ by $\xi_v=t^{b_v}$, where $t^{-\infty}:=0$. Since S is a PID with a unique prime element, t, the S-module $H_i(T_L,S_\xi)$ decomposes as

$$H_i(T_L,S_\xi)=S^{\rho_i}\oplus\bigoplus_{j\geqslant 1}\bigl(S/t^jS\bigr)^{\varepsilon^i_j},\tag{37}$$

for all $i\geqslant 0$. The next result shows, in particular, that the multiplicities $e^i_j(f)$ do not depend on f.

Proposition 5.6. *With notation as above, $r_i=\rho_i$ and $e^i_j(f)=\varepsilon^i_j$, for all $i\geqslant 0$ and $j\geqslant 1$.*

Proof. Since S is the (t)-adic completion of $\Bbbk[t]$, we have a canonical ring map $S\to\widehat{\Lambda}$, $t\mapsto f$. It follows from (33) that

$$C_\bullet(T_L,\widehat{\Lambda}_\gamma)=\widehat{\Lambda}\otimes_S C_\bullet(T_L,S_\xi).\tag{38}$$

Using the Universal Coefficients Theorem (over S) and decomposition (37), we find $\widehat{\Lambda}$-isomorphisms

$$\begin{aligned}H_i(T_L,\widehat{\Lambda}_\gamma)&=\bigl(\widehat{\Lambda}\otimes_S H_i(T_L,S_\xi)\bigr)\oplus\operatorname{Tor}^S_1\bigl(\widehat{\Lambda},H_{i-1}(T_L,S_\xi)\bigr)\\&=\widehat{\Lambda}^{\rho_i}\oplus\Bigl(\bigoplus_{j\geqslant 1}\bigl(\widehat{\Lambda}/f^j\widehat{\Lambda}\bigr)^{\varepsilon^i_j}\Bigr)\oplus\Bigl(\bigoplus_{j\geqslant 1}\ker\bigl(\lambda_{f^j}:\widehat{\Lambda}\to\widehat{\Lambda}\bigr)^{\varepsilon^{i-1}_j}\Bigr).\end{aligned}\tag{39}$$

By Corollary 4.3, the third summand above vanishes. Using Corollary 5.5, our claim follows by comparing the $\widehat{\Lambda}$-decompositions (39) and (29). □

5.6. Realizability

Before proceeding to the combinatorial step of our algorithm, we discuss the possible torsion that can occur in the $\Bbbk\mathbb{Z}$-module $H_*(T_L,\Bbbk\mathbb{Z}_\chi)$.

Proposition 5.7. *Let L be a simplicial complex, and let f be a polynomial in $\Bbbk[t]$, irreducible and different from t. Suppose that, for every vertex $v\in\mathsf{V}$ such that $m_v\neq 0$, the polynomial f does not divide $t^{m_v}-1$. Then $H_*(T_L,\Bbbk\mathbb{Z}_\chi)$ has trivial f-primary part.*

Proof. Let $(b_v)_{v\in\mathsf{V}}$ be the vector defined by (34). Our assumption implies that $b_v=0$, if $m_v\neq 0$. Define a $\Bbbk$-valued vector $(\delta_v)_{v\in\mathsf{V}}$ by $\delta_v=1$, if $m_v\neq 0$, and $\delta_v=0$, otherwise. Let γ and ξ be the vectors defined in Sections 5.4 and 5.5. Using formula (33) for the boundary map ∂^ξ, we see that $H_i(T_L,S_\xi)$ is a free S-module of rank equal to $\dim_\Bbbk H_i(T_L,\Bbbk_\delta)$, for all $i\geqslant 0$. The claim follows from Proposition 5.6. □

Conversely, we have the following.

Proposition 5.8. *Let f be an irreducible polynomial in $\Bbbk[t]$ dividing $t^m - 1$, for some $m \geqslant 1$. Then, for any $i > 0$, there exists a simplicial complex L and a homomorphism $\chi : G_L \twoheadrightarrow \mathbb{Z}$ such that $H_i(T_L, \Bbbk\mathbb{Z}_\chi)$ has non-trivial f-primary part.*

Proof. Let L be the cone $v_0 * K$, where $K = \Delta_\Gamma$ is a flag triangulation of S^{i-1}. Define $\chi(v) = 1$ on the vertices of K, and $\chi(v_0) = m$. By the Künneth formula, the $\Bbbk\mathbb{Z}$-module $H_i(T_L, \Bbbk\mathbb{Z}_\chi)$ has $M = \Bbbk\mathbb{Z}/(t^m - 1) \otimes_{\Bbbk\mathbb{Z}} H_i(N_\Gamma, \Bbbk)$ as a direct summand. By [28, Proposition 7.1], the module M has $\Bbbk\mathbb{Z}/(t^m - 1)$ as a direct $\Bbbk\mathbb{Z}$-summand. Clearly, $\Bbbk\mathbb{Z}/(t^m - 1)$ has non-trivial f-primary part, and we are done. $\square$

5.7. The combinatorial step

The computation of $T_f(H_i(T_L, \Bbbk\mathbb{Z}_\chi))$ described in Section 5.5 depends only on the simplicial complex L, and on the vector $b = (b_v(\chi, f))$. More precisely, given the vector $\xi = (t^{b_v})_{v\in \mathsf{V}} \in S^{\mathsf{V}}$ as above, the torsion part of the S-decomposition (37) can be computed from the boundary map $\partial^\xi_{i+1} : S \otimes C_{i+1} \to S \otimes C_i$, given by

$$\partial^\xi(1 \otimes c_\sigma) = \sum_{r=1}^{i+1} (-1)^{r-1} t^{b_{v_r}} \otimes c_{\sigma\setminus\{v_r\}}. \tag{40}$$

Indeed, consider the split exact sequence of S-modules,

$$0 \longrightarrow \ker\bigl(\partial^\xi_i\bigr)/\operatorname{im}\bigl(\partial^\xi_{i+1}\bigr) \longrightarrow \operatorname{coker}\bigl(\partial^\xi_{i+1}\bigr) \longrightarrow \operatorname{im}\bigl(\partial^\xi_i\bigr) \longrightarrow 0. \tag{41}$$

It is readily seen that $\operatorname{Tors} H_i(T_L, S_\xi) = \operatorname{Tors}\operatorname{coker}(\partial^\xi_{i+1})$. Since S is a PID, the matrix of ∂^ξ_{i+1} can be diagonalized, using elementary row and column operations. In other words, with respect to convenient bases over S, the matrix of ∂^ξ_{i+1} can be written as $0 \oplus D_\xi$, where

$$D_\xi = \operatorname{diag}\bigl(t^{a_k}\bigr), \quad \text{with } a_k \geqslant 0. \tag{42}$$

Clearly,

$$\varepsilon^i_j = \bigl|\{k \mid a_k = j\}\bigr|. \tag{43}$$

Using Proposition 5.6, we conclude that $T_f(H_i(T_L, \Bbbk\mathbb{Z}_\chi)) = \bigoplus_k (\Lambda/f^{a_k}\Lambda)$.

The next theorem summarizes our two-step algorithm for computing the $\Bbbk\mathbb{Z}$-torsion in $H_i(T^\chi_L, \Bbbk)$, for all $i \geqslant 0$.

Theorem 5.9. *Fix an irreducible polynomial $f \in \Bbbk[t]$, $f \neq t$. To compute the f-primary part $T_f(H_i(T^\chi_L, \Bbbk)) = \bigoplus_{j\geqslant 1} (\Lambda/f^j\Lambda)^{e^i_j(f)}$, proceed as follows.*

(1) *Compute the vector $b = (b_v)_{v\in\mathsf{V}}$, with $b_v = b_v(\chi, f)$ given by* (34).
(2) *With this vector as input, define the boundary map ∂^ξ as in* (40), *and compute the multiplicities ε^i_j as in* (43), *using the diagonalized matrix D_ξ from* (42).

Finally, set $e^i_j(f) = \varepsilon^i_j$.

5.8. The Bestvina–Brady covers

Particularly simple is the case when χ is the diagonal homomorphism $\nu : G_L \twoheadrightarrow \mathbb{Z}$, taking each $v \in \mathsf{V}$ to 1. The $\Bbbk\mathbb{Z}$-module structure on the homology of the resulting Bestvina–Brady cover, T_L^ν, can be made completely explicit.

Corollary 5.10. *For each $i > 0$, there is an isomorphism of Λ-modules,*

$$H_i\big(T_L^\nu, \Bbbk\big) = \Lambda^{\dim_{\Bbbk} \widetilde{H}_{i-1}(L,\Bbbk)} \oplus \big(\Lambda/(t-1)\Lambda\big)^{\dim_{\Bbbk} B_{i-1}(L,\Bbbk)},$$

where $B_\bullet(L, \Bbbk)$ are the simplicial boundaries of L.

Proof. By Theorem 5.1 the free part of $H_i(T_L^\nu, \Bbbk)$ has rank $\beta_i(\Bbbk\langle L\rangle, \mathsf{V})$. By Proposition 3.6, this equals $\dim_{\Bbbk} \widetilde{H}_{i-1}(L, \Bbbk)$.

By Proposition 5.7, the torsion part is all $(t-1)$-primary. The arithmetic step (34) is very simple: it gives $\xi_v = t$, for all $v \in \mathsf{V}$. Therefore, the diagonal matrix D_ξ has size $\dim_{\Bbbk} B_{i-1}(L, \Bbbk)$, with all diagonal entries equal to t. The conclusion follows. □

When applied to a flag complex $L = \Delta_\Gamma$, Corollary 5.10 recovers Proposition 7.1 from [28].

6. Trivial monodromy test

In this section, we give a combinatorial test for deciding whether $H_{\leqslant r}(T_L, \Bbbk\mathbb{Z}_\chi) := \bigoplus_{i=0}^r H_i(T_L, \Bbbk\mathbb{Z}_\chi)$ has trivial $\Bbbk\mathbb{Z}$-action.

6.1. Supports and primes

First, we need to establish some notation. Let $\chi : G_L \twoheadrightarrow \mathbb{Z}$ be an epimorphism. This amounts to specifying integers $\chi(v) = m_v$ for each vertex $v \in \mathsf{V}$, with the proviso that $\gcd\{m_v \mid v \in \mathsf{V}\} = 1$. In other words, the epimorphism $\chi : G_L \twoheadrightarrow \mathbb{Z}$ is encoded by a graph $\Gamma = L^{(1)}$, equipped with a vertex-labeling function $m : \mathsf{V} \to \mathbb{Z}$, subject to the above coprimality condition. A simple example of such a vertex-labeled graph is given in diagram (53).

In this setup, the support sets from (31) can be written as

$$\mathsf{V}_q(\chi) = \big\{v \in \mathsf{V} \bigm| m_v \neq 0 \pmod q\big\}, \tag{44}$$

for $q = 0$ or q a prime. Clearly, $\mathsf{V}_q(\chi) \subset \mathsf{V}_0(\chi) \subset \mathsf{V}$. Furthermore, $\mathsf{V}_q(\chi) \neq \emptyset$, since the m_v's are coprime. Also define

$$\mathcal{P}(\chi) = \big\{q \text{ prime} \bigm| \mathsf{V}_q(\chi) \subsetneq \mathsf{V}_0(\chi)\big\}. \tag{45}$$

Obviously, this is a finite (possibly empty) set, consisting of all the prime factors of the non-zero m_v's.

Now let $\Bbbk$ be a field, of characteristic p. For an element $a \in \Bbbk \setminus \{0, 1\}$, the homomorphism $\chi_a := \mathrm{ev}_a \circ \chi : G_L \to \Bbbk^\times$ is given by $\chi_a(v) = a^{m_v}$. Let

$$z_{\chi_a} = \sum_{v \in \mathsf{V}} \bigl(a^{m_v} - 1\bigr) v^* \in \Bbbk\langle L\rangle^1 \tag{46}$$

be the corresponding cohomology class. Set $\mathsf{W}_a = \operatorname{supp}(z_{\chi_a})$. Clearly, $v \in \mathsf{W}_a$ if and only if $a^{m_v} \neq 1$ in $\Bbbk$. Thus,

$$\mathsf{W}_a = \bigl\{ v \in \mathsf{V}_0(\chi) \bigm| \operatorname{ord}(a) \nmid m_v \bigr\}. \tag{47}$$

Lemma 6.1. *With notation as above,*

(1) *If* $\operatorname{ord}(a) = \infty$, *then* $\mathsf{W}_a = \mathsf{V}_0(\chi)$.
(2) *If* $\operatorname{ord}(a) < \infty$, *then there is a prime* $q \neq p$ *such that* $\mathsf{W}_a \supset \mathsf{V}_q(\chi)$.

Proof. If a has infinite order, then clearly $\mathsf{W}_a = \{v \in \mathsf{V} \mid m_v \neq 0\}$, which equals $\mathsf{V}_0(\chi)$. If a has finite order d (necessarily, $d > 1$, since $a \neq 1$), then $\mathsf{W}_a = \{v \in \mathsf{V}_0(\chi) \mid d \nmid m_v\}$. We claim there is a prime $q \neq p$ such that $q \mid d$. If $p = 0$, this is clear. If $p > 0$ and $d = p^s$, then $0 = a^d - 1 = (a-1)^{p^s}$, which forces $a = 1$, a contradiction. Hence, $\mathsf{V}_q(\chi) = \{v \in \mathsf{V}_0(\chi) \mid q \nmid m_v\} \subset \mathsf{W}_a$. □

6.2. Triviality test for the monodromy

We are now ready to state and prove the main result of this section.

Theorem 6.2. *Let* $\Bbbk$ *be a field, with* $\operatorname{char}\Bbbk = p$. *The* $\Bbbk\mathbb{Z}$*-module* $H_{\leqslant r}(T_L, \Bbbk\mathbb{Z}_\chi)$ *has trivial* $\mathbb{Z}$*-action if and only if*

$(\dagger)_r$ $\beta_i(\Bbbk\langle L\rangle, \mathsf{V}_p(\chi)) = 0$, *for all* $i \leqslant r$;
$(\ddagger)_r$ $\beta_i(\Bbbk\langle L\rangle, \mathsf{V}_q(\chi)) = 0$, *for all* $q \in \mathcal{P}(\chi)$ *with* $q \neq p$, *and for all* $i \leqslant r$.

Proof. Everything depends only on $p = \operatorname{char}\Bbbk$, and not on $\Bbbk$ itself, so we may as well assume $\Bbbk$ is algebraically closed. It follows that, up to units, the only irreducible polynomials in $\Lambda = \Bbbk\mathbb{Z}$ are of the form $t - a$, with $a \in \Bbbk^\times$. Thus, we may write

$$H_i(T_L, \Lambda_\chi) = \Lambda^{r_i} \oplus T_1^i \oplus \bigoplus_{a \neq 1} T_a^i, \tag{48}$$

where T_a denotes $(t-a)$-primary part.

Let $\chi_\Bbbk \in \Bbbk\langle L\rangle^1 = H^1(T_L, \Bbbk)$ be the cohomology class corresponding to the composite $G_L \xrightarrow{\chi} \mathbb{Z} \to \Bbbk$. Clearly, $\operatorname{supp}(\chi_\Bbbk) = \mathsf{V}_p(\chi)$; thus, $\beta_i(\Bbbk\langle L\rangle, \chi_\Bbbk) = \beta_i(\Bbbk\langle L\rangle, \mathsf{V}_p(\chi))$, by Lemma 3.4. It follows from [29, Proposition 9.4] that

$$\begin{aligned} \Lambda^{r_i} \oplus T_1^i \text{ has trivial } \mathbb{Z}\text{-action}, \ \forall i \leqslant r \quad &\Leftrightarrow \quad r_i = 0 \text{ and } T_1^i = \bigl(\Lambda/(t-1)\bigr)^{e_i}, \quad \forall i \leqslant r \\ &\Leftrightarrow \quad \text{condition } (\dagger)_r \text{ is satisfied.} \end{aligned} \tag{49}$$

Now fix $a \neq 0$ or 1. Since $t-1$ and $t-a$ are coprime, $t-1$ acts trivially on T_a if and only if $T_a = 0$. So we are left with proving the following two claims.

Claim 1. *Suppose condition $(\dagger)_r$ is satisfied. If $T_a^{\leqslant r}=0$, for all $a \in \Bbbk \setminus \{0,1\}$, then $\beta_i(\Bbbk\langle L\rangle, \mathsf{V}_q(\chi))=0$, for all $i \leqslant r$, and all $q \in \mathcal{P}(\chi)$ with $q \neq p$.*

The proof is by induction on r, with the case $r=0$ clear; indeed, $\mathsf{V}_q(\chi) \neq \emptyset$ implies $\beta_0(\Bbbk\langle L\rangle, \mathsf{V}_q(\chi))=0$.

Assume the claim is true for $r-1$. Suppose condition $(\dagger)_r$ is satisfied, and $T_a^{\leqslant r}=0$, for all $a \in \Bbbk \setminus \{0,1\}$. Using (49), we deduce that

$$H_i(T_L, \Lambda_\chi) = \big(\Lambda/(t-1)\big)^{e_i}, \quad \text{for all } i \leqslant r. \tag{$(*)_r$}$$

Fix an element a as above. Since $a \neq 1$, the Universal Coefficient Theorem, together with $(*)_r$, gives $H_r(T_L, \Bbbk_a)=0$. Hence, by Lemmas 3.11 and 3.4, $\beta_r(\Bbbk\langle L\rangle, \mathsf{W}_a)=0$.

Now fix a prime $q \in \mathcal{P}(\chi)$, $q \neq p$. Since $\Bbbk = \bar{\Bbbk}$, we may find an element $a \in \Bbbk^\times \setminus \{1\}$ with $\operatorname{ord}(a)=q$. Clearly, $\mathsf{W}_a = \mathsf{V}_q(\chi)$, and so, by the above, $\beta_r(\Bbbk\langle L\rangle, \mathsf{V}_q(\chi))=0$. This finishes the proof of Claim 1.

Claim 2. *Suppose conditions $(\dagger)_r$ and $(\ddagger)_r$ are satisfied. Then $T_a^i=0$, for all $i \leqslant r$, and all $a \in \Bbbk \setminus \{0,1\}$.*

The proof is by induction on r, with the case $r=0$ clear; indeed, $H_0(T_L, \Bbbk\mathbb{Z}_\chi)=\Bbbk$, and so $T_a^0=0$.

Assume the claim is true for $r-1$, and suppose conditions $(\dagger)_r$ and $(\ddagger)_r$ are satisfied. Using (49), we deduce that $(*)_{r-1}$ holds, and

$$H_r(T_L, \Lambda_\chi) = \big(\Lambda/(t-1)\big)^{e_r} \oplus \bigoplus_{b \in \Bbbk \setminus \{0,1\}} T_b^r. \tag{50}$$

Now fix an element a in $\Bbbk \setminus \{0,1\}$. We need to show $T_a^r=0$. Write

$$T_a^r = \bigoplus_{j \geqslant 1} \big(\Lambda/(t-a)^j \Lambda\big)^{\varepsilon_j}. \tag{51}$$

The Universal Coefficient Theorem, together with $(*)_{r-1}$ and (50) yields $H_r(T_L, \Bbbk_a) = \bigoplus_{j \geqslant 1} \Bbbk^{\varepsilon_j}$. So it is enough to show $H_r(T_L, \Bbbk_a)=0$. In view of Lemma 3.11, it remains to show that

$$\beta_r\big(\Bbbk\langle L\rangle, \mathsf{W}_a\big)=0. \tag{52}$$

By Lemma 6.1, there are two possibilities to consider.

(i) $\mathsf{W}_a = \mathsf{V}_0(\chi)$. Recall $\mathsf{V}_0(\chi) \supset \mathsf{V}_p(\chi)$. By Lemma 3.5 and condition $(\dagger)_r$, we have $\beta_r(\Bbbk\langle L\rangle, \mathsf{W}_a) \leqslant \beta_r(\Bbbk\langle L\rangle, \mathsf{V}_p(\chi))=0$.
(ii) $\mathsf{W}_a \supset \mathsf{V}_q(\chi)$, for some prime $q \neq p$. Here, there are two sub-cases.
   (a) $q \in \mathcal{P}(\chi)$. Then $\beta_r(\Bbbk\langle L\rangle, \mathsf{W}_a) \leqslant \beta_r(\Bbbk\langle L\rangle, \mathsf{V}_q(\chi))=0$.
   (b) $q \notin \mathcal{P}(\chi)$. Then $\mathsf{W}_a \supset \mathsf{V}_q(\chi) = \mathsf{V}_0(\chi) \supset \mathsf{V}_p(\chi)$, and so $\beta_r(\Bbbk\langle L\rangle, \mathsf{W}_a) \leqslant \beta_r(\Bbbk\langle L\rangle, \mathsf{V}_p(\chi))=0$.

Thus, in all cases equality (52) holds. This finishes the proof of Claim 2, and thereby ends the proof of the theorem. $\square$

Remark 6.3. Let X be a connected CW-complex of finite type. Let $\nu : \pi_1(X) \twoheadrightarrow A$ be an epimorphism onto an abelian group A. Denote by X^ν the Galois A-cover of X corresponding to $\ker(\nu)$. For a coefficient field $\Bbbk$, consider the following properties of X^ν:

(1) $H_{\leqslant r}(X^\nu, \Bbbk)$ has trivial A-action.
(2) $\dim_{\Bbbk} H_{\leqslant r}(X^\nu, \Bbbk) < \infty$.
(3) $\nu_{\Bbbk} \notin \bigcup_{i=1}^{r} \mathcal{R}_1^i(X, \Bbbk)$ (when $A = \mathbb{Z}$).

It is readily seen that (1) $\Rightarrow$ (2). For $A = \mathbb{Z}$, the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2) follow from [29, Proposition 9.4]. As illustrated by the following example (inspired by [29, Example 8.5]), neither of these implications can be reversed.

Example 6.4. Consider the cover $T_\Gamma^\chi \to T_\Gamma$ defined by the weighted graph

$$\overset{1}{\circ} \frac{\qquad}{} \overset{2}{\circ} \frac{\qquad}{} \overset{1}{\circ} \tag{53}$$

Clearly, $T_\Gamma \simeq S^1 \times (S^1 \vee S^1)$. If $\operatorname{char} \Bbbk \neq 2$, then $\chi_{\Bbbk} \notin \mathcal{R}_1^1(T_\Gamma, \Bbbk)$, by Corollary 3.9, yet the $\mathbb{Z}$-action on $H_1(T_\Gamma^\chi, \Bbbk)$ is non-trivial, by Theorem 6.2. If $\operatorname{char} \Bbbk = 2$, then $\dim_{\Bbbk} H_{\leqslant 1}(T_\Gamma^\chi, \Bbbk) < \infty$, by Corollary 5.2, but $\chi_{\Bbbk} \in \mathcal{R}_1^1(T_\Gamma, \Bbbk)$, again by Corollary 3.9. Note also that the corresponding Artin kernel, $N_\chi = \pi_1(T_\Gamma^\chi)$, is finitely presented, as may be checked by using test (b) from Section 5.1.

7. Cohomology ring and finiteness properties

We are now in a position to compute the (truncated) cohomology ring of an arbitrary Galois $\mathbb{Z}$-cover of a toric complex, in the case when the monodromy action is trivial (up to a fixed degree).

7.1. Cohomology ring of a toric cover

Let T_L be a toric complex, $\chi : G_L \twoheadrightarrow \mathbb{Z}$ an epimorphism, and $\pi : T_L^\chi \to T_L$ the corresponding Galois $\mathbb{Z}$-cover. As before, we denote by $\chi_{\Bbbk} \in H^1(T_L, \Bbbk)$ the cohomology class determined by χ. Let $(\chi_{\Bbbk})$ be the ideal of $H^*(T_L, \Bbbk)$ it generates. By Proposition 4.4(1), the induced homomorphism between cohomology rings, $\pi^* : H^*(T_L, \Bbbk) \to H^*(T_L^\chi, \Bbbk)$, factors through a ring map,

$$\bar{\pi}^* : H^*(T_L, \Bbbk)/(\chi_{\Bbbk}) \to H^*\big(T_L^\chi, \Bbbk\big). \tag{54}$$

Theorem 7.1. *Let $\Bbbk$ be a field, with* $\operatorname{char} \Bbbk = p$. *Fix an integer $r \geqslant 1$, and suppose*

(†) $\beta_i(\Bbbk\langle L\rangle, \mathsf{V}_p(\chi)) = 0$,
(‡) $\beta_i(\Bbbk\langle L\rangle, \mathsf{V}_q(\chi)) = 0$, *for all $q \in \mathcal{P}(\chi)$ with $q \neq p$,*

for all $i \leqslant r$. Then $\bar{\pi}^* : H^{\leqslant r}(T_L, \Bbbk)/(\chi_{\Bbbk}) \to H^{\leqslant r}(T_L^\chi, \Bbbk)$ is a ring isomorphism.

Proof. By Theorem 6.2, the hypothesis is equivalent to $H_{\leqslant r}(T_L, \Bbbk\mathbb{Z}_\chi)$ having trivial $\mathbb{Z}$-action. The conclusion follows from Proposition 4.4(2). $\square$

In the particular case of Bestvina–Brady covers, $\mathsf{V}_q(\nu) = \mathsf{V}_0(\nu) = \mathsf{V}$, for all primes q. In view of Proposition 3.6, we recover from Theorem 7.1 the following result of Leary and Saadetoğlu (see Theorem 13 from [21]).

Corollary 7.2. *Let $\pi : T_L^\nu \to T_L$ be the Bestvina–Brady cover associated to L. If $\widetilde{H}_{<r}(L, \Bbbk) = 0$, then $\bar{\pi}^* : H^{\leqslant r}(T_L, \Bbbk)/(\nu_\Bbbk) \to H^{\leqslant r}(T_L^\nu, \Bbbk)$ is a ring isomorphism.*

The case when L is a simply-connected flag complex, $r = 2$, and $\Bbbk = \mathbb{Q}$ was first proved in [28, Theorem 1.3], by completely different methods.

7.2. Finiteness properties of Artin kernels

Recall that a group G is of type FP_r ($r \leqslant \infty$) if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$ of the trivial G-module $\mathbb{Z}$, with P_i finitely generated for all $i \leqslant r$. The FP_r condition obviously implies that the homology groups $H_i(G, \mathbb{Z})$ are finitely generated, for all $i \leqslant r$, but the converse is far from true, in general.

When coupled with the main result from [23], as restated in a more convenient form in [5], our Corollary 5.2 yields a remarkable property of Artin kernels: the fact that, within this class of groups, the finiteness property FP_r may be detected by the corresponding finiteness property for homology with trivial field coefficients.

Theorem 7.3. *An Artin kernel, $N_\chi = \ker(\chi : G_\Gamma \twoheadrightarrow \mathbb{Z})$, is of type FP_r if and only if $\dim_\Bbbk H_{\leqslant r}(N_\chi, \Bbbk) < \infty$, for any field $\Bbbk$.*

Proof. Set $L = \Delta_\Gamma$ and $\mathsf{W} = \mathsf{V}_0(\chi)$. To prove the non-trivial implication, assume $\dim_\Bbbk H_{\leqslant r}(N_\chi, \Bbbk) < \infty$, for any field $\Bbbk$. Then, by Corollary 5.2(3),

$$\widetilde{H}_i\big(\mathrm{lk}_{L_\mathsf{W}}(\sigma), \Bbbk\big) = 0,$$

for all $\sigma \in L_{\mathsf{V}\setminus\mathsf{W}}$ and $i \leqslant r - 1 - |\sigma|$. Consequently, $\widetilde{H}_i(\mathrm{lk}_{L_\mathsf{W}}(\sigma), \mathbb{Z}) = 0$, for all σ and i as above. By [5, Theorem 14], the group N_χ is of type FP_r. $\square$

For Bestvina–Brady covers and groups, one can say more.

Corollary 7.4. *Let L be a finite simplicial complex on vertex set V, and let $\nu : G_L \to \mathbb{Z}$ be the homomorphism sending each generator $v \in \mathsf{V}$ to 1. For a field $\Bbbk$, and an integer $r \geqslant 1$, the following are equivalent:*

(i) *The $\Bbbk\mathbb{Z}$-module $H_i(T_L^\nu, \Bbbk)$ is trivial, for all $i \leqslant r$.*
(ii) *The $\Bbbk$-vector space $H_i(T_L^\nu, \Bbbk)$ is finite-dimensional, for all $i \leqslant r$.*
(iii) *$\nu_\Bbbk \notin \mathcal{R}_1^i(T_L, \Bbbk)$, for all $i \leqslant r$.*
(iv) *$\widetilde{H}_i(L, \Bbbk) = 0$, for all $i < r$.*

If, in addition, $L = \Delta_\Gamma$ *is a flag complex, conditions* (i)–(iv) *hold over fields* $\Bbbk$ *of arbitrary characteristic if and only if* $N_\Gamma = \ker(\nu)$ *is of type* FP_r.

Proof. For the implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii), see Remark 6.3. The implication (ii) $\Rightarrow$ (i) follows from Corollary 5.10, while the equivalence (ii) $\Leftrightarrow$ (iv) follows from Corollary 5.2, since $\mathsf{V}_0(\nu) = \mathsf{V}$. Finally, the claim about flag complexes follows from Theorem 7.3. □

8. Holonomy Lie algebra

In this section, we study a certain graded Lie algebra, $\mathfrak{h}(A)$, attached to a strongly graded-commutative algebra A. In the process, we relate the non-resonance properties of an element $a \in A^1$ to the graded ranks of the metabelian quotient of $\mathfrak{h}(A/aA)$.

8.1. Holonomy Lie algebra of an algebra

Let A be a connected, graded, graded-commutative algebra over a field $\Bbbk$, with graded pieces A^i, $i \geqslant 0$. We shall assume that $\dim_\Bbbk A^1 < \infty$, and $a^2 = 0$, for all $a \in A^1$. This last condition (which is automatically satisfied if $\mathrm{char}\,\Bbbk \neq 2$) insures that the multiplication map in degree 1 descends to a linear map $\mu : A^1 \wedge A^1 \to A^2$.

Let $A_i = (A^i)^\#$ be the dual $\Bbbk$-vector space, and let $\mathrm{Lie}(A_1)$ be the free Lie algebra on A_1, graded by bracket length. Let $\nabla : A_2 \to A_1 \wedge A_1 = \mathrm{Lie}_2(A_1)$ be the dual of μ. In the spirit of K.T. Chen's approach from [9], define the *holonomy Lie algebra* of A as the quotient of the free Lie algebra on A_1 by the ideal generated by the image of the comultiplication map,

$$\mathfrak{h}(A) = \mathrm{Lie}(A_1)/(\mathrm{im}\,\nabla). \tag{55}$$

Clearly, $\mathfrak{h} = \mathfrak{h}(A)$ inherits a natural grading from the free Lie algebra, compatible with the Lie bracket; let $\mathfrak{h}_s$ be the sth graded piece. By construction, $\mathfrak{h}$ is a finitely presented graded Lie algebra, with generators in degree 1, and relations in degree 2.

Note that $\mathfrak{h}(A)$ depends only on the degree 2 truncation of A, namely $A^{\leqslant 2} = \bigoplus_{i \leqslant 2} A^i$. For the purpose of defining the holonomy Lie algebra of A, we may assume A is generated in degree 1. Indeed, if we let $E = \bigwedge A^1$ be the exterior algebra on A^1, and we set $\bar{A} = E/(K + E^{\geqslant 3})$, where $K = \ker(\mu)$, then the algebra $\bar{A}$ is generated by $\bar{A}^1 = A^1$, and clearly $\mathfrak{h}(\bar{A}) = \mathfrak{h}(A)$.

So assume A is generated in degree 1, i.e, the morphism $q_A : E = \bigwedge A^1 \to A$ extending the identity on A^1 is surjective. In this case, the map ∇ is injective. Thus, we may view A_2 as a subspace of $A_1 \wedge A_1$, and arrive at the following identifications:

$$\mathfrak{h}_1 = A_1, \qquad \mathfrak{h}_2 = (A_1 \wedge A_1)/A_2, \qquad \mathfrak{h}_3 = \mathrm{Lie}_3(A_1)/[A_1, A_2]. \tag{56}$$

8.2. Homological algebra interpretation

Since $\mathfrak{h}$ is generated in degree 1, its derived Lie subalgebra, $\mathfrak{h}'$, coincides with $\mathfrak{h}_{\geqslant 2} = \bigoplus_{s \geqslant 2} \mathfrak{h}_s$. Let $\mathfrak{h}''$ be the second derived Lie subalgebra of $\mathfrak{h}$, and let

$$\mathfrak{g}(A) = \mathfrak{h}(A)/\mathfrak{h}''(A) \tag{57}$$

be the maximal metabelian quotient Lie algebra of $\mathfrak{h}$. It is readily checked that $\mathfrak{g}' = \mathfrak{h}'/\mathfrak{h}''$. Moreover, $\mathfrak{g}_s = \mathfrak{h}_s$, for $s \leqslant 3$.

Viewing A as an E-module via q_A, and $\Bbbk$ as the trivial E-module $E/E^{\geqslant 1}$, we may form the bigraded vector space $\operatorname{Tor}^E_*(A, \Bbbk)_*$, where the first grading comes from a free resolution of A over E, and the second one comes from the grading on E. Then:

$$\mathfrak{g}_s(A)^\sharp = \operatorname{Tor}^E_{s-1}(A, \Bbbk)_s, \tag{58}$$

for all $s \geqslant 2$. As noted in Proposition 2.3 from [27], equality (58) follows easily from work of Fröberg and Löfwall (Theorem 4.1(ii) in [18]).

8.3. Resonance and holonomy

As above, let A be an algebra with A^1 finite-dimensional, $a^2 = 0$ for all $a \in A^1$, and A generated in degree 1. Let a be a non-zero element in A^1, and consider the quotient algebra $B = A/aA$. The next result equates the graded ranks of $\mathfrak{g}'(A)$ and $\mathfrak{g}'(B)$ in a certain range, prescribed by the non-resonance properties of a.

Theorem 8.1. *Suppose $a \notin \bigcup_{i=1}^r \mathcal{R}^i_1(A)$, for some $r \geqslant 1$. Then*

$$\mathfrak{g}_s(A) \cong \mathfrak{g}_s(B), \quad \text{for } 2 \leqslant s \leqslant r+1.$$

Proof. Set $F = \bigwedge B^1$. Let $\phi : A \to B$ be the projection map, and let $\psi : E \to F$ be the extension of $\phi^1 : A^1 \to B^1$ to exterior algebras. We then have a commuting square

$$\begin{tikzcd} E \arrow[r, two heads, "\psi"] \arrow[d, "q_A"] & F \arrow[d, "q_B"] \\ A \arrow[r, two heads, "\phi"] & B \end{tikzcd} \tag{59}$$

with $F = E/aE$ and $A \otimes_E F = B$. This square gives rise to a "change of rings" spectral sequence,

$$E^2_{s,t} = \operatorname{Tor}^F_s\big(\operatorname{Tor}^E_t(A, F), \Bbbk\big) \quad \Rightarrow \quad \operatorname{Tor}^E_{s+t}(A, \Bbbk). \tag{60}$$

Note that the E^2-term has an extra grading, ${}_*E^2_{s,t}$, coming from the degree grading on E. This extra grading is preserved by the differentials, and is compatible with the internal grading on $\operatorname{Tor}^E(A, \Bbbk)$.

A free resolution of the E-module F is given by

$$\cdots \longrightarrow E[t] \xrightarrow{a} E[t-1] \longrightarrow \cdots \longrightarrow E[1] \xrightarrow{a} E \xrightarrow{\psi} F \longrightarrow 0. \tag{61}$$

Here, the free module in position t is regraded as $E[t]$, with $E[t]^i = E^{i-t}$; this is done so that all boundary maps have degree 0. Therefore, $\operatorname{Tor}^E_t(A, F)$ is the homology in degree t of the chain complex

$$\cdots \longrightarrow A[t] \xrightarrow{a} A[t-1] \longrightarrow \cdots \longrightarrow A[1] \xrightarrow{a} A \xrightarrow{\phi} B \longrightarrow 0, \tag{62}$$

In internal degree i, this homology group coincides with $H^{i-t}(A, \cdot a)$, for $t > 0$. Thus,

$$\mathrm{Tor}_t^E(A, F)_i = H^{i-t}(A, \cdot a), \quad \text{for } t > 0. \tag{63}$$

By assumption, $a \notin \mathcal{R}_1^i(A)$, that is, $H^i(A, \cdot a) = 0$, for $i \leqslant r$. Hence, by (63), we have $\mathrm{Tor}_t^E(A, F)_i = 0$ for $i \leqslant r + t$ and $t > 0$. It follows that ${}_iE_{s,t}^2 = 0$, for all $s \geqslant 0$, $t > 0$ and $i \leqslant r + 1$, and so

$$\mathrm{Tor}_{s-1}^E(A, \Bbbk)_s = {}_sE_{s-1,0}^{\infty} = {}_sE_{s-1,0}^2 = \mathrm{Tor}_{s-1}^F(B, \Bbbk)_s, \tag{64}$$

for all $1 \leqslant s \leqslant r + 1$. Invoking (58) finishes the proof. □

Corollary 8.2. *Let $a \in A^1$, and set $B = A/aA$.*

(1) *If $a \notin \mathcal{R}_1^1(A)$, then $\mathfrak{h}_2(A) = \mathfrak{h}_2(B)$.*
(2) *If $a \notin \mathcal{R}_1^1(A) \cup \mathcal{R}_1^2(A)$, then $\mathfrak{h}_3(A) = \mathfrak{h}_3(B)$.*

Let $\mathbb{L}_m = \mathrm{Lie}(\Bbbk^m)$ be the free Lie $\Bbbk$-algebra on m generators, graded by bracket length.

Corollary 8.3. *Suppose $A^1 = \Bbbk^n$, $n \geqslant 1$, and $A^{\geqslant 3} = 0$. If $\mathcal{R}_1^1(A) \cup \mathcal{R}_1^2(A) \neq A^1$, then $\mathfrak{g}'(A) \cong \mathbb{L}_{n-1}'/\mathbb{L}_{n-1}''$.*

Proof. Pick $a \in A^1 \setminus (\mathcal{R}_1^1(A) \cup \mathcal{R}_1^2(A))$, and set $B = A/aA$. Non-resonance of a in degree 1 implies $a \neq 0$; hence, $B^1 = \Bbbk^{n-1}$. Non-resonance of a in degree 2 implies $A^2 = aA^1$; in particular, A is generated in degree 1, and $B^{\geqslant 2} = 0$. Hence, $\mathfrak{h}(B) = \mathbb{L}_{n-1}$, by definition (55). The claim then follows from Theorem 8.1. □

9. Lie algebras associated to Artin kernels

In this section, we study three graded Lie algebras associated to an Artin kernel $N_\chi = \ker(\chi : G_L \twoheadrightarrow \mathbb{Z})$: the associated graded Lie algebra $\mathrm{gr}(N_\chi)$, the Chen Lie algebra $\mathrm{gr}(N_\chi/N_\chi'')$, and the holonomy Lie algebra $\mathfrak{h}(N_\chi)$.

9.1. Associated graded and Chen Lie algebras

We start with a classical construction from the 1930s, due to P. Hall and W. Magnus, see [22]. Let G be a group. The lower central series (LCS) of G is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = (\gamma_k G, G)$, where $(x, y) = xyx^{-1}y^{-1}$. The *associated graded Lie algebra*, $\mathrm{gr}(G)$, is the direct sum of the successive LCS quotients,

$$\mathrm{gr}(G) = \bigoplus\nolimits_{k \geqslant 1} \gamma_k G/\gamma_{k+1} G, \tag{65}$$

with Lie bracket induced from the group commutator. For a field $\Bbbk$, we shall write $\mathrm{gr}(G) \otimes \Bbbk = \bigoplus_{k \geqslant 1} \mathrm{gr}_k(G) \otimes \Bbbk$.

By construction, the Lie algebra $\mathrm{gr}(G)$ is generated by $\mathrm{gr}_1(G) = G_{\mathrm{ab}}$. Thus, if G is a finitely generated group, then $\mathrm{gr}(G)$ is a finitely generated Lie algebra. Moreover, the derived Lie subalgebra, $\mathrm{gr}'(G)$, coincides with $\bigoplus_{k\geqslant 2} \mathrm{gr}_k(G)$.

In [8], K.T. Chen introduced a useful variation on this theme. Let $G' = \gamma_2 G$ be the derived group, and $G'' = (G')'$ the second derived group. Note that $H_1(G,\mathbb{Z}) = G/G'$ is the maximal abelian quotient of G, whereas G/G'' is the maximal metabelian quotient. The *Chen Lie algebra* of G is simply $\mathrm{gr}(G/G'')$. Though a coarser invariant that $\mathrm{gr}(G)$, the Chen Lie algebra captures some subtle phenomena, in its own distinctive manner, more closely tied to commutative algebra. We refer to [26] for more on this subject.

9.2. Associated graded of an Artin kernel

Let Γ be a finite simple graph, with vertex set V, and denote by $G = G_\Gamma$ the corresponding right-angled Artin group. Let $\chi : G \twoheadrightarrow \mathbb{Z}$ be an epimorphism, and set $N = \ker(\chi)$. Denoting by $\iota : N \to G$ the inclusion map, we have a split exact sequence

$$1 \longrightarrow N \xrightarrow{\ \iota\ } G \xrightarrow{\ \chi\ } \mathbb{Z} \longrightarrow 1. \tag{66}$$

Lemma 9.1. *Suppose $H_1(N,\mathbb{Q})$ has trivial $\mathbb{Q}\mathbb{Z}$-action. Then:*

(1) *N is finitely generated.*
(2) *$H_1(N,\Bbbk)$ has trivial $\Bbbk\mathbb{Z}$-action, for any field $\Bbbk$.*
(3) *$H_1(N,\mathbb{Z})$ has trivial $\mathbb{Z}\mathbb{Z}$-action.*
(4) *The restriction of ι to derived subgroups, $\iota' : N' \to G'$, is an isomorphism.*
(5) *The induced homomorphism $\iota_* : H_1(N,\mathbb{Z}) \to H_1(G,\mathbb{Z})$ is injective. In particular, $H_1(N,\mathbb{Z})$ is a free abelian group, of rank $|\mathsf{V}| - 1$.*

Proof. (1) By Theorem 6.2 and Corollary 5.2, the induced subgraph $\Gamma_{\mathsf{V}_0(\chi)}$ is connected and dominant. Hence, by [24], the group N is finitely generated.

(2) By Theorem 6.2, our hypothesis means that $\beta_i(\mathbb{Q}\langle L\rangle, \mathsf{W}) = 0$, for $i = 0, 1$, and

$$\mathsf{W} = \mathsf{V}_0(\chi), \quad \text{or} \quad \mathsf{W} = \mathsf{V}_q(\chi), \quad \text{with } q \in \mathcal{P}(\chi). \tag{67}$$

Now let $\Bbbk$ be a field of characteristic p. Again by Theorem 6.2, we have to check that $\beta_i(\Bbbk\langle L\rangle, \mathsf{W}) = 0$, for $i = 0, 1$, where either $\mathsf{W} = \mathsf{V}_p(\chi)$, or $\mathsf{W} = \mathsf{V}_q(\chi)$, with $q \in \mathcal{P}(\chi)$ and $q \neq p$. Using Remark 3.1 and formula (22), we see that it is enough to check $\beta_i(\mathbb{Q}\langle L\rangle, \mathsf{W}) = 0$, for $i = 0, 1$, and W as just above. There are 3 cases to consider:

- $\mathsf{W} = \mathsf{V}_q(\chi)$, with $q \in \mathcal{P}(\chi)$, $q \neq p$.
- $\mathsf{W} = \mathsf{V}_p(\chi)$, and $p \in \mathcal{P}(\chi)$.
- $\mathsf{W} = \mathsf{V}_p(\chi)$, and $p \notin \mathcal{P}(\chi)$, in which case $\mathsf{V}_p(\chi) = \mathsf{V}_0(\chi)$, by definition (45).

Clearly, all three cases are covered by (67), and we are done.

(3) Follows from (1), (2), and the Universal Coefficient Theorem.

(4) Plainly, $\iota(N') \subseteq G'$. The triviality of the $\mathbb{Z}\mathbb{Z}$-action on $N/N' = H_1(N,\mathbb{Z})$ forces $\iota(N') = G'$. Thus, $\iota' : N' \to G'$ is an isomorphism.

(5) It follows from (3) and [17, Lemma 3.4] that (66) remains exact upon abelianization. Recalling that $H_1(G,\mathbb{Z})$ is a free abelian group of rank $|\mathsf{V}|$ finishes the proof. $\square$

For a homomorphism $\alpha : G \to H$, let $\bar{\alpha} : G/G'' \to H/H''$ be the induced homomorphism on maximal metabelian quotients, and $\mathrm{gr}(\alpha) : \mathrm{gr}(G) \to \mathrm{gr}(H)$ the induced morphism of graded Lie algebras. The proof of the next result is similar to (and generalizes) the proofs of Propositions 4.2 and 5.4, and Theorem 5.6 from [28].

Proposition 9.2. *Suppose $H_1(N,\mathbb{Q})$ has trivial $\mathbb{Q}\mathbb{Z}$-action. Then, we have split exact sequences of graded Lie algebras,*

$$0 \longrightarrow \mathrm{gr}(N) \xrightarrow{\mathrm{gr}(\iota)} \mathrm{gr}(G) \xrightarrow{\mathrm{gr}(\chi)} \mathrm{gr}(\mathbb{Z}) \longrightarrow 0, \tag{68}$$

$$0 \longrightarrow \mathrm{gr}(N/N'') \xrightarrow{\mathrm{gr}(\bar{\iota})} \mathrm{gr}(G/G'') \xrightarrow{\mathrm{gr}(\bar{\chi})} \mathrm{gr}(\mathbb{Z}) \longrightarrow 0. \tag{69}$$

Proof. By Lemma 9.1(3), the abelianization $N/N' = H_1(N,\mathbb{Z})$ is a trivial $\mathbb{Z}\mathbb{Z}$-module. Applying the gr functor to (66), and making use once again of Lemma 3.4 from Falk and Randell [17], yields (68).

By Lemma 9.1(4), $\iota(N') = G'$. It follows that $\iota(N'') = G''$. Hence, there is an exact sequence $1 \to N/N'' \xrightarrow{\bar{\iota}} G/G'' \xrightarrow{\bar{\chi}} \mathbb{Z} \to 1$. Applying the gr functor as above yields (69). $\square$

Consequently, $\mathrm{gr}'(N) \cong \mathrm{gr}'(G)$ and $\mathrm{gr}'(N/N'') \cong \mathrm{gr}'(G/G'')$, as graded Lie algebras. Using the computation of $\mathrm{gr}(G)$ and $\mathrm{gr}(G/G'')$ from [27], we obtain the following immediate corollary, which generalizes Theorems 5.1 and 5.2 from [28].

Corollary 9.3. *Suppose $H_1(N,\mathbb{Q})$ has trivial $\mathbb{Z}$-action. Then, both $\mathrm{gr}(N)$ and $\mathrm{gr}(N/N'')$ are torsion-free, with graded ranks, ϕ_k and θ_k, given by*

$$\prod_{k=1}^{\infty}\bigl(1-t^k\bigr)^{\phi_k} = \frac{P(-t)}{1-t}, \tag{70}$$

where $P(t) = \sum_{k\geqslant 0} f_k(\Gamma)t^k$ is the clique polynomial *of Γ, with $f_k(\Gamma)$ equal to the number of k-cliques of Γ, and*

$$\sum_{k=2}^{\infty}\theta_k t^k = Q\left(\frac{t}{1-t}\right), \tag{71}$$

where $Q(t) = \sum_{j\geqslant 2}(\sum_{\mathsf{W}\subset\mathsf{V}:|\mathsf{W}|=j}\tilde{b}_0(\Gamma_{\mathsf{W}}))t^j$ is the cut polynomial *of Γ.*

9.3. Holonomy Lie algebra of an Artin kernel

Suppose G is a finitely generated group. Then $A = H^*(G;\Bbbk)$ is a connected, graded, graded-commutative $\Bbbk$-algebra, with A^1 finitely generated. Assuming $\operatorname{char}\Bbbk \neq 2$, we may define the holonomy Lie algebra of G, with coefficients in $\Bbbk$, to be $\mathfrak{h}(G,\Bbbk)=\mathfrak{h}(A)$; we will simply write $\mathfrak{h}(G)=\mathfrak{h}(G,\mathbb{Q})$.

The obvious identification $A_1=\mathrm{gr}_1(G)\otimes\mathbb{Q}$ extends to a Lie algebra map, $\mathrm{Lie}(A_1)\twoheadrightarrow \mathrm{gr}(G)\otimes\mathbb{Q}$, which in turn factors through an epimorphism $\Psi_G:\mathfrak{h}(G)\twoheadrightarrow \mathrm{gr}(G)\otimes\mathbb{Q}$; see [26] for further details and references.

Consider now a right-angled Artin group, $G=G_\Gamma$. In this case, it is known that $\Psi_G:\mathfrak{h}(G)\to \mathrm{gr}(G)\otimes\mathbb{Q}$ is an isomorphism; see [27]. Let $\chi:G\twoheadrightarrow\mathbb{Z}$ be an epimorphism, and $N=\ker(\chi)$ the corresponding Artin kernel.

Lemma 9.4. *Suppose $H_1(N,\mathbb{Q})$ has trivial $\mathbb{Q}\mathbb{Z}$-action. Then:*

(1) $\mathfrak{h}_1(\iota):\mathfrak{h}_1(N)\to\mathfrak{h}_1(G)$ *is injective.*
(2) $\mathfrak{h}_k(\iota):\mathfrak{h}_k(N)\to\mathfrak{h}_k(G)$ *is surjective, for all $k\geqslant 2$.*

Proof. Part (1). The map $\mathfrak{h}_1(\iota)$ may be identified with the induced homomorphism $\iota_*: H_1(N,\mathbb{Q})\to H_1(G,\mathbb{Q})$, which is injective, by triviality of the $\mathbb{Q}\mathbb{Z}$-action on $H_1(N,\mathbb{Q})$.

Part (2). Consider the commuting diagram

$$\begin{tikzcd} \mathfrak{h}(N) \arrow[r, "\mathfrak{h}(\iota)"] \arrow[d, "\Psi_N"] & \mathfrak{h}(G) \arrow[d, "\Psi_G"] \\ \mathrm{gr}(N)\otimes\mathbb{Q} \arrow[r, "\mathrm{gr}(\iota)"] & \mathrm{gr}(G)\otimes\mathbb{Q} \end{tikzcd} \tag{72}$$

and fix a degree $k\geqslant 2$. By Proposition 9.2, $\mathrm{gr}_k(\iota)$ is an isomorphism. Thus, $\mathfrak{h}_k(\iota)$ is surjective. $\square$

The next theorem identifies (under certain conditions) the holonomy Lie algebra $\mathfrak{h}(N)$ as a Lie subalgebra of $\mathfrak{h}(G)$.

Theorem 9.5. *Let $N=\ker(\chi:G\twoheadrightarrow\mathbb{Z})$. Suppose $H_i(N,\mathbb{Q})$ has trivial $\mathbb{Q}\mathbb{Z}$-action, for $i=1,2$. Then, we have a split exact sequence of graded Lie algebras,*

$$0\longrightarrow \mathfrak{h}(N)\xrightarrow{\mathfrak{h}(\iota)}\mathfrak{h}(G)\xrightarrow{\mathfrak{h}(\chi)}\mathfrak{h}(\mathbb{Z})\longrightarrow 0. \tag{73}$$

In particular, the restriction of $\mathfrak{h}(\iota)$ to derived Lie subalgebras, $\mathfrak{h}'(\iota):\mathfrak{h}'(N)\to\mathfrak{h}'(G)$, is an isomorphism of graded Lie algebras.

Proof. Let $A=H^{\leqslant 2}(G,\mathbb{Q})$, so that $\mathfrak{h}(G)=\mathfrak{h}(A)$. Consider the element $a=\chi_{\mathbb{Q}}\in A^1$, and set $B=A/aA$. By Theorem 7.1, $B\cong H^{\leqslant 2}(N,\mathbb{Q})$. Therefore, $\mathfrak{h}(N)=\mathfrak{h}(B)$.

Since $H_{\leqslant 2}(N,\mathbb{Q})$ has trivial $\mathbb{Q}\mathbb{Z}$-action, a does not belong to $\mathcal{R}^1_1(G,\mathbb{Q})\cup\mathcal{R}^2_1(G,\mathbb{Q})$; see Remark 6.3. Hence, by Corollary 8.2, $\mathfrak{h}_2(G)=\mathfrak{h}_2(N)$ and $\mathfrak{h}_3(G)=\mathfrak{h}_3(N)$. It follows from Lemma 9.4 that $\mathfrak{h}_2(\iota)$ and $\mathfrak{h}_3(\iota)$ are isomorphisms.

Set $n=b_1(G)$, and pick bases $\{x_1,\dots,x_{n-1}\}$ for B_1 and $\{x_1,\dots,x_{n-1},y\}$ for A_1. Since $\mathfrak{h}_2(\iota):(B_1\wedge B_1)/B_2\to(A_1\wedge A_1)/A_2$ is an isomorphism, we may identify A_2/B_2 with $A_1\wedge$

$A_1/B_1 \wedge B_1$, which is a vector space with basis the cosets represented by $\{x_1 \wedge y, \dots, x_{n-1} \wedge y\}$. Thus, we may decompose A_2 as

$$A_2 = B_2 \oplus \operatorname{span}\{x_1 \wedge y - z_1, \dots, x_{n-1} \wedge y - z_{n-1}\},$$

where $z_i \in B_1 \wedge B_1$. Now define a linear map,

$$\alpha : B_1 \to B_1 \wedge B_1, \quad \alpha(x_i) = z_i,$$

and extend it to a degree 1 Lie algebra derivation, $\tilde{\alpha} : \operatorname{Lie}(B_1) \to \operatorname{Lie}(B_1)$. Given an element $s \in B_2 \subset \operatorname{Lie}_2(B_1)$, we have $\mathfrak{h}_3(\iota)(\tilde{\alpha}(s)) = [\mathfrak{h}_2(\iota)(s), y]$, and this vanishes in $\mathfrak{h}_3(A)$. Since $\mathfrak{h}_3(\iota)$ is an isomorphism, $\tilde{\alpha}(s) = 0$ in $\mathfrak{h}_3(B)$. In other words, $\tilde{\alpha}(B_2) \subset [B_1, B_2]$. Hence, $\tilde{\alpha}$ factors through a degree 1 Lie derivation, $\tilde{\alpha} : \mathfrak{h}(B) \to \mathfrak{h}(B)$. Since the graded Lie algebra $\mathfrak{h}(\mathbb{Z})$ is freely generated by y, the Lie algebra $\mathfrak{h}(A)$ splits as a semidirect product, $\mathfrak{h}(A) = \mathfrak{h}(B) \rtimes_{\tilde{\alpha}} \mathfrak{h}(\mathbb{Z})$, and we are done. □

For the Bestvina–Brady groups $N_\Gamma = \ker(\nu : G_\Gamma \twoheadrightarrow \mathbb{Z})$, the above results take a particularly simple form.

Corollary 9.6. *Let Γ be a finite, connected graph. If $H_1(\Delta_\Gamma, \mathbb{Q}) = 0$, then the inclusion map $\iota : N_\Gamma \hookrightarrow G_\Gamma$ induces a group isomorphism, $\iota' : N'_\Gamma \xrightarrow{\simeq} G'_\Gamma$, and an isomorphism of graded Lie algebras, $\mathfrak{h}'(\iota) : \mathfrak{h}'(N_\Gamma) \xrightarrow{\simeq} \mathfrak{h}'(G_\Gamma)$.*

Proof. By Corollary 7.4, $H_{\leqslant 2}(N_\Gamma, \mathbb{Q})$ has trivial $\mathbb{Q}\mathbb{Z}$-action. The conclusions follow from Lemma 9.1(4) and Theorem 9.5. □

This corollary generalizes [28, Lemma 6.2], proved under the more restrictive hypothesis $\pi_1(\Delta_\Gamma) = 0$.

10. Formality properties of Artin kernels

In this section we give certain conditions guaranteeing the 1-formality of a finitely-generated Artin kernel N_χ. These conditions may be satisfied, even when N_χ does not admit a finite presentation.

10.1. Malcev Lie algebras and 1-formality

In [31, Appendix A], Quillen defines a *Malcev Lie algebra* to be a rational Lie algebra E, endowed with a decreasing, complete, $\mathbb{Q}$-vector space filtration, $E = F_1 E \supseteq F_2 E \supseteq \dots$, with the property that $[F_s E, F_r E] \subseteq F_{s+r} E$, for all $s, r \geqslant 1$, and such that the associated graded Lie algebra, $\operatorname{gr}(E)$, is generated by $\operatorname{gr}_1(E)$.

An example of a Malcev Lie algebra is $\widehat{\mathfrak{h}}(G)$, the completion with respect to bracket length filtration of $\mathfrak{h}(G)$, the rational holonomy Lie algebra of a finitely generated group G. Clearly, $\operatorname{gr}(\widehat{\mathfrak{h}}(G)) = \mathfrak{h}(G)$.

Also in [31], Quillen associates to a group G a pronilpotent, rational Lie algebra, $\mathfrak{m}(G)$, called *the* Malcev Lie algebra of G. This functorial construction yields a Malcev Lie algebra with the crucial property that $\operatorname{gr}(\mathfrak{m}(G)) \cong \operatorname{gr}(G) \otimes \mathbb{Q}$, as graded Lie algebras.

Assume now G is finitely generated. Following Sullivan [35], we say G is 1-*formal* if $\mathfrak{m}(G) \cong \widehat{\mathfrak{h}}(G)$, as filtered Lie algebras. Equivalently, $\mathfrak{m}(G)$ is filtered Lie isomorphic to the degree completion of a quadratic Lie algebra. This fact is proved in [12, Lemma 2.9] for finitely presented groups, but the proof given there works as well in this wider generality.

If G is 1-formal, the map $\Psi_G : \mathfrak{h}(G) \to \mathrm{gr}(G) \otimes \mathbb{Q}$ is an isomorphism. Moreover, there is a (non-canonical) filtered Lie isomorphism, $\kappa_G : \mathfrak{m}(G) \xrightarrow{\simeq} \widehat{\mathfrak{h}}(G)$, with the property that $\mathrm{gr}_1(\kappa_G) = \mathrm{id}_{G_{\mathrm{ab}} \otimes \mathbb{Q}}$; see [12, Lemma 2.10], with the same proviso as above.

10.2. 1-*Formality of Artin kernels*

In [19], Kapovich and Millson showed that all finitely generated (in particular, right-angled) Artin groups are 1-formal. In [28], we showed that finitely presented Bestvina–Brady groups are 1-formal. We now generalize this last result to a wider class of (not necessarily finitely presented) Artin kernels.

As before, let $G = G_\Gamma$ be a right-angled Artin group, $\chi : G \twoheadrightarrow \mathbb{Z}$ an epimorphism, and $N = \ker(\chi)$ the corresponding Artin kernel.

Theorem 10.1. *Suppose $H_i(N, \mathbb{Q})$ has trivial $\mathbb{Q}\mathbb{Z}$-action, for $i = 1, 2$. Then N is finitely generated and* 1*-formal.*

Proof. Finite generation of N is assured by Lemma 9.1(1). As a functor from finitely generated groups to filtered Lie algebras, Malcev completion is right-exact. Applying the functor $\mathfrak{m}$ to the exact sequence (66), we get an exact sequence of filtered Lie algebras,

$$\mathfrak{m}(N) \xrightarrow{\mathfrak{m}(\iota)} \mathfrak{m}(G) \xrightarrow{\mathfrak{m}(\chi)} \mathfrak{m}(\mathbb{Z}) \longrightarrow 0. \tag{74}$$

The exactness of (68) and the natural isomorphism $\mathrm{gr}(\mathfrak{m}(\bullet)) \cong \mathrm{gr}(\bullet) \otimes \mathbb{Q}$ imply that $\mathrm{gr}(\mathfrak{m}(\iota))$ is injective. Since the filtration on $\mathfrak{m}(N)$ is complete, we conclude that $\mathfrak{m}(\iota)$ is injective; that is, sequence (74) is also exact on the left.

Let $\mathfrak{h}(\chi) : \mathfrak{h}(G) \twoheadrightarrow \mathfrak{h}(\mathbb{Z})$ be the morphism induced by χ at the level of holonomy Lie algebras. Passing to completions, we obtain a filtered Lie morphism, $\widehat{\mathfrak{h}}(\chi) : \widehat{\mathfrak{h}}(G) \twoheadrightarrow \widehat{\mathfrak{h}}(\mathbb{Z})$, whose kernel, denoted by $\mathfrak{K}$, we equip with the induced filtration.

Since G and $\mathbb{Z}$ are 1-formal, we have filtered Lie isomorphisms, $\kappa_G : \mathfrak{m}(G) \xrightarrow{\simeq} \widehat{\mathfrak{h}}(G)$ and $\kappa_{\mathbb{Z}} : \mathfrak{m}(\mathbb{Z}) \xrightarrow{\simeq} \widehat{\mathfrak{h}}(\mathbb{Z})$, normalized in degree 1, as explained in Section 10.1. Clearly, $\mathrm{gr}_{>1}(\mathbb{Z}) \otimes \mathbb{Q} = 0$, which implies that $F_2\mathfrak{m}(\mathbb{Z}) = 0$. Using the normalization property, we infer that $\kappa_{\mathbb{Z}} \circ \mathfrak{m}(\chi) = \widehat{\mathfrak{h}}(\chi) \circ \kappa_G$. We then have the following commuting diagram in the category of filtered Lie algebras:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}(N) \arrow[r, "\mathfrak{m}(\iota)"] \arrow[d] & \mathfrak{m}(G) \arrow[r, "\mathfrak{m}(\chi)"] \arrow[d, "\kappa_G"] & \mathfrak{m}(\mathbb{Z}) \arrow[r] \arrow[d, "\kappa_{\mathbb{Z}}"] & 0 \\ 0 \arrow[r] & \mathfrak{K} \arrow[r] & \widehat{\mathfrak{h}}(G) \arrow[r, "\widehat{\mathfrak{h}}(\chi)"] & \widehat{\mathfrak{h}}(\mathbb{Z}) \arrow[r] & 0 \end{tikzcd} \tag{75}$$

Since $\mathrm{gr}(\mathfrak{m}(\iota))$ is injective, the filtration of $\mathfrak{m}(N)$ is induced from $\mathfrak{m}(G)$. Hence, $\mathfrak{m}(N) \cong \mathfrak{K}$, as filtered Lie algebras. Now note that $\mathfrak{K}$ is the kernel of $\mathfrak{h}(\chi) : \mathfrak{h}(G) \to \mathfrak{h}(\mathbb{Z})$, completed with respect to degree filtration. By Theorem 9.5, $\ker(\mathfrak{h}(\chi)) = \mathfrak{h}(N)$. Hence, $\mathfrak{K} \cong \widehat{\mathfrak{h}}(N)$, as filtered Lie algebras, and we are done. □

10.3. Non-finitely presented, 1*-formal groups*

We now apply the above machinery to the Bestvina–Brady groups N_Γ. In [28, Proposition 6.1], we proved the following: If Δ_Γ is simply-connected (equivalently, if N_Γ is finitely presented), then N_Γ is 1-formal. We may strengthen that result, as follows.

Corollary 10.2. *Let Γ be a finite, connected graph. If $H_1(\Delta_\Gamma, \mathbb{Q}) = 0$, then N_Γ is finitely generated and* 1*-formal.*

Proof. By Corollary 7.4, $H_{\leqslant 2}(N_\Gamma, \mathbb{Q})$ has trivial $\mathbb{Q}\mathbb{Z}$-action. The conclusion follows from Theorem 10.1. □

We conclude with some examples of finitely generated Bestvina–Brady groups which are 1-formal, yet admit no finite presentation.

Example 10.3. Let $L = \Delta_\Gamma$ be a flag triangulation of the real projective plane, $\mathbb{RP}^2$. Clearly, Γ is connected. On the other hand, $H_1(L, \mathbb{Z}) = \mathbb{Z}_2$, and so, by [2], N_Γ is not finitely presented. But $H_1(L, \mathbb{Q}) = 0$, and so, by Corollary 10.2, N_Γ is 1-formal.

Example 10.4. Let $L = \Delta_\Gamma$ be a flag triangulation of a spine of the Poincaré homology sphere. In [2], Bestvina and Brady noted the following facts about the group N_Γ: it is of type FP_∞ (since $\widetilde{H}_*(L, \mathbb{Z}) = 0$), but not finitely presented (since $\pi_1(L) \neq 0$). Our Corollary 10.2 shows that N_Γ is 1-formal.

Before finishing, we cannot but recall from [2] the following striking alternative about this group: either N_Γ is a counterexample to the Eilenberg–Ganea conjecture, or the Whitehead conjecture is false. It would be interesting to know whether the formality property of N_Γ can play a role in deciding the Bestvina–Brady alternative.

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