Necessary and Sufficient Conditions for Optimality for Singular Control Problems; A Transformation Approach

JASON L. SPEYER*

Analytical Mechanics Associates, Inc., Cambridge, Massachusetts

AND

DAVID H. JACOBSON†

Division of Engineering of Applied Physics
Harvard University, Cambridge, Massachusetts

Submitted by R. Bellman

Necessary and sufficient conditions for optimality for singular control problems are presented for the case where the extremal path is totally singular. These conditions are obtained by transforming the perturbed states of the accessory minimum problem in the calculus of variations to a new set of state variables in which the dimension of the state space can be reduced. The accessory problem in the reduced state space is nonsingular, and thus previously established necessary and sufficient conditions are applicable. It is shown that Jacobson's sufficiency conditions for singular control problems applied to the total transformed state space are equivalent to those of the nonsingular accessory problem thereby establishing necessity as well as sufficiency for Jacobson's conditions. In fact, since the transformation is nonsingular, necessity as well as sufficiency of Jacobson's conditions is established in the original space. In a companion paper a limit argument is used to establish necessity of Jacobson's sufficient conditions without the need to transform to a reduced state space.

I. INTRODUCTION

Tests of local optimality of extremal paths containing singular arcs are not yet complete. Kelley [1, 2] generalized the Legendre-Clebsch condition and Jacobson added an additional necessary condition [3] as well as sufficiency conditions [4]. If all available necessary conditions are passed but

* Supported by NASA Contract NAS 12-656. Presently with the Charles Stark Draper Laboratory, Massachusetts Institute of Technology, Cambridge, Mass.
† Partially Supported by NASA Grant NGR 22-007-068 and ONR Contract N00014-67-A-0298-0006.
the sufficiency test fails then doubt still remains as to the optimality of the singular arc.

A transformation approach was suggested by Kelley [5] which allows analysis of singular problems in a reduced state space. This approach has the practical shortcoming of requiring that the solution of a system of nonlinear differential equations, required for synthesis of the transformation, be obtained in “closed form.”

In this paper, Kelley’s transformation is applied to the perturbed state variables involved in the accessory minimum problem about a singular extremal. Fortunately, the transformation can be obtained in closed form allowing the structure of the accessory problem to be studied in a reduced state space. This is quite attractive; the accessory problem becomes nonsingular, so that existing necessary and sufficient conditions are applicable [6, 7, 11]. The generalized Legendre-Clebsch condition is found directly without the need of special variations [2]. Goh [12] and McDanell and Powers [13] have independently obtained similar results applying Goh’s transformation to the accessory problem.

Jacobson’s sufficiency conditions, applied to the transformed problem, are equivalent to the necessary conditions for this transformed nonsingular problem. That is, they are equivalent to the generalized Legendre-Clebsch condition and to the Jacobi condition in the form of a matrix Riccati equation. Jacobson’s conditions are now seen to be both necessary and sufficient. Since the transformation is invertible (the determinant of the Jacobian is one) and since Jacobson’s conditions are coordinate independent, Jacobson’s conditions are necessary and sufficient for optimality of singular problems in the original state space. A companion paper [8] demonstrates from a limit approach that Jacobson’s sufficiency conditions are also necessary. The limit approach has the advantage that necessity is established without transforming to a reduced state space (If the problem is singular of order higher than one, then reduction of the state space to achieve a nonsingular problem requires repeated application of the transformation technique; this is cumbersome especially if there are multiple control variables).

If the original control variable is unbounded, the conditions apply only to weak variations of the control variable of the transformed problem. However, if the original control variable is bounded, the restriction to a weak minimum in the transformed variables loses significance since only weak variations of the control variable of the transformed problem can be obtained.

Control problems without terminal constraints are considered first. The results generalize easily to the case where constraints on the terminal states are present. Jacobson’s sufficient conditions with terminal constraints are given here in slightly different form than that of [4].
II. Problem Formulation

Consider a dynamical system described by a set of first order nonlinear differential equations
\[ \dot{x} = f(x, u, t), \quad x(t_0) \text{ given}, \]  
where
\[ f(x, u, t) = f_1(x, t) + f_u(x, t) u, \]  
x is an \(n\)-dimensional state vector, \(u\) is an \(m\)-dimensional control vector, and \(t\) is the independent variable. \(f\) is a known nonlinear \(n\)-vector function of the state and time \(t\) and a linear function of the control.

The problem is to find a control program \(u(\cdot)\) in the fixed time interval \([t_0, t_f]\) which minimizes the cost functional
\[ V(x_0, t_0) = F[x(t_f), t_f] + \int_{t_0}^{t_f} [L_1(x, t) + u^T L_2(x, t)] dt, \]  
subject to the dynamic constraints (1), the terminal constraints
\[ \psi[x(t_f), t_f] = 0 \]  
and the control constraints
\[ u(\cdot) \in U, \]  
where
\[ U = \{ u(\cdot) : |u_i(t)| \leq 1, \ t \in [t_0, t_f], i = 1, \ldots, m \}. \]  

Here \(F\) and \(L_1\) are known scalar functions, \(\psi\) is a known \(p\)-vector function and \(L_2(x, t)\) is a known \(m\)-vector function. The functions \(f, L_1, L_2, F\) and \(\psi\) are assumed to be as many times continuously differentiable as needed in subsequent sections.

III. First Order Necessary Conditions For Totally Singular Problems

Along an extremal path Pontryagin’s Minimum Principle yields the following necessary conditions
\[ \lambda^T = -H_x, \quad \lambda^T(t_f) = [F_x + u^T \psi]_{t=t_f} \]  
\[ \text{\footnote{The partial derivative of some scalar function } Q(x) \text{ by a column vector } x \text{ is a row vector } Q_x = (\partial Q/\partial x).} \]
and

\[ u = \arg \min_{|u_i| \leq 1} H \tag{8} \]

where \( H \) the variational Hamiltonian is defined as

\[ H(x, u, \lambda, t) = L_1(x, t) + u^T L_2(x, t) + \lambda^T f(x, u, t). \tag{9} \]

Here \( \lambda \) is an \( n \)-vector Lagrange multiplier associated with the dynamics (1) and \( \nu \) is a \( p \)-vector Lagrange multiplier associated with the terminal constraints (4).

The variational Hamiltonian (9) is linear in the control. Using (8) the two classes of arcs that can occur are called "bang-bang" and "singular". On a bang-bang arc \( H_{u_i} \neq 0; \; i = 1, \ldots, m \) (except at a finite number of switch times). Bang-bang arcs will not be considered here.

On a singular arc

\[ H_{u_i} = 0, \quad i = 1, \ldots, m \tag{10} \]

for a finite time interval. This means that along a singular arc \( H \) is explicitly independent of the control \( u \). Here, the totally singular arc is considered, extending over the entire interval \([t_0, t_f] \). Extremal paths which are combinations of "bang-bang" arcs and singular arcs are not considered. All the controls are considered to be singular simultaneously.

IV. The Accessory Minimum Problem (Without Terminal Constraints)

The accessory minimum problem (second variation) is a secondary minimization problem within the context of the original problem. This new problem is; find a control deviation which minimizes the second variation of the augmented performance index expanded about an extremal path. From [7] the expression for the second variation is

\[ \delta^2 V = \frac{1}{2} \delta x^T F_{xx} \delta x \bigg|_{t=t_f} + \frac{1}{2} \int_{t_0}^{t_f} \left[ \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x + \delta x^T H_{zu} \delta u \right] dt. \tag{11} \]

The control deviation, \( \delta u(\cdot) \), in the interval \([t_0, t_f] \) is to be found which minimizes (11) subject to

\[ \delta \dot{x} = f_x \delta x + f_u \delta u; \quad \delta x(t_0) = 0 \tag{12} \]

for \( \delta u(\cdot) \) sufficiently small to justify a second order expansion of \( V \) (it is shown later that much weaker conditions justify (11)), and such that

\[ [u(\cdot) + \delta u(\cdot)] \in U, \tag{13} \]
where \( \bar{u}(\cdot) \) is the extremal control. It is assumed that \( \bar{u}(\cdot) \) never touches the boundary of \( U \) and in fact remains a first order variation away from \( U \) in all the components of \( \delta u(\cdot) \).

To simplify the forthcoming algebra convert the terminal function to integral form as

\[
\frac{1}{2} \delta x^T F_{xx} \delta x \bigg|_{t_f} = \frac{1}{2} \int_{t_0}^{t_f} \left( \delta x^T (t) \left[ F_{xx}(t_f) f_x(t) + f_x^T(t) F_{xx}(t_f) \right] \delta x(t) + \delta x^T(t) F_{xx}(t_f) f_u(t) + \delta u^T(t) f_u^T(t) F_{xx}(t_f) \delta x(t) \right) dt.
\]

(14)

Let

\[
A(t) = H_{xx}(t) + F_{xx}(t_f) f_x(t) + f_x^T(t) F_{xx}(t_f)
\]

(15)

and

\[
B(t) = H_{uu}(t) + f_u^T(t) F_{xx}(t_f).
\]

(16)

Then (11) becomes simply

\[
\delta^2 V = \frac{1}{2} \int_{t_0}^{t_f} [\delta x^T A \delta x + \delta u^T B \delta x + \delta x^T B^T \delta u] dt.
\]

(17)

The accessory minimum problem retains the same singular form as the original problem; namely, the control enters linearly into the cost (17) and dynamics (12).

V. TRANSFORMATION OF PERTURBED STATE VARIABLES

1. Mayer Form of Accessory Problem

Kelley's transformation approach is applicable to the Mayer form of the accessory minimum problem. An additional state variable, \( x_a \), whose dynamics are described by the integrand of (17), augments the present state vector as

\[
\delta x_a = \frac{1}{2} \delta x^T A \delta x + \delta x^T B^T \delta u; \quad \delta x_a(t_0) = 0,
\]

\[
\delta x = f_x \delta x + f_u \delta u; \quad \delta x(t_0) = 0.
\]

(18)

For simplicity we will assume that the control is a scalar although the extension to a vector control is straightforward but notationally cumbersome.

2. Kelley's Transformation

Introduce a new set of variables \((x_0, z)\) as

\[
x_0 = \gamma_0(\delta x_0, \delta x), \quad z = \gamma(\delta x_0, \delta x)
\]

(19)
where \( x_0 \) is a scalar and \( z \) is an \( n - 1 \) vector. \((x_0, z)\) are to be found such that their dynamic equations are not functions of the control:

\[
\begin{align*}
\dot{z}_0 &= \frac{\partial y_0}{\partial x_0} \left[ \frac{1}{2} \delta x^T A \delta x \right] + \frac{\partial y_0}{\partial x} f_u \delta x \\
\dot{z} &= \frac{\partial y}{\partial x_0} \left[ \frac{1}{2} \delta x^T A \delta x \right] + \frac{\partial y}{\partial x} f_u \delta x
\end{align*}
\]

where \((y_0, y)\) are chosen so that the coefficients of the control variable \( \delta u \) vanish

\[
\begin{align*}
\frac{\partial y_0}{\partial x_0} B \delta x + \frac{\partial y_0}{\partial x} f_u &= 0 \\
\frac{\partial y}{\partial x_0} B \delta x + \frac{\partial y}{\partial x} f_u &= 0.
\end{align*}
\]


The transformation requires solving the homogeneous quasilinear partial differential equation (21). Equation (21) can be solved by the method of characteristics [9] which are described by first order ordinary differential equations

\[
\begin{align*}
\frac{d \delta x_0}{ds} &= B(t) \delta x \\
\frac{d \delta x}{ds} &= f_u(t)
\end{align*}
\]

in which \( s \) is a parameter and \( t \) is regarded as fixed. These equations can be solved by changing the independent variable from \( s \) to one of the states, say \( \delta x_n \). If we denote the \( n - 1 \) subvectors of the above \( n \)-vectors less the element pertaining to \( \delta x_n \) by bold characters (i.e. \( f_u = (f_u^T, f_{u, u}^n) \) and \( \delta x^T = (\delta x^T, \delta x_n) \)) then

\[
\begin{align*}
\frac{d \delta x_0}{d \delta x_n} &= \frac{B \delta x}{f_u^n} \\
\frac{d \delta x}{d \delta x_n} &= \frac{f_u}{f_u^n}
\end{align*}
\]

where \( f_u^n \) is assumed non-zero. Fortunately (23) can be solved in closed form in terms of \( n \) integrating constants \( C = (C_0, C) \) as

\[
\begin{align*}
\delta x &= \frac{f_u}{f_u^n} \delta x_n + C \\
\delta x_0 &= \frac{(B f_u)}{2(f_u^n)^2} \delta x_n^2 + \left(BC\right) \frac{\delta x_n}{f_u^n} + C_0.
\end{align*}
\]
It is easily seen that

$$C = \gamma(\delta x_0, \delta x)$$  \hspace{1cm} (25)

form \(n\) mutually independent solutions to the homogeneous partial differential equation (21). Thus using (19), (24) and (25) the desired transformation is

$$z_0 = \delta x_0 - \frac{Bf_u}{2(f_u^n)^2} \delta x_n - \left( Bz \right) \delta x_n$$

$$z = \delta x - \frac{f_u}{f_u^n} \delta x_n$$

$$z_n = \delta x_n$$  \hspace{1cm} (26)

Although Eqs. (26a) are defined according to (25), (26b) is an arbitrary choice being chosen because \(f_u^n \neq 0\) over the domain of interest. The above transformation is nonsingular since the Jacobian determinant

$$\det \left( \frac{\partial (z_0, \ldots, z_n)}{\partial (\delta x_0, \ldots, \delta x_n)} \right) = 1.$$  \hspace{1cm} (27)

4. Dynamic Equations of Transformed Variables

The dynamic equations in the variables \((z_0, z)\) are determined by differentiating (26) with respect to time and using (18)

$$\dot{z}_0 = \frac{\delta x^T A \delta x}{2} + \delta u^T B \delta x - \frac{d}{dt} \left[ \frac{Bf_u}{2(f_u^n)^2} \delta x_n^2 - \frac{Bf_u}{(f_u^n)^2} [f_u^n \delta x + f_u^n \delta u] \right]$$

$$\dot{z}_n = f_u \delta x + f_u \delta u - \frac{d}{dt} \left[ \left( \frac{f_u}{f_u^n} \right) \left( f_u^n \delta x + f_u^n \delta u \right) \right]$$

$$\left[ \dot{z}_0 \dot{z}_n \right] = \left( \frac{f_u}{f_u^n} \right) \left[ f_u^n \delta x + f_u^n \delta u \right] - \frac{d}{dt} \left[ \left( \frac{f_u}{f_u^n} \right) \delta x_n \right].$$  \hspace{1cm} (28)

The transformation was designed so that terms involving \(\delta u\) in \(\dot{z}_0\) and \(\dot{z}\) are identically zero. Furthermore, terms involving \(z_n^2\) in (28) reduce to the form

$$\alpha = (-1) \frac{\partial}{\partial u} \left[ \frac{d^2}{dt^2} H_u \right]/(f_u^n)^2$$  \hspace{1cm} (30)

\(^2\text{Consider an} \ n\text{-column vector} \ p(x). \ \text{The partial derivative} \ p_u(x) = \frac{\partial p}{\partial x} \text{is partitioned as}

$$p_u = \begin{bmatrix} p_x & p_x^n \\ p_x^n & p_{x^n} \end{bmatrix}$$
which is the form of the generalized Legendre-Clebsch condition. If this term is zero, then additional transformations must be made until the strong form of this condition appears. For now we assume that $\alpha \neq 0$ for all $t \in [t_0, t_f]$.

Let us now make the following additional identifications:

$$C = \{ A - 2B^T f_u^n / f_u^n \},$$

$$D^T = \left\{ f_u^n A + f_u^n A_1^T - \frac{B}{f_u^n} \left[ f_x \right] - \frac{Bf_x f_u}{(f_u^n)^2} - \frac{d}{dt} \left[ \frac{B}{f_u^n} \right] \right\},$$

$$J = \{ f_x - f_u f_x^n / f_u^n \},$$

$$K = \left\{ \left( f_x f_{x_u} f_u^n / (f_u^n)^2 - \frac{d}{dt} (f_u f_u^n) \right) \right\},$$

$$W = [ f_x f_u^n + f_{x_u} ],$$

where $A$ of (15) is partitioned as

$$A = \begin{bmatrix} A & A_1 \\ A_1^T & A_{11} \end{bmatrix},$$

and $A_{11}$ is here a scalar.

Finally we write the dynamical system of the transformed variables as

$$\dot{z}_0 = \frac{1}{2} \left\{ [z^T, z_n] \begin{bmatrix} C & D \\ D^T & \alpha \end{bmatrix} [z, z_n] \right\},$$

$$\dot{z} = f_z + K z_n,$$

and

$$\dot{z}_n = (f_x^n) z + W z_n + f_u^n \delta u.$$

VI. NECESSARY AND SUFFICIENT CONDITIONS IN THE TRANSFORMED VARIABLE SPACE OF REDUCED DIMENSIONS

1. Optimization Problem in Transformed Space

The accessory problem can now be stated in the transformed state space as find the control deviation $\delta u(\cdot)$ to minimize

$$\delta^2 V = \frac{1}{2} [z, z_n] \begin{bmatrix} 0 & B^T / f_u^n \\ B / f_u^n & B f_u / (f_u^n)^2 \end{bmatrix} \begin{bmatrix} z_n \end{bmatrix} + \frac{1}{2} \int_{t_0}^{t_f} [z, z_n] \begin{bmatrix} C & D \\ D^T & \alpha \end{bmatrix} [z, z_n] dt,$$

(40)
subject to (37) and (38). Since $\delta u$ only enters into the $z_n$ equation and thereby affects the cost only through $z_n$, the state space can be reduced by making $z_n$ the control variable. This is done while assuming that a $\delta u$ can always be found to give the required $z_n$ while meeting the constraint $u(\cdot) + \delta u(\cdot) \in U$; see Section VI-4. Note that $z_n$ appears in the terminal function of (40).

2. Accessory Minimum Problem in Reduced Space

With the control variable $z_n$ the accessory minimum problem is to be solved by defining a variational Hamiltonian

$$\mathcal{H} = \frac{1}{2} [z, z_n] \begin{bmatrix} C & D \\ D^T & \alpha \end{bmatrix} [z, z_n] + \rho^T (Jz + Kz_n), \quad (41)$$

and also

$$\Phi = \left[ \frac{Bf_u}{2(f_u^{-1})^2}, z_n^2 \right] \left[ \frac{Bz}{f_u^n}, z_n \right]_{t = t_f}, \quad (42)$$

where $\rho$ is an $n - 1$ vector Lagrange multiplier associated with (37). Both $\mathcal{H}$ and $\Phi$ are to be minimized with respect to $z_n$. The result of operating on (42) is

$$z_n(t_f) = -\frac{(Bz)f_u^n}{(Bf_u)}. \quad (43)$$

Note that a minimum is assured if

$$Bf_u > 0. \quad (44)$$

This is the strong form of Jacobson's necessary condition evaluated at the terminal time [3]. Eliminating $z_n$, defined by (43), in $\Phi$ of (42) is

$$\Phi = -\frac{1}{2} \frac{(Bz)^2}{(Bf_u)} \leq 0. \quad (45)$$

The Euler-Lagrange equations described in canonical form from (41) are

$$\dot{\rho} = -\frac{\partial \mathcal{H}}{\partial z} = -Cz - Dz_n - J^T \rho \quad (46)$$

with boundary condition

$$\rho(t_f) = -\frac{B^T Bz(t_f)}{(Bf_u)}. \quad (47)$$

Also

$$0 = \mathcal{H}_{z_n} = D^T z + \alpha z_n + \rho^T K, \quad (48)$$
which implies that

\[ z_n = - \frac{1}{\alpha} [D^T z + \rho^T K]. \tag{49} \]

As previously recognized \( (30) \), \( \alpha(f_u^{(n)})^2 \) is the generalized Legendre-Clebsch condition in which \( \alpha \geq 0 \). The necessity of this condition is obtained directly without the need of special variations \( [2] \). It is assumed that \( \alpha > 0 \), otherwise additional transformations must be made. The number of transformations needed defines the order of the singular arc.

Using (49) to eliminate \( z_n \) in (37) and (46), the \((2n - 2)\) set of first order differentials equations results

\[
\begin{bmatrix}
\dot{z} \\
\dot{\rho}
\end{bmatrix} = \begin{bmatrix}
J - \frac{KD^T}{\alpha}, & -\frac{KK^T}{\alpha} \\
- C + \frac{DD^T}{\alpha}, & \left( J - \frac{KD^T}{\alpha} \right)^T
\end{bmatrix} \begin{bmatrix}
z \\
\rho
\end{bmatrix};
\]

\[
\begin{bmatrix}
z(t_f) \\
\rho(t_f)
\end{bmatrix} = \begin{bmatrix}
I \\
-\frac{B'B}{Bf_u}
\end{bmatrix} z(t_f).
\]

3. Conjugate Point Conditions (Jacobi Test)

The conjugate point test can be made in two ways. First, a matrix solution can be found for (50) using terminal conditions

\[
\begin{bmatrix}
Z(t_f) \\
Y(t_f)
\end{bmatrix} = \begin{bmatrix}
I \\
-\frac{B'B}{Bf_u}
\end{bmatrix}.
\]

A conjugate point occurs \([10, 11]\) when

\[
\det[Z(t)] = 0. \tag{52}
\]

The second way is to develop a Riccati equation for the symmetric \((n - 1) \times (n - 1)\) matrix \( P \) which forms the relation

\[
\rho = Pz. \tag{53}
\]
P is propagated by the Riccati equation

\[ \dot{P} + \left[ C - \frac{DD^T}{\alpha} \right] + \left[ J - \frac{KD^T}{\alpha} \right]^T P + P \left[ J - \frac{KD^T}{\alpha} \right] - \frac{P KK^T P}{\alpha} = 0, \]  

(54)

with the terminal boundary conditions

\[ P(t_f) = -\frac{B^TB}{Bf_u}. \]  

(55)

The conjugate point occurs at the time when \( P \) becomes infinite [6, 11].

4. Admissible Control Variations

From (43) and (49), \( z_n \), in general, is discontinuous at the terminal time. This demands a richer class of control variables than that defined by (5), (6). However, this richer class of discontinuous controls \( \{z_n\} \) can be approximated closely enough by a subset of continuous controls so that conclusions which are based upon the assumption that the richer class is feasible are valid.

If all the coefficients in Section VI-2 are continuous then \( z_n \) found from the accessory problem is continuous except at the terminal point. If \( z \) is small enough then there exists an admissible \( \delta u \) which will produce the desired \( z_n \) everywhere except at the terminal point (By differentiating (49) and using (50), (53) and (38) it is easy to relate \( \delta u \) to \( z \)). At the terminal point a discontinuous \( z_n \) is approximated by the dominant term of (38) integrated over the time interval \( \Delta T \), as

\[ z_n(t_f) \approx z_n(t_f - \Delta T) + f_u \delta u \Delta T. \]  

(56)

Here the order of \( z \) and \( \Delta T \) is higher than \( \delta u \) so that the error in the second variational performance index (40) caused by approximating the discontinuity in \( z_n(t_f) \) by (56) is negligible (third order). Thus the conclusions of Section VI-5 which are based upon Section VI-3 (where discontinuous \( z_n(t_f) \) is allowed) are valid. McDanell and Powers [13] appear to have overlooked this point with the result that they restrict attention to a much smaller class of singular problems than is necessary.

If weak variations in \( \delta u \) are made, additional (higher-order) terms in the Taylor Series expansion of the performance index (11) and dynamics (12) are negligible. However, if strong variations in \( \delta u \) are made, the error in neglecting the term \( \delta x^T H_{xu} \delta x \delta u \) and \( f_{uu} \delta x \delta u \) must be considered since \( \delta x^T H_{xu} \delta x \) is the same order as \( \delta x^T H_x \delta x \), and \( f_{uu} \delta x \) is of the same order as \( f_x \delta x \). In Section IX, Kelley's transformation is applied to a performance index which includes third order terms. In this example the matrix \( f_{uu} \) is zero. The result is that the term \( \delta x^T H_{xu} \delta x \delta u \) transforms into third order terms in \( x \).

If the generalized Legendre-Clebsch condition and the Jacobson necessary
condition hold in the strong form, these third order terms can be neglected. If $f_{ux} \neq 0$, the term $f_{ux}\delta x \delta u$ transforms into second order terms in $x$ in the transformed dynamics and into third order terms in the performance index. Thus, this term is also negligible. Thus, there is no need to restrict variations in $\delta u$ to be weak, so that strong variations in $\delta u$ are permitted.

5. Necessary and Sufficient Condition

Reference [6, 7] prove that $\delta^2 V$ for $\alpha > 0$ will be positive definite for all $x$ if and only if the interval $[t_0, t_f]$ contain no conjugate points. From the transformed accessory minimum problem, if the following strengthened necessary conditions are satisfied:

1. $a! > 0$; the strengthened generalized Legendre-Clebsch condition.
2. $Bf_{u} > 0$; strengthened Jacobson necessary condition at the terminal time.$^3$
3. No conjugate point exists in the interval $t \in [t_0, t_f]$,

then we have sufficiency for a weak local minimum in the transformed space.

If the control $u$ is unbounded, then strong variations in $x_u$ could be induced. Therefore, for an unbounded control variation the above (strengthened) conditions are necessary and sufficient for a weak minimum. A generalized Weierstrass condition might be found by including higher order terms in the expansion of the cost function. However, if the control is bounded as in (6), then only weak variations (first order variations) in $x_u$ can be obtained. Thus the restriction to a weak minimum in the transformed space $x$ loses significance [2], (i.e. the conditions are sufficient for $V$ to have a strong relative minimum with respect to $u(\cdot)$). Section IX considers this question in more detail.

VII. Relationship to Other Sufficiency Conditions
For Singular Problems

Jacobson [4] gives sufficient conditions for a weak relative minimum for the accessory problem (11) and (12). These conditions are that there exist a real symmetric, bounded, matrix function of time, $P(\cdot)$ such that

$$ II_{ux} + f_u^T P = 0, \quad \forall t \in [t_0, t_f], \quad (57) $$

$$ \dot{P} + H_{xx} + f_x^T P + P f_x = \bar{M}(t) > 0, \quad \forall t \in [t_0, t_f], \quad (58) $$

Note that if $Bf_u = 0$ and $B = 0$ at the terminal time, we may still obtain sufficiency by restricting attention to the class of problems where $f_u(x, t) = f_u(t)$ and $F[x(t_f)]$ is quadratic in $x(t_f)$; the boundary condition is $P(t_f) = 0$. (cf. [13] where $F[x(t_f)]$ is assumed linear in $x(t_f)$ and $f_u(x, t) = f_u(t)$). This restriction insures that higher order terms will not dominate.
and

\[- P(t_f) + F_{xx} = \bar{G}(t_f) > 0. \quad (59)\]

These conditions are now applied to the transformed accessory minimum problem described by (40) subject to (37) and (38). A weakened form of these conditions is shown to be equivalent to those of Section VI establishing that the above weakened sufficiency conditions are both necessary and sufficient. Jacobson's sufficiency conditions become

\[- P(t_f) + \bar{G}(t_f) > 0. \quad (59)\]

From (56) noting that \( P \) is symmetric

\[ P_{mn} = P_{nm} = \cdots = P_{nn} = 0, \quad \forall t \in [t_0, t_f]. \quad (63)\]

This implies that the time derivatives of these variables are also zero. From (61) using (63), the partitioned matrix results

\[ \begin{bmatrix} 0 & B^T f_u^n \\ B f_u^n & \frac{B f_u}{(f_u^n)^2} \end{bmatrix} = G(t_f) > 0. \quad (62)\]

Also from (62) using (63), the partitioned matrix results

\[ \begin{bmatrix} - P & B^T f_u^n \\ B f_u^n & \frac{B f_u}{(f_u^n)^2} \end{bmatrix} = \bar{G}(t_f) > 0. \quad (65)\]

The necessary and sufficiency conditions of Section VI structured in the form of (64) and (65) are

\[ \begin{bmatrix} \dot{P} + C + J^T P + PJ, & D + KP \\
\dot{D}^T + K^T P, & \alpha \end{bmatrix} = M > 0. \quad (64)\]

\[ \begin{bmatrix} (D + PK)/\alpha^{1/2} \\ \alpha^{1/2} \end{bmatrix} \left[ \begin{bmatrix} \frac{(D + PK)^T}{\alpha^{1/2}} \\ \alpha^{1/2} \end{bmatrix}, \frac{(D + PK)^T}{\alpha^{1/2}}, \frac{(D + PK)^T}{\alpha^{1/2}} \right] \geq 0 \quad (66)\]
and
\[
\begin{bmatrix}
-P & B^T f_u^m \\
B^m f_u & \frac{B f_u}{(f_u^m)^{3/2}}
\end{bmatrix} = \left[ B^T (B f_u)^{1/2} / f_u^m \right] \left[ B / (B f_u)^{1/2} \right] \frac{(B f_u)^{1/2}}{f_u^m} \geq 0
\]

respectively.

The sufficiency conditions given in [4] are equivalent to the necessary and sufficiency conditions of the previous section where \(M\) and \(G\) are positive semi-definite matrices of rank one. This insures that the generalized Legendre-Clebsch condition and the Jacobson's necessary condition are satisfied in strong form as
\[
\begin{bmatrix} 0 & f_u^m \\ f_u^m & M \end{bmatrix} \begin{bmatrix} 0 \\ f_u^m \end{bmatrix} = \alpha > 0,
\]

\[
\begin{bmatrix} 0 & f_u^m \\ f_u^m & G(t_f) \end{bmatrix} \begin{bmatrix} 0 \\ f_u^m \end{bmatrix} = B f_u > 0.
\]

Since the transformation is nonsingular and since Jacobson's conditions are coordinate independent, Jacobson's conditions are also necessary in the original state space. In fact, if the generalized Legendre-Clebsch condition is satisfied with strict inequality with only one transformation (first order singular arc) and the Jacobson necessary condition is satisfied in the strong form, then in the original state space only the strong forms
\[
f_u^T M f_u > 0,
\]

\[
f_u^T G(t_f) f_u > 0,
\]

respectively, need be satisfied; clearly (70), (71) are weaker than \(\bar{M} > 0, \ G > 0\).

VIII. Example. The following example is designed to illustrate the new necessary and sufficient conditions. The problem is: find the control \(u\) to minimize
\[
V = \int_{t_0}^{t_f} u [B_1 x_1 + B_2 x_2] \, dt
\]
subject to the dynamics
\[
\dot{x}_1 = -x_2, \quad x_1(t_0) = 0,
\]

\[
\dot{x}_2 = u, \quad x_2(t_0) = 0.
\]

\(\dagger\) This is easy to prove. Let \(y = \theta(t)x\) where \(\theta(t)\) is an \(n \times n\) nonsingular matrix function of time; then equation (57) in the \(y\) space can be written as \(H_u \theta^{-1} + f_u^T \theta P = 0\). Multiplying this by \(\theta\) yields \(H_u \theta + f_u^T P = 0\), where \(P = \theta^T \theta\). Similarly for inequalities (58) and (59).
Applying the Kelley transformation, the new set of state variables are
\[ x_0 = B_1 z_1 z_2 + \frac{B_2}{2} z_2^2 + z_0, \]  
\[ x_1 = z_1, \]  
and
\[ x_2 = z_2. \]

The problem in the transformed space is to minimize
\[ V = \frac{1}{2} [z_1 \ z_2] \begin{bmatrix} 0 & B_1 \\ B_1 & B_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \int_{t_0}^{t_f} B_1 z_2^2 \, dt \]  
subject to
\[ \dot{z}_1 = -z_2 \]  
and
\[ \dot{z}_2 = u. \]

Using \( z_2 \) as the control variable, minimizing (78) subject to (79) is a non-singular problem. The strong form of the generalized Legendre–Clebsch condition is
\[ B_1 > 0. \]

The strong form of the Jacobson necessary condition [3] at the terminal time is
\[ B_2 > 0. \]

We now test to see if the generalized Jacobi condition is satisfied. Define the variational Hamiltonian as
\[ H = B_1 z_2^2 - \lambda z_2 \]
and
\[ \Phi(t_f) = \left[ B_1 z_1 z_2 + \frac{B_2}{2} z_2^2 \right]_{t=t_f}. \]

By minimizing \( \Phi(t_f) \)
\[ z_2 = -\frac{B_1 z_1}{B_2} \]
then
\[ \Phi(t_f) = -\frac{(B_1 z_1)^2}{2B_2}. \]
For this problem the Riccati equation is

$$\dot{P} - \frac{P^2}{2B_1} = 0, \quad P(t_f) = -\frac{B_1^2}{B_2}. \quad (87)$$

The solution of (87) is

$$P(t) = \frac{2B_1}{(t_f - t) - \frac{2B_2}{B_1}}. \quad (88)$$

A conjugate point occurs at

$$(t_f - t) = \frac{2B_2}{B_1}. \quad (89)$$

Thus in the interval $0 \leq t_f - t < 2B_2/B_1$ the arc $x_1(t) = 0, x_2(t) = 0, u(t) = 0$ is a local minimum.

**IX. Strong Minimum For Bounded Control**

If strong variations in the control deviations $\delta u$ are allowed, higher-order terms should be considered in the expansion of the performance index (11) as

$$\delta^2 V = \frac{1}{2} \delta x^T F_{xx} \delta x \bigg|_{t=t_f} + \frac{1}{2} \int_{t_0}^{t_f} \left[ \delta x^T H_{xx} \delta x + 2 \delta u H_{ux} \delta x + \delta u \delta x^T H_{xux} \delta x \right] dt \quad (90)$$

Here we assume that $f_{ux} = 0$. The additional term of the expansion, $\delta u \delta x^T H_{xux} \delta x$ maybe of the same order as $\delta x^T H_{xx} \delta x$. If $H_{ux} \neq 0$ then $\delta u H_{ux} \delta x$ is an order lower than $\delta u \delta x^T H_{xux} \delta x$ and thus dominates in the expansion. Actually this additional term can be neglected under much weaker conditions. To show this, the Kelley transformation is applied to the accessory minimum problem using (90) now as the performance index. The term $\delta x^T H_{xux} \delta x \delta u$ adds only third order terms to the performance index in the transformed state variables, $z$ (the control variable $\delta u$ only enters the problem through the $z_n$ equation). With $\delta u$ bounded, variations in $z_n$ are weak; therefore, the second order terms will dominate the third order terms if both the generalized Legendre-Clebsch condition and Jacobson’s necessary condition evaluated at the terminal time are invoked in the strong form. Thus, for strong but bounded control variations, $\delta u$, the previous necessary and sufficiency conditions still hold. If $\delta u$ is unbounded, $z_n$ is not restricted to weak variations in $z_n$ but our analysis is: thus our conditions in this case are for a weak minimum.
To illustrate the above we expand upon the example of Section VIII. The performance index is

\[ I' = J \{ -a_1w_1^2 + a_2w_2^2 + \frac{1}{2}x_1^2 + 4x_1 + 4x_2 + B_1x_1 + B_2x_2 + B_3x_1^2 + B_4x_2^2 + B_5x_1x_2 \} dt \]

with dynamics

\[ \dot{x}_1 = -x_2 \]

and

\[ \dot{x}_2 = u. \]

Using the Kelley transformation the problem becomes

\[ V = \left[ B_1z_1z_2 + B_2z_2^2 + B_3z_1^2z_2 + B_4z_3^2 + B_5z_1z_2^2 \right] t_{t_f} \]

\[ + \int_{t_0}^{t_f} \left[ a_1z_1^2 + a_2z_1x_2 + a_3z_2^2 + B_1z_1x_2^2 + 2B_3z_1z_2^2 + B_5z_3^2 \right] dt, \]

\[ \dot{z}_1 = -z_2, \]

\[ \dot{z}_2 = u, \]

\[ |u| \leq 1. \]

If the strong form of the generalized Legendre-Clebsch is invoked, \([\alpha_3 + B_1] > 0\), then the terms under the integral are dominated by the second order terms. If the strong form of Jacobson's necessary condition is invoked, \(B_2 > 0\), then the second order terms dominate the terminal function. These conditions guarantee that both the variational Hamiltonian and the terminal function have a minimum with respect to \(z_n\).

X. Constrained Terminal State for Totally Singular Arcs

Terminal constraints (4) can be handled with little additional difficulty. Adjoin (4) to the performance index (3) with a Lagrange multiplier \(p\)-vector \(v\). The second variation is properly adjusted if \(F_{xx} \) is replaced by \(F_{xx} + v^T\psi_{xx} \) in the performance index (11) and subsequently in (15) and (16). The accessory minimum problem in the transformed state space now becomes; Minimize (40) subject to the differential constraints (37), (38) and

\[ \psi_x \delta x \bigg|_{t=t_f} = \left[ \psi_x z + \psi_x f_u z_n \right]_{t=t_f} = 0. \]
Let us partition $\psi_x$ as

$$
\psi_x = \begin{bmatrix} \psi_x^p \\ \psi_x^n \end{bmatrix},
$$

where $\psi_x$ is a $(p-1) \times (n-1)$ matrix, $\psi_x^p$ is an $(n-1)$ vector, $\psi_x^n$ is a $(p-1)$ vector and $\psi_x^n$ is a scalar. In the spirit of Section VI one of the constraints is eliminated by choosing $z_n$ to satisfy the $p$-th constraint as

$$
z_n = - \frac{\int_u \psi_x^p z}{\psi_x^n f_u}
$$

where it is assumed that $\psi_x^n f_u \neq 0$. The number of constraints is now reduced by one where (98) is written as

$$
\begin{bmatrix} \psi_x - \frac{\psi_x f_u \psi_x^p}{\psi_x^n f_u} \end{bmatrix} z = 0.
$$

With the control variable $z_n$, the variational Hamiltonian for the accessory minimum problem in the reduced transformed space is given by (41) and $\Phi(t_f)$ is given by (42). However, $\Phi(t_f)$ is rewritten using (100)

$$
\Phi(t_f) = z^T \left[ \frac{(\psi_x^p)^T B f_u (\psi_x^p)}{2(\psi_x^n f_u)^2} - \frac{(\psi_x^p)^T B}{\psi_x^n f_u} \right] z.
$$

The accessory minimum problem in a reduced transformed state space with terminal constraints (101) is nonsingular. Reference [7] shows that the Jacobi test requires solving

$$
\dot{P} + \left[ \mathcal{C} - \frac{\mathcal{D} \mathcal{T}}{\alpha} \right] + \left[ J - \frac{K \mathcal{D} \mathcal{T}}{\alpha} \right]^T P + P \left[ J - \frac{K \mathcal{D} \mathcal{T}}{\alpha} \right] - \frac{P K K^T P}{\alpha} = 0
$$

with boundary condition

$$
P(t_f) = \left[ \frac{(\psi_x^p)^T B f_u (\psi_x^p)}{2(\psi_x^n f_u)^2} - \frac{(\psi_x^p)^T B}{\psi_x^n f_u} \right].
$$

Note that only the symmetric part contributes to $\Phi(t_f)$ in (102). The following additional equations must also be solved

$$
\dot{R} + \left[ \left( J - \frac{K \mathcal{D} \mathcal{T}}{\alpha} \right) - \frac{P K K^T}{\alpha} \right] R = 0,
$$

with boundary conditions

$$
R(t_f) = \left[ \psi_x - \frac{\psi_x f_u \psi_x^p}{\psi_x^n f_u} \right].
$$
and finally
\[
\dot{Q} + R^T \frac{K^T K}{\alpha} R = 0; \quad Q(t_f) = 0. \tag{107}
\]

Reference [7] shows that a conjugate point occurs when
\[
[P - RQ^{-1}R^T] \to \infty. \tag{108}
\]

XI. JACOBSON’S SUFFICIENCY CONDITION WITH CONSTRAINTS

Sufficiency conditions for optimality for constrained totally singular arcs, equivalent to those given by Jacobson [4], are developed below. In the transformed state space a weakened form of these conditions is equivalent to the necessary and sufficient conditions established in Section X. Since the transformation is nonsingular (20), we obtain necessary and sufficient condition in the original state space.


The second variational performance index (11), adjusted for constraints by replacing \( F_{xx} \) by \( F_{xx} + \nu^T \psi_{xx} \) is augmented by the linearized differential equations (12) and by the linearized terminal constraints, \( \delta \psi = \psi \delta x(t_f) \), with an \( n \)-vector multiplier \( \delta \lambda \) and a \( p \)-vector multiplier \( dv \), respectively, as

\[
\delta^2 V = \left( \frac{1}{2} \delta x^T (F_{xx} + \nu^T \psi_{xx}) \delta x + dv^T (\psi \delta x - \delta \psi) \right)_{t=t_f} + \int_{t_0}^{t_f} [\delta x^T A \delta x + \delta u B \delta x + \delta \lambda^T (f_{x} \delta x + f_{u} \delta u - \delta x)] dt. \tag{109}
\]

If \( \delta \lambda \), chosen as
\[
\delta \lambda(t) = \frac{1}{2} P(t) \delta x(t) + R(t) dv, \tag{110}
\]
and \( \delta \psi \), chosen as
\[
\delta \psi = R^T(t) \delta x(t) + Q(t) dv, \tag{111}
\]
are introduced into (109), if the terms \( \delta x^T P \delta x \) and \( dv^T R^T \delta x \) are integrated by parts, and if the identically zero quantity
\[
dv^T \left[ -Q(t_f) + Q(t_0) + \int_{t_0}^{t_f} Q dt \right] dv = 0 \tag{112}
\]
is added to (109), the augmented performance index takes the form
\[
\delta^aV = \frac{1}{2} \left[ \delta x^T \delta v^T \right] \left[ \begin{array}{cc}
(p_{xx} + \nu^T \phi_{xx} - P) & (\psi_x^T - R) \\
(\psi_x - R^T) & -Q
\end{array} \right] \left[ \begin{array}{c}
\delta x \\
\delta v
\end{array} \right]_{t=0}^{t_1}
+ \left[ \delta x^T \delta v^T \right] \left[ \begin{array}{c}
P \\
0
\end{array} \right] \left[ \begin{array}{c}
\delta x \\
\delta v
\end{array} \right]_{t=0}^{t_1}
+ \frac{1}{2} \int_{t_0}^{t_1} [\delta x^T, \delta v^T, \delta u] \left[ \begin{array}{ccc}
P_{xx} + \nu^T \phi_{xx} & \psi_x & -R^T \\
R & -Q & R^T \\
H_{ux} + f_u^T P & f_u^T R & 0
\end{array} \right] \left[ \begin{array}{c}
\delta x \\
\delta v \\
\delta u
\end{array} \right] dt,
\] (113)
where \( P \) is an \( n \times n \) symmetric matrix, \( R \) is an \( n \times p \) matrix and \( Q \) is a \( p \times p \) matrix.

We compare trajectories with the same initial and terminal conditions, \( \delta x(t_0) = 0 \) and
\[
\delta \psi = \psi \delta x(t_1) = 0,
\] (114)
to determine conditions for \( \delta^aV > 0 \). Using (114) and \( \delta v = -Q^{-1}R^T \delta x \), obtained from (111), equation (113) reduces to
\[
\delta^aV = \frac{1}{2} \left[ \delta x \left( p_{xx} + \nu^T \phi_{xx} - \hat{P} \right) \right] \delta x \right|_{t=0}^{t_1}
+ \frac{1}{2} \int_{t_0}^{t_1} [\delta x^T \delta u] \left[ \begin{array}{c}
\hat{P} + Pf_x + f_x^T P + H_{xx}, \\
R + f_x^T R, \\
H_{ux} + f_u^T P,
\end{array} \right] \left[ \begin{array}{c}
\delta x \\
\delta u
\end{array} \right] dt,
\] (115)
where
\[
\hat{P} = P - RQ^{-1}R^T.
\] (116)

From (114), \( \delta \psi \) of the \( \delta \psi \)'s are linearly related to \( n - p \) of the remaining \( \delta \psi \)'s, called \( \delta x^{n-p} \). Partition \( \psi_x = [S_1; S_2] \) so that \( S_1 \) is a nonsingular \( p \times p \) matrix (i.e., assume \( \phi_x \) has rank \( p \)). Then \( \delta \psi \) can be written in terms of \( \delta x^{n-p} \) as
\[
\delta \psi(t_1) = \left[ -S_1^{-1} S_2 \right] \delta x^{n-p} = S \delta x^{n-p}.
\] (117)
The terminal function, \( \Phi \), of (115) is now
\[
\Phi = \frac{1}{2} \left( \delta x^{n-p} \right)^T S^T \left( p_{xx} + \nu^T \phi_{xx} - \hat{P} \right) S \delta x^{n-p}.
\] (118)

Jacobson’s sufficiency condition for \( \delta^aV > 0 \) for the constrained singular arc [4], analogous to those given in Section VII for the unconstrained arc, are then
\[
H_{ux} + f_u^T \hat{P} = 0,
\] (119)
\[
\hat{P} + Pf_x + f_x^T \hat{P} + H_{xx} = M^1(t) > 0,
\] (120)
and
\[ S^T[F_{xx} + v^T \psi_{xx} - \dot{P}] S = G(t_f) > 0. \] (121)

Equation (119) annihilates the cross term \( \delta u \delta x \) in (115) while (120) and (121) insure that the remaining quadratic terms in (115) and (118) are positive.

2. Equivalence Between Jacobson's Sufficiency Conditions and the Necessary Conditions of Section X

Jacobson's sufficiency conditions for \( \delta V > 0 \) can be written in the variables of Section X by using (113) directly but imposing (117) on the terminal function. Then an enlarged set of sufficiency conditions (equivalent to those of (119) to (121) if \( P, Q \) and \( R \) are combined through (116)) is constructed as

\[ H_{ux} + f_u^T P = 0, \] (122)

\[ f_u^T R = 0, \] (123)

\[ \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix} + \begin{bmatrix} f_x^T & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} f_x^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix} + \begin{bmatrix} H_{xx} & 0 \\ 0 & 0 \end{bmatrix} = N > 0. \] (124)

\[ \begin{bmatrix} S^T(F_{xx} + v^T \psi_{xx} - P) S, & S^T(\psi_x^T - Q) \end{bmatrix} = G(t_f) > 0. \] (125)

Note that (124), (125) and (112) imply that \( -Q(t_0) > 0 \).

These conditions, applied to the transformed accessory minimum problem described by the performance index (40) subject to (37), (38) and the transformed constraints (98), become\(^5\)

\[ [0, f_x^n] P = 0, \] (126)

\[ [0, f_x^n] R = 0, \] (127)

\[ \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix} + \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} K & W \\ W^T & 0 \end{bmatrix} + \begin{bmatrix} J^T(f_x^n)^T \\ 0 \end{bmatrix} \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix} \]

\[ + \begin{bmatrix} C & D \\ 0 & 0 \end{bmatrix} = N > 0, \] (128)

\(^5\) Using (127), \( R \) is partitioned as

\[ R = \begin{bmatrix} R & R_p \\ 0 & 0 \end{bmatrix} \]

where \( R \) is an \( (n - 1) \times (p - 1) \) matrix and \( R_p \) is an \( (n - 1) \) vector.
From (126) (note \( P \) is symmetric)

\[
P_{n_1} = P_{n_2} = \cdots = P_{n_n} = 0 \quad \forall t \in [t_n, t_f]
\]  

(130)

and from (127)

\[
R_{n_1} = R_{n_2} = \cdots = R_{n_p} = 0 \quad \forall t \in [t_0, t_f]
\]  

(131)

This implies that the time derivatives of these variables are also zero.

In Section X, the \( p \)-th constraint was already used to linearly relate \( z_n \) to \( z \). Let us then suppose that the remaining \( p - 1 \) constraints relate \( z \) to \( z^{n-p} \) \((z \) has \( n - 1 \) elements) as

\[
z = Sz^{n-p}.
\]  

(132)

Then \( S \) becomes

\[
S = \begin{bmatrix} I \\ -f_u^n \psi_x^p \\
\psi_x^n f_u \end{bmatrix}.
\]  

(133)

Using (133), (130) and (131) in (129), we have\(^6\)

\[
\begin{bmatrix}
\begin{bmatrix} 0 & B^T f_u^n \\
B f_u^n & \frac{B f_u}{(f_u^n)^2} \\
& \\
\psi_x^n f_u & \\
\psi_x^n f_u \end{bmatrix}
- P, \\
- P \\
S^T \left[ \begin{bmatrix} (\psi_x)^T f_u^n \end{bmatrix} \right] - R^T, \\
- R^T \\
\left[ \begin{bmatrix} \psi_x f_u \end{bmatrix} \right] \\
- R_p^T \\
\end{bmatrix} S,
\end{bmatrix}
\]  

(134)

\[= G > 0.\]

\(^6\) Let

\[
Q = \begin{bmatrix} Q & Q_1 \\
Q_1^T & Q_{11} \end{bmatrix}
\]

where \( Q \) is a \((p - 1) \times (p - 1)\) matrix, \( Q_1 \) is a \((p - 1)\) vector and \( Q_{11} \) is a scalar.
If the quantities obtained in Section X are used for $P(t_f)$, $R(t_f)$, and $Q(t_f)$ where we chose

$$R_p(t_f) = 0, \quad Q_1(t_f) = 0, \quad Q_{11}(t_f) = 0,$$

then (134) reduces to

$$G(t_f) = 0.$$  (136)

We have assumed that at least one element of $\psi_x f_u$ is non zero (i.e., at least one of the constraints depends upon $x_n$). If this is not the case, then, in general, the strong form of Jacobson's necessary condition must be invoked; here $G(t_f)$ would have rank one.

Furthermore, if we manipulate the differential equations of Section X into the form of (128), we obtain a weakened form for the Jacobson sufficiency conditions

$$L = P + P^T + J^T P + C, \quad D + PK, \quad \dot{R} + J^T R, \quad R_p + J^T R_p$$

$$= \left[ \begin{array}{cccc}
\dot{P} + P^T + J^T P + C, & D + PK, & \dot{R} + J^T R, & R_p + J^T R_p \\
D^T + K^T P, & \alpha, & K^T R, & K^T R_p \\
R^T + R^T J, & R^T K, & \dot{Q}, & \dot{Q}_1 \\
R + R^T J, & R_p^T K, & \dot{Q}_1^T, & \dot{Q}_{11}
\end{array} \right] = N(t) \geq 0.$$  (137)

The boundary conditions (135) imply that $R_p(t), Q_1(t)$ and $Q_{11}(t)$ are zero. The necessary and sufficient conditions of Section X are equivalent to a weakened form of Jacobson's sufficiency condition where $G(t_f) = 0$ and $N(t) \geq 0$. $N(t)$ must be positive semi-definite of rank one to insure that $\alpha$ is positive for first order singular arcs. Since the transformation is nonsingular and Jacobson's conditions are coordinate independent, we conclude that in the original space the sufficiency conditions (124) and (125) can be weakened to $G(t_f) = 0$ and $N(t) \geq 0$.

Similarly, the sufficiency condition (120) can be weakened by requiring only that the

$$M^1(t) \geq 0$$  (138)

and

$$f_u^T M^1(t) f_u > 0.$$  (139)

This is similar to the unconstrained case which is discussed in Section VII.
XI. Conclusions

A transformation proposed by Kelley is applied to the accessory minimum problem of a totally singular optimal control problem. One of the transformed states can be chosen as a control variable which causes the accessory problem (in reduced state space) to be nonsingular. Conditions corresponding to the generalized Legendre-Clebsch condition and Jacobson necessary condition (for the unconstrained problem) can be identified. In addition a generalized Jacobi test is found for problems with and without terminal constraints. Thus enough necessary conditions are obtained to imply sufficiency for totally singular arcs.

Furthermore, Jacobson's sufficiency conditions for singular control problems applied in the transformed state space are equivalent to the Riccati equation, the generalized Legendre-Clebsch condition and for unconstrained problems, the Jacobson necessary condition. This demonstrates that these conditions are also necessary. Since the transformation is nonsingular and Jacobson's conditions are coordinate independent, necessity as well as sufficiency for Jacobson's conditions [4] is established in the original state space for first order arcs for constrained as well as unconstrained problems. If the problem is singular of order higher than one (i.e., if \( \alpha = 0 \)) then the transformation technique must be used repeatedly until a nonsingular accessory problem is obtained. This can be tedious especially if there are multiple controls. Furthermore, the transformation technique requires that the parameters of the accessory problem (second-variation) \( H_{xx}, H_{ux}, f_x, f_u \) be many times differentiable with respect to \( t \). A companion paper [8] proves directly the necessity and sufficiency of Jacobson's conditions [4] and circumvents the above difficulties.

REFERENCES


