The Singular Limit Dynamics of Semilinear Damped Wave Equations*

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INTRODUCTION

In this paper we consider the qualitative dynamics of a one-dimensional semilinear damped wave equation in the singular limit corresponding to the vanishing of the coefficient of the second-order time derivative, in which limit the equation turns into a semilinear diffusion one. Henceforth, the coefficient just mentioned will be denoted by $\varepsilon^2$, so that the limit under consideration corresponds to $\varepsilon \to 0$. By combining our result with that of Henry [10] and Angenent [1], we shall be able to conclude that if the stationary states are hyperbolic then, for sufficiently small values of $\varepsilon$, the global attractor of the considered damped wave equation is equivalent to that of the limiting diffusion equation, i.e., there is a homeomorphism between both global attractors which preserves orbits together with their orientation.

Our approach to the problem is based upon the fact that for small values of $\varepsilon$ the global attractor is contained in a certain finite-dimensional invariant manifold of class $C^1$. For the limiting diffusion equation the existence of such invariant manifolds, or at least Lipschitzian manifolds with analogous properties, has been treated by several authors including Henry [9, Chap. 6 and particularly p. 166], Mañé [13], Mora [14], Foias, et al. [6], Constantin, et al. [5], and Mallet-Paret and Sell [12]. The case of $C^1$ manifolds, which is the one that interests us, is considered in particular by Mañé [13] and Mallet-Paret and Sell [12]. For a semilinear damped wave equation like the one considered here, the existence of $C^1$ manifolds with the properties above has been established by Mora [15] under the condition that $\varepsilon$ be sufficiently small. On the other hand, for large values of $\varepsilon$, Mora and Solà-Morales [16] have given an example where generically the global attractor ceases to be contained in a finite-dimensional invariant manifold of class $C^1$.

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The core of the present work consists in a detailed study of the way that the above mentioned manifolds depend on $\varepsilon$ as this parameter tends to zero. Specifically, it will be shown that there exist an integer $n$ and a real number $\delta > 0$ such that, for every $\varepsilon$ belonging to the interval $[0, \delta)$, the global attractor of the corresponding dynamical system is contained in an invariant manifold of class $C^1$ and dimension $n$, and that for $\varepsilon \to 0$ both this manifold and the vector field on it converge in the $C^1$ topology towards the ones corresponding to $\varepsilon = 0$.

Some independent results about this problem have been obtained recently by Chow and Lu [4], whose paper contains a general study of the existence of smooth invariant manifolds containing the global attractor for a class of problems which in particular includes those considered here. Concerning the limiting behaviour of such manifolds in the singular limit described above, they obtain a result of convergence essentially in the $C^0$ topology. Another related work is that of Hale and Raugel [8], who centre their attention directly on the global attractor and show that for $\varepsilon \to 0$ this set converges in the Hausdorff topology towards the one corresponding to $\varepsilon = 0$. Although this property is weaker than the one obtained here, their result applies to the more general case of several space variables. Also, a different result on the same problem has been announced recently by Babin and Vishik [3].

1. The Equations and Some Preliminaries

Our results will apply specifically to the following problem, where $u$ is a function of $x \in (0, L)$ and $t \in \mathbb{R}$ with values in $\mathbb{R}$,

\begin{align*}
v^2 u_{tt} + 2 u u_t &= \beta u_{xx} + f(x, u) + q(x) \quad (1.1) \\
|u|_{x=0} &= \rho_0, \quad |u|_{x=L} = \rho_L \quad (1.2)_D \\
|u|_{t=0} &= u_0, \quad |u_t|_{t=0} = \varepsilon v_0 \quad (1.3)
\end{align*}

or the analogous one where $(1.2)_D$ is replaced by

\begin{align*}
|u|_{x=0} &= \sigma_0, \quad |u|_{x=L} = \sigma_L \quad (1.2)_N
\end{align*}

Henceforth, the boundary conditions $(1.2)_D$ or $(1.2)_N$ will be referred to as $(1.2)_B$, where $B$ stands for either $D$ or $N$. In the preceding equations, $\varepsilon$ is a real parameter which we consider to vary right up to the value $\varepsilon = 0$, $\alpha$ and $\beta$ are fixed positive real parameters, $f$ is a function $(0, L) \times \mathbb{R} \to \mathbb{R}$, $q$ is a fixed function of $x \in (0, L)$, $\rho_0, \rho_L, \sigma_0, \sigma_L$ are real numbers, and the initial data $u_0$ and $v_0$ are given functions of $x \in (0, L)$. The function $f$ is assumed to satisfy the following conditions:
(f1) \( f(\cdot, u) \) belongs to the Sobolev space \( H^1(0, L) \) for every \( u \in \mathbb{R} \), and in the case \( B = D \) \( f \) satisfies the condition \( f(0, \rho_0) = f(L, \rho_L) = 0 \); \( f(x, \cdot) \) is of class \( C^{2+\eta} \) for every \( x \in (0, L) \), \( f_{xx}(x, \cdot) \) is of class \( C^{1+\eta} \) for almost every \( x \in (0, L) \), and for every bounded open interval \( J \subset \mathbb{R} \), the quantities

\[
\sup_{x \in (0, L)} \| f(x, \cdot) \|_{C^{2+\eta}} \quad \text{and} \quad \int_0^L \| f_x(x, \cdot) \|_{C^{1+\eta}}^2 \, dx
\]

are both finite.

\((f^*)\) \( c := \lim \sup_{|u| \to \infty} \sup_{x \in (0, L)} \frac{f(x, u)}{u} \leq \begin{cases} \beta \pi^2/L^2, & \text{if } B = D \\ 0, & \text{if } B = N \end{cases} \) \( (1.4) \)

Concerning the function \( q \), we assume simply that it belongs to \( L_2(0, L) \).

**Remark.** The case \( B = D \) with \( f(0, \rho_0) \) or \( f(L, \rho_L) \) not equal to zero can be reduced to the preceding one by letting \( \rho : (0, L) \to \mathbb{R} \) be any smooth function satisfying \( \rho(0) = \rho_0 \), and \( \rho(L) = \rho_L \), and changing \( f(x, u) \) and \( q(x) \) respectively by \( f(x, u) - f(x, \rho(x)) \) and \( q(x) + f(x, \rho(x)) \).

Let \( u^* \) be the solution in \( H^2(0, L) \) of the equation \( \beta u_{xx} + q(x) = 0 \) with the nonhomogeneous boundary conditions \( (1.2)_B \). By switching over to the new variable \( u := u - u^* \), the problem reduces to the homogeneous case \( q = 0, \rho_0 = \rho_L = 0, \sigma_0 = \sigma_L = 0 \); the role of \( f \) is now played by the function \( \tilde{f}(x, \tilde{u}) := f(x, u^*(x) + \tilde{u}) \), which can be verified to inherit properties (f1) and \((f^*)\) from \( f \). Furthermore, by suitably rescaling time and space and dividing Eq. (1.1) by a constant, the problem can be normalized to \( 2x = 1, \beta = 1, L = \pi \). Henceforth, problem (1.1), (1.2)_\( B \), (1.3) will always be considered in this particular normalized homogeneous form.

The preceding problem will be dealt with as a particular case of an abstract second order evolution problem on a Hilbert space. In the following, this space is denoted by \( E \), and its inner product and the corresponding norm are denoted respectively by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \). Together with this space we assume to be given a self-adjoint linear operator \( A \) having numerical range bounded from below and compact resolvent. The first of these two conditions means that

\[
\lambda_1 := \inf_{u \in \text{Dom}(A)} \frac{\langle Au, u \rangle}{\langle u, u \rangle} > - \infty. \quad (1.5)
\]

As is well known, \( \lambda_1 \) coincides with the smallest element of the spectrum of \( A \). Associated with the space \( E \) and the operator \( A \), we consider also the Hilbert spaces \( E^x \) \( (x = 0, \frac{1}{2}, 1) \) each of which consists of the domain of the
operator \((A - \xi I)^\alpha\) endowed with the inner product associated with the norm

\[ \|u\|_{\alpha, \xi} := \|(A - \xi I)^\alpha u\|, \]

(1.6)

where \(\xi\) is a real number smaller than \(\lambda_1\) (different choices of \(\xi < \lambda_1\) result in the same vector space with different but equivalent inner products). In the following, the inner product corresponding to norm (1.6) will be denoted by \(\langle \cdot, \cdot \rangle_{\alpha, \xi}\), and if the specific value of \(\xi\) is not relevant we shall use simply the notations \(\|\cdot\|_\alpha\) and \(\langle \cdot, \cdot \rangle_\alpha\).

The abstract problem under consideration has the following form, where \(u\) stands now for a function of \(t \in \mathbb{R}\) with values in the Hilbert space \(E\), and the dot denotes differentiation with respect to \(t\):

\[ \varepsilon^2 \ddot{u} + \dot{u} + Au = Fu \]

(1.7)

\[ u(0) = u_0, \quad \varepsilon \dot{u}(0) = \varepsilon v_0. \]

(1.8)

Here, \(\varepsilon\) is the real parameter which we consider to vary right up to the value \(\varepsilon = 0\), and \(F\) is a nonlinear operator satisfying the following conditions (for the definition of the spaces \(C_{bd}\), see the end of this section):

\((F1)\) \(F\) maps \(E^{1/2}\) to itself, and this mapping belongs to \(C_{bd}^{1+\eta}(E^{1/2}, E^{1/2})\) for some \(\eta > 0\).

\((F2)\) There exists a functional \(\Gamma: E^{1/2} \to \mathbb{R}\) belonging to \(C_{bd}^{2+\eta}(E^{1/2}, \mathbb{R})\) such that

\[ D\Gamma(u)v = \langle Fu, v \rangle, \quad \forall u, v \in E^{1/2}. \]

\((F^*)\) There exists a real number \(\kappa < \lambda_1\) and finite constants \(C_1\) and \(C_2\) such that

\[ \Gamma(u) \leq \frac{1}{2} C_1 + \frac{1}{2} \kappa \langle u, u \rangle, \quad \forall u \in E^{1/2} \]

\[ \langle Fu, u \rangle \leq C_2 + \kappa \langle u, u \rangle, \quad \forall u \in E^{1/2}. \]

Problem (1.1), (1.2), (1.3) can be viewed as a particular case of the preceding one where \(E = L^2 := L^2(0, L)\), and \(A\) and \(F\) are the operators given by

\[ Au = -u_{xx} \]

(1.9)

\[ Fu = f(\cdot, u(\cdot)) \]

(1.10)

with domains \(E^1\) and \(E^{1/2}\) respectively equal to \(H^2_B\) and \(H^1_B\). Here \(H^k_B\) \((k = 1, 2)\) denote the closures in the Sobolev spaces \(H^k := H^k(0, L)\) of the
set \( \{ u; (0, L) \to \mathbb{R} | u \in C^\infty([0, L]) \} \) and satisfies the boundary conditions (1.2). Conditions (f1), (f*) on \( f \) ensure that \( F \) satisfies conditions (F1), (F2), (F*) with \( \Gamma \) given by

\[
\Gamma(u) = \int_0^L \gamma(x, (u(x))) \, dx, \quad \text{where} \quad \gamma(x, u) := \int_0^u f(x, w) \, dw, \quad (1.11)
\]

and with \( \kappa \) being a number slightly greater than the constant \( c \) of (f*).

As is well known, the hypotheses on \( A \) imply that \(-A\) is the generator of an analytic semigroup on \( E \). This fact together with condition (F1) on \( F \) determines that problem (1.7), (1.8) with \( \varepsilon = 0 \) fits in the standard theory of semilinear evolution equations of parabolic type, which ensures that it generates a (for the moment local) semidynamical system of class \( C^{1+\eta} \) on \( E^{1/2} \).

As usual, for \( \varepsilon \neq 0 \) we shall take as state variable the pair \((u, \dot{u}) =: (u, v) =: U\), which variable takes values in \( E^{1/2} \times E \). In terms of this variable, problem (1.7), (1.8) takes the first-order form

\[
\ddot{U} + \mathbb{A}_\varepsilon U = \mathbb{F}_\varepsilon U \quad (1.12)
\]

\[
U(0) = U_0, \quad (1.13)
\]

where \( \mathbb{A}_\varepsilon \) and \( \mathbb{F}_\varepsilon \) denote the operators on \( E^{1/2} \times E \) given by

\[
\mathbb{A}_\varepsilon(u, v) = (-v, \varepsilon^{-2}(Au + v)) \quad (1.14)
\]

\[
\mathbb{F}_\varepsilon(u, v) = (0, \varepsilon^{-2}Fu) \quad (1.15)
\]

with domains respectively equal to \( E^1 \times E^{1/2} \) and \( E^{1/2} \times E \). It is a standard fact that \(-\mathbb{A}_\varepsilon\) is the generator of a group on \( E^{1/2} \times E \). On the other hand, condition (F1) implies that \( \mathbb{F}_\varepsilon \) maps \( E^{1/2} \times E \) to itself and that this mapping belongs to \( C^{1+\eta}(E^{1/2} \times E, E^{1/2} \times E) \). With this, the problem fits in the standard theory of semilinear evolution equations of hyperbolic type, which ensures that it generates a (for the moment local) dynamical system of class \( C^{1+\eta} \) on \( E^{1/2} \times E \).

Condition (F2) provides these problems with a global Lyapunov functional. For \( \varepsilon = 0 \) this functional is given essentially by

\[
\Phi_0(u) = \frac{1}{2} \langle Au, u \rangle - \Gamma(u) \quad \text{for} \quad u \in E^1. \quad (1.16)
\]

By rewriting \( \langle Au, u \rangle \) as \( \langle (A - \xi I)^{1/2} u, (A - \xi I)^{1/2} u \rangle + \xi \langle u, u \rangle = \|u\|_{1/2, \xi}^2 + \xi \langle u, u \rangle \), one sees immediately that \( \Phi_0 \) extends uniquely to a functional on \( E^{1/2} \) of class \( C_0^{\text{bdd}}(E^{1/2}, \mathbb{R}) \), and in fact of class \( C_0^{2+\eta}(E^{1/2}, \mathbb{R}) \). The derivative of \( \Phi_0 \) along (strict) solutions of (1.7) with \( \varepsilon = 0 \) is easily verified to be

\[
\frac{d}{dt} \Phi_0(u) = -\|\dot{u}\|^2 \leq 0. \quad (1.17)
\]
On the other hand, condition (F*) implies that the level sets of $\Phi_0$, 
$\{u \mid \Phi_0(u) \leq c\} \ (c \in \mathbb{R})$, are bounded in $E^{1/2}$. Specifically, one finds that for 
every $\xi \in [\kappa, \lambda_1)$
\[ \|u\|^2_{1/2, \zeta} \leq C_1 + 2\Phi_0(u), \quad \forall u \in E^{1/2}. \]  
(1.18)

Combined with the decreasing character of $\Phi_0$, this fact provides an a priori bound on the solutions as $t \to +\infty$, which implies that the corresponding semidynamical system is global.

For $\varepsilon \neq 0$, the Lyapunov functional is given by
\[ \Phi_\varepsilon(u, v) := \Phi_0(u) + \frac{1}{2} \varepsilon^2 \|v\|^2. \]  
(1.19)

From the preceding remarks on $\Phi_0$, it follows immediately that $\Phi_\varepsilon$ belongs to $C^2_{\text{bdd}}\!\!(E^{1/2} \times E, \mathbb{R})$, and that its level sets are bounded in $E^{1/2} \times E$. Specifically, for every $\xi \in [\kappa, \lambda_1)$,
\[ \|u\|^2_{1/2, \zeta} + \varepsilon^2 \|v\|^2 \leq C_1 + 2\Phi_\varepsilon(u, v), \quad \forall (u, v) \in E^{1/2} \times E. \]  
(1.20)

In the present case, the derivative of $\Phi_\varepsilon$ along (strict) solutions of (1.12) is given by
\[ \frac{d}{dt} \Phi_\varepsilon(u, v) = -\|v\|^2 \leq 0. \]  
(1.21)

Similarly as in the case $\varepsilon = 0$, by combining (1.20) with the decreasing character of $\Phi_\varepsilon$, we obtain an a priori bound on solutions as $t \to +\infty$. Furthermore, in the present case we can introduce (1.20) in the right-hand side of (1.21) to obtain the inequality
\[ \frac{d}{dt} \Phi_\varepsilon(u, v) \geq -2\varepsilon^{-2}(\Phi_\varepsilon(u, v) + \frac{1}{2}C_1). \]  
(1.22)

By solving this inequality for $\Phi_\varepsilon(u, v) + \frac{1}{2}C_1$ and introducing the result in (1.20), we obtain an exponential bound on solutions as $t \to -\infty$, namely
\[ \|u\|^2_{1/2, \zeta} + \varepsilon^2 \|v\|^2 \leq (C_1 + 2\Phi_\varepsilon(u_0, v_0)) \exp(-2\varepsilon^{-2}t), \quad \forall t \leq 0. \]  
(1.23)

Therefore, we can conclude that in the present case the dynamical system is global both in positive and negative time.

It is a well-known fact that, both for $\varepsilon = 0$ and for $\varepsilon \neq 0$, the preceding problem has a compact global attractor in the sense of Babin and Vishik [2] and Hale [7]. In the following this set will be denoted by $\mathcal{A}_\varepsilon$. We recall that $\mathcal{A}_\varepsilon$ consists of all initial states for which the solution is defined and bounded on $(-\infty, 0]$. In Section 3 we shall make use of the fact that there exists a finite constant $C$ independent of $\varepsilon$ such that, for every $s$, $\mathcal{A}_\varepsilon$
is contained in the bounded and positively invariant set \( \Phi_0 < C \) (i.e., for \( \varepsilon = 0 \) it is contained in the set \( \{ u \in E^{1/2} | \Phi_0(u) < C \} \), and for \( \varepsilon \neq 0 \) it is contained in the set \( \{ (u, v) \in E^{1/2} \times E | \Phi_\varepsilon(u, v) < C \} \)). This is a consequence of the fact that the set of stationary states is contained in such a set \( \{ \Phi_\varepsilon < C \} \). The fact that \( C \) can be taken independent of \( \varepsilon \) follows from the fact that the stationary states are independent of \( \varepsilon \), and that for \( \varepsilon \neq 0 \) they have \( v = 0 \) and therefore they have \( \Phi_\varepsilon \) equal to \( \Phi_0 \).

We end this section by making precise our notation concerning nonlinear operators between Banach spaces. In the following, \( X \) and \( Y \) stand for Banach spaces, \( W \) stands for a domain (i.e., an open and connected subset) of \( X \), and \( F \) stands for a mapping \( W \to Y \). Henceforth, the sup norm of such a mapping will be denoted as \( \sup F \), and its Hölder–Lipschitz seminorms will be denoted as \( \text{Lip}_\eta F \) \((0 < \eta \leq 1)\), with the usual abbreviation \( \text{Lip}_1 F =: \text{Lip} F \). As usual, for \( k \) integer and nonnegative, \( C^k(W, Y) \) will denote the Banach space consisting of the mappings \( F: W \to Y \) which are \( k \) times (Fréchet) differentiable and such that the functions \( D^i F: W \to L^i(X, Y) \) \((0 \leq i \leq k)\) are bounded and uniformly continuous, the corresponding norm being given by \( \| F \|_{C^k} := \sum_{i=0}^k \sup D^i F \). Also, for \( k \) integer and nonnegative, and \( \eta \) real in the interval \( 0 < \eta \leq 1 \), \( C^{k, \eta}(W, Y) \) will denote the Banach space consisting of the mappings \( F: W \to Y \) which, besides belonging to \( C^k(W, Y) \), have the property that \( \text{Lip}_\eta D^k F \) is finite, the corresponding norm being given by \( \| F \|_{C^{k, \eta}} := \| F \|_{C^k} + \text{Lip}_\eta D^k F \). Alternatively, the spaces \( C^{k, \eta} \) will also be denoted as \( C^{k+\eta} \) if \( \eta \neq 1 \), and as \( C^{(k+1)-} \) if \( \eta = 1 \). Finally, the notations \( C^{k,\text{bdd}}(W, Y) \) and \( C^{k,\text{bdd}}(W, Y) \) will denote the spaces consisting of the mappings \( F: W \to Y \) such that, for every bounded domain \( w \subset X \) with \( \bar{w} \subset W \), the restriction of \( F \) to \( w \) belongs respectively to \( C^k(w, Y) \) and \( C^{k, \eta}(w, Y) \). For a mapping \( F \) of this type, the sup norm and Hölder–Lipschitz seminorms of the restriction of \( F \) to \( W \cap \{ x | \| x \|_X < R \} \) \((R > 0)\) will be denoted respectively as \( \sup(F|R) \) and \( \text{Lip}_\eta(F|R) \).

2. Setting of the Problem and Main Results

By following the practice which is common in similar cases, in order to look for attracting invariant manifolds of (1.7), we shall decompose the state variable into fast and slow components, and we shall consider (1.7) as a (finite) perturbation of a linear system where the fast and slow components are mutually decoupled. The desired attracting invariant manifolds will then be sought for as graphs of mappings giving the fast components as a function of the slow ones. In Mora [15], this was done for \( \varepsilon \neq 0 \) by working on the first-order system (1.12) as a perturbation of the one corresponding to \( F = 0 \), and decomposing the variable according to the spectrum of \( \mathbb{A}_\varepsilon \). Here, we shall adopt a somewhat different approach, the
differences lying in both the manner of decomposing the variable and of choosing the "unperturbed" linear system. In particular, here we shall consider \( U \) as decomposed into \( u \) and \( \dot{u} \), which in its turn will be decomposed according to the spectrum of \( A \); this will have the advantage that the decomposition will not depend on \( \varepsilon \).

Let \( \lambda_k \) \((k = 1, 2, \ldots)\) denote the eigenvalues of \( A \) arranged in a non-decreasing sequence. Let us now take a positive integer \( n \) such that \( \lambda_n < \lambda_{n+1} \), and consider the orthogonal decomposition invariant by \( A \), \( E = E_1 \oplus E_2 \), where \( E_1 \) and \( E_2 \) denote the closed linear subspaces of \( E \) generated respectively by the first \( n \) eigenfunctions and the rest of them. In the following, the orthogonal projections of \( E \) onto \( E_i \), \((i = 1, 2)\) will be denoted as \( P_i \), and the corresponding parts of \( A \) will be denoted as \( A_i \).

Parallel to this decomposition of \( E \), the spaces \( E^a \), \((a = 1, 2)\) also decompose orthogonally as \( E^a = E_1^a \oplus E_2^a \), where \( E_1^a = P_i E^a \). The spaces \( E_1, E_1^{1/2}, E_1^{1} \) consist all in the same \( n \)-dimensional vector space provided with different but equivalent inner products; specifically, the corresponding norms are related as follows

\[
(\lambda_1 - \xi)\|u\| \leq (\lambda_1 - \xi)^{1/2} \|u\|_{1/2, \xi} \leq \|u\|_{1, \xi} \leq (\lambda_n - \xi)\|u\|,
\]

where \( \xi < \lambda_1 \) is the constant which appears in the definition of the spaces \( E_1^{1/2}, E_1^1 \), and their inner products. According to this fact, in the future the spaces \( E_1^{1/2} \) and \( E_1^1 \) will be distinguished from \( E_1 \) only when the specific inner product plays a significant role.

Let us now introduce the preceding decomposition in Eq. (1.7). Henceforth, the components of \( u \) in \( E_1 \) and \( E_2 \) will be denoted respectively as \( u_1 \) and \( u_2 \). By applying the projections \( P_1 \) and \( P_2 \), Eq. (1.7) transforms itself into a system for \( u_1 \) and \( u_2 \), which we shall write as

\[
\varepsilon^2 \ddot{u}_1 + \dot{u}_1 = P_1 F(u_1 + u_2) - A_1 u_1
\]

\[
\varepsilon^2 \ddot{u}_2 + \dot{u}_2 + A_2 u_2 = P_2 F(u_1 + u_2),
\]

where the term \( A_1 u_1 \) has been moved to the right-hand side to reflect the fact that in the future this system will be considered as a perturbation of the one which is obtained when its right-hand sides are set to zero. For \( \varepsilon = 0 \), the state variable is thus decomposed into the two components \( u_1 \) and \( u_2 \), which are to be considered as taking values respectively in \( E_1^{1/2} \) and \( E_2^{1/2} \). For \( \varepsilon \neq 0 \), the state variable will be considered as decomposed into the four components \( u_1, u_2, \dot{u}_1, \dot{u}_2 \), with values respectively in \( E_1^{1/2}, E_2^{1/2}, E_1, E_2 \).

It is known that if the gap between \( \lambda_{n+1} \) and \( \text{max}(\lambda_n, 0) \) is large enough, then the global attractor of the parabolic system (1.7) \((\varepsilon = 0)\) is contained
in a local invariant manifold of class $C^1$ and dimension $n$, $M_0$, which is given by a relation of the type

$$u_2 = h_0(u_1)$$  \hspace{1cm} (2.4)

with $h_0$ belonging to $C^1(W_1, E_2^{1/2})$ and in fact to $C^1(W_1, E_1)$, where $W_1$ is a certain bounded domain in $E_1$. Here, this result will be obtained together with an extension to small nonzero values of $\varepsilon$. Specifically, it will be shown that, under the same gap condition there exists an $\bar{\varepsilon} > 0$ such that, for $\varepsilon \in (0, \bar{\varepsilon})$, the global attractor of the hyperbolic system (1.7) is also contained in a local invariant manifold of class $C^1$ and dimension $n$, $M_\varepsilon$, which will be described by a set of relations giving $u_2, \dot{u}_1$, and $\dot{u}_2$ as functions of $u_1$,

$$u_2 = h_\varepsilon(u_1)$$  \hspace{1cm} (2.5)

$$\dot{u}_1 = k_\varepsilon(u_1)$$  \hspace{1cm} (2.6)

$$\dot{u}_2 = l_\varepsilon(u_1),$$  \hspace{1cm} (2.7)

where $h_\varepsilon, k_\varepsilon,$ and $l_\varepsilon$ will belong respectively to $C^1(W_1, E_2^{1/2}), C^1(W_1, E_1)$, and $C^1(W_1, E_2)$, and in fact $h_\varepsilon$ belongs to $C^1(W_1, E_1)$.

Although these manifolds $M_\varepsilon$ are possibly normally hyperbolic, we shall not enter into this question, which on the other hand does not play any role in the development below.

Our main objective consists in showing that, as $\varepsilon \to 0$, both the manifold $M_\varepsilon$ and the vector field on it converge in the $C^1$ topology towards their analogous for $\varepsilon = 0$. Certainly, according to the preceding paragraph, $M_\varepsilon$ ($\varepsilon \neq 0$) are submanifolds of $E_2^{1/2} \times E$, while $M_0$ is a submanifold of $E_1$. In order that the problem of comparing $M_\varepsilon$ ($\varepsilon \neq 0$) with $M_0$ be correctly posed, this last manifold will be considered as embedded in $E_2^{1/2} \times E$ by taking $\dot{u}_1$ and $\dot{u}_2$ as determined by Eqs. (2.2), (2.3) (with $\varepsilon = 0$) together with (2.4):

$$\dot{u}_1 = P_1 F(u_1 + h_0(u_1)) - A_1 u_1 =: k_0(u_1)$$  \hspace{1cm} (2.8)

$$\dot{u}_2 = P_2 F(u_1 + h_0(u_1)) - A_2 h_0(u_1) =: l_0(u_1).$$  \hspace{1cm} (2.9)

Notice that (2.9) has indeed a meaning since $h_0$ takes values in $E_2^{1/2}$. In fact, from the properties of $h_0$ stated above, it is obvious that $k_0$ and $l_0$ will belong respectively to $C^1(W_1, E_1)$ and $C^1(W_1, E_2)$. On the other hand, one should notice also that relations (2.6) and (2.8), besides being part of the specification of $M_\varepsilon$, also give the evolution equation for the flow on $M_\varepsilon$. In other words, $k_\varepsilon$ is the projection on $E_1$ of the vector field on $M_\varepsilon$. Thus, concerning the vector field on $M_\varepsilon$, our objective is to prove that, as $\varepsilon \to 0$, $k_\varepsilon$ converges towards $k_0$ in the space $C^1(W_1, E_1)$. 
The main result of the paper is contained in the following

**THEOREM 2.1.** Let us consider problem (1.7), (1.8) with $A$ and $F$ satisfying the hypotheses stated in Section 1. There exists a constant $l$ such that if $\lambda_n$ and $\lambda_{n+1}$ satisfy the conditions

\[
\lambda_{n+1} - \lambda_n > 4l \tag{2.10}
\]
\[
\lambda_{n+1} > 2l \tag{2.11}
\]

then there exist $\hat{E} > 0$ and a bounded domain $W_1$ in $E_1$ such that:

(a) For $\varepsilon = 0$, the global attractor $\mathcal{A}_0$ is contained in $M_0$, an inflowing local invariant submanifold of $E^{1/2}$ of class $C^1$ and dimension $n$, which has form (2.4) with $h_0 \in C^1(W_1, E^1_{1/2})$.

(b) For every $\varepsilon \in (0, \hat{E})$, the global attractor $\mathcal{A}_\varepsilon$ is contained in $M_\varepsilon$, an inflowing local invariant submanifold of $E^{1/2} \times E$ of class $C^1$ and dimension $n$, which has form (2.5)-(2.7), with $h_\varepsilon \in C^1(W_1, E^1_{1/2})$, $k_\varepsilon \in C^1(W_1, E_1)$, and $l_\varepsilon \in C^1(W_1, E_2) \cap C^0(W_1, E^1_{1/2})$.

(c) Let $k_0$ and $l_0$ be defined by (2.8) and (2.9). Then, as $\varepsilon \to 0$, $h_\varepsilon$ converges towards $h_0$ in the space $C^1(W_1, E^1_{1/2})$, $k_\varepsilon$ converges towards $k_0$ in the space $C^1(W_1, E_1)$, and $l_\varepsilon$ converges towards $l_0$ in both spaces $C^1(W_1, E_2)$ and $C^0(W_1, E^1_{1/2})$.

(d) For any $\varepsilon \in [0, \hat{E})$, the solutions lying in $M_\varepsilon$ are twice continuously differentiable with respect to time, with $\hat{u}$ given by a relation of the type

\[
\hat{u} = m_\varepsilon(u_1), \tag{2.12}
\]

where $m_\varepsilon \in C^0(W_1, E)$. As $\varepsilon \to 0$, $m_\varepsilon$ converges towards $m_0$ in the space $C^0(W_1, E)$.

In particular this result implies that

**COROLLARY 2.2.** Under the hypotheses of Theorem 2.1, then, for $\varepsilon \in [0, \hat{E})$, the solutions lying in the global attractor $\mathcal{A}_\varepsilon$ have $u$, $\hat{u}$, and $\hat{u}$ bounded independently of $\varepsilon$ respectively in the spaces $E^1$, $E^1$, and $E$.

Remark. If $F$ belongs to $C^1(E^{1/2}, E^{1/2})$, i.e., it is globally bounded and has a globally bounded derivative, then the constant $l$ appearing in (2.10), (2.11) is given simply by $l = \text{Sup } DF$. When $F$ belongs only to $C^1_{\text{bou}}(E^{1/2}, E^{1/2})$, then $l$ depends on the bounds on $F$ and $DF$ in a certain ball containing the global attractor.

In the application to problem (1.1), (1.2)$_B$, (1.3), conditions (2.10), (2.11) reduce to

\[
2n + 1 > 4l, \quad \text{if } B = D
\]
\[
2n - 1 > 4l, \quad \text{if } B = N
\]
which will always be satisfied if \( n \) is taken large enough. Since it is known (Henry [10], Angenent [1]) that for \( \varepsilon = 0 \) this system is Morse–Smale whenever the stationary states are all hyperbolic, the standard theory of Morse–Smale (Palis and Smale [17]) allows to conclude that

**Corollary 2.3.** Let us consider problem (1.1), (1.2), (1.3) with \( f \) satisfying the hypotheses stated in Section 1, and assume that the stationary states are all hyperbolic. Then, for \( \varepsilon \) small enough, the flow on \( M_\varepsilon \) is equivalent to that on \( M_0 \). In particular, the flow on \( A_\varepsilon \) is equivalent to that on \( A_0 \).

**3. Modifying the Equation Far from the Attractor**

As is usual in similar circumstances, we shall begin by modifying the equation far from the attractor so that we can deal with global invariant manifolds instead of local ones.

According to the penultimate paragraph of Section 1, there exists a constant \( C \) independent of \( \varepsilon \) such that, for \( \varepsilon = 0 \), the global attractor \( A_0 \) is contained in the bounded open set \( W := \{ u \in E^{1/2} | \Phi_0(u) < C \} \), and for any \( \varepsilon \neq 0 \) the corresponding global attractor \( A_\varepsilon \) is contained in the cylinder \( W := W \times E \). In the following we shall modify the equation for \( u \) outside a neighbourhood of \( \text{Cl} W \) in such a way that, for every \( \varepsilon \) in a neighbourhood of zero, the modified flow will have a global invariant manifold \( M_\varepsilon'' \) of form (2.4) or (2.5)–(2.7) with \( W_1 = E_1 \), which manifold will have the property of containing all the solutions which stay defined and bounded as \( t \to -\infty \); and furthermore, when \( \varepsilon \) tends to 0, the manifolds \( M_\varepsilon'' \) and the corresponding vector fields will have convergence properties completely analogous to those appearing in Theorem 2.1. The desired local invariant manifolds for the original flows can then be taken as \( M_0 := M_0'' \cap W \), and for \( \varepsilon \neq 0 \), \( M_\varepsilon := \{ (u, \dot{u}) \in M_\varepsilon'' | u_1 \in W_1 \} \), where \( W_1 := P_1 M_0 \). By using the fact that for \( \varepsilon = 0 \) the flow at the boundary of \( M_0 \) points strictly towards the interior, one checks easily that, for \( \varepsilon \) small enough, \( M_\varepsilon \) will be an inflowing local invariant manifold of the original flow with the property of containing the global attractor \( A_\varepsilon \). Obviously, in this situation the convergence properties of \( M_\varepsilon \) as \( \varepsilon \) tends to 0 will follow immediately from the analogous properties of \( M_\varepsilon'' \).

Let us now describe the way we shall modify the equation. Our modification will operate outside an open ball in \( E^{1/2} \) centred at the origin and containing \( \text{Cl} W \). In the following, such a ball is denoted by \( B \), and its radius is denoted by \( R \).

In the first place, we shall modify \( F \) outside \( B \) so that the modified function is globally bounded with derivative globally bounded and globally
Hölder. This can be accomplished by taking this modified function, which we shall denote by $F^m$, as

$$F^m(u) := \sigma(\|u\|_{1/2}/R) F(u),$$

(3.1)

where $\sigma$ is a function $\mathbb{R}_+ \to [0, 1]$ of class $C^{1+\eta}$ with $\sigma(r) = 1$ for $r \leq 1$, and $\sigma(r) = 0$ for $r$ greater than some $\rho < +\infty$. With this definition, one easily checks that

**Lemma 3.1** If $F \in C_{\text{bd}}^{1+\eta}(E^{1/2}, E^{1/2})$, then $F^m \in C^{1+\eta}(E^{1/2}, E^{1/2})$; in particular one has the following bound:

$$\sup D F^m \leq \sup (DF|_R) + \frac{\max|\sigma'|}{R} \sup (F|_R) =: l.$$ (3.2)

Although one could possibly proceed with only the aid of this modification, we shall still perform another one which will have the advantage of making the treatment of Eq. (2.2) fairly simpler. Namely, the linear operator $A_1$ appearing in that equation will be replaced by a non linear function $A^m_1$ of class $C^{1+\eta}$ which for $u \in B$ will coincide with $A_1$, but for $u$ outside $B$ it will deviate from linearity so as to remain globally bounded. Of course, we will also require the derivative of $A^m_1$ to be globally bounded; in fact, for later development it will be essential that this derivative be bounded exactly by the norm of the linear operator $A_1$, namely $\lambda_n$. Such a modification can be accomplished by taking $A^m_1$ as

$$A^m_1(u) := \sigma(\|u\|_{1/2}/R) A_1 u_1,$$

(3.3)

where $\sigma$ is a cut-off function like that used in the definition of $F^m$ with the additional property that

$$|r\sigma'(r) + \sigma(r)| \leq 1, \quad \forall r \geq 0.$$ (3.4)

For instance, one can take $\sigma(r) = r^{-1} \int_0^r \phi$, with $\phi$ as shown in Fig. 3.1.
Lemma 3.2. The function $A^m_l$ defined above belongs to $C^{1+\gamma}(E_i^{1/2}, E_i^{1/2})$ and it satisfies the following bound:

$$\sup \| \mathcal{D}^m_l \|_{L(E_i^{1/2}, E_i^{1/2})} = \lambda_\alpha.$$  \hspace{1cm} (3.5)

Proof. In the following we use the notation $\| u_i \|_{1/2}/R = r_i$. By moving the scalar factor $\sigma(r_i)$ across the linear operator $A_i$, our objective reduces to showing that the auxiliary function $J^m$ defined by

$$J^m(u_i) := \sigma(r_i) u_i$$

belongs to $C^{1+\gamma}(E_i^{1/2}, E_i^{1/2})$ and satisfies the bound

$$\sup \mathcal{D}^m = 1.$$

One verifies easily that $J^m \in C^{1+\gamma}(E_i^{1/2}, E_i^{1/2})$, its derivative being given by

$$\mathcal{D}^m(u_i) v_1 = \frac{\sigma'(r_i)}{R} \left\langle \frac{u_i}{\| u_i \|_{1/2}} , v_1 \right\rangle_{1/2} u_1 + \sigma(r_i) v_1.$$

In order to verify that $\sup \mathcal{D}^m = 1$, we put $v_1 = p_1 + q_1$ with $p_1$ and $q_1$ respectively parallel and orthogonal to $u_1$ in $E_i^{1/2}$. By proceeding in this way, we obtain that

$$\mathcal{D}^m(u_i) v_1 = \sigma'(r_i) r_i p_1 + \sigma(r_i) p_1 + \sigma(r_i) q_1,$$

from which follows that

$$\| \mathcal{D}^m(u_i) v_1 \|_{1/2} = \left[ (r_i \sigma'(r_i) + \sigma(r_i))^2 \| p_1 \|_{1/2}^2 + \sigma(r_i)^2 \| q_1 \|_{1/2}^2 \right]^{1/2} \leq \left[ \| p_1 \|_{1/2}^2 + \| q_1 \|_{1/2}^2 \right]^{1/2} = \| v_1 \|_{1/2},$$

where we have used the fact that $\sigma$ satisfies (3.4). \hfill \Box

By performing the preceding modifications, system (2.2), (2.3) is transformed into

$$\varepsilon^2 \ddot{u}_1 + \dot{u}_1 = G_1(u),$$  \hspace{1cm} (3.6)

$$\varepsilon^2 \ddot{u}_2 + \dot{u}_2 + A_2 u_2 = G_2(u),$$  \hspace{1cm} (3.7)

where $G_i$ ($i = 1, 2$) denote the functions from $E_i^{1/2}$ to $E_i^{1/2}$ given by

$$G_1(u) := P_1 F^m(u) - A^m_1(u_1)$$  \hspace{1cm} (3.8)

$$G_2(u) := P_2 F^m(u).$$  \hspace{1cm} (3.9)
From the properties of $F$ and $A^n$ it follows that the functions $G_i$ belong to $C^{1+\eta}(E^{1/2}, E^{1/2})$, and their derivatives $D_{u_i}G_i$ are bounded as

\begin{align*}
\text{sup} \quad D_{u_i}G_i &\leq l + \lambda_i \quad (3.10) \\
\text{sup} \quad D_{u_2}G_1 &\leq l \quad (3.11) \\
\text{sup} \quad D_{u_2}G_2 &\leq l \quad (3.12) \\
\text{sup} \quad D_{u_1}G_2 &\leq l \quad (3.13)
\end{align*}

where $l$ is the constant appearing in (3.2).

4. Plan of the Proof

Our objective consists now in studying system (3.6), (3.7) to obtain the global invariant manifolds which in Section 3 were denoted as $M^\varepsilon$. Since these are the only invariant manifolds considered in the remainder of the paper, from now on we shall denote them simply as $M$ instead of $M^\varepsilon$.

We recall that, for every $\varepsilon$, $M$ should contain all (mild) solutions which stay defined and bounded as $t \to -\infty$. In order to obtain such manifolds we shall use the classical method of Lyapunov and Perron in the special form as it appears for instance in Vanderbauwhede and Van Gils [18]. The main idea consists in looking for $M$ as consisting not only of all solutions which stay bounded as $t \to -\infty$, but more generally all solutions which satisfy an exponential growth condition of the form

$$\|u(t)\|_{1/2} = O(e^{-\mu t}) \quad \text{as} \quad t \to -\infty,$$

where $\mu$ will be a positive real number belonging to the interval $(\lambda_n, \lambda_{n+1})$. Admitting these extra solutions will result in the set $M$ being a differentiable manifold.

The solutions of (3.6), (3.7) which satisfy the growth condition (4.1) will be obtained as fixed points of certain mappings $u^0 \mapsto u$ which result from solving the pair of nonhomogeneous linear equations

\begin{align*}
\varepsilon^2 \ddot{u}_1 + \dot{u}_1 &= G_1(u^0) =: f_1 \\
\varepsilon^2 \ddot{u}_2 + A_2u_2 &= G_2(u^0) =: f_2
\end{align*}

with the additional condition that $u = u_1 + u_2$ satisfies (4.1). As will be seen below, it turns out that, for $\mu \in (0, \lambda_{n+1})$, and $\varepsilon$ small enough, the set of solutions of (4.2), (4.3) which satisfy (4.1) is parametrized by $u_1(0) \in E_1$. Thus, by adding an initial condition of the form $u_1(0) = x$, we obtain a different mapping $u^0 \mapsto u$ for every $x \in E_1$. By applying a suitable version of
the parametrized contraction theorem, we shall obtain that, under certain conditions to be stated thereafter ((8.13)-(8.15)), each of these mappings has a unique fixed point. The totality of these fixed points will give us the set $M_\varepsilon$ we are looking for, which in fact will be a manifold parametrized by $x \in E_1$. Finally, the behaviour of $M_\varepsilon$ as $\varepsilon \to 0$ will also be taken care of by our specific version of the parametrized contraction theorem based on a previous detailed study of the behaviour of the solutions as $\varepsilon \to 0$ of the nonhomogeneous linear equations (4.2), (4.3) with the additional conditions mentioned above.

5. SPACES OF CURVES WITH EXPONENTIALLY WEIGHTED NORMS

In order to deal with curves satisfying exponential growth conditions like (4.1), we shall make use of certain fairly standard exponentially weighted norms and the corresponding Banach spaces. In this section we introduce these spaces, together with some basic facts about them which are used in the forthcoming development.

Given a Banach space $X$, in the following $\mathbb{R}$ will denote the linear space of continuous functions from $\mathbb{R}_- := (-\infty, 0]$ to $X$, and for every $\mu \in \mathbb{R}$, $J_\mu$ will denote the mapping from $\mathbb{R}$ to itself defined by $(J_\mu x)(t) := e^{\mu t} x(t)$. The basic spaces we shall deal with in the future are the spaces $\tilde{X}_\mu$ defined as follows: for every $\mu \in \mathbb{R}$, $\tilde{X}_\mu$ is the Banach space consisting of the continuous functions $x \in X$ such that $J_\mu x$ is bounded and uniformly continuous, the norm in $\tilde{X}_\mu$ being given by

$$
\|x\|_{\tilde{X}_\mu} := \sup_{t \leq 0} \|J_\mu x(t)\|_X = \sup_{t \leq 0} e^{\mu t} \|x(t)\|_X. \tag{5.1}
$$

In particular, $\tilde{X}_0$ is the classical Banach space of bounded and uniformly continuous functions from $\mathbb{R}_-$ to $X$ with the norm of the uniform convergence. Obviously, for every $\mu \in \mathbb{R}$, $J_\mu$ establishes an isomorphism from $\tilde{X}_\mu$ to $\tilde{X}_0$. On the other hand, it is also obvious that if $\mu \leq \nu$ then $\tilde{X}_\mu$ is continuously embedded in $\tilde{X}_\nu$ with embedding constant equal to 1. Finally, we also remark here that, for $\mu \geq 0$, the space $X$ is also continuously embedded in $\tilde{X}_\mu$ with embedding constant equal to 1, the embedding being given by the operator $K: X \to \tilde{X}$ which to every $x_0 \in X$ assigns the function constantly equal to $x_0$.

Now let a continuous mapping $F: X \to Y$ be given, where $X$ and $Y$ are both Banach spaces and let us consider the Nemytskii operator $\tilde{F}: \tilde{X} \to \tilde{Y}$ defined by $\tilde{F}(x) := F \circ x$. A simple argument shows that

**Lemma 5.1.** If $F$ is bounded and Lipschitzian, then, for every $\mu \geq 0$, $\tilde{F}$ maps $\tilde{X}_\mu$ to $\tilde{Y}_\mu$, and the mapping $\tilde{F}_\mu: \tilde{X}_\mu \ni x \mapsto \tilde{F}(x) \in \tilde{Y}_\mu$ is also bounded and Lipschitzian, with $\text{Sup} \tilde{F}_\mu \leq \text{Sup} F$ and $\text{Lip} \tilde{F}_\mu \leq \text{Lip} F$. 


Let us now assume that $F$ is (Fréchet) differentiable with $DF$ bounded and uniformly continuous. In this case, it turns out that the mappings $\bar{F}_\mu : \bar{Y}_\mu \to \bar{Y}_\mu$ ($\mu \geq 0$) are not necessarily differentiable. However, it holds that

**Lemma 5.2.** If $F$ is (Fréchet) differentiable with $DF : X \to L(X, Y)$ bounded and uniformly continuous, then, for every $\mu, v$ with $v > \mu$ and $v > 0$, the mapping $\bar{F}_{\mu,v} : \bar{X}_\mu \ni x \mapsto \bar{F}(x) \in \bar{Y}_v$ is (Fréchet) differentiable, its derivative being given by $D\bar{F}_{\mu,v}(x)h = DF(x(\cdot))h(\cdot)$, and $D\bar{F}_{\mu,v} : \bar{X}_\mu \to L(\bar{X}_\mu, \bar{Y}_v)$ is bounded and uniformly continuous.

For the proof of this fact we refer the reader to Vanderbauwhede and Van Gils [18, Lemma 5]. Finally, one verifies easily that

**Lemma 5.3.** If, besides the hypotheses of Lemma 5.2, one assumes also that $DF : X \to L(X, Y)$ satisfies a Hölder–Lipschitz condition of exponent $\eta$ $(0 < \eta < 1)$, then, for every $\mu, v$ with $v \geq (1 + \eta)\mu$ and $v > 0$, $D\bar{F}_{\mu,v}$ also satisfies a Hölder–Lipschitz condition of exponent $\eta$, with $\text{Lip}_\eta D\bar{F}_{\mu,v} \leq \text{Lip}_\eta DF$.

In the forthcoming development we shall also make use of a notion of uniform equicontinuity adapted to the exponentially weighted norms. By definition, a subset $\mathcal{F}$ of $\bar{X}$ will be called $\bar{X}_\mu$-uniformly equicontinuous if and only if $J_\mu \mathcal{F}$ is uniformly equicontinuous in the usual sense, i.e., for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{x \in \mathcal{F}} \sup_{t, s \in \mathbb{R}_- \atop |t - s| < \delta} \| e^{\mu t} x(t) - e^{\mu s} x(s) \|_X < \varepsilon.$$ 

Parallel to this terminology, sometimes we shall also use the expression “$\mathcal{F}$ is $\bar{X}_\mu$-bounded” for “$\mathcal{F}$ is a bounded subset of $\bar{X}_\mu$.” In the following lemma, we state without proof several elementary facts about $\bar{X}_\mu$-uniform equicontinuity which will be needed in Section 9:

**Lemma 5.4.** (a) If $\mathcal{F} \subset \bar{X}$ is $\bar{X}_\mu$-bounded and $\bar{X}_\mu$-uniformly equicontinuous, then, for every $v \geq \mu$, $\mathcal{F}$ is also $\bar{X}_v$-bounded and $\bar{X}_v$-uniformly equicontinuous.

(b) If $\mathcal{F} \subset \bar{X}$ consists of functions of class $C^1$, and both $\mathcal{F}$ and $\mathcal{F} := \{ \dot{x} \mid x \in \mathcal{F} \}$ are $\bar{X}_\mu$-bounded, then $\mathcal{F}$ is $\bar{X}_\mu$-uniformly equicontinuous; in the case $\mu = 0$, the hypothesis that $\mathcal{F}$ be $\bar{X}_\mu$-bounded is not required.

(c) If $F : X \to Y$ is uniformly continuous and $\mathcal{F} \subset \bar{X}$ is $\bar{X}_0$-uniformly equicontinuous, then $\bar{F}(\mathcal{F}) := \{ \bar{F}(x) \mid x \in \mathcal{F} \}$ is $\bar{Y}_0$-uniformly equicontinuous.
(d) For every $\mu, v \in \mathbb{R}$, if $\mathcal{F} \subset \mathcal{X}$ is $\mathcal{X}_\mu$-bounded and $\mathcal{X}_\mu$-uniformly equicontinuous, and $\mathcal{B} \subset L(X, Y)$ is $L(X, Y)_{\tilde{\nu}}$-bounded and $L(X, Y)_{\tilde{\nu}}$-uniformly equicontinuous, then $\mathcal{B} \mathcal{F} := \{ Bx : t \mapsto B(t)x(t) | B \in \mathcal{B}, x \in \mathcal{F} \}$ is $\mathcal{Y}_{\tilde{\nu}}$-bounded and $\mathcal{Y}_{\tilde{\nu}}$-uniformly equicontinuous.

6. Resolution of the Nonhomogeneous Linear Equations in Exponentially Weighted Spaces

In this section we state two fundamental lemmas containing our results on the resolution of the nonhomogeneous linear equations (4.2), (4.3) in spaces of the type $\mathcal{E}_{\tilde{\nu}} := (E_{\tilde{\nu}}^\tau)$, particularly including the limiting behaviour of the corresponding resolvent operators as $\varepsilon \to 0$. The part of the lemmas which concerns a fixed value of $\varepsilon$ is a rather standard step of the particular method that we have adopted to obtain the invariant manifolds $M_\nu$. The part which concerns the behaviour as $\varepsilon \to 0$ contains the core of the present paper in its aspect of dealing with an infinite-dimensional singular limit problem. The proof of this part involves certain narrow estimates which are rather laborious to obtain. Because of their length, the proofs will be deferred to Section 10.

The first lemma refers to an equation of the form (4.2), namely

$$\varepsilon^2 \ddot{u} + \dot{u} = f,$$

where $u$ and $f$ stand for functions of $t \in \mathbb{R}_+$ with values in a general Banach space $X$, and $\varepsilon$ is a real parameter varying right up to $\varepsilon = 0$.

**Lemma 6.1.** Let $\mu$ be a real number $> 0$, and let $\varepsilon$ be restricted to the interval $0 \leq \varepsilon < \mu^{-1/2}$. Then:

(a) For every $f \in \mathcal{X}_\mu$ and every $x \in X$, Eq. (1) has a unique solution $u$ belonging to $\mathcal{X}_\mu$ and satisfying the additional condition $u(0) = x$. This solution and its time derivative are given, respectively, by

$$u = Kx + R_\varepsilon f$$
$$\dot{u} = \dot{R}_\varepsilon f,$$

where $K$ is the embedding operator from $X$ to $\mathcal{X}_\mu$ alluded to in Section 5, and both $R_\varepsilon$ and $\dot{R}_\varepsilon$ are bounded linear operators from $\mathcal{X}_\mu$ to itself. The operators $R_\varepsilon$ and $\dot{R}_\varepsilon$ are bounded independently of $\varepsilon$ when $\varepsilon$ varies in a neighbourhood of $\varepsilon = 0$; specifically they satisfy the estimates

$$\| R_\varepsilon f \| \mathcal{X}_\mu \leq \mu^{-1} (1 + \mu^2)^{-1} \| f \| \mathcal{X}_\mu$$
$$\| \dot{R}_\varepsilon f \| \mathcal{X}_\mu \leq (1 + \mu^2)^{-1} \| f \| \mathcal{X}_\mu.$$
(b) When \( \varepsilon \to 0 \), \( R_\varepsilon \) converges to \( R_0 \) in the norm topology; specifically one has that
\[
\| R_\varepsilon - R_0 \|_{L(F_\mu, T_\mu)} = O(\varepsilon^2). \tag{6.5}
\]
At the same time, \( \hat{R}_\varepsilon \) converges to \( \hat{R}_0 \) in the strong topology, and in fact convergence is uniform on \( \mathcal{F} \subset \tilde{X}_\mu \) whenever \( \mathcal{F} \) is \( \tilde{X}_\mu \)-bounded and \( \tilde{X}_\mu \)-uniformly equicontinuous.

Remark. The condition of \( \mathcal{F} \) being \( \tilde{X}_\mu \)-bounded and \( \tilde{X}_\mu \)-uniformly equicontinuous is slightly weaker than that of its being a compact subset of \( \tilde{X}_\mu \).

The second lemma refers to an equation of the form (4.3), namely
\[
\varepsilon^2 \ddot{u} + \dot{u} + Au = f, \tag{II}_\varepsilon
\]
where \( u \) and \( f \) take now values in a Hilbert space \( E \), and \( A \) is a self-adjoint linear operator on \( E \) with numerical range bounded from below and compact resolvent. In the following, \( \xi_1 \) denotes the greatest lower bound of the numerical range of \( A \), and \( E^a \) \((a = 0, \frac{1}{2}, 1)\) denote the fractional spaces associated with \( A \). According to our particular needs in the future application to Eq. (4.3), in the following lemma we consider the case where \( f \in E^{1/2}_\mu \).

**Lemma 6.2.** Let \( \mu \) be a real number \(< \xi_1 \). If \( \mu > 0 \), we also assume that \( \varepsilon \) is restricted to the interval \( 0 \leq \varepsilon < (2\mu)^{-1/2} \); if \( \mu \leq 0 \), we allow \( \varepsilon \) to have any value in the whole interval \( 0 \leq \varepsilon < +\infty \). Under these assumptions, then:

(a) For every \( f \in E^{1/2}_\mu \), Eq. (II)_\varepsilon has a unique mild solution \( u \) belonging to \( \tilde{E}^{1/2}_\mu \), which is in fact a strict solution. This solution and its time derivative are given, respectively, by
\[
\begin{align*}
\dot{u} &= \hat{S}_\varepsilon f, \tag{6.6} \\
\ddot{u} &= \hat{S}_\varepsilon f, \tag{6.7}
\end{align*}
\]
where \( S_\varepsilon \) is a bounded linear operator from \( E^{1/2}_\mu \) to \( \tilde{E}^{1}_\mu \), and \( \hat{S}_\varepsilon \) is a bounded linear operator from \( \tilde{E}^{1/2}_\mu \) to \( E^{1}_\mu \). As operators between these spaces, \( S_\varepsilon \) and \( \hat{S}_\varepsilon \) are bounded independently of \( \varepsilon \) when \( \varepsilon \) varies in a neighbourhood of \( \varepsilon = 0 \). Besides, for \( S_\varepsilon \) as an operator from \( E^{1/2}_\mu \) to \( \tilde{E}^{1/2}_\mu \), one specifically has a bound of the form
\[
\| S_\varepsilon f \|_{\tilde{E}^{1/2}_\mu} \leq K_\mu(\varepsilon)(\xi_1 - \mu)^{-1} \| f \|_{E^{1/2}_\mu} \tag{6.8}
\]
with \( K_\mu(0) = 1 \) and \( K_\mu(\varepsilon) \to 1 \) as \( \varepsilon \to 0 \).

(b) When \( \varepsilon \to 0 \), \( S_\varepsilon \) converges to \( S_0 \) in the norm topology of bounded linear operators from \( E^{1/2}_\mu \) to \( \tilde{E}^{1}_\mu \); specifically one has that
\[
\| S_\varepsilon - S_0 \|_{L(E^{1/2}_\mu, \tilde{E}^{1}_\mu)} = O(\varepsilon^{1/3}). \tag{6.9}
\]
At the same time, $\hat{S}_e$ converges to $\hat{S}_0$ in the strong topology of bounded linear operators from $\bar{E}_0^{1/2}$ to $\bar{E}_\mu$, and in fact convergence is uniform on $\mathcal{F} \subset \bar{E}_\mu^{1/2}$ whenever $\mathcal{F}$ is $\bar{E}_\mu^{1/2}$-bounded and $\bar{E}_\mu^{1/2}$-uniformly equicontinuous.

Remark. The fact that $S_\varepsilon f$ is a strict solution of (II)$_\varepsilon$ means that, for $\varepsilon = 0$, $S_0 f$ is continuous in $\mathcal{E}^1$ and continuously differentiable in $\mathcal{E}$, and, for $\varepsilon \neq 0$, $S_\varepsilon f$ is continuous in $\mathcal{E}^1$, continuously differentiable in $\mathcal{E}^{1/2}$, and twice continuously differentiable in $\mathcal{E}$. In fact, for $\varepsilon \neq 0$, $\hat{S}_e$ is a bounded linear operator from $\bar{E}_\mu^{1/2}$ to $\bar{E}_\mu^{1/2}$, and $\hat{S}_e$ (the operator $f \mapsto \bar{u}$) is a bounded linear operator from $\bar{E}_\mu^{1/2}$ to $\bar{E}_\mu$; however, in these spaces the norms of $\hat{S}_e$ and $\hat{S}_e$ will hardly be bounded as $\varepsilon \to 0$.

7. The Mappings $T_e$: A Contraction Theorem with Parameters

Having at our disposal the results of Sections 5 and 6, we can now make the plan which was drawn up in Section 4 more explicit. Clearly, for instance by applying Lemma 5.1, it is ensured that, for every $u^0$ belonging to $\bar{E}_\mu^{1/2}$, $\bar{G}_i(u^0)$ belongs to $\bar{E}_i^{1/2}$ ($i = 1, 2$). Therefore, by applying Lemmata 6.1 and 6.2, we obtain that, if $0 < \mu < \lambda_{n, 1}$ and $0 < \varepsilon < (2\mu)^{-1/2}$, then, for every $x \in E_1$, the system (4.2), (4.3) has a unique solution $u = u_1 + u_2$ belonging to $\bar{E}_\mu^{1/2}$ and satisfying $u_i(0) = x$; namely

$$u = Kx + R_\varepsilon \bar{G}_1(u^0) + S_\varepsilon \bar{G}_2(u^0) =: T_e(u^0, x). \quad (7.1)$$

According to Section 4, for every $\varepsilon$ in a neighbourhood of 0, $M_\varepsilon$ will consist of the totality of curves $u \in \bar{E}_\mu^{1/2}$ which are fixed points of the mappings $T_e(\cdot, x): \bar{E}_\mu^{1/2} \to \bar{E}_\mu^{1/2}$, where $x$ varies over $E_1$. Our objective now is to obtain these fixed points by means of the contraction theorem; more precisely, by using a suitable version of the contraction theorem with parameters, we aim at showing that, for every $\varepsilon$ in a neighbourhood of 0 and every $x \in E_1$, the mapping $T_e(\cdot, x)$ has a unique fixed point $Y_e(x) \in \bar{E}_\mu^{1/2}$, which is a $C^1$ function of $x \in E_1$, and that for $\varepsilon \to 0$ this function $Y_e$ converges in the $C^1$ norm towards $Y_0$. To this end, we need a version of the contraction theorem with parameters which takes care of our double parameter situation, and also the fact that, according to the observations of Section 5, the mappings $\bar{G}_i$ ($i = 1, 2$) will not be of class $C^1$ from $\bar{E}_\mu^{1/2}$ to $\bar{E}_i^{1/2}$, but only from $\bar{E}_\mu^{1/2}$ to $\bar{E}_i^{1/2}$ with $\nu > \mu$, and accordingly $T_e(\cdot, x)$ will not be of class $C^1$ from $\bar{E}_\mu^{1/2}$ to itself, but only from $\bar{E}_\mu^{1/2}$ to $\bar{E}_i^{1/2}$ with $\nu > \mu$.

In the following we give a version of the contraction theorem with parameters which suits the situation just described. It essentially consists of a theorem of the type of Irwin [11, Theorem C.10] covering differentiability with respect to one parameter and continuity with respect to
another, with the variation that differentiability involves a pair of Banach spaces one embedded into the other, as in Vanderbauwhede and Van Gils [18, Theorem 3].

Let $U$ and $X$ be Banach spaces, and $\Sigma$ a topological space. Let $T_\varepsilon (\varepsilon \in \Sigma)$ be a family of mappings $U \times X \to U$ such that

$$(H1) \quad T_\varepsilon (\cdot, x): U \to U \text{ is a contraction on } U \text{ uniformly with respect to both } x \in X \text{ and } \varepsilon \in \Sigma; \text{ i.e., there exists } \kappa < 1 \text{ such that}$$

$$\|T_\varepsilon(u, x) - T_\varepsilon(v, x)\|_U \leq \kappa \|u - v\|_U \quad (\forall x \in X, \forall \varepsilon \in \Sigma). \quad (7.2)$$

By the classical contraction theorem, each of the mappings $T_\varepsilon(\cdot, x)$ ($x \in X, \varepsilon \in \Sigma$) has a unique fixed point, which we shall denote as $Y_\varepsilon(x)$. Let us assume also that

$$(H2) \quad T_\varepsilon(u, \cdot): X \to U \text{ is Lipschitzian from } X \text{ to } U \text{ uniformly with respect to both } u \in U \text{ and } \varepsilon \in \Sigma; \text{ i.e., there exists } Q < +\infty \text{ such that}$$

$$\|T_\varepsilon(u, x) - T_\varepsilon(u, y)\|_U \leq Q \|x - y\|_X \quad (\forall u \in U, \forall \varepsilon \in \Sigma). \quad (7.3)$$

It is well known and very elementary to show that in such a case the mappings $Y_\varepsilon : X \to U$ are also Lipschitzian from $X$ to $U$ uniformly with respect to $\varepsilon \in \Sigma$; specifically, one obtains that

$$\text{Lip } Y_\varepsilon \leq (1 - \kappa)^{-1} Q \quad (\forall \varepsilon \in \Sigma). \quad (7.4)$$

Let us now make the further assumption that

$$(H3) \quad \text{For some fixed } \varepsilon_0 \in \Sigma, \text{ the mappings } T_\varepsilon - T_{\varepsilon_0} (\varepsilon \in \Sigma) \text{ are bounded, and } T_\varepsilon - T_{\varepsilon_0} \text{ converges uniformly to } 0 \text{ as } \varepsilon \to \varepsilon_0.$$

It is immediately shown that in such a case the mappings $Y_\varepsilon - Y_{\varepsilon_0} (\varepsilon \in \Sigma)$ are also bounded, and that $Y_\varepsilon - Y_{\varepsilon_0}$ also converges uniformly to $0$ as $\varepsilon \to \varepsilon_0$; specifically, one obtains that

$$\text{Sup}(Y_\varepsilon - Y_{\varepsilon_0}) \leq (1 - \kappa)^{-1} \text{Sup}(T_\varepsilon - T_{\varepsilon_0}). \quad (7.5)$$

By following Vanderbauwhede and Van Gils [18], we now assume that $U$ is continuously embedded in another Banach space $\bar{U}$ through a certain linear operator $J$, and we consider the $\bar{U}$-valued mappings $\bar{T}_\varepsilon := JT_\varepsilon$ and $\bar{Y}_\varepsilon := JY_\varepsilon$.

**Theorem 7.1** Besides the foregoing hypotheses (H1), (H2), and (H3), let us assume also that the mappings $\bar{T}_\varepsilon : U \times X \to \bar{U} (\varepsilon \in \Sigma)$ satisfy the following conditions:

$$(J1) \quad \text{For every } \varepsilon \in \Sigma, \bar{T}_\varepsilon \text{ is (Fréchet) differentiable with } D\bar{T}_\varepsilon : U \times X \to L(U \times X, \bar{U}) \text{ bounded and uniformly continuous, and there exist mappings}$$
$d_uT_e: U \times X \to L(U, U)$, $d_uT_e: U \times X \to L(\bar{U}, \bar{U})$, and $d_xT_e: U \times X \to L(X, U)$ such that

$$D_u\tilde{T}_e(u, x) = J d_uT_e(u, x) = \tilde{d}_uT_e(u, x)J$$

(7.6)

$$D_x\tilde{T}_e(u, x) = J d_xT_e(u, x)$$

(7.7)

$$\|d_uT_e(u, x)\|_{L(U, U)} \leq \kappa$$

(7.8)

$$\|\tilde{d}_uT_e(u, x)\|_{L(U, U)} \leq \kappa$$

(7.9)

$$\|d_xT_e(u, x)\|_{L(X, U)} \leq Q$$

(7.10)

where $\kappa < 1$ and $Q < +\infty$ are the same constants that appear in (H1) and (H2).

(J2) $D\tilde{T}_e$ converges uniformly to $D\tilde{T}_{e_0}$ as $e \to e_0$.

Then, the mappings $\bar{Y}_e: X \to \bar{U}$ ($e \in \Sigma$) have the following properties:

(K1) For every $e \in \Sigma$, $\bar{Y}_e$ is (Fréchet) differentiable with $D\bar{Y}_e: X \to L(X, \bar{U})$ bounded and uniformly continuous, and there exist mappings $dY_e: X \to L(X, U)$ such that

$$D\bar{Y}_e(x) = J dY_e(x)$$

(7.11)

$$\|dY_e(x)\|_{L(X, U)} \leq (1 - \kappa)^{-1} Q.$$  

(7.12)

Specifically, $D\bar{Y}_e$ and $dY_e$ are given, respectively, by

$$D\bar{Y}_e(x) = (I_U - d_uT_e(Y_e(x), x))^{-1} D_xT_e(Y_e(x), x)$$

(7.13)

$$dY_e(x) = (I_U - d_uT_e(Y_e(x), x))^{-1} d_xT_e(Y_e(x), x).$$

(7.14)

(K2) $D\bar{Y}_e$ converges uniformly to $D\bar{Y}_{e_0}$ as $e \to e_0$.

Proof. We begin by remarking that (7.8) and (7.9) together with the fact that $\kappa < 1$ imply that the linear operators

$$L_e(x) = (I_U - d_uT_e(Y_e(x), x))^{-1}$$

$$\bar{L}_e(x) = (I_{\bar{U}} - \bar{d}_uT_e(Y_e(x), x))^{-1}$$

are both well defined with

$$\|L_e(x)\|_{L(U, U)} \leq (1 - \kappa)^{-1}$$

(7.15)

$$\|\bar{L}_e(x)\|_{L(\bar{U}, \bar{U})} \leq (1 - \kappa)^{-1}.$$  

(7.16)

In order to verify that $\bar{Y}_e$ is differentiable with $D\bar{Y}_e$ given by the right-hand of (7.13), we observe that
By virtue of (7.16) and the differentiability of $\tilde{T}_e$, it is clear that when $y \to x$ the last expression is $o\left(\|Y_e(y) - Y_e(x)\|_U + \|y - x\|_X\right)$, and taking into account that $Y_e$ is Lipschitzian (7.4), we obtain that it is $o(\|y - x\|_X)$, as required for Fréchet differentiability. The fact that $DY_e(x)$ satisfies (7.11) with $dY_e(x)$ given by (7.14) follows easily from (7.13) by using (7.6) and (7.7). The bound (7.12) follows immediately from (7.14) by using (7.16) and (7.10). In order to complete the proof of (K1) it remains only to verify that, by using (7.13), the difference $D\tilde{Y}_e(x) - D\tilde{Y}_e(y)$ can be transformed as follows:

$$
D\tilde{Y}_e(x) - D\tilde{Y}_e(y)
= \tilde{L}_e(x)[D_x \tilde{T}_e(Y_e(x), x) - D_x \tilde{T}_e(Y_e(y), y)] + (D_x \tilde{T}_e(Y_e(x), x) - D_x \tilde{T}_e(Y_e(y), y)) dY_e(x)
$$

By using this last expression and taking into account the bounds (7.16) and (7.12), the uniform continuity of $D\tilde{Y}_e$ is easily derived from the analogous property of $D\tilde{T}_e$ expressed in (J1) and the Lipschitz property of $Y_e$ (7.4). Finally, in order to prove (K2), we transform the difference $D\tilde{Y}_e(x) - D\tilde{Y}_e(y)$ similarly as above to obtain

$$
D\tilde{Y}_e(x) - D\tilde{Y}_e(y)
= \tilde{L}_e(x)[(D_u \tilde{T}_e(Y_e(x), x) - D_u \tilde{T}_e(Y_e(y), y))] dY_e(x)
+ (D_x \tilde{T}_e(Y_e(x), x) - D_x \tilde{T}_e(Y_e(y), y))].
$$

By taking norms on this equality we obtain that

$$
\|D\tilde{Y}_e(x) - D\tilde{Y}_e(y)\|_{L(X, U)}
\leq (1 - \kappa)^{-1} \left[(1 - \kappa)^{-1} Q\|D_u \tilde{T}_e(Y_e(x), x) - D_u \tilde{T}_e(Y_e(y), y)\|_{L(U, U)}
+ \|D_x \tilde{T}_e(Y_e(x), x) - D_x \tilde{T}_e(Y_e(y), x)\|_{L(X, U)}\right].
$$
Now, the uniform continuity of $DT_{e_0}$ allows us to write

$$
\|D_i T_e(Y_e(x), x) - D_i T_{e_0}(Y_{e_0}(x), x)\|
\leq \|D_i T_e(Y_e(x), x) - D_i T_{e_0}(Y_{e_0}(x), x)\| + \omega_i(\|Y_e(x) - Y_{e_0}(x)\|),
$$

where $i$ stands for either $u$ or $x$ and $\omega_i$ denotes the modulus of continuity of $D_i T_{e_0}$. Using these facts, the uniform convergence of $D_\varepsilon Y_e$ towards $D_\varepsilon Y_{e_0}$ as $\varepsilon \to \varepsilon_0$ derives immediately from the analogous properties of $D_\varepsilon T_\varepsilon$ (72) and $Y_\varepsilon$ (7.5).

8. Applying the Contraction Theorem

In the following we proceed with the application of Theorem 7.1 to the specific mappings $T_{\varepsilon}$ introduced at the beginning of Section 7. In this application, the spaces $U$, $\tilde{U}$ will be taken respectively in the form $U = \tilde{E}_{\mu}^{1/2}$, $\tilde{U} = \tilde{E}_{\mu}^{1/2}$ with $\mu < \tilde{\mu}$, where, besides this inequality, for the moment $\mu$ and $\tilde{\mu}$ are restricted only to belong to the interval $(0, \lambda_{n+1})$. On the other hand, $X$ will be the space $E_1$, and $\Sigma$ will be taken in the form $\Sigma = [0, \tilde{\varepsilon}_1)$, where for the moment $\tilde{\varepsilon}_1$ will be restricted only by the inequality $0 < \tilde{\varepsilon}_1 < (2\mu)^{-1/2}$. In order to simplify the notation, in the following the complementary subspaces of $U = \tilde{E}_{\mu}^{1/2}$ and $\tilde{U} = \tilde{E}_{\mu}^{1/2}$ given respectively by $\tilde{E}_{i,\mu}^{1/2}$ ($i = 1, 2$) and $\tilde{E}_{i,\mu}^{1/2}$ ($i = 1, 2$) will be denoted as $U_i$ and $\tilde{U}_i$; on the other hand, the operators that $\tilde{G}_i$ ($i = 1, 2$) determines from $U = \tilde{E}_{\mu}^{1/2}$ to $U_i = \tilde{E}_{i,\mu}^{1/2}$ and from $U = \tilde{E}_{\mu}^{1/2}$ to $\tilde{U}_i = \tilde{E}_{i,\mu}^{1/2}$, which in the notation of Section 5 would be expressed respectively by $\tilde{G}_{i,\mu}$ and $\tilde{G}_{i,\mu,\tilde{\mu}}$, will here be denoted respectively by $\Gamma_i$ and $\tilde{\Gamma}_i$. With this notation, the mappings $T_{\varepsilon}: U \times X \to U$ to which we want to apply Theorem 7.1 are given by

$$
T_{\varepsilon}(u, x) = Kx + R_{\varepsilon}\Gamma_{\varepsilon}(u) + S_{\varepsilon}\Gamma_{\varepsilon}(u),
$$

where $K$, $R_{\varepsilon}$, and $S_{\varepsilon}$ are considered respectively as operators from $X = E_1$ to $U_1 = \tilde{E}_{1,\mu}^{1/2}$, from $U_1 = \tilde{E}_{1,\mu}^{1/2}$ to itself, and from $U_2 = \tilde{E}_{2,\mu}^{1/2}$ to itself; on the other hand, the mappings $\tilde{T}_e = J T_\varepsilon$ ($J$ is the embedding operator from $U = \tilde{E}_{\mu}^{1/2}$ to $\tilde{U} = \tilde{E}_{\mu}^{1/2}$) to which reference is made in Theorem 7.1 are immediately seen to be given by

$$
T_{\varepsilon}(u, x) = \tilde{K}x + \tilde{R}_{\varepsilon}\tilde{\Gamma}_{\varepsilon}(u) + \tilde{S}_{\varepsilon}\tilde{\Gamma}_{\varepsilon}(u),
$$

where $\tilde{K}$, $\tilde{R}_{\varepsilon}$, and $\tilde{S}_{\varepsilon}$ denote the operators analogous to $K$, $R_{\varepsilon}$, and $S_{\varepsilon}$ but operating respectively from $X = E_1$ to $\tilde{U}_1 = \tilde{E}_{1,\mu}^{1/2}$, from $\tilde{U}_1 = \tilde{E}_{1,\mu}^{1/2}$ to itself, and from $\tilde{U}_2 = \tilde{E}_{2,\mu}^{1/2}$ to itself.

According to Lemmata 5.3, 5.4, and 5.5, it is ensured that the mappings $\Gamma_i = \tilde{G}_{i,\mu}$ are bounded and Lipschitzian, and that the mappings $\tilde{\Gamma}_i = \tilde{G}_{i,\mu,\tilde{\mu}}$
are Fréchet differentiable with $D\Gamma_i: U \to L(U, \bar{U}_i)$ bounded and uniformly continuous; specifically, for every $u \in U$, $D\Gamma_i(u)$ can be expressed in the forms

$$D\Gamma_i(u) = Jd\Gamma_i(u) = \partial\Gamma_i(u) J,$$  \hspace{1cm} (8.3)

where $d\Gamma_i(u)$ and $\partial\Gamma_i(u)$ are the bounded linear operators from $U = E_{1/2}^{1/2}$ to $U_i = E_{1/2}^{1/2}$ and from $\bar{U} = \bar{E}_{1/2}^{1/2}$ to $\bar{U}_i = \bar{E}_{1/2}^{1/2}$, determined by the mapping $h \mapsto DG_i(u(\cdot))h(\cdot)$. On the other hand, by Lemmata 6.1 and 6.2 we know that $R_\varepsilon$, $\bar{R}_\varepsilon$, $S_\varepsilon$, and $\bar{S}_\varepsilon$ are bounded linear operators, and that for $\varepsilon > 0$ they converge in norm towards $R_0$, $\bar{R}_0$, $S_0$, $\bar{S}_0$. By combining these facts, we obtain that (a) the mappings $T_\varepsilon$ ($\varepsilon \in \Sigma$) are Lipschitzian; (b) the mappings $T_\varepsilon - T_0$ ($\varepsilon \in \Sigma$) are bounded, and $T_\varepsilon - T_0$ converges uniformly to 0 as $\varepsilon \to 0$; (c) the mappings $\bar{T}_\varepsilon$ ($\varepsilon \in \Sigma$) are Fréchet differentiable with $D\bar{T}_\varepsilon: U \times X \to L(U \times X, \bar{U})$ bounded and uniformly continuous, and for every $(u, x) \in U \times X$, $D_u \bar{T}_\varepsilon(u, x)$ and $D_x \bar{T}_\varepsilon(u, x)$ satisfy (7.6) and (7.7) with

$$d_u T_\varepsilon(u, x) = R_\varepsilon d\Gamma_1(u) + S_\varepsilon d\Gamma_2(u) \quad \hspace{1cm} (8.4)$$

$$d_u T_\varepsilon(u, x) = R_\varepsilon d\Gamma_1(u) + S_\varepsilon d\Gamma_2(u) \quad \hspace{1cm} (8.5)$$

$$d_x T_\varepsilon(u, x) = K; \quad \hspace{1cm} (8.6)$$

and that (d) the derivative mapping $D\bar{T}_\varepsilon$ converges uniformly to $D\bar{T}_0$ as $\varepsilon \to 0$.

With this, we have obtained all the hypotheses of Theorem 7.1 except the quantitative ones, namely the contractive character of $T_\varepsilon(\cdot, x): U \to U$ and the bounds (7.8) and (7.9). In order to check these properties, we will have to bring into play the estimates (3.10)–(3.13) on the derivatives of $G_i$, and the bounds provided by Lemmata 6.1 and 6.2 on the linear operators $R_\varepsilon$, $S_\varepsilon$ and their parallels $\bar{R}_\varepsilon$, $\bar{S}_\varepsilon$, namely

$$\|R_\varepsilon\|_{L(E_{1/2}^{1/2}, E_{1/2}^{1/2})} \leq \mu^{-1}(1 - \mu\varepsilon^2)^{-1} \quad \hspace{1cm} (8.7)$$

$$\|S_\varepsilon\|_{L(E_{1/2}^{1/2}, E_{1/2}^{1/2})} \leq K_\mu(\varepsilon)(\lambda_{n+1} - \mu)^{-1}, \quad \hspace{1cm} (8.8)$$

where $K_\mu(0) = 1$, and $K_\mu(\varepsilon) \to 1$ as $\varepsilon \to 0$, and similarly for $\bar{R}_\varepsilon$, $\bar{S}_\varepsilon$ with $\mu$ replaced by $\bar{\mu}$. In order to obtain better results, it will be convenient that for the moment we keep both components of $u = u_1 \perp u_2$ apart, so that we can afterwards choose as norm in $U$ the particular combination of the norms in $U_1$ and $U_2$ that better suits our purposes. To this end, in the following the components of $T_\varepsilon(u, x) =: \hat{u}$ in $U_1$ and $U_2$ will be denoted respectively as $\hat{u}_1$ and $\hat{u}_2$; thus, $\hat{u}_1$ and $\hat{u}_2$ are given respectively by $\hat{u}_1 = Kx + R_\varepsilon \Gamma_1(u)$ and $\hat{u}_2 = S_\varepsilon \Gamma_2(u)$. Let $u, v$ be two arbitrary elements of $U$. By using (3.10)–(3.13) and (8.7), one obtains immediately that

$$\|\hat{u}_1 - \hat{v}_1\|_{U_1} \leq (1 - \mu\varepsilon^2)^{-1} \left[ \frac{\lambda_n + I}{\mu} \|u_1 - v_1\|_{U_1} + \frac{I}{\mu} \|u_2 - v_2\|_{U_2} \right], \quad \hspace{1cm} (8.9)$$
On the other hand, from (3.10)-(3.13) and (8.8), it follows that

\[ \| \bar{u}_2 - \bar{v}_2 \|_{U_2} \leq K_\mu(\epsilon) \left[ \frac{l}{\lambda_{n+1} - \mu} \| u_1 - v_1 \|_{U_1} + \frac{l}{\lambda_{n+1} - \mu} \| u_2 - v_2 \|_{U_2} \right] , \]  

(8.10)

where we recall that \( K_\mu(0) = 1 \), and \( K_\mu(\epsilon) \to 1 \) as \( \epsilon \to 0 \). In view of these inequalities, we shall choose the norm in \( U \) to be given as follows:

\[ \|u\|_U := \max(\|u_1\|_{U_1}, \|u_2\|_{U_2}). \]  

(8.11)

With this choice, the combination of (8.9) and (8.10) gives us that

\[ \| T_\epsilon(u, x) - T_\epsilon(v, x) \|_U \leq \max \left( (1 - \mu \epsilon^2)^{-1} \frac{\lambda_n + 2l}{\mu} K_\mu(\epsilon) \frac{2l}{\lambda_{n+1} - \mu} \right) \|u - v\|_U. \]  

(8.12)

In order that the coefficient of the right-hand side of (8.12) be less than one, it suffices that \( \lambda_n, \lambda_{n+1}, l, \mu \) satisfy the inequalities

\[ \lambda_n + 2l < \mu < \lambda_{n+1} - 2l \]

and that \( \epsilon \) be small enough. On the other hand, when estimating \( d_u T_\epsilon \) and \( d_u \bar{T}_\epsilon \), one finds that they are bounded also by the coefficient of the right-hand side of (8.12), except that in the case of \( d_u \bar{T}_\epsilon \) the number \( \bar{\mu} \) appears instead of \( \mu \).

With this we have finally completed the whole set of hypotheses of Theorem 7.1. By applying that theorem to the preceding situation, we arrive at the following result (the parameter \( \mu \) of the statement corresponds to the one that until now was called \( \bar{\mu} \)).

**Theorem 8.1.** Let us assume that \( \lambda_n, \lambda_{n+1}, \) and \( l \) (the constant appearing in (3.2)) satisfy the inequalities

\[ \lambda_{n+1} - \lambda_n > 4l \]  

(8.13)

\[ \lambda_{n+1} > 2l \]  

(8.14)

and let \( \mu \) be any real number in the interval

\[ \max(0, \lambda_n + 2l) < \mu < \lambda_{n+1} - 2l. \]  

(8.15)

Then, there exists an \( \bar{\epsilon}_1 > 0 \) such that, for every \( \epsilon \in [0, \bar{\epsilon}_1] \) and every \( x \in E_1 \), system (3.6), (3.7) has a unique solution \( u \) belonging to \( E_1^{1/2} \) and satisfying the initial condition \( u_1(0) = x \). This solution is given by

\[ u = Y_\epsilon(x), \]  

(8.16)
where $Y_\varepsilon$ is equal to $K$ plus a mapping belonging to $C^1(E_1, \tilde{E}^{1/2}_{\mu})$, and, for $\varepsilon \to 0$, $Y_\varepsilon - K$ converges in this space towards $Y_0 - K$.

Besides this, in the following it will be important to bear in mind the fact that $Y_\varepsilon(x)$ is a fixed point of $T_\varepsilon(\cdot, x)$, i.e., that it satisfies the equation

$$Y_\varepsilon(x) = Kx + R_\varepsilon \mathcal{G}_1(Y_\varepsilon(x)) + S_\varepsilon \mathcal{G}_2(Y_\varepsilon(x))$$  (8.17)

(in fact, to obtain Theorem 8.1 we already have had to use this equation to derive the fact that $Y_\varepsilon - K$ is bounded).

9. Further Regularity, the Time Derivatives, and the Manifolds $M_\varepsilon$

Until now, the linear operators $S_\varepsilon$ have been considered only as acting from $E_\varepsilon^{1/2}$ to itself. However, Lemma 6.2 tells us that in fact they are still bounded from $E_\varepsilon^{1/2}$ to $E_\varepsilon^{1/2}$, and furthermore that the convergence of $S_\varepsilon (\varepsilon \to 0)$ towards $S_0$ remains true in the norm of bounded linear operators between these spaces. As it is easily seen, if one introduces the results of Theorem 8.1 into the right-hand side of (8.17), and then applies the preceding stronger information about the operators $S_\varepsilon$, one obtains that

**Theorem 9.1.** Under the conditions of Theorem 8.1, there exists an $\tilde{\varepsilon}_2 > 0$ such that, for every $\varepsilon \in [0, \tilde{\varepsilon}_2)$, $Y_\varepsilon - K$ belongs to $C^1(E_1, \tilde{E}^{1/2}_{\mu})$, and, for $\varepsilon \to 0$, $Y_\varepsilon - K$ converges towards $Y_0 - K$ in this space.

**Proof:** Take $\mu'$ in the interval $\max(0, \lambda_n + 2l) < \mu' < \mu$, apply Theorem 8.1 to $\mu'$, and take $\tilde{\varepsilon}_2(\mu') := \tilde{\varepsilon}_1(\mu')$. According to Theorem 8.1, for every $\varepsilon \in [0, \tilde{\varepsilon}_2)$, $Y_\varepsilon - K$ belongs to $C^1(E_1, \tilde{E}^{1/2}_{\mu})$, and for $\varepsilon \to 0$ it converges towards $Y_0 - K$ in this space. On the other hand, Lemma 5.2 guarantees that $\mathcal{G}_i (i = 1, 2)$ define mappings of class $C^1(\tilde{E}^{1/2}_{\mu}, \tilde{E}^{1/2}_{\mu})$, and finally, Lemmata 6.1 and 6.2 guarantee that $R_\varepsilon$ and $S_\varepsilon$ belong to $L(\tilde{E}^{1/2}_{\mu}, \tilde{E}^{1/2}_{\mu})$ ($i$ equal respectively to 1 and 2), and for $\varepsilon \to 0$ they converge in the norm of these spaces respectively towards $R_0$ and $S_0$. This immediately implies the statement of Theorem 9.1.

In the following, the first component of $Y_\varepsilon(x)$, i.e., its component in $\tilde{E}^{1/2}_{1,\mu} \simeq E_{1,\mu}$, will be denoted by $\psi_\varepsilon(x)$,

$$\psi_\varepsilon(x) := \tilde{P}_1 Y_\varepsilon(x) = Kx + R_\varepsilon \mathcal{G}_1 Y_\varepsilon(x).$$  (9.1)

This projection will play an important role in what follows; in fact, $\psi_\varepsilon$ will give the finite-dimensional flow as projected from $M_\varepsilon$ to $E_\varepsilon$. Obviously, the above properties of $Y_\varepsilon$ imply that, under the conditions of Theorem 8.1, $\psi_\varepsilon$ belongs to $C^1(E_1, \tilde{E}^{1/2}_{1,\mu})$, and that for $\varepsilon \to 0$, $\psi_\varepsilon$ converges towards $\psi_0$ in this space.
By virtue of Lemmata 6.1 and 6.2, for every \( \varepsilon \in [0, \bar{\varepsilon}) \) and every \( x \in \mathbb{E}_1 \), the function \( Y_\varepsilon(x) \) is continuously differentiable with respect to time in the space \( \mathbb{E} \), and its derivative function, which we shall denote by \( \dot{Y}_\varepsilon(x) \), belongs to \( \mathbb{E}_\mu \) and is given by

\[
\dot{Y}_\varepsilon(x) = \dot{\mathcal{R}}_\varepsilon \mathcal{G}_1 Y_\varepsilon(x) + \dot{\mathcal{S}}_\varepsilon \mathcal{G}_2 Y_\varepsilon(x),
\]

where \( \dot{\mathcal{R}}_\varepsilon \) and \( \dot{\mathcal{S}}_\varepsilon \) are the bounded linear operators from \( \mathbb{E}_i^{1/2} \) to \( \mathbb{E}_i^{1/2} \) (\( i \) equal respectively to 1 and 2) whose properties are discussed in Lemmata 6.1 and 6.2. As in Theorem 9.1, if one introduces the results of Theorem 8.1 into the right-hand side of (9.2), one obtains that under the conditions of Theorem 8.1, there exists an \( \bar{\varepsilon}_3 > 0 \) such that, for every \( \varepsilon \in [0, \varepsilon_3) \), \( \dot{Y}_\varepsilon \) belongs to \( C^1(\mathbb{E}_1, \mathbb{E}_\mu) \).

Before entering into the question of the convergence of \( \dot{Y}_\varepsilon (\varepsilon \to 0) \) towards \( \dot{Y}_0 \), let us see how \( Y_\varepsilon \) and \( \dot{Y}_\varepsilon \) determine the invariant manifold \( M_\varepsilon \). Specifically, we consider \( M_\varepsilon \subset \mathbb{E}_1 \times \mathbb{E} \) defined as the set of the initial values of the pair of functions \( (Y_\varepsilon(x), \dot{Y}_\varepsilon(x)) \) where \( x \) varies over \( \mathbb{E}_1 \). In the following, the initial values of the functions \( Y_\varepsilon(x) \) and \( \dot{Y}_\varepsilon(x) \) will be denoted respectively as \( y_\varepsilon(x) \) and \( z_\varepsilon(x) \),

\[
y_\varepsilon(x) := Y_\varepsilon(x; 0) \quad \text{(9.3)}
\]

\[
z_\varepsilon(x) := \dot{Y}_\varepsilon(x; 0) \quad \text{(9.4)}
\]

(here and in what follows, the value of \( Y_\varepsilon(x) \) at time \( t \) is denoted by \( Y_\varepsilon(x; t) \), and analogously for \( \dot{Y}_\varepsilon, \psi_\varepsilon, \) and \( \dot{\psi}_\varepsilon \)). Thus, \( M_\varepsilon \) is given by

\[
M_\varepsilon := \{ (y_\varepsilon(x), z_\varepsilon(x)) \in \mathbb{E}_1^{1/2} \times \mathbb{E} \mid x \in \mathbb{E}_1 \}. \quad \text{(9.5)}
\]

According to Lemma 6.1, we have that \( P_1 y_\varepsilon(x) = x \); the other components of \( y_\varepsilon \) and \( z_\varepsilon \) are the mappings \( h_\varepsilon, k_\varepsilon, l_\varepsilon \) appearing in the description of \( M_\varepsilon \) used in Section 2,

\[
h_\varepsilon(x) := P_2 y_\varepsilon(x) \quad \text{(9.6)}
\]

\[
k_\varepsilon(x) := P_1 z_\varepsilon(x) \quad \text{(9.7)}
\]

\[
l_\varepsilon(x) := P_2 z_\varepsilon(x). \quad \text{(9.8)}
\]

According to these expressions, the properties of the mappings \( h_\varepsilon, k_\varepsilon, l_\varepsilon \) will follow trivially from those of \( y_\varepsilon \) and \( z_\varepsilon \), which in turn will follow from those of \( Y_\varepsilon \) and \( \dot{Y}_\varepsilon \). In particular, the obtained properties of \( Y_\varepsilon \) and \( \dot{Y}_\varepsilon \) imply immediately that, for every \( \varepsilon \in [0, \bar{\varepsilon}_2) \), \( y_\varepsilon \in C^1(\mathbb{E}_1, \mathbb{E}_1) \) and \( z_\varepsilon \in C^1(\mathbb{E}_1, \mathbb{E}) \), and furthermore \( y_\varepsilon (\varepsilon \to 0) \) converges to \( y_0 \) in the above space \( C^1(\mathbb{E}_1, \mathbb{E}_1) \). These smoothness properties of \( y_\varepsilon \) and \( z_\varepsilon \) together with the fact that \( y_\varepsilon \) has the form \( y_\varepsilon(x) = x + h_\varepsilon(x) \) with \( x \in \mathbb{E}_1 \) and \( h_\varepsilon(x) \in \mathbb{E}_2 \)
show that $M_\varepsilon$ is indeed a submanifold of class $C^1$ in $E^{1/2} \times E$ (in fact in $E^1 \times E$).

Obviously, for every $t \in \mathbb{R}$ the function $u : \mathbb{R} \ni \tau \mapsto Y_\varepsilon(x; t + \tau) \in E^{1/2}$ is also, like $Y_\varepsilon$ itself, a solution of (3.6), (3.7) belonging to $E^{1/2}_{\mu'}$, and it satisfies the initial condition $u_1(0) = \psi_\varepsilon(x; t)$. Therefore, by uniqueness, one must have

$$Y_\varepsilon(x; t + \tau) = Y_\varepsilon(\psi_\varepsilon(x; t); \tau),$$

and consequently, by differentiating with respect to $\tau$,

$$\dot{Y}_\varepsilon(x; t + \tau) = \dot{Y}_\varepsilon(\psi_\varepsilon(x; t); \tau).$$

By evaluating these relations at $\tau = 0$, we obtain that

$$Y_\varepsilon(x; t) = y_\varepsilon(\psi_\varepsilon(x; t)) \quad (9.9)$$

$$\dot{Y}_\varepsilon(x; t) = z_\varepsilon(\psi_\varepsilon(x; t)). \quad (9.10)$$

These relations express the fact that the manifold $M_\varepsilon$ is indeed invariant, with the flow on $M_\varepsilon$ being given precisely by the corresponding lift of $\psi_\varepsilon$ from $E_1$ to $M_\varepsilon$. Furthermore, by projecting (9.10) onto $E_1$ we obtain that

$$\dot{\psi}_\varepsilon(x; t) = k_\varepsilon(\psi_\varepsilon(x; t)); \quad (9.11)$$

i.e., the flow $\psi_\varepsilon$ is indeed generated by the vector field $k_\varepsilon$.

Our goal now is to prove the convergence of $Y_\varepsilon (\varepsilon \to 0)$ towards $Y_0$ in the space $C^1(E_1, E)$. This will imply the convergence of $z_\varepsilon$ towards $z_0$ in the space $C^1(E_1, E)$, and consequently the convergence of $k_\varepsilon$ and $l_\varepsilon$ towards $k_0$ and $l_0$ in the analogous space with $E$ replaced respectively by $E_1$ and $E_2$.

We begin by decomposing $Y_\varepsilon - Y_0$ as

$$V_\varepsilon = R_\varepsilon \mathcal{G}_1(Y_\varepsilon - Y_0) + S_\varepsilon \mathcal{G}_2(Y_\varepsilon - Y_0)$$

$$W_\varepsilon = (R_\varepsilon - R_0) \mathcal{G}_1 Y_0 + (S_\varepsilon - S_0) \mathcal{G}_2 Y_0.$$
(a1) \( \{ G_i(Y_0(x; \cdot)) | x \in E_1 \} \) is \( \bar{E}_{i,\mu}^{1/2} \)-bounded and \( \bar{E}_{i,\mu}^{1/2} \)-uniformly equicontinuous.

(a2) \( \{ DG_i(Y_0(x; \cdot)) D_x Y_0(x; \cdot) | x \in E_1 \} \) is \( L(E_1, E_i^{1/2}) \)-bounded and \( L(E_1, E_i^{1/2}) \)-uniformly equicontinuous.

By making use of Lemma 5.4 and the facts that \( G_i: E_i^{1/2} \to E_i^{1/2} \) is bounded and has a bounded derivative, and that \( Y_0: E_1 \to \bar{E}_{\mu}^{1/2} \) has a bounded derivative, these conditions reduce to the following ones:

(b1) \( \{ Y_0(x; \cdot) | x \in E_1 \} \) is \( \bar{E}_{\mu}^{1/2} \)-uniformly equicontinuous.

(b2) \( \{ D_x Y_0(x; \cdot) | x \in E_1 \} \) is \( L(E_1, E_i^{1/2}) \)-uniformly equicontinuous.

At this point, one could try to proceed by obtaining suitable bounds on \( \hat{Y}_0 \) and \( D_x \hat{Y}_0 \) directly from Eq. (9.2); however, this yields only a weaker version of the preceding conditions, namely the space of values would not be \( E_i^{1/2} \) but \( E \). In order to attain our objective, we proceed indirectly by first using relation (9.9) with \( \varepsilon = 0 \),

\[
Y_0(x; t) = y_0(\psi_0(x; t))
\]  (9.12)

together with what is obtained from it by differentiating with respect to \( x \), namely

\[
D_x Y_0(x; t) = D_y(y_0(\psi_0(x; t))) D_x \psi_0(x; t).
\]  (9.13)

By applying Lemma 5.4 again, conditions (b1), (b2) reduce in this way to the following ones:

(c1) \( \{ \psi_0(x; \cdot) | x \in E_1 \} \) is \( \bar{E}_{1,0} \)-uniformly equicontinuous.

(c2) \( \{ D_x \psi_0(x; \cdot) | x \in E_1 \} \) is \( L(E_1, E_1) \)-uniformly equicontinuous.

By virtue of Lemma 5.4, part (b), this will be guaranteed if we verify that

(d1) \( \{ \psi_0(x; \cdot) | x \in E_1 \} \) is \( \bar{E}_{1,0} \)-bounded.

(d2) \( \{ D_x \psi_0(x; \cdot) | x \in E_1 \} \) is \( L(E_1, E_1) \)-bounded.

(d3) \( \{ D_x \psi_0(x; \cdot) | x \in E_1 \} \) is \( L(E_1, E_1) \)-bounded.

Now, condition (d1) is an immediate consequence of relation (9.11) with \( \varepsilon = 0 \),

\[
\psi_0(x; t) = k_0(\psi_0(x; t))
\]  (9.14)

and the fact that \( k_0: E_1 \to E_1 \) is bounded. On the other hand, condition (d2) follows from the fact that \( Y_0: E_1 \to \bar{E}_{\mu}^{1/2} \) has a bounded derivative. Finally, (d3) can be obtained for instance by differentiating (9.14) with respect to \( x \) and then using (d2) together with the fact that \( k_0: E_1 \to E_1 \) has a bounded derivative.
Further information on $\dot{Y}_\varepsilon$ can be obtained from the following relation, which stems from (9.9) by differentiating with respect to $t$:

$$\dot{Y}_\varepsilon(x; t) = D_y(\psi_\varepsilon(x; t)) \dot{\psi}_\varepsilon(x; t).$$

(9.15)

By using the above results on the smoothness and convergence of $\psi_\varepsilon$, $\dot{\psi}_\varepsilon$, and $y_\varepsilon$ as $\varepsilon \to 0$, one obtains that $\dot{Y}_\varepsilon \in C^0(E_1, \bar{E}_\mu)$, and that, as $\varepsilon \to 0$, $\dot{Y}_\varepsilon$ converges towards $\dot{Y}_0$ in this space.

More surprisingly, an analogous argument starting from (9.10) shows that for every $\varepsilon \in [0, \bar{\varepsilon}_2]$ (including $\varepsilon = 0$) and every $x \in E_1$, $Y_\varepsilon(x)$ is twice continuously differentiable with respect to $t$ in the space $E$, and the second derivative $\ddot{Y}_\varepsilon(x)$ is given by

$$\ddot{Y}_\varepsilon(x; t) = D_{zz}(\psi_\varepsilon(x; t)) \ddot{\psi}_\varepsilon(x; t),$$

(9.16)

from which the above results on $\psi_\varepsilon$, $\dot{\psi}_\varepsilon$, and $z_\varepsilon$ imply that $\dot{Y}_\varepsilon \in C^0(E_1, \bar{E}_\mu)$, and that, as $\varepsilon \to 0$, $\dot{Y}_\varepsilon$ converges towards $\dot{Y}_0$ in this space.

Summing up the results obtained in the present section, we can state the following

**Theorem 9.2.** Under the conditions of Theorem 8.1, there exists an $\bar{\varepsilon}_3 > 0$ such that, for every $\varepsilon \in [0, \bar{\varepsilon}_3)$ and every $x \in E_1$, the solution of (3.6), (3.7), $u = Y_\varepsilon(x)$, obtained in Theorem 8.1 is twice continuously differentiable with respect to time in the space $E$, its first and second time derivatives being given by

$$\dot{u} = \dot{Y}_\varepsilon(x),$$

(9.17)

$$\ddot{u} = \ddot{Y}_\varepsilon(x),$$

(9.18)

where $\dot{Y}_\varepsilon$ belongs to $C^1(E_1, \bar{E}_\mu)$ $\cap$ $C^0(E_1, \bar{E}_\mu)$ and for $\varepsilon \to 0$ it converges towards $\dot{Y}_0$ in both spaces, and $\ddot{Y}_\varepsilon$ belongs to $C^0(E_1, \bar{E}_\mu)$ and for $\varepsilon \to 0$ it converges in this space towards $\ddot{Y}_0$.

As we have been indicating in this and the preceding sections, this concludes the proof of Theorem 2.1. In fact, in the case where $F$ itself (without modifying it) belongs to $C^{1+\eta}(E^{1/2}, \bar{E}^{1/2})$, Theorems 8.1, 9.1, and 9.2 give a stronger result, namely the realization of $M_\varepsilon$ as a submanifold of class $C^1$ in $\bar{E}_\mu^{1/2} \times \bar{E}_\mu$, and in fact in $\bar{E}_\mu^{1/2} \times \bar{E}_\mu$.

**10. Proofs of Lemmata 6.1 and 6.2**

In this last section we give the proofs of Lemmata 6.1 and 6.2 on the resolution of the nonhomogeneous linear equations (I)$_\varepsilon$ and (II)$_\varepsilon$ in spaces of curves with exponentially weighted norms.
10.1. A Fundamental Principle

We begin by recording a fundamental lemma which, besides being used in the proofs below, will also help illustrate the gist of the matter. This lemma refers to an equation of the form

\[ \dot{u} + Au = f, \quad (10.1) \]

where \( u \) and \( f \) stand for functions of \( t \in \mathbb{R}_- \) with values in a Banach space \( X \), and \( -A \) is the generator of a semigroup on \( X \) satisfying the bound

\[ \|e^{-At}\| \leq Me^{-\alpha t}, \quad \forall t \geq 0. \quad (10.2) \]

**Lemma 10.1.** If \( \mu < \alpha \) then, for every \( f \in \tilde{X}_\mu \), (10.1) has a unique mild solution \( u \) belonging to \( \tilde{X}_\mu \), namely the function

\[ u(t) = \int_{-\infty}^{t} e^{-A(t-s)}f(s) \, ds =: (Qf)(t), \quad (10.3) \]

and the linear operator \( Q : \tilde{X}_\mu \to \tilde{X}_\mu \) is bounded with

\[ \|Qf\|_{\tilde{X}_\mu} \leq M(\alpha - \mu)^{-1} \|f\|_{\tilde{X}_\mu}. \quad (10.4) \]

**Proof.** A function \( u : \mathbb{R}_- \to X \) is a mild solution of (10.1) if and only if it satisfies the following equation:

\[ u(t) = e^{-A(t-r)}u(r) + \int_{r}^{t} e^{-A(t-s)}f(s) \, ds, \quad \forall r \forall t \text{ with } -\infty < r < t. \quad (10.5) \]

Let us assume that \( f \in \tilde{X}_\mu \) and that \( u \) is a solution of (10.5) belonging also to \( \tilde{X}_\mu \). Under these conditions, the right-hand side of (10.5) is easily seen to have a well-defined limit when \( r \to -\infty \), namely, the quantity given by (10.3). Indeed, the bound

\[ \|e^{-A(t-s)}f(s)\|_X \leq Me^{-\alpha t}e^{(\alpha - \mu)s} \|f\|_{\tilde{X}_\mu}, \quad \forall s \leq t, \quad (10.6) \]

ensures that the arising improper integral is convergent, and that the analogous bound with \( f \) replaced by \( u \) shows that the first term of the right-hand side of (10.5) tends to zero. Therefore, by passing to the limit \( r \to -\infty \), relation (10.5) implies that \( u \) must coincide with the function given by (10.3).

Conversely, for every \( f \in \tilde{X}_\mu \), one verifies easily that (10.3) is indeed a solution of (10.5), and by using (10.6) one obtains that it indeed belongs to \( \tilde{X}_\mu \) with the bound (10.4).
10.2. Proof of Lemma 6.1

Step 1. Let us begin by considering the limiting case \( \varepsilon = 0 \). In this case, Eq. (I)_\varepsilon reduces simply to

\[
\dot{u} = f. \tag{I_0}
\]

Obviously, for every \( f: \mathbb{R} \to X \) continuous and every \( x \in X \), this equation has a unique solution satisfying \( u(0) = x \), namely the function \( u \) given by (6.1) with

\[
(R_0 f)(t) := \int_0^t f(s) \, ds. \tag{10.7}
\]

From this formula, one immediately verifies that if \( \mu > 0 \), then, for every \( f \) belonging to \( \mathcal{X}_\mu \), \( R_0 f \) belongs also to \( \mathcal{X}_\mu \) and it satisfies the bound (6.3) with \( \varepsilon = 0 \). Finally, in this case the operator \( \hat{R}_0 \) is simply the identity, which certainly satisfies (6.4) with \( \varepsilon = 0 \).

Step 2. Let us now proceed with the case \( \varepsilon \neq 0 \). We shall begin by proving the uniqueness of the solution as stated in the lemma. By linearity, it suffices to verify that, for the homogeneous problem \( \varepsilon^2 \dot{u} + \dot{u} = 0 \), \( u(0) = 0 \), the only solution belonging to \( \mathcal{X}_\mu \) is \( u = 0 \); indeed, the general solution of this problem is \( u(t) = (1 - \exp(-t/\varepsilon^2))z \) (\( z \in X \)), but, due to the inequality \( \mu < \varepsilon^{-2} \), this function will not belong to \( \mathcal{X}_\mu \) unless \( z = 0 \).

Let us now rewrite (I)_\varepsilon as the first-order system

\[
\begin{align*}
\dot{u} &= v 
\tag{10.8}
\end{align*}
\]

\[
\varepsilon^2 \dot{v} + v = f. \tag{10.9}
\]

Since \( \mu < \varepsilon^{-2} \), Lemma 10.1 ensures that, for every \( f \in \mathcal{X}_\mu \), Eq. (10.9) has a unique solution \( v \) belonging to \( \mathcal{X}_\mu \). Let us take this function \( v \) and introduce it in the right-hand side of (10.8). According to Step 1 of the present proof, this equation has a unique solution \( u \) belonging to \( \mathcal{X}_\mu \) and satisfying \( u(0) = x \). Obviously, by uniqueness, this function \( u \) is the solution \( R_\varepsilon f \) of (I)_\varepsilon which we were going after, and the function \( v \) previously obtained from (10.9) is its time derivative \( \hat{R}_\varepsilon f \). According to Lemma 10.1 and Step 1 above, \( \hat{R}_\varepsilon f \) is given by

\[
(\hat{R}_\varepsilon f)(t) = \varepsilon^{-2} \int_{-\infty}^t \exp(\varepsilon^{-2}(t - s)) f(s) \, ds, \tag{10.10}
\]

and \( R_\varepsilon f = R_0 \hat{R}_\varepsilon f \) is given by

\[
(R_\varepsilon f)(t) = \varepsilon^{-2} \int_{-\infty}^t \int_0^\tau \exp(-\varepsilon^{-2}(\tau - s)) f(s) \, ds \, d\tau, \tag{10.11}
\]
or, by interchanging the order of integration and performing the integration with respect to \( \tau \),

\[
(R_\varepsilon f)(t) = -\int_t^0 f(s) \, ds - \int_{-\infty}^t \exp(-\varepsilon^{-2}(t-s)) f(s) \, ds
\]

\[
+ \int_{-\infty}^0 \exp(\varepsilon^{-2}s) f(s) \, ds
\]

(10.12)

i.e.,

\[
R_\varepsilon f = R_0 f - \varepsilon^2(\dot{R}_\varepsilon f - K\mathcal{E}\dot{R}_\varepsilon f),
\]

(10.13)

where \( \mathcal{E} \) stands for the evaluation at time \( t = 0 \) as an operator from \( \vec{X}_\mu \) to \( X \), and \( K \) is the embedding operator from \( X \) to \( \vec{X}_\mu \) alluded to in Section 5. Finally, the bounds (6.3) and (6.4) are an immediate consequence of those provided by Lemma 10.1 and Step 1. This completes the proof of part (a) of Lemma 6.1.

**Step 3.** In order to prove part (b), we notice that (10.13) leads to the inequality

\[
\|R_\varepsilon f - R_0 f\|_{\vec{X}_\mu} \leq \varepsilon^2(\|\dot{R}_\varepsilon f\|_{\vec{X}_\mu} + \|K\mathcal{E}\dot{R}_\varepsilon f\|_{\vec{X}_\mu}).
\]

By using the fact that both \( \mathcal{E}: \vec{X}_\mu \to X \) and \( K: X \to \vec{X}_\mu \) have norm equal to 1, this implies that

\[
\|R_\varepsilon f - R_0 f\|_{\vec{X}_\mu} \leq 2\varepsilon^2\|\dot{R}_\varepsilon f\|_{\vec{X}_\mu},
\]

which combined with (6.4) yields (6.5).

Finally, we concern ourselves with the convergence of \( \dot{R}_\varepsilon \) towards \( \dot{R}_0 \) as \( \varepsilon \to 0 \). From expression (10.10) and the identity \( \dot{R}_0 f = f \), we deduce that

\[
e^\mu t \| (\dot{R}_\varepsilon f)(t) - (\dot{R}_0 f)(t) \|_X
\]

\[
\leq \varepsilon^{-2} \int_{-\infty}^t \exp(- (\varepsilon^{-2} - \mu)(t-s)) e^{\mu s} \| f(s) - f(t) \|_X \, ds
\]

\[
=: \int_{-\infty}^t \varphi_\varepsilon(s) \, ds.
\]

(10.14)

Note that when \( \varepsilon \to 0 \), the functions \( s \mapsto \varepsilon^{-2} \exp(- (\varepsilon^{-2} - \mu)(t-s)) \) tends to a Dirac's measure located at \( s = t \). As usual in similar circumstances, we shall decompose the integral into two parts, \( \int_{-\infty}^{-\varepsilon} \varphi_\varepsilon(s) \, ds \) and \( \int_{-\varepsilon}^t \varphi_\varepsilon(s) \, ds \). The first integral can be bounded as follows:

\[
\int_{-\infty}^{-\varepsilon} \varphi_\varepsilon(s) \, ds \leq (1 - \mu \varepsilon^{-2})^{-1} \exp(-(\varepsilon^{-2} - \mu)\varepsilon) \cdot 2\| f \|_{\vec{X}_\mu}.
\]

(10.15)
Concerning the second integral, by using the inequality
\[ e^{\mu s} \|f(s) - f(t)\|_X \leq \|e^{\mu s}f(s) - e^{\mu t}f(t)\|_X + (1 - e^{-\mu t}) \|f\|_{X_{\mu}}, \quad \text{for} \quad t - s \leq \tau, \]
and defining
\[ \omega_{\mu}(f; \tau) := \sup_{|t - s| < \tau} \|e^{\mu s}f(s) - e^{\mu t}f(t)\|_X, \]
we obtain that
\[ \int_{t-\tau}^{t} \varphi_{\mu}(s) \, ds \leq (1 - \mu e^{-2})^{-1} [\omega_{\mu}(f; \tau) + (1 - e^{-\mu \tau}) \|f\|_{X_{\mu}}]. \quad (10.16) \]
Here, \( \omega_{\mu}(f; \tau) \) tends to zero as \( \tau \to 0 \), because, by definition, the fact that \( f \in \tilde{X}_{\mu} \) implies that \( J_{\mu}f \) is uniformly continuous. The idea is now to first take \( \tau \) small enough to make (10.16) as small as desired, and afterwards to take \( \varepsilon \) small enough to make (10.15) as small as desired. In this way one immediately obtains the convergence of \( R_{\varepsilon} \) towards \( R_0 \) in the strong topology, and in view of the way \( f \) appears in the right-hand sides of (10.15) and (10.16), it is clear that this convergence will be uniform on \( \mathscr{S} \subset \tilde{X}_{\mu} \) whenever \( \mathscr{S} \) is \( \tilde{X}_{\mu} \)-bounded and \( \tilde{X}_{\mu} \)-uniformly equicontinuous.

10.3. Proof of Lemma 6.2

In the following, \( e_k \) and \( \xi_k \) \((k = 1, 2, \ldots)\) denote respectively a complete orthogonal system of eigenvectors of \( A \) and the corresponding sequence of eigenvalues; we assume that the ordering is such that the sequence of eigenvalues is nondecreasing, and that the eigenvectors are normalized in \( E^{1/2} \). Furthermore, \( \xi \) denotes the real number smaller than \( \xi_1 \), which determines the particular inner product used in the spaces \( E^2 \).

**Step 1.** Let us begin by considering the limiting case \( \varepsilon = 0 \). As is well known, the hypotheses on \( A \) imply that \(-A\) is the generator of a semigroup \( e^{-\lambda t} \) \((t \geq 0)\) on \( E \) which in particular has the following properties:

(i) For \( t \geq 0 \), the operators \( e^{-\lambda t} \) map \( E^{1/2} \) to itself with the following bound:
\[ \|e^{-\lambda t}x\|_{1/2} \leq e^{-\xi_1 t\|x\|_{1/2}}. \quad (10.17) \]

(ii) For \( t > 0 \), the operators \( e^{-\lambda t} \) map \( E^{1/2} \) to \( E^1 \) with the following bound:
\[ \|e^{-\lambda t}x\|_1 \leq [(\xi_1 - \xi) + (et)^{-1}]^{1/2} e^{-\xi_1 t\|x\|_{1/2}}. \quad (10.18) \]

By applying Lemma 10.1 and using these properties, one obtains that if \( \mu < \xi_1 \), then, for every \( f \in \tilde{E}_{1/2} \), (II), has a unique mild solution \( u \) belonging to \( \tilde{E}_{1/2} \), namely
\[ u(t) = \int_{-\infty}^{t} e^{A(t-s)}f(s) \, ds =: (S_0 f)(t), \quad (10.19) \]
and the linear operator $S_0$ maps $\bar{E}^{1/2}_\mu$ to $\bar{E}^1_\mu$ with the following bounds:

\[
\|S_0f\|_{\bar{E}^{1/2}_\mu} \leq (\xi_1 - \mu)^{-1}\|f\|_{\bar{E}^{1/2}_\mu},
\]

\[
\|S_0f\|_{\bar{E}^1_\mu} \leq \left[\frac{(\xi_1 - \xi)^{1/2}}{\xi_1 - \mu} + \frac{1}{\sqrt{e(\xi_1 - \mu)^{1/2}}}\right]\|f\|_{\bar{E}^{1/2}_\mu}.
\]

By an usual argument of semigroup theory, the fact that $S_0f$ is continuous in $E^1$ implies that it is in fact a strict solution. Finally, the statement that $\dot{S}_0$ is a bounded operator from $\bar{E}^{1/2}_\mu$ to $\bar{E}^1_\mu$ follows trivially from the relation

\[
\dot{S}_0f = f - AS_0f
\]

and the fact that $S_0$ is bounded from $\bar{E}^{1/2}_\mu$ to $\bar{E}^1_\mu$.

**Step 2.** Let us now consider the case $\varepsilon \neq 0$. As in Section 10.2, we begin by proving the statement of uniqueness. By linearity, it suffices to show that the homogeneous equation $\varepsilon^2 \dot{u} + \dot{u} + Au = 0$ has no mild solution belonging to $\bar{E}^{1/2}_\mu$ other than $u = 0$. By resolving $u$ into its components $u_k$ with respect to the Hilbert basis $e_k$ ($k = 1, 2, \ldots$), this reduces to showing that, for every $k = 1, 2, \ldots$, the differential equation

\[
\varepsilon^2 \ddot{u}_k + \dot{u}_k + \xi_k u_k = 0
\]

has no solution belonging to $\bar{R}^k_\mu$ other than $u_k = 0$. This will be true if the exponentials $\exp(-\alpha_k^\mu t)$ which appear in the general solution of this equation always have $\text{Re} \alpha_k^\mu > \mu$. This happens when $\mu < \min(\xi_1, 1/2\varepsilon^2)$, which is the condition stated at the beginning of the lemma.

Proceeding as in Section 1, we now rewrite (11), as a first-order equation for $U := (u, \dot{u})$, which variable is considered here as taking values in $E^1 \times E^{1/2}$. This equation reads

\[
\dot{U} + A_\varepsilon U = F_\varepsilon,
\]

where $A_\varepsilon$ is given by (1.14), with domain equal to $E^{3/2} \times E^1$, and $F_\varepsilon := (0, \varepsilon^{-2}f)$. Our objective is to apply Lemma 10.1 to this equation. In order to admit values of $\mu$ as large as possible, we must have an estimate of the semigroup $e^{-A_\varepsilon t}$ ($t \geq 0$) in the form (10.2) with $\alpha$ as large as possible. Such estimates can be obtained by examining the numerical range of $A_\varepsilon$ with respect to an appropriate equivalent inner product on $E^1 \times E^{1/2}$ (which depends on $\varepsilon$). Specifically, we use the inner product associated with the norm

\[
\|U\|^2_\ast := \|(A - \xi I)^{1/2} u\|^2_{1/2} + \left\|\frac{1}{2\varepsilon} u + \varepsilon v\right\|^2_{1/2},
\]
with

\[
\xi = \begin{cases} 
2\xi_1 - \frac{1}{4\varepsilon^2}, & \text{if } \xi_1 < \frac{1}{4\varepsilon^2} \\
\xi_1 - \delta, & \text{if } \xi_1 = \frac{1}{4\varepsilon^2} \\
\frac{1}{4\varepsilon^2}, & \text{if } \xi_1 > \frac{1}{4\varepsilon^2},
\end{cases}
\]

where, in the case \(\xi_1 = 1/4\varepsilon^2\), \(\delta\) stands for a strictly positive real number destined to be chosen sufficiently small. By proceeding as in Mora and Solà-Morales [16, Sect. 2.1], one obtains that, in this norm, the operators \(e^{-A_{11}t} (t \geq 0)\) satisfy a bound of form (10.2) with \(M = 1\) and

\[
\alpha = \begin{cases} 
\frac{\xi_1}{1/2 + \sqrt{1/4 - \varepsilon^2\xi_1}}, & \text{if } \xi_1 < \frac{1}{4\varepsilon^2} \\
\frac{1}{2\varepsilon^2} - \sqrt{2\varepsilon^2}, & \text{if } \xi_1 = \frac{1}{4\varepsilon^2} \\
\frac{1}{2\varepsilon^2}, & \text{if } \xi_1 > \frac{1}{4\varepsilon^2},
\end{cases}
\]

i.e., in the case \(\xi_1 \neq 1/4\varepsilon^2\) \(\alpha\) is equal exactly to the least real part of the spectrum of \(A_{11}\), and in the case \(\xi_1 = 1/4\varepsilon^2\) it is smaller but it can be made arbitrarily near to it by taking \(\delta\) sufficiently small. Now, the hypothesis that \(\mu < \min(\xi_1, 1/2\varepsilon^2)\) implies that the preceding quantity \(\alpha\) is, or can be made, greater than \(\mu\), as required in Lemma 10.1. By applying that lemma, and translating the result to Eq. (II), one obtains that for every \(f \in \tilde{E}^{1/2}_\mu\), (II) \(e\) has a unique solution belonging to \(\tilde{E}^1_\mu\) with derivative belonging to \(\tilde{E}^{1/2}_\mu\), and this solution and its derivative are given respectively by \(S_{e}f\) and \(\hat{S}_{e}f\), where \(S_{e}\) is a bounded linear operator from \(\tilde{E}^{1/2}_\mu\) to \(\tilde{E}^1_\mu\), and \(\hat{S}_{e}\) is a bounded linear operator from \(\tilde{E}^{1/2}_\mu\) to \(\tilde{E}^{1/2}_\mu\).

By carrying out the details of the preceding application of Lemma 10.1, one obtains that

\[
\|S_{e}f\|_{\tilde{E}^1_\mu} \leq \frac{C(e)}{e(\xi_1 - \mu)} \|f\|_{\tilde{E}^{1/2}_\mu}, \quad (10.24)
\]

\[
\|\hat{S}_{e}f\|_{\tilde{E}^{1/2}_\mu} \leq \frac{C(e)}{e^2(\xi_1 - \mu)} \|f\|_{\tilde{E}^{1/2}_\mu}, \quad (10.25)
\]

where \(C(e) \to 1\) as \(e \to 0\). In order to obtain information on \(\hat{S}_{e}\) as an operator from \(\tilde{E}^{1/2}_\mu\) to \(\tilde{E}^1_\mu\) rather than from \(\tilde{E}^{1/2}_\mu\) to itself, we shall make use of the identity

\[
\hat{S}_{e}f = \hat{R}_{e}(f - AS_{e}f), \quad (10.26)
\]
where \( \hat{R}_e \) denotes the bounded linear operator from \( \bar{E}_\mu \) to \( E_\mu \) which Lemma 6.1 associates with Eq. (1) of \( \varepsilon \) considered now in the space \( X = E \). Using this identity and applying bounds (10.24) and (6.4), we obtain that
\[
\| \hat{S}_e \|_{L(E_{1/2}^{1/2}, E_\mu)} = O(\varepsilon^{-1}).
\] (10.27)

In view of (10.24) and (10.27), we are still far from obtaining the uniform boundedness and the convergence as \( \varepsilon \to 0 \) of \( S_e \) and \( \hat{S}_e \) as bounded linear operators respectively from \( \hat{E}_{1/2}^{1/2} \) to \( E_\mu \) and from \( \bar{E}_{1/2}^{1/2} \) to \( E_\mu \). In order to analyse this question, we shall need to treat separately the different proper modes of the solution.

**Step 3.** Before entering into a detailed analysis by proper modes, we show that all the pending properties will be settled once we prove the convergence of \( S_e \) (\( \varepsilon \to 0 \)) towards \( S_0 \) in the norm of bounded linear operators from \( \hat{E}_\mu \) to \( E_\mu \).

Certainly, this property will immediately imply the uniform boundedness of the operators \( S_e \) as stated in part (a) of the lemma, and combined with bound (10.20) for \( S_0 \) it will immediately give the estimate (6.8).

Let us now concern ourselves with the operators \( \hat{S}_e : \hat{E}_\mu \to E_\mu \). As we said above, we shall base our analysis upon identity (10.26). Obviously, in view of this identity, the uniform boundedness of these operators for small \( \varepsilon \) will be an immediate consequence of the analogous properties of \( S_e : \hat{E}_\mu \to E_\mu \) and \( \hat{R}_e : \hat{E}_\mu \to E_\mu \) (6.4). In order to study the convergence of \( \hat{S}_e \) towards \( \hat{S}_0 \) as \( \varepsilon \to 0 \), we take the difference \( \hat{S}_e f - \hat{S}_0 f \) as given by (10.26) and rewrite it as follows:
\[
\hat{S}_e f - \hat{S}_0 f = (\hat{R}_e - \hat{R}_0)(f - AS_0 f) - \hat{R}_e A(S_e f - S_0 f).
\] (10.28)

Since the operators \( \hat{R}_e \) are uniformly bounded for small \( \varepsilon \) (6.4), it is obvious that the convergence of \( S_e \) towards \( S_0 \) in the norm of bounded linear operators from \( \hat{E}_{1/2}^{1/2} \) to \( E_\mu \) will imply that the second term on the right-hand side of (10.28) tends to 0 uniformly with respect to \( f \in \mathcal{F} \) whenever \( \mathcal{F} \) is \( \hat{E}_{1/2}^{1/2} \)-bounded. In order to deal with the remaining term, we shall base our analysis upon the property of convergence of \( \hat{R}_e \) towards \( \hat{R}_0 \) which was established in Lemma 6.1(b). According to these observations, the proof of the statement of Lemma 6.2(b) concerning the convergence of \( \hat{S}_e \) towards \( \hat{S}_0 \) will reduce to showing that if \( \mathcal{F} \subset \hat{E}_{1/2}^{1/2} \) is \( \hat{E}_{1/2}^{1/2} \)-bounded and \( \hat{E}_{1/2}^{1/2} \)-uniformly equicontinuous, then \( \{ f - AS_0 f \mid f \in \mathcal{F} \} \subset E_\mu \) is \( E_\mu \)-bounded and \( \hat{E}_{1/2}^{1/2} \)-uniformly equicontinuous. We will prove only the main intermediate result, namely, that \( \{ S_0 f \mid f \in \mathcal{F} \} \subset \hat{E}_\mu \) is \( \hat{E}_\mu \)-uniformly equicontinuous. With this purpose, we rewrite (10.19) in the form
\[
(S_0 f)(t) = \int_0^\infty e^{-A\tau} f(t - \tau) \, d\tau,
\]
to obtain that
\[ e^{\mu t}(S_0 f)(t) - e^{\mu s}(S_0 f)(s) \]
\[ = \int_0^\infty e^{\mu t} e^{-\lambda t} (e^{\mu(t-\tau)} f(t-\tau) - e^{\mu(s-\tau)} f(s-\tau)) \, d\tau. \]

Now, by using (10.18) and the fact that \( \mu < \xi_1 \), we obtain that there exists a finite constant \( K \) such that
\[ \|e^{\mu t}(S_0 f)(t) - e^{\mu s}(S_0 f)(s)\|_1 \leq K \sup_{\tau \in [0, T]} \|e^{\mu(t-\tau)} f(t-\tau) - e^{\mu(s-\tau)} f(s-\tau)\|_{1/2}, \]
which immediately yields the desired result.

**Step 4.** In this step we establish the convergence of \( S_\varepsilon \) towards \( S_0 \) in the norm of bounded linear operators from \( \tilde{E}^{1/2}_\mu \) to \( \tilde{E}^1_\mu \) with the specific estimate (6.9). According to the preceding remarks, this will complete the proof of Lemma 6.2. In order to simplify the notation, from now on we shall write \( u^\varepsilon \) instead of \( S_\varepsilon f \).

We begin by noticing that the norm of \( S_\varepsilon - S_0 \) in \( L(\tilde{E}^{1/2}_\mu, \tilde{E}^1_\mu) \), which in principle is given by
\[ \|S_\varepsilon - S_0\|_{L(\tilde{E}^{1/2}_\mu, \tilde{E}^1_\mu)} = \sup_{f \in \tilde{E}^{1/2}_\mu} \frac{\|e^{\mu t} u(t) - u^0(t)\|_1}{\|f\|_{\tilde{E}^{1/2}_\mu}}, \]
\[ (10.29) \]
can in fact be computed simply from the values of \( u^\varepsilon \) and \( u^0 \) at time \( t = 0 \) according to the formula
\[ \|S_\varepsilon - S_0\|_{L(\tilde{E}^{1/2}_\mu, \tilde{E}^1_\mu)} = \sup_{f \in \tilde{E}^{1/2}_\mu} \frac{\|u^\varepsilon(0) - u^0(0)\|_1}{\|f\|_{\tilde{E}^{1/2}_\mu}}. \]
\[ (10.30) \]
Obviously, the right-hand side of (10.30) is less than or equal to (10.29). In order to prove the converse inequality we use a time-shift argument based upon the following facts: for every \( t \leq 0 \) and every \( f \in \tilde{E}^{1/2}_\mu \), the translated function \( f(t + \cdot) \colon \mathbb{R} \ni s \mapsto f(t + s) \in E^{1/2} \) belongs also to \( \tilde{E}^{1/2}_\mu \) with \( \|f(t + \cdot)\|_{\tilde{E}^{1/2}_\mu} \leq e^{-\mu t} \|f\|_{\tilde{E}^{1/2}_\mu} \); for every \( t \leq 0 \) and every \( f \in \tilde{E}^{1/2}_\mu \), the function \( u^\varepsilon(t + \cdot) \) is the unique solution in \( \tilde{E}^{1/2}_\mu \) of (II)_\varepsilon with \( f \) replaced by \( f(t + \cdot) \). According to (10.30), in the following our objective consists in analysing the behaviour of \( \|u^\varepsilon(0) - u^0(0)\|_1 \) as \( \varepsilon \to 0 \).

Let us now decompose \( u^\varepsilon \) and \( f \) as \( u^\varepsilon(t) = \sum u_k^\varepsilon(t) e_k \) and \( f(t) = \sum f_k(t) e_k \), where \( u_k^\varepsilon(t) = \langle u^\varepsilon(t), e_k \rangle_{1/2} \) and \( f_k(t) = \langle f(t), e_k \rangle_{1/2} \) (\( k = 1, 2, \ldots \)). Obviously, for every \( k = 1, 2, \ldots \), the functions \( u_k^\varepsilon \) and \( f_k \) belong to \( \tilde{H}_\mu \) and they satisfy the equation
\[ e^{\xi t} u_k^\varepsilon + \xi_k u_k^\varepsilon + \xi_k u_k^\varepsilon = f_k. \]
\[ (10.31) \]
In terms of the coefficients $u_k^e$ ($k = 1, 2, \ldots$), the quantity $\|u'(0) - u^0(0)\|_1$ is given by

$$\|u'(0) - u^0(0)\|_1^2 = \sum_{k=1}^{\infty} (\xi_k - \xi)(u_k^e(0) - u_k^0(0))^2$$

(10.32)

(we recall that from the beginning of Section 10.3 the eigenvectors have been assumed to be normalized in $E^{1/2}$, not in $E$). In order to analyse the behaviour of this sum as $\varepsilon \to 0$, we shall decompose it into two parts according to whether $\xi_k$ is smaller or greater than $\varepsilon^{-\gamma}$, where $\gamma$ will be a fixed real number in the interval $0 < \gamma < 2$. The specific value of $\gamma$ will be chosen at the end of the calculation in order to optimize the bound. Notice that this decomposition depends on $\varepsilon$ and, since $\gamma > 0$, as $\varepsilon \to 0$ every eigenvalue $\xi_k$ eventually gets in the first part. Obviously, we can write the inequality

$$\|u'(0) - u^0(0)\|_1^2 \leq \sum_{\xi_k < \varepsilon^{-\gamma}} (\xi_k - \xi)(u_k^e(0) - u_k^0(0))^2 + \sum_{\xi_k > \varepsilon^{-\gamma}} (\xi_k - \xi)(u_k^e(0) - u_k^0(0))^2.$$ 

(10.33)

We are going to show that each of the three terms of the right-hand side of (10.33) tends to zero. The origin of this strategy, and in particular of the restriction $\gamma < 2$, is related to the fact that, for $\varepsilon > 0$, the modes with $\xi_k < (2\varepsilon)^{-2}$ are overdamped and this makes them appropriate for comparison with parabolic behaviour, while those with $\xi_k > (2\varepsilon)^{-2}$ are oscillatory. From now on we always assume $\varepsilon$ small enough so as to have the inequality $\varepsilon^{-\gamma} < (2\varepsilon)^{-2}$.

In the following the initial value $u_k^e(0)$ will be determined from the condition that $u_k^e$ must belong to $\mathcal{F}_\mu$ by making use of the Laplace transform. In order to use this transform in the standard way, we invert the sign of the time variable and introduce the functions $u_k^e(t) := u_k^e(-t)$ and $g_k(t) := f_k(-t)$ ($t \in \mathbb{R}_+$), which transforms the preceding equation into

$$\varepsilon^2 \ddot{v}_k^e - \dot{v}_k^e + \xi_k v_k^e = g_k.$$

Indicating by a caret the image function under the Laplace transform, we obtain the general solution of this equation in the form

$$\dot{v}_k^e(z) = \frac{\hat{g}_k(z) - v_k^e(0) + \varepsilon^2(zv_k^e(0) + \dot{v}_k^e(0))}{\xi_k - z + \varepsilon^2 z^2}.$$ 

(10.34)

Now, imposing $u_k^e \in \mathcal{F}_\mu$ implies that $\hat{v}_k^e$ must be analytic in the half-plane $H_\mu := \{ z \mid \text{Re } z > \mu \}$. As we see next, this uniquely determines the initial
value $v_k^0(0)$ (and also $\hat{v}_k^0(0)$ if $\varepsilon > 0$). Let us consider first the case $\varepsilon = 0$. Since $f \in \mathbb{R}_\mu$, $\hat{g}_k$ and consequently the numerator of (10.34) are analytic in $H_\mu$. However, for $\varepsilon = 0$, the denominator of (10.34) vanishes at $z = \zeta_k$, which belongs to $H_\mu$ because of the hypothesis that $\mu < \xi_1$. Therefore, a necessary (and sufficient) condition for $\hat{v}_k^0$ to be analytic in $H_\mu$ is that the numerator of (10.34) vanishes also at $z = \zeta_k$. This determines $v_k^0(0)$ or, which is the same, $u_k^0(0)$ as

$$u_k^0(0) = \hat{g}_k(\zeta_k).$$

(10.35)

In the case $\varepsilon > 0$, the denominator has two roots, namely

$$z_\pm = \frac{1 \pm \sqrt{1 - 4\varepsilon^2 \zeta_k}}{2\varepsilon^2}. \quad (10.36)$$

For $\zeta_k > (2\varepsilon)^{-2}$ these roots are complex with real part equal to $1/(2\varepsilon^2)$, and for $\zeta_k < (2\varepsilon)^{-2}$ they are real with $z_+ > z_- > \zeta_k$; furthermore, as $\varepsilon$ tends to 0, $z_-$ converges towards $\zeta_k$, while $z_+$ tends to $+\infty$. In particular, owing to the hypothesis that $\mu < \min(\xi_1, 1/(2\varepsilon^2))$ we have that both $z_+$ and $z_-$ belong also to $H_\mu$. Therefore, in order that $\hat{v}_k^0$ be analytic in $H_\mu$ the numerator of (10.34) must vanish at both $z_+$ and $z_-$, which in particular gives

$$u_k^0(0) = -\frac{1}{\varepsilon^2} \frac{\hat{g}_k(z_+) - \hat{g}_k(z_-)}{z_+ - z_-} \quad (10.37)$$

with the obvious limiting value

$$u_k^0(0) = -\frac{1}{\varepsilon^2} \hat{g}_k(z_+) \quad (10.38)$$

if $z_+ = z_-$, i.e., if $\zeta_k = (2\varepsilon)^{-2}$.

Let us now introduce the expressions (10.35), (10.37), and (10.38) in (10.33). By using some simple inequalities and making some new arrangements we obtain that

$$\|u^\varepsilon(0) - u^0(0)\|^2 \leq \sum_{\xi_k < \varepsilon^{-\gamma}} F_k^1 + \sum_{\varepsilon^{-\gamma} < \xi_k < \varepsilon^{-\gamma}} F_k^2 + \sum_{\xi_k > \varepsilon^{-\gamma}} F_k^3$$

$$+ \sum_{\xi_k > \varepsilon^{-\gamma}} F_k^4 + \sum_{\varepsilon^{-\gamma} < \xi_k < \varepsilon^{-\gamma}} F_k^5 + \sum_{\xi_k > \varepsilon^{-\gamma}} F_k^6, \quad (10.39)$$

where

$$F_k^1 = 3 \frac{\zeta_k - \xi}{\varepsilon^4(z_+ - z_-)^2} (\hat{g}_k(z_-) - \hat{g}_k(\zeta_k))^2, \quad (10.40)$$

$$F_k^2 = 3(\xi_k - \zeta) \left(\frac{1}{\varepsilon^2(z_+ - z_-)} - 1\right)^2 (\hat{g}_k(\zeta_k))^2, \quad (10.41)$$
\[ F_k^3 = 3 \frac{\xi_k - \xi}{\epsilon^4 (z_+ - z_-)^2} (\hat{g}_k(z_+))^2, \]  
\[ F_k^4 = 2(\xi_k - \xi)(\hat{g}_k(\xi_k))^2, \]  
\[ F_k^5 = 2 \frac{\xi_k - \xi}{\epsilon^4} \times \left\{ \frac{\hat{g}_k(z_+) - \hat{g}_k(z_-)}{z_+ - z_-} \right\}^2, \quad \text{for } \xi_k \neq (2\epsilon)^{-2} \]  
\[ F_k^6 = \frac{\xi_k - \xi}{\epsilon^4 |z_+ - z_-|^2} (|\hat{g}_k(z_+)|^2 + |\hat{g}_k(z_-)|^2). \]  

In the following we proceed to bound each of the six terms in (10.39).

**First bound.** Having in mind that \( z_- \) converges towards \( \xi_k \), we apply the mean-value theorem to the function \( \hat{g}_k \) to obtain that

\[ \hat{g}_k(z_-) - \hat{g}_k(\xi_k) = \hat{g}_k'(\xi)(z_- - \xi_k), \quad \text{where } \xi_k < \xi < z_-. \]

By using the inequality \( \sqrt{1 - x} \geq 1 - x/2 - x^2/2 \quad (\forall x \leq 1) \), one obtains that

\[ z_- - \xi_k \leq 4\epsilon^2 \xi_k^2. \]

On the other hand, starting from the formula for the derivative of the Laplace transform and using the Schwarz inequality, we have that

\[ |\hat{g}_k'(\xi)| = \left| \int_0^\infty t e^{-\xi t} g_k(t) \, dt \right| \leq \int_0^\infty t e^{-\xi t} |g_k(t)| \, dt \]

\[ \leq \left( \int_0^\infty t^2 e^{-(\xi_k - \mu)t} \, dt \right)^{1/2} \left( \int_0^\infty e^{-(\xi_k - \mu)t} e^{-2\mu t} |g_k(t)|^2 \, dt \right)^{1/2} \]

\[ = \frac{\sqrt{2}}{(\xi_k - \mu)^{3/2}} \left( \int_0^\infty e^{-(\xi_k - \mu)t} e^{-2\mu t} |g_k(t)|^2 \, dt \right)^{1/2}. \]

By collecting these bounds together and replacing \( \epsilon^4 (z_+ - z_-)^2 \) by its value

\[ \epsilon^4 (z_+ - z_-)^2 = 1 - 4\epsilon^2 \xi_k, \]

we obtain that

\[ F_k^1 \leq 24 \frac{\xi_k^4 (\xi_k - \xi)}{(1 - 4\epsilon^2 \xi_k)(\xi_k - \mu)^3} \int_0^\infty e^{-(\xi_k - \mu)t} e^{-2\mu t} |g_k(t)|^2 \, dt, \]

which, taking into account that we are dealing with the case \( \xi_1 \leq \xi_k \leq \epsilon^{-\gamma} \), gives

\[ F_k^1 = O(\epsilon^{4-\gamma}) \int_0^\infty e^{-(\xi_1 - \mu)t} e^{-2\mu t} |g_k(t)|^2 \, dt. \]
Here and in the following the Landau symbol has the same meaning as usual with the additional connotation that the function of $\varepsilon$ to which it refers is understood to be independent of $k$. Finally, summing over $k$ for $\xi_k \leqslant \varepsilon^{-\gamma}$, we obtain that

$$
\sum_{\xi_k \leqslant \varepsilon^{-\gamma}} F_k^1 = O(\varepsilon^{4-2\gamma}) \int_0^\infty e^{-(\xi_1 - \mu)t} e^{-2\mu t} \left( \sum_{\xi_k \leqslant \varepsilon^{-\gamma}} |g_k(t)|^2 \right) dt
= O(\varepsilon^{4-2\gamma}) \int_0^\infty e^{-(\xi_1 - \mu)t} dt \|f\|_{Z^\mu}^{2/2},
$$
i.e.,

$$
\sum_{\xi_k \leqslant \varepsilon^{-\gamma}} F_k^1 = O(\varepsilon^{4-2\gamma}) \|f\|_{Z^\mu}^{2/2}. \tag{10.46}
$$

*Second bound.* By using the inequality $|1 - \sqrt{1 - x}| \leqslant |x|$ ($\forall x \leqslant 1$), one obtains that

$$
\left| \frac{1}{\varepsilon^2 (z_+ - z_-)} - 1 \right| = \left| \frac{1 - \sqrt{1 - 4\varepsilon^2 \xi_k}}{\sqrt{1 - 4\varepsilon^2 \xi_k}} \right| \leqslant \frac{4\varepsilon^2 |\xi_k|}{\sqrt{1 - 4\varepsilon^2 \xi_k}}.
$$

On the other hand,

$$
|\tilde{g}_k(\xi_k)| \leqslant \int_0^\infty e^{-\xi_k |t|} |g_k(t)| dt
\leqslant \left( \int_0^\infty e^{-(\xi_k - \mu)t} dt \right)^{1/2} \left( \int_0^\infty e^{-(\xi_k - \mu)t} e^{-2\mu t} |g_k(t)|^2 dt \right)^{1/2}
= \frac{1}{(\xi_k - \mu)^{1/2}} \left( \int_0^\infty e^{-(\xi_k - \mu)t} e^{-2\mu t} |g_k(t)|^2 dt \right)^{1/2}.
$$

Therefore, we obtain that

$$
F_k^2 \leqslant 48\varepsilon^4 \frac{\xi_k^2 (\xi_k - \xi)}{(1 - 4\varepsilon^2 \xi_k)(\xi_k - \mu)} \int_0^\infty e^{-(\xi_k - \mu)t} e^{-2\mu t} |g_k(t)|^2 dt,
$$

which, by introducing the inequality $\xi_1 \leqslant \xi_k \leqslant \varepsilon^{-\gamma}$, gives

$$
F_k^2 = O(\varepsilon^{4-2\gamma}) \int_0^\infty e^{-(\xi_1 - \mu)t} e^{-2\mu t} |g_k(t)|^2 dt.
$$

By proceeding as in the first bound, one finally derives that

$$
\sum_{\xi_k \leqslant \varepsilon^{-\gamma}} F_k^2 = O(\varepsilon^{4-2\gamma}) \|f\|_{Z^\mu}^{2/2}. \tag{10.47}
$$
**Third bound.** By proceeding similarly as above, but now using the fact that \( z_+ > 1/2\varepsilon^2 \), we obtain that

\[
|\hat{g}_k(z_+)| \leq \int_0^{\infty} e^{-t/(2\varepsilon^2)} |g_k(t)| \, dt \leq \sqrt{2} \varepsilon \left( \int_0^{\infty} e^{-t/(2\varepsilon^2)} |g_k(t)|^2 \, dt \right)^{1/2}.
\]

Introducing this in the expression of \( F_k^3 \), we have that

\[
F_k^3 \leq 6\varepsilon^2 \frac{\xi_k - \xi}{1 - 4\varepsilon^2 \xi_k} \int_0^{\infty} e^{-t/(2\varepsilon^2)} |g_k(t)|^2 \, dt
\]

\[
= O(\varepsilon^{2-\gamma}) \int_0^{\infty} e^{-t/(2\varepsilon^2)} |g_k(t)|^2 \, dt.
\]

Finally, summing over \( k \) for \( \xi_k \leq \varepsilon^{-\gamma} \), we obtain that

\[
\sum_{\xi_k \leq \varepsilon^{-\gamma}} F_k^3 = O(\varepsilon^{2-\gamma}) \int_0^{\infty} e^{-(1/2\varepsilon^2 - 2\mu)} \left( \sum_{\xi_k \leq \varepsilon^{-\gamma}} |g_k(t)|^2 \right) \, dt
\]

\[
= O(\varepsilon^{2-\gamma}) \int_0^{\infty} e^{-(1/2\varepsilon^2 - 2\mu)} \, dt \|f\|_{L^2/2}^{1/2},
\]

which, performing the integration, yields

\[
\sum_{\xi_k \leq \varepsilon^{-\gamma}} F_k^3 = O(\varepsilon^{4-\gamma}) \|f\|_{L^2/2}^{1/2}.
\]

(10.48)

**Fourth bound.** Here \( \xi_k \) ranges over \( \xi_k > \varepsilon^{-\gamma} \). In this situation we use the inequality

\[
|\hat{g}_k(\xi_k)| \leq \int_0^{\infty} e^{-\xi_k t} |g_k(t)| \, dt
\]

\[
\leq \left( \int_0^{\infty} e^{-\xi_k t} \, dt \right)^{1/2} \left( \int_0^{\infty} e^{-\xi_k t} |g_k(t)|^2 \, dt \right)^{1/2}
\]

\[
= (\xi_k)^{-1/2} \left( \int_0^{\infty} e^{-\xi_k t} |g_k(t)|^2 \, dt \right)^{1/2}.
\]

Introducing this in the expression of \( F_k^4 \), we obtain that

\[
F_k^4 \leq 2 \frac{\xi_k - \xi}{\xi_k} \int_0^{\infty} e^{-\xi_k t} |g_k(t)|^2 \, dt,
\]

which, taking into account the inequality \( \xi_k > \varepsilon^{-\gamma} \), gives

\[
F_k^4 = O(1) \int_0^{\infty} e^{-\varepsilon^{-\gamma} t} |g_k(t)|^2 \, dt.
\]
Finally, by proceeding similarly as in the preceding case, we obtain

$$\sum_{\xi_k > \epsilon^{-\gamma}} F_k^e = O(1) \int_0^\infty e^{-(\epsilon^{-\gamma} - \frac{2\mu}{\epsilon})t} dt \|f\|_{L^{1/2}_\mu},$$

i.e.,

$$\sum_{\xi_k > \epsilon^{-\gamma}} F_k^e = O(\epsilon^\gamma) \|f\|_{L^{1/2}_\mu}.$$  \hfill (10.49)

**Fifth bound.** Here $\xi_k$ ranges over $\epsilon^{-\gamma} < \xi_k < \epsilon^{-2}$. By applying the mean value theorem, we are led to estimate $|\hat{g}'(\xi)|$ for $\xi$ in the line segment between $z_-$ and $z_+$. Now, in the case $\xi_k \leq (2\epsilon)^{-2}$ $\xi$ is real and greater than $\xi_k$ which in its turn is greater than $\epsilon^{-\gamma}$; on the other hand, in the case $\xi_k > (2\epsilon)^{-2}$ $\xi$ is complex with real part equal to $1/2\epsilon^2$, which is also greater than $\epsilon^{-\gamma}$. Therefore, we can write

$$|\hat{g}'(\xi)| = \left| \int_0^\infty t e^{-\xi t} g_k(t) \, dt \right| \leq \left( \int_0^\infty t^2 e^{-\xi t} \, dt \right)^{1/2} \left( \int_0^\infty e^{-\xi t} |g_k(t)|^2 \, dt \right)^{1/2},$$

$$= (2\epsilon^{3\gamma})^{1/2} \left( \int_0^\infty e^{-\xi t} |g_k(t)|^2 \, dt \right)^{1/2}.$$  \hfill (10.50)

Introducing this in the expression of $F_k^5$, we obtain that

$$F_k^5 \leq 4\epsilon^{3\gamma} (\xi_k - \xi) \int_0^\infty e^{-\epsilon^{-\gamma} t} |g_k(t)|^2 \, dt.$$

Now, taking into account that $\xi_k$ is presently restricted to be less than or equal to $\epsilon^{-2}$, we derive that

$$F_k^5 = O(\epsilon^{3\gamma} - 6) \int_0^\infty e^{-\xi t} |g_k(t)|^2 \, dt.$$

Finally, by proceeding as in the preceding cases, we obtain that

$$\sum_{\epsilon^{-\gamma} < \xi_k \leq \epsilon^{-2}} F_k^5 = O(\epsilon^{4\gamma - 6}) \|f\|_{L^{1/2}_\mu}. \hfill (10.50)$$

**Sixth bound.** Here $\xi_k$ ranges over $\xi_k > \epsilon^{-2}$. The roots $z_+$ and $z_-$ are now complex with real part equal to $1/2\epsilon^2$. Therefore,

$$|\hat{g}_k(z_\pm)| \leq \int_0^\infty e^{-t(2\epsilon^2)} |g_k(t)| \, dt \leq \sqrt{2} \epsilon \left( \int_0^\infty e^{-t(2\epsilon^2)} |g_k(t)|^2 \, dt \right)^{1/2}.$$
By substituting this in the expression of $F^6_k$ and replacing $\varepsilon^4 |z_+ - z_-|^2$ by its present value

$$\varepsilon^4 |z_+ - z_-|^2 = 4\varepsilon^2 \xi_k - 1,$$

we obtain that

$$F^6_k \leq 16\varepsilon^2 \frac{\xi_k - \xi}{4\varepsilon^2 \xi_k - 1} \int_0^\infty e^{-t/(2\varepsilon^2)} |g_k(t)|^2 dt,$$

which, on account of the inequality $\xi_k > \varepsilon^{-2}$, gives

$$F^6_k = O(1) \int_0^\infty e^{-t/(2\varepsilon^2)} |g_k(t)|^2 dt.$$

Finally, by proceeding as in the preceding cases, we obtain that

$$\sum_{\xi_k > \varepsilon^{-2}} F^6_k = O(\varepsilon^2) \|f\|_{L^{1/2}}^2. \quad (10.51)$$

By introducing estimates (10.46)–(10.51) in (10.39), and applying (10.30), we obtain finally that $\|S - S_0\|_{L^1(F_{1/2}^1, E_{1/2}^1)} = O(\varepsilon^p)$ with $p = \frac{1}{2} \min \{4 - 2\gamma, 4 - \gamma, 4\gamma - 6, 2\}$. As $\gamma$ varies in the interval $(0, 2)$, the best value of $p$ that one obtains is $p = \frac{1}{4}$, which corresponds to $\gamma = \frac{3}{2}$. This establishes the estimate (6.9).