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Irreducible polynomials and full elasticity in rings of integer-valued polynomials

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Abstract

Let D be a unique factorization domain and S an infinite subset of D. If $f(X)$ is an element in the ring of integer-valued polynomials over S with respect to D (denoted $\mathrm{Int}(S, D)$), then we characterize the irreducible elements of $\mathrm{Int}(S, D)$ in terms of the fixed-divisor of $f(X)$. The characterization allows us to show that every nonzero rational number n/m is the leading coefficient of infinitely many irreducible polynomials in the ring $\mathrm{Int}(\mathbb{Z}) = \mathrm{Int}(\mathbb{Z}, \mathbb{Z})$. Further use of the characterization leads to an analysis of the particular factorization properties of such integer-valued polynomial rings. In the case where $D = \mathbb{Z}$, we are able to show that every rational number greater than 1 serves as the elasticity of some polynomial in $\mathrm{Int}(S, \mathbb{Z})$ (i.e., $\mathrm{Int}(S, \mathbb{Z})$ is fully elastic).

Keywords: Integer-valued polynomial; Irreducible element; Elasticity of factorization

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1. Introduction

A great deal of recent literature has been devoted to the study of integral domains and monoids where factorization of elements into irreducible elements is not unique. Given an integral domain (or more generally a commutative cancellative monoid) D, let $\mathcal{I}(D)$ represent the set of irreducible elements of D, $\mathcal{U}(D)$ the set of units of D and $D^\bullet = D - \{0\}$ the multiplicative monoid of D. We say that D is *atomic* if every element of $D^\bullet$ can be written as a product of elements from $\mathcal{I}(D)$. Given a nonzero nonunit x of D, the principal object of interest in the study of non-unique factorizations is

$$\mathcal{L}(x) = \{n \mid \exists \alpha_1, \ldots, \alpha_n \in \mathcal{I}(D) \text{ with } x = \alpha_1 \cdots \alpha_n\}$$

or the *set of lengths of* x. If D is atomic and $|\mathcal{L}(x)| < \infty$ for all $x \in D^\bullet$, then D is called a *bounded factorization domain* (BFD). Given a BFD D and $x \in D^\bullet$ with $\mathcal{L}(x) = \{n_1, \ldots, n_t\}$ where $n_i \leqslant n_{i+1}$ for $1 \leqslant i \leqslant t-1$, set

$$\rho(x) = \frac{\max \mathcal{L}(x)}{\min \mathcal{L}(x)} = \frac{n_t}{n_1}.$$

While $\rho(x)$ describes the local character of non-unique factorizations, this function can be extended to the global descriptor

$$\rho(D) = \sup\{\rho(x) \mid x \in D^\bullet\}.$$

The value $\rho(x)$ is known as the *elasticity* of x and $\rho(D)$ as the *elasticity* of D. An extensive amount of literature is devoted to the study of elasticity (see [1] for a survey). In particular, if D is the ring of integers in an algebraic number field, then the elasticity of D can be bounded above using the class number [10]. Moreover, an algorithm exists for computing the elasticity of any Krull monoid with finite divisor class group [7].

This paper continues the study begun in [2,5] (which is summarized in [6]) concerning factorization properties of rings of integer-valued polynomials. If D is an integral domain with quotient field K and S is a subset of D, then the ring of integer-valued polynomials over D with respect to S is defined by

$$\operatorname{Int}(S, D) = \{f(X) \mid f(X) \in K[X] \text{ with } f(s) \in D \text{ for all } s \in S\}$$

(if $S = D$, then we use the notation $\operatorname{Int}(D, D) = \operatorname{Int}(D)$). In [2, Proposition 1.7], it is shown that if D is an integral domain and S an infinite subset of D such that

(1) $\operatorname{Int}(S, D)$ is atomic;
(2) there exists a discrete valuation v on K;
(3) there exists a principal prime ideal $\mathfrak{M}$ in D with $|D/\mathfrak{M}| < \infty$,

then $\rho(\operatorname{Int}(S, D)) = \infty$. In fact, there is no known example of an atomic ring of integer-valued polynomials with finite elasticity.

Our interest in further studying factorization properties of $\operatorname{Int}(S, D)$ came from a close examination of the elasticity arguments in [2,5]. In both these papers, it is shown that

$\rho(\mathrm{Int}(S, D))$ gets arbitrarily large without actually computing the elasticity of an integer-valued polynomial (in these arguments, only lower bounds on $\rho(f(X))$ are used). Moreover, while some conditions are presented in these papers to ensure that certain polynomials in $\mathrm{Int}(S, D)$ are irreducible (such as Theorem 2.1 and Corollary 2.2 in [2] or Propositions VI.3.4, VI.3.7, Corollary VI.3.8 and Proposition VI.3.10 in [6]), no characterization of these irreducible elements is offered (no matter how strong a hypothesis is placed on D). In this paper, we deal with the case where D is a unique factorization domain and S is an infinite subset of D (under the UFD hypothesis, [2, Proposition 1.1 and Theorem 1.2] implies that $\mathrm{Int}(S, D)$ is atomic if and only if $|S| = \infty$). We give in Section 2 a characterization, in terms of the *fixed divisor* of an element $f(X)$ of $\mathrm{Int}(S, D)$, of the irreducible elements of $\mathrm{Int}(S, D)$. We use this characterization in Section 3 to show that polynomials in $D[X]$ whose fixed divisors are 1 (which we call *image primitive*) have unique factorization in $\mathrm{Int}(S, D)$. We further show in this section that if n/m is any nonzero rational, then n/m is the leading coefficient of infinitely many irreducible polynomials in $\mathrm{Int}(\mathbb{Z})$. We use the characterization of irreducibles to examine in Section 4 factorization properties in the specific case where $S \subseteq \mathbb{Z} = D$. We compute $\rho(f(X))$ for a large class of polynomials in $\mathrm{Int}(S, \mathbb{Z})$ and these calculations are used to show that $\mathrm{Int}(S, \mathbb{Z})$ has *full elasticity* (i.e., every rational greater than or equal to 1 can serve as the elasticity of some integer-valued polynomial). This property was recently introduced in a paper co-authored by the first author [8] where it is shown that certain rings of algebraic integers satisfy this property (if $\rho(D) < \infty$, then only elasticities less than or equal to $\rho(D)$ can be attained), while non-cyclic numerical monoids do not.

The notation we use will be consistent with that of [6] and any undefined terminology can be found there. As outlined above, we assume throughout the remainder of this paper that D is a unique factorization domain and S is an infinite subset of D. If $f(X) = \sum_{i=0}^{t} f_i X^i$ is a polynomial in $D[X]$, then set $c(f) = \gcd\{f_0, \ldots, f_t\}$ to be the content of f. Our work in Section 3 will be heavily dependent on the *binomial polynomials*. These are defined for each $n \geqslant 1$ as,

$$\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!}$$

and $\binom{X}{0} = 1$. The set $\{\binom{X}{n}\}_{n\geqslant 0}$ is a basis of $\mathrm{Int}(\mathbb{Z})$ as a free $\mathbb{Z}$-module. Since $\binom{n}{n} = 1$ for all $n \geqslant 0$, each $\binom{X}{n}$ is image primitive (see Definition 2.1). Moreover, for $n > 0$ each $\binom{X}{n}$ is irreducible in $\mathrm{Int}(\mathbb{Z})$ (see [6, Corollary VI.3.5]).

2. On irreducible polynomials and fixed divisors in Int(S, D)

We open with two definitions.

Definition 2.1. Let D be a unique factorization domain, $S \subseteq D$ and $f(X) \in \mathrm{Int}(S, D)$. The *fixed divisor* of f over S, denoted $\mathrm{d}(S, f)$, is defined as

$$\mathrm{d}(S, f) = \gcd\big\{f(s) \mid s \in S\big\}.$$

If $\mathrm{d}(S, f) = 1$, then we call $f(X)$ *image primitive* over S.

The fixed divisor has been an object of much recent research, especially in the context where D is a Dedekind domain (see the papers of Bhargava [3,4] or Wood [11]). The monograph of Narkiewicz [9] contains a chapter on this subject, and can be used as a general reference. While Gauss' Lemma indicates that the content behaves nicely under our hypothesis (i.e., $c(fg) = c(f)c(g)$), the same is not the case for the fixed divisor. While we do not have in general that $\mathrm{d}(S, fg) = \mathrm{d}(S, f)\mathrm{d}(S, g)$, we can show the following.

Lemma 2.2. *Let $f(X)$ be a nonzero polynomial in* $\mathrm{Int}(S, D)$. *Suppose $f_1(X), \ldots, f_k(X)$ are nonzero polynomials in* $\mathrm{Int}(S, D)$ *with*

$$f(X) = f_1(X) \cdots f_k(X)$$

then

(1) $\mathrm{d}(S, f_1(X)) \cdots \mathrm{d}(S, f_k(X)) \mid \mathrm{d}(S, f(X))$,
(2) *if $f_1(X) = f_2(X) = \cdots = f_k(X)$, then* $\mathrm{d}(S, f(X)) = \mathrm{d}(S, (f_1(X))^k) = (\mathrm{d}(S, f_1(X)))^k$.

Proof. (1) Let $m = \mathrm{d}(S, f_1(X)) \cdots \mathrm{d}(S, f_k(X))$. Then for all $z \in D$, $f(z) = f_1(z) \cdots f_k(z)$ and since $m \mid f_1(z) \cdots f_k(z)$, $m \mid f(z)$. Thus $m \mid \mathrm{d}(S, f(X))$. For (2), set $n = \mathrm{d}(S, (f_1(X))^k)$. By (1), $m \mid n$. Since for all $z \in D$, $(f_1(z))^k = (f_1^k)(z)$, $n \mid (f_1(z))^k$ and hence $n \mid m$. Thus $n = m$. □

We state the relationship between the image primitive and primitive conditions in $D[X]$. The proof is left to the reader, and we note that simple examples in the case where $D = \mathbb{Z}$ show that the converse does not hold.

Lemma 2.3. *If $f(X) \in D[X]$ is image primitive with respect to S, then $f(X)$ is primitive in $D[X]$.*

The following statements involving image primitive polynomials in $\mathrm{Int}(S, D)$ will be important in our later arguments.

Lemma 2.4. *Let $f(X) \in \mathrm{Int}(S, D)$ be of degree $r \geqslant 1$. The following hold.*

(1) *If $f(X)$ is irreducible in* $\mathrm{Int}(S, D)$, *then $f(X)$ is image primitive.*
(2) *If $f(X)$ is image primitive over S, and $f(X) = f_1(X)f_2(X) \cdots f_w(X)$, with each $f_j(X) \in \mathrm{Int}(S, D)$, then each $f_j(X)$ is also image primitive over S. Moreover, if $\deg(f_i(X)) = 0$ for some $i \in [1, k]$, then $f_i(X)$ is a unit in* $\mathrm{Int}(S, D)$.

Proof. Assertion (2) follows directly from Lemma 2.2. For (1), assume that $\mathrm{d}(S, f) = m$. Then

$$f(X) = \mathrm{d}(S, f)\frac{f(X)}{\mathrm{d}(S, f)} = m\frac{f(X)}{m},$$

with $f(X)/m \in \mathrm{Int}(S,\mathbb{Z})$. Then $f(X)$ irreducible in $\mathrm{Int}(S,\mathbb{Z})$ implies $m = \mathrm{d}(S, f) = \pm 1$. □

Elementary examples in the case where $D = \mathbb{Z}$ show that the converse of Lemma 2.4(1) is not true. Notice that part (2) of the lemma implies that a nonunit image primitive polynomial in $\mathrm{Int}(S, D)$ cannot be divisible in $\mathrm{Int}(S, D)$ by a nonunit constant polynomial. In the special case where $S = D = \mathbb{Z}$, we can give a better description of the fixed divisor.

Lemma 2.5. *Let $F(X) \in \mathrm{Int}(\mathbb{Z})$ have degree r, so that $F(X) = F_0 + F_1\binom{X}{1} + \cdots + F_r\binom{X}{r}$, where $F_i \in \mathbb{Z}$ and $F_r \neq 0$. Then*

$$\mathrm{d}(\mathbb{Z}, F) = \gcd\big(F(0), F(1), \ldots, F(r)\big) = \gcd(F_0, F_1, \ldots, F_r).$$

Proof. That $\mathrm{d}(\mathbb{Z}, F) = \gcd(F(0), F(1), \ldots, F(r))$ is well known (see for example [2, Lemma 2.7]). We show that $\mathrm{d}(\mathbb{Z}, F) = \gcd(F_0, F_1, \ldots, F_r)$. Clearly, $\gcd(F_0, \ldots, F_r) \mid \mathrm{d}(\mathbb{Z}, f)$. On the other hand, using the fact $\mathrm{Int}(\mathbb{Z})$ is a free $\mathbb{Z}$-module generated by the binomial polynomials, we infer that every coefficient F_i must be divisible by $\mathrm{d}(\mathbb{Z}, f)$. Hence, $\mathrm{d}(\mathbb{Z}, f) \mid \gcd(F_0, \ldots, F_r)$, completing the proof. □

If D is a unique factorization domain with quotient field K and $f(X)$ is a primitive polynomial in $D[X]$, then $f(X)$ is irreducible in $K[X]$ if and only if $f(X)$ is irreducible in $D[X]$. This result fails if the polynomial ring $K[X]$ is replaced by $\mathrm{Int}(D)$. It can be recovered by applying the image primitive condition.

Theorem 2.6. *Let D be a unique factorization domain with quotient field K and $f(X)$ a primitive polynomial in $D[X]$. $f(X)$ is irreducible in $\mathrm{Int}(S, D)$ if and only if $f(X)$ is irreducible and image primitive in $D[X]$.*

Proof. ($\Rightarrow$) If $f(X)$ is irreducible in $\mathrm{Int}(S, D)$, then $f(X)$ is image primitive by Lemma 2.4(1). Suppose $f(X) = f_1(X)f_2(X)$ is a proper factorization of $f(X)$ in $D[X]$. Since the units of $\mathrm{Int}(S, D)$ and $D[X]$ agree (see [5, Lemma 1.1]), this is also a proper factorization of $f(X)$ in $\mathrm{Int}(S, D)$.

($\Leftarrow$) If $f(X)$ is irreducible and image primitive in $D[X]$, $f(X)$ is irreducible in $K[X]$. Suppose $f(X) = f_1(X)f_2(X)$ is a proper factorization of $f(X)$ in $\mathrm{Int}(S, D)$. Since $f(X)$ is image primitive, both $f_1(X)$ and $f_2(X)$ are nonconstant, which contradicts the irreducibility of $f(X)$ over $K[X]$. □

We show that image primitive polynomials can be expressed uniquely in terms of quotients.

Lemma 2.7. *Let $f(X)$ be image primitive in $\mathrm{Int}(S, D)$. There exists a unique (up to associates) primitive polynomial $f^*(X) \in D[X]$ and unique (up to associate) $n \in D$ such that*

$$f(X) = \frac{f^*(X)}{n}. \qquad (\star)$$

Proof. Write $f(X) = h(X)/m$, where $h(X) \in D[X]$ and $m \in D$. If $h(X)$ is not primitive in $D[X]$, then write $h(X) = c(h(X))h_1(X)$ with $h_1(X)$ primitive in $D[X]$. So

$$f(X) = \frac{c(h(X))h_1(X)}{n}.$$

Since $f(X)$ is image primitive, $\mathrm{d}(S, c(h(X))h_1(X)) = n$. Now $\mathrm{d}(S, c(h(X))h_1(X)) = c(h(X))\mathrm{d}(S, h_1(X))$. So

$$f(X) = \frac{c(h(X))h_1(X)}{c(h(X))\mathrm{d}(S, h_1(X))} = \frac{h_1(X)}{\mathrm{d}(S, h_1(X))}.$$

Setting $f^*(X) = h_1(X)$ and $n = \mathrm{d}(h_1(X))$ yields the desired representation. Now, suppose

$$\frac{f^*(X)}{n} = \frac{f^{**}(X)}{n^*}$$

with $f^*(X)$, $f^{**}(X)$ primitive and n, n^* in D. Then unique factorization in $D[X]$ and $n^* f^*(X) = n f^{**}(X)$ yield $f^*(X) = f^{**}(X)$ and $n = n^*$. □

Lemma 2.7 is vital in determining exactly which elements of $\mathrm{Int}(S, D)$ are irreducible. The next theorem explores this for nonconstant polynomials.

Theorem 2.8. *Let $f(X)$ be a nonconstant primitive polynomial in $D[X]$. The following statements are equivalent.*

(a) $\dfrac{f(X)}{\mathrm{d}(S, f(X))}$ *is irreducible in* $\mathrm{Int}(S, D)$.

(b) *Either $f(X)$ is irreducible in $D[X]$ or for every pair of nonconstant polynomials $f_1(X)$, $f_2(X)$ in $D[X]$ with $f(X) = f_1(X)f_2(X)$, $\mathrm{d}(S, f(X)) \nmid \mathrm{d}(S, f_1(X)) \times \mathrm{d}(S, f_2(X))$.*

Proof. (a) ⇒ (b) Suppose (a) holds and that $f(X)$ is not irreducible in $D[X]$. Assume there exists a pair $f_1(X)$ and $f_2(X)$ with $f(X) = f_1(X)f_2(X)$ and $\mathrm{d}(S, f(X)) \mid \mathrm{d}(S, f_1(X))\mathrm{d}(S, f_2(X))$. Then $u \cdot \mathrm{d}(S, f_1(X))\mathrm{d}(S, f_2(X)) = \mathrm{d}(S, f(X))$ where u is a unit of D. Hence

$$\frac{f(X)}{\mathrm{d}(S, f(X))} = u^{-1}\frac{f_1(X)f_2(X)}{\mathrm{d}(S, f_1(X))\mathrm{d}(S, f_2(X))} = u^{-1}\frac{f_1(X)}{\mathrm{d}(S, f_1(X))} \cdot \frac{f_2(X)}{\mathrm{d}(S, f_2(X))}$$

is a nontrivial factorization of $f(X)/\mathrm{d}(S, f(X))$ in $\mathrm{Int}(S, D)$, a contradiction.

(b) ⇒ (a) If $f(X)$ is irreducible in $D[X]$ then clearly $f(X)/\mathrm{d}(S, f(X))$ is irreducible in $\mathrm{Int}(S, D)$. Suppose that the second statement in condition (b) holds and that $f(X)/\mathrm{d}(S, f(X))$ is not irreducible. Then

$$\frac{f(X)}{\mathrm{d}(S, f(X))} = h_1(X)h_2(X)$$

with $h_1(X), h_2(X)$ nonunits in $\mathrm{Int}(S, D)$. Since $f(X)/\mathrm{d}(S, f(X))$ is image primitive, both $h_1(X)$ and $h_2(X)$ are nonconstant and image primitive by Lemma 2.4(2). By Lemma 2.7, write each

$$h_i(X) = \frac{z_i(X)}{m_i}$$

with $z_i(X)$ primitive in $D[X]$ and $m_i \in D$. Thus

$$\frac{f(X)}{\mathrm{d}(S, f(X))} = \frac{z_1(X)}{m_1} \cdot \frac{z_2(X)}{m_2}.$$

By using unique factorization in $D[X]$ and the fact that $m_1 m_2 f(X) = \mathrm{d}(S, f(X)) z_1(X) \times z_2(X)$, we obtain that $f(X) = z_1(X) z_2(X)$ and $m_1 m_2 = \mathrm{d}(S, f(X))$. Moreover, the image primitive condition implies that $\mathrm{d}(S, f(X)) = m_1 m_2 = \mathrm{d}(S, z_1(X))\mathrm{d}(S, z_1(X))$, contradicting condition (b). □

The following characterization of irreducible elements in $\mathrm{Int}(S, D)$ follows directly from Theorem 2.8 and [6, Lemma VI.3.1(ii)].

Corollary 2.9. *Let $f(X)$ be a nonunit in* $\mathrm{Int}(S, D)$. *$f(X)$ is irreducible in* $\mathrm{Int}(S, D)$ *if and only if*

(1) $\deg(f(X)) = 0$ *and $f(X)$ is irreducible in D or*
(2) $\deg(f(X)) > 0$, *$f(X)$ is image primitive in* $\mathrm{Int}(S, D)$ *and when expressed in the form* ($\star$) *either*
   (a) *$f^*(X)$ is irreducible in $D[X]$ and $n = \mathrm{d}(S, f^*)$ or*
   (b) *$n = \mathrm{d}(S, f^*)$ and for every factorization $f^*(X) = f_1(X) f_2(X)$ into non-units of $D[X]$, $n \nmid \mathrm{d}(S, f_1^*)\mathrm{d}(S, f_2^*)$.*

3. On irreducible polynomials and their leading coefficients

Arguments surrounding infinite elasticity in [2,5] tend to focus on polynomials of the type $(X - i_1) \cdots (X - i_t)$ where the elements $i_1, \ldots, i_t$ are chosen in some careful manner. In particular, in $\mathrm{Int}(\mathbb{Z})$ the observation that

$$n \cdot \binom{X}{n} = \binom{X}{n-1}\big(X - (n-1)\big)$$

leads to an elementary proof that $\rho(\mathrm{Int}(\mathbb{Z})) = \infty$. It is interesting to note that such arguments center on the fact that the choice of $i_1, \ldots, i_t$ forces $\mathrm{d}((X - i_1) \cdots (X - i_t)) \neq 1$. We open this section by showing that this approach was wise, since image primitive polynomials in $D[X]$ factor uniquely as products of irreducible elements of $\mathrm{Int}(S, D)$.

Theorem 3.1. *Let $f(X) \in D[X]$ be of degree $d \geqslant 1$. If $f(X)$ is image primitive, then $f(X)$ factors uniquely as a product of irreducible elements of* $\mathrm{Int}(S, D)$.

Proof. Since $D[X]$ is a UFD, let $f(X) = q_1(X)\cdots q_t(X)$, with each $q_i(X)$ irreducible in $D[X]$. By Lemma 2.4(2), each $q_i(X) \in D[X]$ is also image primitive and hence each is primitive by Lemma 2.3. Suppose $f(X)$ factors in $\mathrm{Int}(S, D)$ as

$$f(X) = j_1(X)j_2(X)\cdots j_r(X),$$

where each $j_i(X)$ is irreducible in $\mathrm{Int}(S, D)$. By Lemma 2.4(2), no $j_i(X)$ is a nonunit of D. Notice also that $j_i(X) \notin K[X] \setminus D[X]$. To see this, we apply Lemma 2.7 to each $j_i(X)$ and obtain

$$j_i(X) = \frac{\tilde{j}_i(X)}{p_i},$$

where $\tilde{j}_i(X)$ is primitive in $D[X]$ and $p_i \in D$. Then

$$f(X) = j_1(X)j_2(X)\cdots j_r(X) = \frac{\tilde{j}_1(X)}{p_1}\frac{\tilde{j}_2(X)}{p_2}\cdots\frac{\tilde{j}_r(X)}{p_r} = \left(\frac{\tilde{j}_1(X)\tilde{j}_2(X)\cdots\tilde{j}_r(X)}{p_1p_2\cdots p_r}\right).$$

By Gauss's Lemma, the product $\tilde{j}_1(X)\tilde{j}_2(X)\cdots\tilde{j}_r(X)$ is a primitive polynomial. Since $f(X)$ is primitive in $D[X]$, we have that $(p_1p_2\cdots p_r)$ divides each coefficient in the polynomial $\tilde{j}_1(X)\tilde{j}_2(X)\cdots\tilde{j}_r(X)$ in D. Since this polynomial is primitive, $(p_1p_2\cdots p_r)$ is a unit. This yields that p_i is a unit for each $i \in [1, r]$ and hence, $j_i(X)$ and $\tilde{j}_i(X)$ are associates in $D[X]$ for each $i \in [1, r]$. So each $j_i(X) \in D[X]$ and

$$f(X) = j_1(X)\cdots j_r(X) = q_1(X)\cdots q_t(X),$$

implies that $r = t$ and for some permutation of the j_i's, each $j_i(X) \sim q_i(X)$. We have thus shown that $f(X)$ factors in $\mathrm{Int}(S, D)$ uniquely as a product of irreducible elements. □

In Example 3.6 we will show that Theorem 3.1 fails if $f(X)$ is chosen to be an arbitrary image primitive polynomial in $\mathrm{Int}(S, D)$. We now use our prior results to show that every rational can serve as the leading coefficient of an irreducible integer-valued polynomial in $\mathrm{Int}(\mathbb{Z})$.

Theorem 3.2. *For every $m, n \in \mathbb{N}$, there are infinitely many irreducible polynomials $f(X) \in \mathrm{Int}(\mathbb{Z})$ with leading coefficient n/m.*

Proof. For ease of notation, let us write the falling factorial polynomials in the following manner:

$$X^{(n)} = X(X-1)\cdots(X-n+1) = \alpha_1^{(n)}X + \alpha_2^{(n)}X^2 + \cdots + \alpha_{n-1}^{(n)}X^{n-1} + \alpha_n^{(n)}X^n.$$

Note that since $X^{(n)}$ is monic for each n, we have $\alpha_i^{(i)} = 1$ for each i. Observe that for every $m \in \mathbb{N}$, there is a least $r \in \mathbb{N}$ for which $m \mid r!$. Let us denote such a pair as $\{m, r\}$. Given

any $m, n \in \mathbb{N}$, we find the pair $\{m, r\}$ and construct a polynomial of degree r fulfilling the theorem.

If $F(X)$ is an arbitrary integer-valued polynomial of degree r, then $F(X)$ can be written in the form

$$\begin{aligned} F(X) &= F_0 + F_1 X + F_2 \binom{X}{2} + \cdots + F_{r-1} \binom{X}{r-1} + F_r \binom{X}{r} \\ &= F_0 + F_1 X + F_2 \frac{X^{(2)}}{2!} + \cdots + F_{r-1} \frac{X^{(r-1)}}{(r-1)!} + F_r \frac{X^{(r)}}{r!}. \end{aligned}$$

Since $m \mid r!$, we have $m\beta = r!$, with some $\beta \in \mathbb{N}$. Let $F_2', \ldots, F_{r-1}'$ be nonnegative integers such that $F_2 = 2!F_2', \ldots, F_{r-1} = (r-1)!F_{r-1}'$ and set $F_r = n\beta$. Clearly, $F(X)$ may be written as

$$F(X) = F_0 + F_1 X + F_2' X^{(2)} + \cdots + F_{r-1}' X^{(r-1)} + \frac{nX^{(r)}}{m}. \tag{1}$$

Now, we can rewrite Eq. (1) as

$$F(X) = \frac{mF_0 + mF_1 X + mF_2' X^{(2)} + \cdots + mF_{r-1}' X^{(r-1)} + nX^{(r)}}{m}.$$

Let us expand about the $X^{(i)}$, so that

$$\begin{aligned} F(X) = {} & \frac{mF_0 + mF_1 X}{m} + \frac{mF_2'(\alpha_1^{(2)} X + \alpha_2^{(2)} X^2)}{m} \\ & + \cdots + \frac{mF_{r-2}'(\alpha_1^{(r-2)} X + \alpha_2^{(r-2)} X^2 + \cdots + \alpha_{r-2}^{(r-2)} X^{r-2})}{m} \\ & + \cdots + \frac{mF_{r-1}'(\alpha_1^{(r-1)} X + \alpha_2^{(r-1)} X^2 + \cdots + \alpha_{r-1}^{(r-1)} X^{(r-1)})}{m} \\ & + \frac{n(\alpha_1^{(r)} X + \alpha_2^{(r)} X^2 + \cdots + \alpha_r^{(r)} X^r)}{m}. \end{aligned}$$

Recalling that $\alpha_i^{(i)} = 1$ for each i and combining like powers of X, we arrive at

$$\begin{aligned} F(X) = {} & \frac{mF_0 + X(mF_1 + mF_2'\alpha_1^{(2)} + \cdots + mF_{r-1}'\alpha_1^{(r-1)} + n\alpha_1^{(r)})}{m} \\ & + \frac{X^2(mF_2' + mF_3'\alpha_2^{(3)} + \cdots + mF_{r-2}'\alpha_2^{(r-2)} + mF_{r-1}'\alpha_2^{(r-1)} + n\alpha_2^{(r)})}{m} \\ & + \cdots + \frac{X^{r-2}(mF_{r-2}' + mF_{r-1}'\alpha_{r-2}^{(r-1)} + n\alpha_{r-2}^{(r)}) + X^{r-1}(mF_{r-1}' + n\alpha_{r-1}^{(r)}) + nX^r}{m}. \end{aligned}$$

Set $F_0 = p$, for some prime integer p such that $\gcd(p, nr!) = 1$. Note that $\gcd(p, m) = 1$ as well since $m \mid r!$, and that $\gcd(p, n\beta) = 1$ since $n\beta \mid nr!$. Hence, $p \mid F_0$ while $p^2 \nmid F_0$ and $p \nmid F_r$. Consider the system of congruences:

$$mF'_{r-1} \equiv \left[-n\alpha_{r-1}^{(r)}\right] \pmod{p}, \tag{2}$$

$$mF'_{r-2} \equiv \left[-mF'_{r-1}\alpha_{r-2}^{(r-1)} - n\alpha_{r-2}^{(r)}\right] \pmod{p}, \tag{3}$$

$$\vdots$$

$$mF'_2 \equiv \left[-mF'_3\alpha_2^{(3)} - \cdots - mF'_{r-2}\alpha_2^{(r-2)} - mF'_{r-1}\alpha_2^{(r-1)} - n\alpha_2^{(r)}\right] \pmod{p}, \tag{4}$$

$$mF_1 \equiv \left[-mF'_2\alpha_1^{(2)} - \cdots - mF'_{r-1}\alpha_1^{(r-1)} - n\alpha_1^{(r)}\right] \pmod{p}. \tag{5}$$

Since $\gcd(p, m) = 1$, Eq. (2) has a solution F'_{r-1}. Using the value F'_{r-1}, we can now recursively solve Eq. (3) for F'_{r-2}. Iterate this process to obtain integers $F'_{r-1}, F'_{r-2}, \ldots, F_1$ which solve the system. Now, set

$$\begin{aligned}
G_1 &= mF_1 + mF'_2\alpha_1^{(2)} + \cdots + mF'_{r-1}\alpha_1^{(r-1)} + n\alpha_1^{(r)},\\
G_2 &= mF'_2 + mF'_3\alpha_2^{(3)} + \cdots + mF'_{r-2}\alpha_2^{(r-2)} + mF'_{r-1}\alpha_2^{(r-1)} + n\alpha_2^{(r)},\\
&\vdots\\
G_{r-2} &= mF'_{r-2} + mF'_{r-1}\alpha_{r-2}^{(r-1)} + n\alpha_{r-2}^{(r)},\\
G_{r-1} &= mF'_{r-1} + n\alpha_{r-1}^{(r)},
\end{aligned}$$

so that

$$F(X) = \frac{mF_0 + G_1X + G_2X^2 + \cdots + G_{r-2}X^{r-2} + G_{r-1}X^{r-1} + nX^r}{m}.$$

By construction, $p \mid mF_0$, $p^2 \nmid mF_0$, $p \mid G_1, \ldots, p \mid G_{r-1}$, and $p \nmid n$. Hence, the numerator of $F(X)$ is irreducible in $\mathbb{Z}[X]$ by an application of Eisenstein's Criterion.

To see that $\mathrm{d}(\mathbb{Z}, F) = 1$, recall that $F_r = n\beta$, and that $F_0 = p$. Since we have chosen p so that $\gcd(p, nr!) = 1$, and $n\beta \mid nr!$, we have that $\gcd(F_0, F_r) = 1$. Hence, $\gcd(F_0, F_1, \ldots, F_{r-1}, F_r) = 1$. By Lemma 2.5, we have $\mathrm{d}(\mathbb{Z}, F) = 1$. Finally, to see that there are infinitely many such irreducible polynomials, notice that in congruence (5) alone, there are infinitely many solutions F_1 that we might have chosen. (For, having found one such solution, say x_0, there are infinitely many integers congruent to x_0 modulo p.) Translating this observation into infinitely many valid G_1 completes the claim. □

Notice that by our method of constructive proof in Theorem 3.2, we make the following claim (which may be of greater intrinsic interest to the reader).

Corollary 3.3. *For every nonzero m and $n \in \mathbb{N}$, there are infinitely-many irreducible polynomials $f(X) \in \mathbb{Z}[X]$ with leading coefficient n for which $\mathrm{d}(\mathbb{Z}, f) = m$.*

We use our results to this point to construct some irreducible elements of $\mathrm{Int}(S,\mathbb{Z})$ which will later be of interest. The following terminology will be required. If m is an integer, then set

$$\mathcal{R}_S(m) = \big\{n \mid 0 \leqslant n \leqslant m-1 \text{ and } \exists s \in S \text{ so that } n \equiv s \pmod m\big\}.$$

We refer to $\mathcal{R}_S(m)$ as the *set of residues of m with respect to S*. If $|\mathcal{R}_S(m)| = t$, then we refer to the integers $b_1, b_2, \ldots, b_t$ as a *complete set of residues of m with respect to S* if

(1) for all i there exists an $n \in \mathcal{R}_S(m)$ such that $b_i \equiv n \pmod m$, and
(2) $b_i \not\equiv b_j \pmod m$ for $i \neq j$.

We say that the integers $c_1, c_2, \ldots, c_k$ *lack a complete set of residues of m with respect to S* if no subset of this sequence forms a complete set of residues of m with respect to S.

Proposition 3.4. *Let p be a prime number. There exists a sequence $i_1, i_2, \ldots, i_t$ of integers such that the polynomial*

$$f_p(X) = \frac{(X-i_1)(X-i_2)\cdots(X-i_t)}{p}$$

is irreducible in $\mathrm{Int}(S,\mathbb{Z})$.

Proof. Suppose

$$\mathcal{R}_S(p) = \{n_1, \ldots, n_t\}.$$

Since S is infinite, there are at most finitely many primes $q \neq p$ such that

$$\big|\mathcal{R}_S(q)\big| \leqslant \big|\mathcal{R}_S(p)\big|.$$

Enumerate the primes q which satisfy this condition by $Q = \{q_1, \ldots, q_k\}$. For each q_j, choose a sequence of integers

$$a_{j,1}, a_{j,2}, \ldots, a_{j,t}$$

which lacks a complete set of residues for q_j with respect to S. For each $1 \leqslant i \leqslant t$, consider the system of residues

$$\begin{aligned} X &\equiv n_i \pmod p, \\ X &\equiv a_{1,i} \pmod{q_1}, \\ X &\equiv a_{2,i} \pmod{q_2}, \\ &\vdots \\ X &\equiv a_{k,i} \pmod{q_k}. \end{aligned} \tag{$*$}$$

By the Chinese Remainder Theorem, each of these systems has a solution which is unique modulo $pq_1 \cdots q_k$. For each i, let c_i be such a solution. By construction, the sequence $c_1, \ldots, c_t$ forms a complete set of residues of p with respect to S and lacks a complete set of residues for all primes $q \neq p$. If $h(X) = (X - c_1)(X - c_2) \cdots (X - c_t)$, then $p \mid h(s)$ for every $s \in S$. Hence $p^r \mid \mathrm{d}(S, h(X))$ for some $r \in \mathbb{N}$. For each prime $q \neq p$, let s_q be an element of S which lacks a residue in $\mathcal{R}_S(q)$. Then $q \nmid h(s_q)$ and $q \nmid \mathrm{d}(S, h(X))$. Thus $\mathrm{d}(S, h(X)) = p^r$. If $r = 1$, then $f_p(X) = (X - c_1)(X - c_2) \cdots (X - c_t)/p$ is irreducible by Theorem 2.8 and setting $i_j = c_j$ for all j completes the argument. Suppose $r > 1$. Let s be an element of S with $s - c_1 \equiv 0 \pmod{p}$. Since $r > 1$, there exists a $v > 1$ so that p^v exactly divides $s - c_1$. If $c_1' = c_1 + pq_1 \cdots q_k$, then c_1' is another solution to $(*)$ for $i = 1$. Now, p exactly divides $s - c_1'$, and the sequence $c_1', c_2, \ldots, c_t$ yields $\mathrm{d}(S, (X - c_1')(X - c_2) \cdots (X - c_t)) = p$. Setting $i_1 = c_1'$ and $i_j = c_j$ for all $j \neq 1$ and, proceeding as in the former case, completes the proof. □

Hence, for each prime p, we let $\mathfrak{I}_S(p)$ denote a sequence $i_1, \ldots, i_t$ so that the polynomial $f_p(X)$ of Proposition 3.4 is irreducible in $\mathrm{Int}(S, \mathbb{Z})$.

Example 3.5. In the proof of Proposition 3.4, the condition that $r = 1$ is vital. To see this, let $S = \{1 + p^2 t, 3 + p^2 t\}_{t=0}^{\infty}$ and $p \geqslant 5$ a prime integer. If $\mathcal{I} = \{1, 3\}$, then $\mathcal{I}$ forms a complete set of residues modulo p with respect to S but an incomplete set of residues modulo any other prime q. But notice that for $f(X) = (X - 1)(X - 3)$,

$$f\big(1 + p^2 t\big) = p^2(t)\big(p^2 t - 2\big), \qquad f\big(3 + p^2 t\big) = p^2(t)\big(p^2 t + 2\big)$$

so that $p \geqslant 5$ implies $\mathrm{d}(S, f) = p^2$. Set $h(X) = (X - 1)(X - 3)/p$. Then

$$\frac{(X-1)(X-3)}{p} = \frac{(X-1)(X-3)}{p^2} \cdot p$$

in $\mathrm{Int}(S, \mathbb{Z})$ so $h(X)$ is not irreducible. Choose $\mathcal{I}' = \{1 + p, 3 + p\}$. In this case,

$$\frac{(X - (1+p))(X - (3+p))}{p}$$

is irreducible in $\mathrm{Int}(S, \mathbb{Z})$, since $\mathrm{d}(S, (X - (1 + p))(X - (3 + p)) = p$.

Example 3.6. While image primitive polynomials in $\mathbb{Z}[X]$ factor uniquely in $\mathrm{Int}(S, \mathbb{Z})$, the same cannot be said for a general image primitive polynomial in $\mathrm{Int}(S, \mathbb{Z})$. Set $S = \mathbb{Z}$ and let $p > q$ be distinct primes. Choose polynomials

$$f_p(X) = \frac{(X - i_1)(X - i_2) \cdots (X - i_p)}{p} \quad \text{and} \quad f_q(X) = \frac{(X - j_1)(X - j_2) \cdots (X - j_q)}{q}$$

using Proposition 3.4 with the following additional restrictions on the sequences $i_1, \ldots, i_p, j_1, \ldots, j_q$:

(1) $j_s \not\equiv j_t \pmod{p}$ for all $s \neq t$.
(2) $j_1 \equiv i_{p-q+1} \pmod{p}, \ldots, j_q \equiv i_p \pmod{p}$.
(3) $i_s \not\equiv j_1 \pmod{q}$ for all s.
(4) $i_1, \ldots, i_p, j_1, \ldots, j_q$ does not form a complete set of residues for any prime $q' \leqslant p+q$, $q' \nmid pq$.

By the proof of Proposition 3.4, $f_p(X)$ and $f_q(X)$ are each image primitive. Moreover, the construction in the proof yields integers s_1 and s_2 such that p exactly divides $(s_1 - i_l)$ and q exactly divides $(s_2 - j_1)$. Set $k(X) = (X - i_1) \cdots (X - i_p)(X - j_1) \cdots (X - j_q)$. For each prime q' distinct from p and q, condition (4) implies that $q' \nmid \mathrm{d}(\mathbb{Z}, k(X))$ and hence $\mathrm{d}(\mathbb{Z}, k(X)) = p^u q^v$ for nonnegative integers u and v. Condition (2) implies that p exactly divides $k(s_1)$ and condition (3) implies that q exactly divides $k(s_2)$. Thus $\mathrm{d}(\mathbb{Z}, k(X)) = pq$ and $h(X) = f_p(X) f_q(X)$ is also image primitive. Now

$$\begin{aligned} h(X) &= \frac{(X - i_1) \cdots (X - i_p)}{p} \cdot \frac{(X - j_1) \cdots (X - j_q)}{q} \\ &= \frac{(X - i_1) \cdots (X - i_{p-q})(X - j_1) \cdots (X - j_q)}{pq} \cdot (X - i_{p-q+1}) \cdots (X - i_p) \end{aligned}$$

are two irreducible factorizations of $h(X)$ in $\mathrm{Int}(\mathbb{Z})$. Since the first has length 2 and the second has length $q + 1$, $\rho(h(X)) > 1$.

4. On elasticity in Int(S, D)

Definition 4.1. Let D be an atomic integral domain. The *set of elasticities* of nonunits in D, is defined as

$$\mathcal{R}(D) = \{\rho(x) \mid x \in D^\bullet\}.$$

If $\rho(D) < \infty$, then D is *fully elastic* if $\mathcal{R}(D) = \mathbb{Q} \cap [1, \rho(D)]$. If $\rho(D) = \infty$, then D is *fully elastic* if $\mathcal{R}(D) = \mathbb{Q} \cap [1, \infty)$.

One can easily generalize this definition in a natural way to an atomic commutative cancellative monoid M. The fully elastic property is studied in detail in [8], where two principle results are derived:

(a) Any numerical monoid S which requires more than one generator is not fully elastic [8, Theorem 2.2].
(b) If D is a ring of integers in finite extension of $\mathbb{Q}$ with class number p^k, where p is a prime, then D is fully elastic [8, Corollary 3.10].

We note that the results from [6] mentioned previously in the introduction indicate that if D is a one-dimensional Noetherian domain and $|S| = \infty$, then $\rho(\mathrm{Int}(S, D)) = \infty$. As a consequence, we focus our attention in this section on the case where $D = \mathbb{Z}$ and argue

for infinite S that $\mathrm{Int}(S,\mathbb{Z})$ is fully elastic. The argument will center on the polynomials $f_p(X)$ constructed in Proposition 3.4 and the related polynomial

$$h_p(X) = (X - i_1)\cdots(X - i_t),$$

where $\mathfrak{I}_S(p) = \{i_1, \ldots, i_t\}$. Note with this notation that $f_p(X) = h_p(X)/p$ and $h_p(X)$ is monic (and hence primitive) in $\mathbb{Z}[X]$. We begin by looking at the nonconstant irreducible divisors of $h_p^k(X) f_p^s(X)$.

Lemma 4.2. *Let S be an infinite subset of $\mathbb{Z}$, p a prime integer, k and s nonnegative integers and $\mathfrak{I}_p(S)$ as in Section* 3. *The only possible nonconstant irreducible divisors in* $\mathrm{Int}(S,\mathbb{Z})$ *of $h_p^k(X) f_p^s(X)$ are*:

(1) $f_p(X)$.
(2) *The monomials* $(X - i_1), \ldots, (X - i_t)$.

Proof. Suppose $q(X)$ is a nonconstant irreducible divisor of $h_p^k(X) f_p^s(X)$ over $\mathrm{Int}(S,\mathbb{Z})$. By Lemma 2.7, $q(X) = f^*(X)/n$ for some primitive $f^*(X)$ in $\mathbb{Z}[X]$ and $n \in \mathbb{Z}$. By the uniqueness of factorization in $\mathbb{Z}[X]$, $f^*(X) \mid \prod_{j=1}^t (X - i_j)^{k+s}$ and hence $f^*(X) = \prod_{j=1}^t (X - i_j)^{n_j}$ where each $0 \leqslant n_j \leqslant k + s$. We have two cases.

Case 1. Some $n_j = 0$. By the choice of $\mathfrak{I}_p(S)$, $\mathrm{d}(S, f^*(X)) = 1$ and hence $n = 1$. Since $q(X)$ is irreducible, $q(X) = (X - i_j)$ for some j.

Case 2. Each $n_j > 0$. In this case, $\mathrm{d}(S, f^*(X)) = p^\alpha$ where $\alpha = \min\{n_j\}_{j=1}^t$. It follows that $\mathrm{d}(S, \prod_{j=1}^t (X - i_j)^{n_j - 1}) = p^{\alpha - 1}$, and hence

$$q(X) = f_p(X) \cdot \frac{\prod_{j=1}^t (X - i_j)^{n_j - 1}}{p^{\alpha - 1}}.$$

Since $q(X)$ is irreducible, each $n_j = 1$ and $\alpha = 1$. □

The last lemma allows us to compute the set of lengths of $h_p^k(X) f_p^s(X)$ for any nonnegative integers k and s.

Lemma 4.3. *Let S, p, k, s and $\mathfrak{I}_p(S)$ be as in Lemma* 4.2. *In* $\mathrm{Int}(S,\mathbb{Z})$,

$$\mathcal{L}\big(h_p^k(X) f_p^s(X)\big) = \big\{2j + (k - j)t + s \mid 0 \leqslant j \leqslant k\big\},$$

for natural numbers k and s, and $|\mathfrak{I}_S(p)| = t$.

Proof. If $n \in \mathbb{Z}$ divides $h_p^k(X) f_p^s(X)$ over $\mathrm{Int}(S, \mathbb{Z})$, then $n \mid \mathrm{d}(S, h_p^{k+s}(X))$ and hence $n = p^\alpha$ for some nonnegative α. By Lemma 4.2, every irreducible factorization of $h_p^k(X) f_p^s(X)$ is of the form

$$h_p^k(X) f_p^s(X) = p^j f_p^w(X)(X - i_1)^{r_1} \cdots (X - i_t)^{r_t}.$$

Obviously, $r_1 = \cdots = r_t$ and hence setting $r_1 = r$ we have

$$h_p^k(X) f_p^s(X) = p^j f_p^w(X)(X - i_1)^r \cdots (X - i_t)^r.$$

Notice that for each $0 \leqslant j \leqslant k$ such a factorization is possible by setting $w = s + j$ and $r = k - j$. Since unique factorization in $\mathbb{Z}[X]$ forces $j + s = w$, for a fixed j both w and r are also fixed. The length of this factorization is $j + w + tr = 2j + t(k - j) + s$, completing the argument. □

Lemma 4.3 immediately yields a computation of the elasticity of $h_p^k(X) f_p^s(X)$.

Corollary 4.4. *In* $\mathrm{Int}(S, \mathbb{Z})$*, for all primes* p *with* $|\mathfrak{I}_S(p)| = t$ *and* $k, s \in \mathbb{N}$*,*

$$\rho\big(h_p^k(X) f_p^s(X)\big) = \frac{kt + s}{2k + s}.$$

We now proceed to show that $\mathrm{Int}(S, \mathbb{Z})$ is fully elastic for every $S \subseteq \mathbb{Z}$, $|S| = \infty$.

Theorem 4.5. *If* $q = t/u \in \mathbb{Q}$ *with* $t > u \geqslant 2$*, then there is a polynomial* $f(X) \in \mathrm{Int}(S, \mathbb{Z})$ *for which* $\rho(f(X)) = t/u$*. Equivalently,* $\mathrm{Int}(S, \mathbb{Z})$ *is fully elastic for any infinite* $S \subseteq \mathbb{Z}$*.*

Proof. Let $q = t/u > 1$ be as given in the theorem. Since S is infinite, choose a prime p' such that $|\mathfrak{I}_S(p')| = j$ where $s = uj - 2t \geqslant 0$. As before, we set $\mathfrak{I}_S(p') = \{i_1, \ldots, i_j\}$,

$$f_{p'}(X) = \frac{(X - i_1) \cdots (X - i_j)}{p'},$$

and $h_{p'}(X) = (X - i_1) \cdots (X - i_j)$. If $k = t - u$, then

$$\frac{kj + s}{2k + s} = \frac{(t - u)j + uj - 2t}{(t - u)2 + uj - 2t} = \frac{tj - 2t}{uj - 2u} = \frac{t(j - 2)}{u(j - 2)} = \frac{t}{u}.$$

By Corollary 4.4,

$$\rho\big(h_{p'}^k(X) f_{p'}^s(X)\big) = \frac{kj + s}{2k + s} = \frac{t}{u}$$

completing the argument. □

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