On the components of $X_0(p^n)$

Robert F. Coleman

Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720, USA

Received 22 October 2003; revised 14 July 2004
Communicated by D. Goss

In fond memory of Arnold Ross

Abstract

We show the ordinary locus of $X_0(p^n)(\mathbb{C}_p)$ is normally the set of $\mathbb{C}_p$-valued points on $2n$ affinoids which correspond to components of the stable model of $X_0(p^n)$. We then show the points on Edixhoven’s “horizontal” components of $X_0(p^2)$ correspond to elliptic curves which are $p$-isogenous to curves which Buzzard calls “too supersingular.”

© 2004 Elsevier Inc. All rights reserved.

Keywords: Stable model; Modular curve; Too supersingular; Canonical subgroup

Fix a prime $p$. Suppose $n \geq 1$. We first present a viewpoint of the ordinary locus of $X_0(p^n)$, slightly different from that taken in Katz–Mazur and Edixhoven [K-M,E] which allows one to see the stable structure of the ordinary locus. Next, we give a moduli-theoretic interpretation of Edixhoven’s (semi)stable model of $X_0(p^2)$ [E].

Edixhoven found the $p$-adic stable model of $X_0(p^2)$ by blowing up the Katz–Mazur–Edixhoven model [K-M,E] at the supersingular points of the reduction. We interpret points on the components of this model in terms of the canonical subgroups of elliptic curves described by Katz in [K-pPMF]. In particular, $(E, C)$ corresponds to a point “on” one of Edixhoven’s “horizontal” components of $X_0(p^2)$ if and only if $E/pC$ is “too supersingular” in the sense of Buzzard [B] i.e., has no canonical subgroup.

E-mail address: coleman@math.berkeley.edu
One can easily add tame level structure which we plan to do in a subsequent article.
These results facilitate a determination of which eigenforms on \( X_0(Np^2) \) give rise to a representations potentially crystalline at \( p \). In a future article, using the results in this paper and the description of the Fontaine monodromy operator given in [C-I], we will show that the representation attached to an eigenform on \( X_0(Np^2) \), \( p > 3 \), which isn’t old and doesn’t come by twisting from a form on \( X_0(Np) \) is potentially crystalline at \( p \).

0. Semi-stable coverings

Our approach to the semi-stable reduction of a curve is encapsulated by the following fact:

Semi-stable coverings of a curve correspond to semi-stable models of the curve.

which is proven in [C-SM, Proposition 2.1] and which we now explain.

First, a wide open is a rigid space conformal to \( \mathbb{C} \setminus D \) where \( \mathbb{C} \) is a smooth complete curve and \( D \) is a finite disjoint union of affinoid disks in \( C \), which contains at least one in each connected component. A wide open disk is the complement of one affinoid disk in \( \mathbb{P}^1 \) (it is conformal to \( B(0,1) \)) and a wide open annulus is conformal to the complement of two disjoint such disks (it is conformal to \( A(r,1) \), where \( r \in |\mathbb{C}_p| \), \( 0 < r < 1 \)). One can also characterize these spaces as smooth one-dimensional rigid spaces \( W \) which contains an affinoid \( X \) such that \( W \setminus X \) is a finite union of annuli and \( H^0(W) \) is isomorphic to \( H^2(W,W-X) \) and is finite dimensional.

By the ends of \( W \), we mean \( \lim X \) Conn. Comp.\((W\setminus X)\) where \( X \) runs over the affinoid subdomains of \( W \). We call an affinoid subdomain \( X \) of \( W \) underlying if the map from Conn. Comp.\((W \setminus X)\) to the set of ends of \( W \) is bijective. A semi-stable covering of a curve \( C \) is a finite-admissible covering \( D \) of \( C \) by connected wide opens such that

(i) if \( U \neq V \in D \), \( U \cap V \) is a finite collection of disjoint wide open annuli,
(ii) if \( T, U, V \in D \) are pairwise distinct, \( T \cap U \cap V = \emptyset \),
(iii) for \( U \in D \), if

\[
Z_U := U \setminus \left( \bigcup_{V \in D, V \neq U} V \right),
\]

\( Z_U \) is a non-empty affinoid whose reduction is irreducible and has at worst regular singular points. We call \( Z_U \) an underlying affinoid of \( D \).

A final object in the category of semistable coverings exists and corresponds to the stable model of \( C \) if and only if the stable model of \( C \) exists and its reduction has at least two components.

1. The vertical components of \( X_0(p^n) \)

For an elliptic curve over \( \mathbb{C}_p \), with ordinary reduction, let \( K(E) \) denote the kernel of reduction and \( K_n(E) \) the cyclic subgroup of order \( p^n \) in \( K(E) \) (more generally,
if $E$ is arbitrary, the canonical cyclic subgroup of order $p^n$, when it exists (see [B, Definition 3.4]).

Let $X_0(p^n)$ be the complete smooth rigid curve over $\mathbb{Q}_p$ associated to the course moduli problem that associates to a scheme $S$ over $\mathbb{Q}_p$ the set of pairs $(E, C)$ where $C$ is a subgroup of $E$ of order $p^n$. There is an (Atkin–Lehner) involution $\alpha$ of $\bigcoprod_{n \geq 0} X_0(p^n)$ which takes the point in $X_0(p^n)$ corresponding to the pair $(E, C)$ to the point corresponding to $(E/C, E[p^n]/C)$. There are also two maps $\pi_f, \pi_v : \bigcoprod_{n \geq 1} X_0(p^n) \to \bigcoprod_{n \geq 0} X_0(p^n)$ which take the point in $X_0(p^n)$ corresponding to the pair $(E, C)$ to the point corresponding to $(E, pC)$ and to $(E/p^{n-1}C, C/p^{n-1}C)$, respectively. We have,

$$\alpha \circ \pi_v = \pi_f \circ \alpha. \quad (1)$$

There are several affinoids to consider in $X_0(p^n)$. First, there are the affinoids $X_{ab}$, $a + b = n$, implicit in the work of Katz–Mazur–Edixhoven whose $\mathbb{C}_p$-valued points correspond to pairs $(E, C)$ where $E$ is a generalized elliptic curve over (the ring of integers in) $\mathbb{C}_p$ with ordinary (this includes multiplicative) reduction and $C \cap K(E) = \text{Ka}(E)$ (The reduction of $C$ modulo $p$ is what is called “$(a, b)$-cyclic” [K-M, 13.4.1]) [K-M, E].

More precisely:

Let $X_0(p)$ denote the model of $X_0(p)$ found by Deligne and Rapoport [D-R, Theorem 1.16]. The formal completion of $X_0(p)$ along the smooth locus of its reduction is an affinoid subdomain of $X_0(p)$ with two connected components. We take $X_{10}$ to be the component containing the cusp $\infty$ and $X_{01}$ the cusp 0. When $p \geq 5$, we could also obtain the more general $X_{ab}$ from the “$(a, b)$-component” of Edixhoven’s model, in this way. (See also [K-M, Section 13.4].)

Now let $\pi_{ab} = \pi_f^b \circ \pi_v^a$. Then, if $a + b = n$

$$X_{ab} = \pi_{a-1,b}^{-1} \pi_{a-1,b}^1 X_{10} \cap \pi_{a,b-1}^{-1} \pi_{a,b-1}^1 X_{01},$$

where we take $\pi_{a-1,b}^{-1} \pi_{a-1,b}^1 X_{10} = \pi_{n-1}^{-1} \pi_{n-1}^1 X_{01} = X_0(p^n)$. The rigid space $X_{ab}$ is an affinoid subdomain of $X_0(p^n)$ because a finite cover of an affinoid is an affinoid and the intersection of two affinoid subdomains of a curve is an affinoid subdomain (see Corollary A.7).

When $a \geq b$ (which we’ll assume until pointed out otherwise), the points on $X_{ab}$ also correspond to pairs $(E, \mathcal{P})$ where $E$ is an elliptic curve with ordinary reduction and $\mathcal{P}$ is a pairing on $K_a := K_a(E)$ onto $\mu_{p^b}$. Indeed, define a pairing $\mathcal{P}$ on $K_a$ by setting

$$\mathcal{P}(A, B) = (R, S)_{E,n}.$$
where $S \in C$, $p^bS = B$ and $R \in K_n$, $p^bR = A$. Here $(\cdot, \cdot)_{E,i}$ denotes the Weil pairing into $\mu_{p^i}$ on $E[p^i]$. On the other hand,

$$C_{\mathcal{P}} = \{ S \in E[p^n] : p^bS \in K_a, (R, S)_{E,n} = \mathcal{P}(p^bR, p^bS), \forall R \in K_n \},$$

is a cyclic group of order $p^n$ such that $C_{\mathcal{P}} \cap K_n = K_a$.

We see that $\mathbb{Z}_p^*$ acts on $X_{ab}$ via $\tau_r : (E, \mathcal{P}) \mapsto (E, \mathcal{P}^r)$, for $r \in \mathbb{Z}_p^*$, $(C_{\mathcal{P}} = \{ T \in C_{\mathcal{P}} : \exists S \in C_{\mathcal{P}}, T \equiv rS \mod K, p^bT = p^bS \}).$

One can use this point of view to show that, over $C_{\mathcal{P}}$, $X_{ab}$ has as many irreducible components as classes mod squares in $(\text{the reduction of } X_{pb})$. This holds if $\text{image of } X_{pb}$ represents a point on $K(E)$.

Now fix $b$ and set $B := B_b$. All the pairs $(E, \mathcal{P})$ such that the class of $\mathcal{P}(P, P)$ for a generator $P$ of $K_a(E)$ equals a given element of $B$ lie on the same component of $X_{ab}$ because the map $X_{ab} \rightarrow B$ which takes the point corresponding to $(E, \mathcal{P})$ to the class of $\mathcal{P}$ is rigid analytic. To see this, let $X_{ab}^1$ be the affinoid above $X_{ab}$ in $X_1(p^n)$. Let $Y$ be the inverse image in $X_{ab}^1$ of a component of $X_{ab}$.

Suppose $(E, Q)$ corresponds to a point $y$ of $Y$, so $p^bQ$ generates $K_a(E) = (Q) \cap K(E)$. Suppose $R \in K(E)$, $p^bR = Q$. Then, $y \mapsto (R, Q)_{E,n}$ is a rigid analytic map from $Y$ to the primitive $p^b$th roots of unity and its image modulo the action of $((-Z/p^bZ)^*)^2$ depends only on the image of $y$ in $X_{ab}$.

For $\beta \in C_{ab}$, call the corresponding component in $X_{ab}$, $X_{ab}^{\beta}$. It is easy to see these components are non-empty. In fact, these components are irreducible. To see this, first, the reduction of $X_{ab}^{\beta}$ is isomorphic (non-canonically) to the quotient of the Igusa curve $Ig(p^b)$ by the group of automorphisms

$$H_b = \{ z_t : z_t(E, Q) = (E, tQ), t \in (Z/p^bZ)^*, t^2 = 1 \}. $$

(In fact, $z_{-1} = z_1$.) Indeed, let $\zeta \in \mu_{p^b}$ represent $\beta$, then there exists a $P \in K_b$ such that $\mathcal{P}(P, P) = \zeta$. Let $Q \in E[p^{2b}]$ such that $(P, Q)_{W_2b} = \zeta$. Then the point $(\tilde{E}, \tilde{Q})$ of $Ig(p^b)$ is well defined up to the action of $H_b$. If $\zeta$ is replaced with $\zeta^2$ we can replace $P$ with $yP$ and $Q$ with $yQ$.

Now, the coup de grace is that $X_{ab}^{\beta}$ is naturally isomorphic to $X_{ab}^{\beta}$. Simply, if $(E, \mathcal{P})$ represents a point on $X_{ab}^{\beta}$, $(E, \mathcal{P}')$ represents a point on $X_{ab}^{\beta}$, where

$$\mathcal{P}'(R, S) = \mathcal{P}(p^{a-b}R, p^{a-b}S)$$

for $R, S \in K_a$. Denote this map from $X_{ab}^{\beta}$ to $X_{ab}^{\beta}$ by $r_{ab}^{\beta}$. 
These components can all be defined over $K_p$, where

$$K_p = \begin{cases} \mathbb{Q}_p(\sqrt{(-1)(p-1)/2}) & \text{if } p \text{ is odd}, \\ \mathbb{Q}_2(\sqrt{-1}, \sqrt{2}) & \text{if } p = 2. \end{cases}$$

To see this, define a function $f_{a,b}$ on $X_{a,b}^1$ with values in $\mu_{p^a}$, by: if $(E, P)$ represents a point $x$ of $X_{a,b}^1$,

$$f_{a,b}(x) = (Q, P)_{E,n},$$

where $Q \in K_n(E)$ and $p^bQ = p^bP$. Clearly, $f_{a,b}$ is a rigid analytic function, and using it, we see $X_{a,b}^1$ has at least $\phi(p^a)$ irreducible components. Now, if $x_{a,b} \in B$, let $\chi_{a,b}$ be the function on the primitive $p^b$th roots of one such that $\chi_{a,b}(x) = \begin{cases} 1 & \text{if } x \in \beta, \\ 0 & \text{otherwise}. \end{cases}$ Then $g_{a,b} = \chi_{a,b} \circ f_{a,b}$ is a rigid analytic function on $X_{a,b}^1$ and if $\tau \in \text{Gal}(\bar{Q}_p/K_p)$, $g_{a,b}(\tau(E, P)) = g_{a,b}(E, P)$.

It follows that $g_{a,b}$ is defined over $K_p$. But also, if $x$ and $y$, in $X_{a,b}^1$, have the same image in $X_0(p^n)$, $g_{a,b}(x) = g_{a,b}(y)$, Thus $g_{a,b}$ is the pullback of a function on $X_{a,b}^1$ defined over $K_p$. We thus see the components $X_{a,b}^1$ are defined over $K_p$.

Also, if $\gamma \in C_{a,b}$, $X_{a,b}^\beta \cong X_{a,b}^\gamma$ because, if $r \in \mathbb{Z}^*_p$ such that $r\beta = \gamma$, $\tau_r$ restricts to an isomorphism from $X_{a,b}^\beta$ onto $X_{a,b}^\gamma$.

There is also a natural map $i_{a,b}^\beta : X_{a,b}^\beta \to X_{b,b}^\beta$.

$$(E, \mathcal{P}) \mapsto (E/K_{a-b}, \mathcal{P}|_{K_a/K_{a-b}}).$$

One can show $t_{b+1, b}$ is the restriction of $\pi_v$ to $X_{b+1, b}$. Also, $t_{b+1, b}^\beta \circ r_{b+1, b}^\beta$ is a lift of Frobenius. Indeed, suppose

$$t_{a, b}^\beta \circ r_{a, b}^\beta (E, \mathcal{P}) = (E/K_{a-b}, \mathcal{P}')$$

and $\phi_e : E \to E/K_c$ is the natural isogeny. Suppose $P, Q \in K_c$, $p^{a-b}R = P$ and $p^{a-b}S = Q$. Then,

$$\mathcal{P}'(\phi_{a-b}(R), \phi_{a-b}(S)) = \mathcal{P}(P, Q).$$
Proposition 1.1. Suppose \((E, \mathcal{P})\) represents a point \(x\) in \(X_{bb}\). Then \(x\) is represented by \((E', \mathcal{P}')\), where \(E' = E/C \mathcal{P}\) and if \(\rho: E \to E'\) is the natural isogeny
\[
\mathcal{P}'(\rho(u), \rho(v)) = \mathcal{P}(p^b u, -p^b v)
\]
if \(u, v \in K_{2b}(E)\).

Proof. First, \(x\) is represented by \((E/C \mathcal{P}, E[2b]/C \mathcal{P})\). Suppose \(A, B \in K_{2b}(E)\).

Let \(P, Q \in K_{2b}(E)\) such that \(p^b P = A\) and \(p^b Q = B\). Suppose \(R \in E[2b]\), \(p^b R \equiv P \mod C \mathcal{P}\) and \(S \in K_{3b}(E)\), \(p^b S = Q\). Write \(p^b T = -p^b P = -A\).

\[
P' = \begin{pmatrix} x \end{pmatrix}_{E', 2b} = \begin{pmatrix} (S, R) \end{pmatrix}_{E, 3b} = \begin{pmatrix} (p^b S, p^b R) \end{pmatrix}_{E, 2b} = \begin{pmatrix} (Q, T) \end{pmatrix}_{E, 2b} = \mathcal{P}(B, -A).
\]

\[\square\]

Corollary 1.1.1. If \(p\) is odd, \(\mathcal{X}(X_{bb}) = X_{bb}^{(\frac{1}{p})\beta}\).

If \(a < b\), we set \(X_{ab}^\beta = \mathcal{X}(X_{ba}^\beta)\).

Suppose \(C\) a curve over \(\mathbb{C}\) which has a stable model \(C\). We call the rigid spaces \(\text{red}^{-1}x\), where \(x\) is a point on \(\overline{C}\), residue classes of \(C\). If the reduction of \(C\) has at least two components, we call the affinoids \(\text{red}^{-1}Z\) in \(C\) where \(Z\) is the smooth locus of a component of \(\overline{C}\) underlying affinoids of \(C\). We define residue classes of affinoids similarly.

Let \(X_0(p^n)\) denote the stable model of \(X_0(p^n)\), when it exists.

Theorem 1.2. If \(p \geq 23\), \(c + d = n\) and \(\beta \in B_{\min(c, d)}\) then \(X := X_0(p^n)\) exists, its reduction has at least two irreducible components and \(X_{cd}^\beta\) is an underlying affinoid of \(X := X_0(p^n)\).

Proof. Since \(p \geq 23\), \(X_0(p)\) and hence \(X_0(p^n)\) has genus at least 2, and since the reduction of \(X_0(p)\) has two components it follows from Proposition 1.5 of [C-SM] that the reduction of \(X\) has at least two irreducible components. We remark that the theorem is true for \(n = 1\). In fact, it follows from [D-R] that \(X_{10}\) and \(X_{01}\) are the underlying affinoids of \(X_0(p)\), in this case.

Now since \(X_{cd}^\beta\) is an affinoid with good reduction, it is either contained in a residue class of \(X\) or is the complement of finitely many residue classes in an underlying affinoid of \(X\). Suppose \(X_{cd}^\beta\) is contained in a residue class \(U\). Then since \(X_{cd}^\beta\) maps finitely onto \(X_{10}\) or \(X_{01}\) via \(\pi_f\) and the reductions of these affinoids are smooth affines with at least three points at infinity, the same is true for the reduction of \(X_{cd}^\beta\). Since
U is a wide open disk or annulus it follows that one of the connected components of \( U \setminus X_{c,d}^B \) is a wide open disk \( D \). The image of \( D \) under \( \pi \) can not be contained in \( X_{10} \) or \( X_{01} \) since some the points in \( D \) correspond to supersingular elliptic curves. It follows that \( \pi(D) \) is disconnected from \( X_{10} \) or \( X_{01} \) which is impossible since \( D \) is connected to \( X_{c,d}^B \). Thus, \( X_{c,d}^B \) is the complement of finitely many residue classes in an underlying affinoid \( Z \) of \( X \). Since these residue classes must be disks, we see they can’t exist using the same argument as above. □

This result should be true for all \( p \), for sufficiently large \( n \).

If \( a \geq b > 0 \) there is a natural map \( p: C_{ab} \rightarrow C_{a,b-1}; (A, \mathcal{P}) \mapsto (A, \mathcal{P}^p) \). If \( a > b > 0 \), there is another natural map \( \sigma: C_{ab} \rightarrow C_{a-1,b}; (A, \mathcal{P}) \mapsto (A/p^{a-1}A, \mathcal{P}) \).

Lemma 1.3. Suppose \( a + b = n > 1 \) and \( \beta \in C_{ab} \). Then \( \pi_f \) restricts to a finite map \( X_{ab}^\beta \rightarrow Y_{ab}^\beta \) where

\[
Y_{ab}^\beta = \begin{cases} 
X_{ab}^\beta \text{ if } a \geq b \geq 1, \\
X_{a,b-1}^\beta \text{ if } a < b \geq 1, \\
X_{a-1,b} \text{ if } a \geq 1 = b + 1.
\end{cases}
\]

The degree of this restriction is

\[
\begin{cases} 
p & \text{if } p > 2 \text{ and } b > 1 \text{ or } p = 2 \text{ and } b \geq 4, \\
(p-1)/2 & \text{if } p > 2 \text{ and } b = 1, \\
1 & \text{otherwise.}
\end{cases}
\]

Proof. First note that \( \pi_f : X_0(p^n) \rightarrow X_0(p^{n-1}) \) is finite of degree \( p \). Also if \( a + b = n \) and \( b > 1 \), \( \pi_f^{-1}X_{a,b-1} = X_{ab} \). We have seen that the irreducible components of \( X_{a,b-1} \) are the \( X_{a,b-1}^\gamma \) for \( \gamma \in C_{a,b-1} \). It is easy to see, using (1), that \( X_{ab}^\beta \) maps to \( Y_{ab}^\beta \).

Suppose now \( p > 2 \) and \( b > 1 \) or \( p = 2 \) and \( b > 3 \). Since \( |C_{a,b-1}| = |C_{ab}| \), this implies that each irreducible component of \( X_{ab} \) maps with degree \( p \) onto an irreducible component of \( X_{a,b-1} \). This proves the lemma in this case.

Now suppose \( p = 2 \) and \( 2 \leq b \leq 3 \). The lemma follows in this case because two irreducible components of \( X_{ab} \) map to each irreducible component of \( X_{a,b-1} \).

Now note that \( \pi_f^{-1}X_{a-1,0} = X_{a-1} \) and \( \pi_f |_{X_{a-1}} \) has degree 1. This completes the proof the lemma when \( p = 2 \) and when \( b = 0 \). So suppose \( p > 2 \) and \( b = 1 \). Then \( \pi_f \) restrict to a finite degree \( p-1 \) map from \( X_{a-1}^\beta \) onto \( X_{a,0}^\beta \). Where \( \beta \in C_{a,1} \) and \( r \in (\mathbb{Z}/p\mathbb{Z})^* \) is a quadratic non-residue. The lemma follows from the fact that \( \pi_f \circ \tau_r = \tau_r \circ \pi_f \) on \( X_{ab} \), \( a \geq b \). □

2. Annuli

Suppose \( r \leq s \in \mathbb{R} \). By the width of an annulus isomorphic to \( A(r,s) \), we mean \( \log_p(s/r) \). If \( A \) is a wide open annulus, denote its width by \( w(A) \).
Lemma 2.1. Suppose \( f : \mathcal{A} \to \mathcal{B} \) is a morphism of wide open annuli such that \( C \) is not contained in any affinoid subdomain. Then \( C \) is an end of \( \mathcal{B} \), \( f : \mathcal{A} \to C \) is finite and \( w(C) = \deg_C(f)w(\mathcal{A}) \).

**Proof.** We can suppose \( \mathcal{A} = A(r, 1) \), \( B = A(s, 1) \) and

\[
\lim_{|x| \to 1} |f(x)| = 1.
\]

Then \( h(T) := f^*(T) \) is a unit in \( A(\mathcal{A}) \). We can write

\[
h(T) = cT^e g(T),
\]

where \( c \in K^*, e \in \mathbb{Z} \) and \( |g(x) - 1| < 1 \) for \( x \in \mathcal{A} \). It follows that \( |c| = 1 \), \( C = A(r^e, 1) \) and \( e = \deg_C(f) \). Thus,

\[
w(C) = -\log_p(r^e) = -\deg(f) \log_p(r) = \deg(f)w(\mathcal{A}).
\]

If \( \mathcal{A} \) is an annulus over \( K \), there are two natural maps \( \Omega^1_{\mathcal{A}/K} \to K \). Indeed, if \( T \) is a parameter on \( \mathcal{A} \) we can write every element \( \omega \) of \( \Omega^1_{\mathcal{A}/K} \to K \) in the form

\[
\left( \sum_{i=-\infty}^{\infty} c_i(\omega, T)T^i \right) \frac{dT}{T},
\]

where \( c_i(\omega, T) \in K \). The map \( \omega \mapsto c_1(\omega, T) \) is a linear map and if \( T' \) is another parameter, there is an \( \varepsilon \in \{\pm 1\} \) such that

\[
c_1(\omega, T) = \varepsilon c_1(\omega, T')
\]

for all \( \omega \in \Omega^1_{\mathcal{A}/K} \) (see [C-RLC, Section II].) We call a choice of one of these two homomorphisms an **orientation** of \( \mathcal{A} \) and if \( \mathcal{A} \) is oriented, we denote the chosen homomorphism by \( \text{Res}_A \).

The following lemma will be used in a future article where we will discuss the crystalline nature of the representation attached to an eigenform.

Lemma 2.2. Suppose \( h : \mathcal{A} \to \mathcal{B} \) is a finite surjective morphism of annuli and suppose \( \mathcal{A} \) and \( \mathcal{B} \) are oriented. Then, if \( \omega \) is a differential on \( \mathcal{B} \) and \( v \) is a differential on \( \mathcal{A} \)

\[
\text{Res}_A h^* \omega = \varepsilon d \text{Res}_B \omega \quad \text{and} \quad \text{Res}_B \text{Tr}_h v = \varepsilon \text{Res}_A v,
\]

where \( d \) is the degree of \( h \) and \( \varepsilon = -1 \) if \( h \) is orientation reversing and \( 1 \) otherwise. If \( h \) is an inclusion such that \( \mathcal{B} \setminus h(\mathcal{A}) \) is a union of annuli, the first formula is still true with \( d = 1 \).
Lemma 2.3. Suppose $h: A \to B$ is morphism of annuli, $C \subseteq B$ is a subannulus at an end of $B$ and $h^{-1}C \to C$ is finite of degree $d$. Then $h(A)$ is an annulus and $h: A \to h(A)$ is finite, étale of degree $d$. If $(d, p) = 1$, $h: A \to h(A)$ is Galois and $A \times h(A)A$ is a disjoint union of $d$ annuli each projecting isomorphically onto $A$.

Proof. Suppose $A = A(t, 1)$ and $B = A(s, 1)$ and $C = A(t, 1)$, $t \geq s$. We can write

$$h(T) = cT^n g(T),$$

where $|g(T) - 1| < 1, s \leq |c| < 1$ and $s \leq |c| r^n \leq 1$. We can suppose $n \geq 0$. The hypothesis about $C$ implies $n = d$ and $|c| = 1$. It follows that $h$ is finite onto $A(r^n, 1)$ of degree $d$.

To prove the last part observe that $g(T)^{1/d}$ makes sense. \(\square\)

By a circle we mean an affinoid isomorphic to $\mathcal{C}_p(T, T^{-1})$, i.e., an annulus conformal to $A[1, 1]$. We call a subannulus $U$ of an annulus $A$ concentric if the connected components of $A \setminus U$ are annuli.

3. Horizontal components

Suppose $p > 3$ is prime. The reduction of Edixhoven’s semi-stable model $X'_2$ of $X_0(p^2)$ (which may be obtained by blowing up the Katz–Mazur–Edixhoven regular model over $\mathbb{Z}_p$ of $X_0(p^2)$ at the supersingular points on its reduction over the extension of $\mathbb{Q}_p^{nr}$ of degree $(p^2 - 1)/2$) has four vertical components $X_{20}, X'_{11}, X_{11}^-$ and $X_{02}$ (as described above (we also let $X'_{20} = X_{20}$ and $X_{02}^\pm = X_{02}$)) and $|SS|$ horizontal components (we’ll frequently use $SS$ to denote $|SS|$, $Z_2(s)$ for $s \in SS$, where $SS$ is the set of supersingular $j$-invariants [E]. (It is stable if $|SS| > 1$.) Moreover, the reductions of any two of these components intersect when and only when one is vertical another is horizontal, in which case, they intersect in one point.

Remark. $X'_2$ is stable if there are at least two supersingular points mod $p$. In general, it may be characterized as follows. Recall, $\pi_f: X_0(p^2) \to X(1)$ is the forgetful map. Let $D \subseteq X(1)$ be the disk around $\infty$ corresponding to elliptic curves with multiplicative reduction. Then $X'_2$ is the minimal semistable model $X$ of $X_0(p^2)$ such that the sections of $\pi_f$ over $D$ factor through embeddings $Spec(\overline{A}_f(D)) \to X$.

An elliptic curve over $\mathbb{C}_p$ is called too supersingular if it has no canonical subgroup and nearly too supersingular if it is $p$-isogenous to a too supersingular curve. Nearly too supersingular curves do have canonical subgroups. (Canonical subgroups of elliptic curves are introduced by Katz [K].) These are subgroup schemes of order $p$. Buzzard defined canonical subgroups of order $p^n$ [B, Definition 3.4.]. If $(E, C)$ is a pair consisting of an elliptic curve $E$ over $\mathbb{C}_p$ with a model with good supersingular reduction and a subgroup $C$ of order $p$, the Buzzard invariant of $(E, C)$ is the positive real number $b(E, C)$ which is characterized by the properties, $b(E, C)$ is the valuation of
the Hasse invariant of the reduction modulo $p$ of a model for $E$ with good reduction when $E$ has a canonical subgroup and it is $C$ and in general

$$b(E, C) + b(E/C, E[p]/C) = 1.$$  

(It is always true that either $C$ is the canonical subgroup of $E$ or $E[p]/C$ is the canonical subgroup of $E/C$.) In particular, $E$ is too supersingular if and only if $b(E, C) = p/(p + 1)$ for one and hence all subgroups $C$ of $E$ of order $p$. If $E$ has a canonical subgroup $K$ of order $p^2$, $b(E/K_1, K_2/K_1) = pb(E, K_1)$ and if $K_1 \neq C$, $b(E/C, E[p]/C) = b(E, C)/p$.

In general, if $H$ is a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ corresponding to $s \in SS$ and $b \in \mathbb{Q}$, the pairs $(E, C)$ such that $E$ reduces to a curve isomorphic to $H$ and $b(E, C) = b$ correspond to the $\mathbb{C}_p$-valued points on a concentric circle, $C_b(s)$, in the wide open annulus $A_1(s)$ in $X_0(p)$ above the singular point of the reduction of $X_0(p)$ corresponding to $s$. Moreover, $A_1(s)\backslash C_b(s) = W_\infty \bigcup W_0$ where $W_\infty$ is a wide open annulus connected to $X_{10}$ of width $b$.

Let $Z_2(s) = \text{red}^{-1}(Z(s)) - \bigcup_{i+j=2} X_i^0$ be the underlying affinoid of $\text{red}^{-1}Z(s)$.

**Theorem 3.1.** The $\mathbb{C}_p$-valued points of $Z_2(s)$ correspond to pairs $(E, C)$ where $E$ is a nearly too supersingular elliptic curve and $C$ is a cyclic subgroup of order $p^2$ and $pC$ is its canonical subgroup or equivalently $Z_2(s)$ is the inverse image under the forgetful map to $X_0(p)$ of $C_{1/(p+1)}(s)$.

This will be proven below.

**Proposition 3.2.** Suppose $f: W \to V$ is a finite map of basic wide opens and $V$ is not a disk. Then if $X$ is a minimal underlying affinoid of $W$, $Z := f(X)$ is a minimal underlying affinoid of $V$ and $X = f^{-1}(Z)$.

**Remark.** A basic wide open which is neither a disk nor an annulus has a unique minimal underlying affinoid. The minimal underlying affinoids in a wide open annulus are the concentric circles. The image of a disk or an annulus under a finite map is a disk or an annulus.

**Proof.** If $V$ is not an annulus let $Y$ be its minimal underlying affinoid. If $V$ is an annulus, the image of $X$ is an affinoid with irreducible reduction so must be contained in a concentric circle of $V$ which we will call $Y$ in this case. Let $E$ be a connected component of $W \backslash X$. Then, because $f$ is finite, $U := f(E)$ is a disk or annulus containing an annulus at an end of $V$ so in the corresponding connected component $D$ of $V \backslash Y$. Claim, it must be contained in $D$. First, suppose $V$ is not an annulus. Then by [C-RLC] (see also, [BL]) $U$ must be contained in $Y$ or in a component of $V \backslash Y$. Since the former is precluded, we have our claim, in this case. If $V$ is an annulus, the claim follows from the fact that if $h$ is a unit on $A(r, 1)$, $h = cT^n$, where $c \in K^*$, $n \in \mathbb{Z}$ and $|g - 1| < 1$ so, in particular if $n \neq 0$, determines a finite map onto $A(|c|T^n, |c|)$. This implies either $U \cap Y$ equals $Y$ or $\emptyset$. Suppose $Y \subset U$. Then there is a proper concentric
wide open subannulus $A$ of $E$ such that $f^{-1}(D \cup Y) \subset A$. It follows that $U - f(A)$ and $Y$ are disconnected but this is impossible since $E - A$ and $X$ are connected and so $f(E) \subseteq D$.

We conclude $f^{-1}Y = X$ and finiteness implies $Y = f(X) = Z$ which concludes the proof. □

It follows that $\pi_f(Z_2(s))$ is a circle.

One thing we may conclude from Lemma 2.1 is that in the situation of Proposition 3.2, $W \setminus X$ maps finitely onto $V \setminus Y$.

There exists a semi-stable model $X_1$ of $X_0(p)$ such that $\pi_f$ extends to a finite morphism $X_2 \to X_1$ (cf. [C-SM]). This amounts to adding components corresponding to the circles $\pi_f(Z_2(s))$, $s \in SS$. Let $G_0(p^n)$ be the oriented graph of the reduction of $X_n$ (one vertex for every irreducible component and one edge for every singular point).

The component $X_{ij}^\pm$ of $\overline{X}_n$ is the irreducible component of the reduction of $X_n$ containing the reduction of $X_{ij}^\pm$.

For every supersingular point $s$ on the reduction of $X(1)$ we have one component $Z_n(s)$ of the reduction of $X_n$ lying over it. Let

$$Z_1(s) = \text{red}^{-1}(Z_1(s) - (X_{10} \cup X_{01})) = \pi_f(Z_2(s))$$

be the underlying circle in $A_1(s) = \text{red}^{-1}Z_1(s)$ and let $A_{ij}^\pm(s)$ be the annulus which is the reduction inverse of the intersection of $Z_n(s)$ and $X_{ij}^\pm$. This annulus has a natural orientation corresponding to the ordered pair $(X_{ij}^\pm, Z_n(s))$. We also put

$$A_{ij}^\pm = \bigcup_s A_{ij}^\pm(s).$$

and $A_{11}(s) = A_{11}(s)^+ \cup A_{11}(s)^-$. Theorem 3.1 is equivalent to the assertion that $Z_1(s)$ is the nearly too supersingular circle $C_{1/(p+1)}$, which we will now prove.

Note that $\pi_f$ has degree $p$ and

$$\pi_f^{-1}A_1(s) \setminus Z_2(s) = A_{20}(s) \cup A_{11}^+(s) \cup A_{11}^-(s) \cup A_{02}(s)$$

**Lemma 3.3.** $w(A_{20}(s)) = w(A_{02}(s))$, and

$$w(A_{10}(s)) + w(A_{01}(s)) = 1.$$

**Proof.** As the Atkin–Lehner involution $\tau$ acts on $X_2$ as $X_2$ is canonical and $\tau(X_{20}) = X_{02}$ we must have $\tau(A_{20}(s)) = A_{02}(s)$ so $w(A_{20}(s)) = w(A_{02}(s))$. The last assertion
follows from the fact that the annulus $A_1(s)$ has width 1 and is the disjoint union of the annuli $A_{10}(s)$ and $A_{01}(s)$ and the circle $Z_1(s)$. □

**Lemma 3.4.** The maps induced by $\pi_f: A_{20}(s) \to A_{10}(s), A_{11}^+(s) \to A_{10}(s), A_{11}^-(s) \to A_{10}(s), A_{11}^+(s) \to A_{10}(s)$ have degrees 1, $(p - 1)/2$, $(p - 1)/2$ and $p$ respectively.

**Proof.** This follows from Lemma 2.3 and the fact that the morphisms $X_{20} \to X_{10}, X_{11}^+ \to X_{10}$ and $X_{20} \to X_{01}$ have, by Lemma 1.3, degrees 1, $(p - 1)/2$ and $p$ respectively. □

**Lemma 3.5.** $w(A_{10}(s)) = w(A_{20}(s))$ and $w(A_{01}(s)) = pw(A_{02}(s))$.

**Proof.** This follows from the previous lemma and Lemma 2.1. □

Thus $w(A_{10}(s)) = 1/(p + 1)$ so $Z_1(s) = C_{1/(p+1)}(s)$. This concludes the proof of Theorem 3.1.

Let $W_{20} = \text{red}^{-1}(X_{20})$ etc.

**Corollary 3.5.1.** The pair $(E, C)$ corresponds to point in $W =: W_{20} \cup W_{11}^- \cup W_{11}^+$ if and only if $E$ has supersingular reduction, $K_2(E)$ exists and $pC = K_1(E)$. It corresponds to point in $W_{20}$ if and only if $C = K_2(E)$, to a point in $W_{11}^-$ if and only if $C \neq K_2(E)$ and the induced pairing $K_1(E) \times K_1(E) \to \mu_p$ is of type $e$.

**Proof.** The first sentence of the corollary is clear since we know $\pi_f^{-1}W_{10} = W$. Next, there is a section $s$ of $\pi_f: W_{20} \to W_{10}$, since if $(E, D)$ corresponds to a point in $W_{10}$, $K_2(E)$ exists and $K_1(E) = D$. Then $s(P)$ will correspond to $(E, K_2(E))$. This is a section because it is when restricted to $X_{10}$ and $W_{20}$ is irreducible. This establishes the corollary for $W_{20}$. Now suppose $(E, C)$ corresponds to a point $P$ in $W_{11}^-$. Then $C \neq K_2(E)$, so we get a pairing $P_P$ on $K_1(E)$ onto $\mu_p$. If $c$ generates $K_1(E)$ and $P_P(c,c) = \zeta^a$, $\zeta(a)$ depends only on $P$ and gives an analytic function on $W_{11}^-$. It must be constant since $W_{11}^-$ is connected. This concludes the proof. □

**Appendix A. Affinoids in curves**

We prove some well-known results about curves for which we don’t know a good reference.

Suppose $K$ is a complete subfield of $\mathbb{C}_p$ with ring of integers $R$. Suppose $C$ is a smooth proper curve over $K$ and $\mathcal{C}$ is a model of $C$ over $R$. If $V$ is a subscheme of the reduction $\overline{C}$ of $C$, let $X(C, V)$ denote generic fiber of the formal completion of $C$ along $V$. If $V$ is a reduced open affine, $X(C, V)$ is an affinoid subdomain of $C$ with reduction $V$. Now suppose $C$ is semi-stable and $S$ is a subset of the set of components $T := T_C$ of $\overline{C}$. Let $Y_S = \bigcup_{Z \in S} Z$, $Y_S^\circ = Y_S \setminus Y_T \setminus S$ and let $X(C, S) = X(C, Y_S^\circ)$. This is an affinoid subdomain if $S \neq T$ of $C$ because if $C_S$ is the blow down of $C$ along $\bigcup_{Z \in S} Z \subseteq Y_S^\circ$ then the image $Y'_S$ of $Y_S^\circ$ in $C_S$ is a reduced open affine in $\overline{C}_S$ and

$$X(C, S) = X(C_S, Y'_S).$$

Of course, $X(C, T) = C$. Also set $S^\infty = Y_S \setminus Y_S^\circ$. 

If $f: \mathcal{T} \to \mathcal{C}$ is a morphism of semi-stable models of $\mathcal{C}$, and $S_{\mathcal{T}} = \{ Z \in T_{\mathcal{T}} : \tilde{f}(Y^\alpha_Z) \subseteq Y^\alpha_S \}$

$$X(\mathcal{T}, S_{\mathcal{T}}) = X(\mathcal{C}, S).$$  \hfill (A.1)

Also, if $E \subseteq \mathcal{C}_p$ is a complete extension of $K$,

$$X(\mathcal{C}_E, S_{\mathcal{R}_E}) = X(\mathcal{C}, S)_E.$$  

If $U$ is another subset of $T$,

$$X(\mathcal{C}, S) \cap X(\mathcal{C}, U) = X(\mathcal{C}, S \cap U) \quad \hfill (A.2)$$

and if $S^\infty \cap U^\infty \subseteq (S \cap U)^\infty$

$$X(\mathcal{C}, S) \cup X(\mathcal{C}, U) = X(\mathcal{C}, S \cup U). \quad \hfill (A.3)$$

If $f: \mathcal{T} \to \mathcal{C}$ is a morphism of semi-stable models of $\mathcal{C}$ and $\tilde{f}^{-1} x \in T_{\mathcal{T}}$ for $x \in S^\infty \cap U^\infty$, $S^\infty_{\mathcal{T}} \cap U^\infty_{\mathcal{T}} \subseteq (S_{\mathcal{T}} \cap U_{\mathcal{T}})^\infty$ so

$$X(\mathcal{C}, S) \cup X(\mathcal{C}, U) = X(\mathcal{T}, S_{\mathcal{T}} \cup U_{\mathcal{T}}). \quad \hfill (A.4)$$

If $\{ W_Z : Z \in T \}$ is the semi-stable covering of $\mathcal{C}$ corresponding to $\mathcal{C}$,

$$X(\mathcal{C}, S) = \bigcup_{Z \in S} W_Z \setminus \bigcup_{Z \in T \setminus S} W_Z.$$  

**Theorem A.1.** If $X$ is an affinoid subdomain of $\mathcal{C}$ and $S$ is a semi-stable model of $\mathcal{C}$ over $R_K$, then there exists a finite extension $E$ of $K$ and a semi-stable model $\mathcal{T}$ of $\mathcal{C}_E$ over $R_E$ mapping to $S_{\mathcal{R}_E}$ and a subset $S$ of $T_{\mathcal{T}}$ so that $X_E = X(\mathcal{C}, S)$.

We first extend scalars to $\mathcal{C}_p$ (one can descend later). We will prove the translation of this theorem into the language of semi-stable coverings. That is, we will regard $S$ as a semi-stable covering and find an appropriate semi-stable refinement $\mathcal{T}$ of $S$. We let $\overline{S}$ denote the reduction of the corresponding semi-stable model. We may and will suppose $X$ is irreducible (equivalently, connected).

By a residue class $U$ of $S$, we mean the subspace of $\mathcal{C}$ corresponding to a point $P$ of $\overline{S}$ and we let $\overline{U} = P$. The space $U$ is a wide open disk or wide open annulus according as $P$ is smooth or singular and we call it either a residue disk or residue annulus. if $W \in S$, let $Z_W$ denote the underlying affinoid in $W$.

If $R \subseteq S$ are rigid spaces let $\mathcal{C}(R, S)$ denote the subspace of $S$ connected to $R$. If $f$ is function on $S$ and $R$ is an affinoid $\| f \|_R$ will denote the sup-norm of the restriction of $f$ to $R$. By a circle we mean an affinoid isomorphic to $\mathcal{C}_p(\mathcal{T}, T^{-1})$, i.e., an
annulus conformal to $A[1,1]$. We call a subannulus $U$ of an annulus $A$ concentric if the connected components of $A\setminus U$ are annuli.

**Lemma A.2.** If $U$ is a wide open disk in $C$, there exists a function $z$ on $C$ with a single pole such that $U = \{x \in C : |z(x)| > 1\}$.

**Proof.** Claim: We can find a semi-stable covering $S$ of $C$ so that $U$ is contained in a residue disk $D$ of $S$. This is clear when $C$ has a model with good reduction. Otherwise, there exists a semi-stable covering $T$ of $C$ such that no element of $T$ is a disk (e.g., the covering corresponding to the stable model if the genus of $C$ is at least 2 and this model has at least two components). Then $U$ must be contained in a residue class of $T$. If it is contained in a residue annulus $A$, it must be contained in a concentric circle $Z$ in $A$ (see [C-RLC, Lemma 3.2]). We can then take $S$ to be

$$\{A\} \cup \{ \text{CC}(W\setminus A, W\setminus Z) : W \in T \}.$$  

It follows using a blowing down argument as above, applied to $S$ that $Y := C\setminus D$ is an affinoid.

Now, let $P \in U$ and suppose $f$ is a function on $C$ with a pole only at $P$. Then

$$D = \{x \in C : |f(x)| > ||f||_Y\}$$

and there exists $r \in |C_p|$, $U = \{x \in D : |f(x)| > r\}$. Suppose $a \in C_p$, $|a| = r$. Take $z = f/a$. □

We will say such a $z$ determines $U$.

**Lemma A.3.** If $D$ is a collection of disjoint wide open disks $D$ in $C$ such that $D \cap X \neq \emptyset$ and $D \setminus X \neq \emptyset$, then $D$ is finite.

**Proof.** For $U \in D$, let $z_U$ be a function on $C$ which determines $U$. Suppose $a_U \in C_p$, $|a_U| = ||z_U||_X > 1$. Let $f_U = (z_U/a_U)|_X$. Then $\overline{f}_U \neq 0$ and $\overline{f}_U \overline{f}_V = 0$ if $V \neq U$ in $D$. Since $\overline{X}$ is reduced of finite type over $\overline{F}_p$, this implies $D$ is finite. □

**Lemma A.4.** If $Z$ is an underlying affinoid of $S$, then either (i) $X \cap Z = \emptyset$, (ii) $X$ is contained in a residue class of $Z$ or (iii) $X \cap Z$ contains all but finitely many residue classes of $Z$.

**Proof.** Suppose neither (i) nor (ii) is true. Let $D$ be a residue disk in $Z$ and $z$ a function on $C$ which determines $D$. Then since $X \setminus D = \{x \in X : |z(x)| \leq 1\}$, $Y := X \setminus D$ is an affinoid. Also $V := C \setminus D$ is an affinoid whose reduction is the blow down of $\overline{S}$ along $T_S \setminus \overline{Z}$ minus $\overline{D}$. Since neither conditions (i) nor (ii) hold, the image of $\overline{Y}$ in $\overline{V}$ is not a point. Since $X$ is connected, $X \cap Z$ cannot be contained in a finite number
greater than one of residue classes, so the image of $Y$ must be a non-empty Zariski open of $V$. The lemma now follows from Lemma A.3. \qed

**Proposition A.5.** If $U$ is a residue class of $S$ and $X \not\subset U$, $U \setminus X = \bigcup (B \cup A)$ where $B$ is a finite set of wide open disks and $A$ is empty if $U$ is a disk, and is either empty or a concentric wide open annulus in $U$ if $U$ is an annulus.

**Proof.** Suppose $D$ is a residue disk of $S$ and neither $D \setminus X$ nor $(D \cap X)$ is empty. Suppose $W$ is the element of $S$ which contains $D$. Let $z$ be a function on $C$ which determines $D$. For each, $Q \in D \setminus X$, let $w_Q = z/(1 - z/z(Q))$ and suppose $a_Q \in C_p$ such that $|a_Q| = r_Q := |w_Q|_X$. Let $B_Q = \{x \in C : |w_Q(x)| > r_Q\}$ which is contained in $D$. Then $B_Q$ is a wide open disk determined by $z_Q := w_Q/a_Q$. $z_Q |_X \in A^0(X)$. We want to prove the collection of disks, $S := \{B_Q : Q \in D \setminus X\}$, is finite.

Suppose $P \in D \setminus X$. Let $A_r := \{A(P, r) = \{x \in D : |z_P(a)| = r\}$. Since $X$ is connected and $X \not\subset D$, $X \cap A_r \neq \emptyset$ and $X \not\subset A_r$ for all $R := r_P \geq r > 1$. Let $D_r = \{x \in D : |z_P(a)| \geq r\}$ and

$$S_r = \{D\} \cup \{W \setminus D_r\} \cup S \setminus \{W\}.$$ 

Then $S_r$ is a semi-stable covering of $C$ and $D_r$ is an underlying affinoid of $S_r$. It follows that $X$ contains all but finitely many residue classes of $B_r$ and hence of $A_r$ for $1 < r \leq r_P$.

Suppose $r_P \geq r_1 > \cdots > r_n > \cdots > 1 \in |C_p|$ and $\{P_i\}$ is a sequence of points such that $P_i \in A_r \setminus X$. Then $|z_{P_i}|_X \neq 0$ and $z_{P_i} |_X \cdot z_{P_j} |_X = 0$, if $i \neq j$. Thus $X$ must contain all but finitely many of the circles $A(P, r)$, $1 > r \geq r_P$ and, by Lemma A.4, in each circle it contains all but finitely many residue disks.

It follows that if $S$ is infinite there exists a sequence of points, $\{Q_i\}$, in $D \setminus X$ and a sequence of $s_j \in |C_p|$ such that $1 > s_{j+1} \geq r_{Q_j}$, $Q_{j+1} \in A(Q_j, s_{j+1})$, $s_{j+1} > |z_{Q_j}(Q_{j+1}) - z_{Q_j}(Q_{j+2})|$ (i.e., all the $Q_i$, $i > j$, lie in the same residue disk of a circle around $B_{Q_j}$). Let $f_i = (z_{Q_i} - z_{Q_j}(Q_{i+1}))$. Then $f_i |_X \in A^0(X)$, $(f_i |_X) \neq 0$ and $(f_j |_X) = 0$, if $i \neq j$. Again, this contradicts the finite typeness of $X$ and establishes the proposition when $U$ is a disk.

**Lemma A.6.** Suppose $A$ is a residue annulus of $S$. Then if $T : A \cong A(R, S)$, $R, S \in |C_p|$, is parameter and $X \not\subset A$, $(R, S) \setminus \{|T(x)| : x \in A \cap X\} = (r, s)$ for some $r, s \in |C_p|$, $R \leq r \leq s \leq S$.

**Proof.** First suppose $\exists W_1, W_2 \in S$, $W_1 \neq W_2$, such that $A$ is a component of $W_1 \cap W_2$. Let $D_i$ be a residue disks in $W_i$ for $i = 1$ or 2. Suppose $z_i$ is a function on $C$ which determines $D_i$. Then $M := C \setminus (D_1 \cup D_2)$ and $X' = X \setminus (D_1 \cup D_2)$ are affinoids. Moreover, $X \cap A = X' \cap A$ and $\overline{M}$ equals the blow down of $S$ along the components which don’t correspond to the $W_i$. In fact, $\overline{M}$ has two components, $Z_1$ and $Z_2$, which correspond to $W_1$ and $W_2$, $z_1 \in A^0(M)$, $(z_1 |_{\overline{M}})$ is not constant but is on $Z_2$. We can and will suppose it vanishes on $Z_2$. 


Let \( g \) be a function on \( \overline{M} \) which vanishes at and only at the singular points of \( \overline{M} \) apart from the one corresponding to \( A \). (Such a function exists because \( \overline{M} \) is an affine curve over a finite field.) Let \( \tilde{g} \in A^0(M) \) be a lifting of \( g \). Let \( N = \{ x \in M : |\tilde{g}(x)| = 1 \} \) and \( Y = \{ x \in X' : |\tilde{g}(x)| = 1 \} \). Then \( N \) and \( Y \) are affinoids, \( A \subset N \), \( N \setminus A \) has two connected components and \( Y \cap A = X \cap A \).

Suppose the restriction of the divisor \( (z_1) \) of \( z_1 \) to \( A \cup Z_{W_1} \setminus D_1 \) is the effective divisor \( D \). There exists a positive integer \( n \) and a function \( f \) on \( C \) such that

\[
(f) = -nD + E,
\]

where \( E \) is an effective divisor supported on \( C \setminus W_1 \). (This is because the points on the Jacobian \( J \) of \( C \) represented by divisors supported on a non-empty open subset of \( C(C_p) \) form an open subgroup of \( J(C_p) \) and such open subgroups have torsion quotients.) We can also suppose \( ||f||_{Z_{W_1}} = 1 \). Let \( h = z_1^nf \). Then \( h \) has poles only at the pole of \( z_1 \), \( ||h||_M = 1 \), \( \overline{h(M)}(A) = 0 \) and \( h \) doesn’t vanish on \( A \cup Z_{W_1} \). It follows that if \( T \) is a parameter on \( A \), \( ||T||_A = 1 \), \( |T(x)| \to 1 \) as \( x \to Z_{W_1} \) \( h = T^n u \), \( n > 0 \).

Suppose \( |T(A)| = (R, 1) \). Then since \( X \) is connected, \( X \cap A \) has at most two components and if \( |T(A \cap X)| \neq (R, 1) \) (which well now suppose), \( Y \) has two components \( Y_1 \) and \( Y_2 \) such that \( Y_i \cap Z_{W_j} \neq \emptyset \) if and only if \( i = j \). Then

\[
|T(A \cap X)| = (R, 1)(r, s),
\]

where \( r = ||h||_{Y_2}^{1/n} \) and \( 1/s = ||1/h||_{Y_1}^{1/n} \).

Now suppose \( A \subset Z_W \). Suppose \( Q \in |C_p| \) and \( R < Q < S \). Let \( Z_Q = \{ x \in u : |T(x)| = Q \} \). Then \( Z_Q \) is a concentric circle of \( A \) and let

\[
S_Q := \{ W \setminus Z_Q \} \cup \{ A \} \cup S \setminus \{ W \}.
\]

Then applying what we just proved to the two components of \( A \setminus Z_Q \) which are residue annuli of \( S_Q \) and components of \( A \cap W \setminus Z_Q \), we see

\[
S := (R, S) \setminus \{ |T(x)| : x \in A \cap X \} = (r_1, s_1) \cup (r_2, s_2) \cup N,
\]

where \( R \leq r_1 \leq s_1 < Q \leq r_2 \leq s_2 < S \) and \( N \) is either empty or \( \{ Q \} \). Suppose \( R < Q' < Q \). We also see

\[
S = (r_1', s_1') \cup (r_2', s_2') \cup N',
\]

where \( R \leq r_1' \leq s_1' < Q' \leq r_2' \leq s_2' < S \) and \( N' \) is either empty or \( \{ Q' \} \). The only possibilities consistent with these two statements are \( S = (r_1, s_2) \), \( (r_2, s_2) \) and \( S = (r_1, s_1) \). \( \square \)
It follows by an argument similar to that used in the analysis of $D \setminus X$ above that $A \setminus X = T^{-1}(r, s) \cup \bigcup B$ where $B$ is a finite collection of wide open disks. This completes the proof of the proposition. □

Now we complete the proof of the theorem. We will make several refinements of $S$.

First we make sure $X$ is not contained in any residue class. Suppose $X$ is contained in a residue class $U$ of $S$. Let $B$ be a closed disk contained in $X$. Let $Y = B$ if $U$ is a disk and the concentric circle in $U$ containing $B$ if $U$ is an annulus. Let $V = U$ if $U$ is a disk and the residue disk of $Y$ containing $B$ otherwise. Let $W'$ be $\text{CC}(W \setminus U, W \setminus Y)$. Let $S'$ equal

$$
\begin{cases}
\{W': W \in S\} \cup \{U\} & \text{if } U \text{ is a disk and} \\
\{W': W \in S\} \cup \{U \setminus B\} \cup \{V\} & \text{otherwise}.
\end{cases}
$$

Clearly, $S'$ is semi-stable and $X$ is contained in no residue class of $S'$.

We will next find a semi-stable refinement $S''$ of $S'$ so that if $A \cap X \neq \emptyset$, $A \subset X$ for each residue annulus $A$ of $S''$. Let $A$ be the set of residue annuli of $S'$. For each $A \in A$ let $C_A$ denote the collection of concentric circles $Z$ in $A$ such that neither $X \cap Z$ nor $Z \setminus X$ is empty. We know from Proposition A.5 that $C_A$ is finite.

Let $S''$ be

$$
\{ \text{CC}(Z_W, W \setminus \bigcup Z'_W) : W \in S' \} \cup \bigcup_{A \in A} \{ \text{CC}(Z, A \setminus \bigcup Y) : Z \in C_A \}.
$$

This is a semi-stable covering with the required properties. Indeed, the spaces in the collection on the left are elements of $S$ with annuli cut out off the residue annuli and the spaces in the collection on the right are annuli which fill in the gaps.

Now we will make a refinement $S'''$ of $S''$ so that if $U \cap X \neq \emptyset$, $U \subset X$ for any residue class $U$ of $S$. For $W \in S''$, let $B_W(X)$ denote the set of residue classes of $S''$ in $W$ so that $X \cap D \neq \emptyset$ and $D \setminus X \neq \emptyset$. It follows from the construction of $S''$ that the elements of $B_W(X)$ are disks and from Lemma A.4 that $B_W(X)$ is finite for each $W \in S''$. For $D \in B_W(X)$, let $D_D$ be the set of connected components of $D \setminus X$. It follows from Proposition A.5 that $D_D$ is a finite collection of wide open disks.

Now suppose $D \in B_W(X)$. For $S \subseteq D_D$, $S \neq \emptyset$, let $B(S)$ denote the smallest closed disk in $D$ containing $\bigcup_{E \in S} E$ and $U(S)$ the largest wide open disk containing $B(S)$ disjoint from $\bigcup_{E \in D \setminus S} E$, if it exists and the empty set if it doesn’t. Let

$$
W_S = U(S) \setminus \bigcup_{T \subseteq S \setminus B(T) \neq B(S)} B(T).
$$

Let $S'''$ be

$$
\{W \setminus \bigcup_{D \in B_W(X)} B(D) : W \in S''\} \cup \{W_S : S \subseteq D_D, W_S \neq \emptyset\}.
$$
This is a semi-stable covering because
\[ D = \bigcup_{S \in \mathcal{D}_D} W_S \]
and if \( T \) and \( S \) are non-empty subsets of \( \mathcal{D}_D \), \( W_S \neq W_T \), \( W_S \cap W_T \neq \emptyset \), either \( W_S \cap W_T \) equals \( U(T) \setminus B(T) \) or \( U(S) \setminus B(S) \), so is an annulus.

Finally, \( \mathcal{T} \) will be a refinement \( S''' \) so that if \( Z \) is a residue annulus or underlying affinoid of \( \mathcal{T} \) and \( X \cap Z \neq \emptyset \), \( Z \subseteq X \). Let \( \mathcal{B} \) be the set of residue disks \( D \) of \( S \) such that \( D \cap X = \emptyset \) but \( Z \cap X \neq \emptyset \) where \( Z \) is the underlying affinoid of \( S''' \) containing \( D \). For each \( D \in \mathcal{B} \), let \( B(D) \) be a closed disk in \( D \). We take \( \mathcal{T} \) to be
\[ \{W \setminus \bigcup_{D \in \mathcal{B}} B(D): W \in S''' \} \cup \mathcal{B}. \]

The point is the residue annuli of \( \mathcal{T} \) are the residue annuli of \( S''' \) and \( \{D \setminus B(D): D \in \mathcal{B}\} \), and the underlying affinoid with respect to \( \mathcal{T} \) of \( W \setminus \bigcup_{D \in \mathcal{B}} B(D) \) is \( Z_W \cap X \) which is an affinoid whose reduction is a Zariski open in \( \overline{Z}_W \).

Then \( X = X(\mathcal{T}, S) \), where \( S \) is the set of components of \( \overline{\mathcal{T}} \) corresponding to the set of \( W \in \mathcal{T} \) such that \( W \cap X \neq \emptyset \). \( \Box \)

**Corollary A.7.** Suppose \( F \) is a complete subfield of \( \mathbb{C}_p \) and \( C \) is a smooth proper curve defined over \( F \). If \( X \) and \( Y \) are affinoid subdomains of \( C \), \( X \cap Y \) is an affinoid subdomain and \( X \cup Y \) either equals \( C \) or is an affinoid subdomain.

**Proof.** We know \( C \) has a semi-stable model over \( \mathcal{R}_K \) where \( K \) is a finite extension of \( F \). By Theorem A.1, there exists a finite extension \( E \) of \( K \), a semi-stable model \( \mathcal{T} \) of \( C \) over \( \mathcal{R}_E \) and a subset \( S \) of \( \mathcal{T}_E \) so that \( X_E = X(C, S) \). Also by this theorem there exists a finite extension \( L \) of \( E \), a semi-stable model \( \mathcal{R} \) of \( C \) over \( \mathcal{R}_L \) mapping to \( S \) and a subset \( U \) of \( \mathcal{T}_E \) so that \( Y_L = X(\mathcal{R}, U) \). It now follows from (1) and (2) that \( Z = X_L \cap Y_L \) is an affinoid subdomain of \( C_L \).

We can assume \( L \) is a Galois extension of \( F \) with Galois group \( G \). let \( \{f_\sigma: \sigma \in G\} \) and \( \{g_\sigma: \sigma \in G\} \) be the natural descent data for \( X_L \) and \( Y_L \). That is, \( f_\sigma: X_L^\sigma \to X_L \) and \( g_\sigma: Y_L^\sigma \to Y_L \) are isomorphisms such that
\[ f_{\sigma t} = f_t \circ f_\sigma^T \quad \text{and} \quad g_{\sigma t} = g_t \circ g_\sigma^T. \]

Now, if \( t_X: X \to C \) and \( t_Y: Y \to C \) are the natural inclusions, \( t_X^\sigma = t_X \circ f_\sigma \) and \( t_Y^\sigma = t_Y \circ g_\sigma \). Also, if \( N \) is an extension of \( L \),
\[ Z(N) = \{(x, y) \in X(N) \times Y(N): t_X(x) = t_Y(y)\} \]
and
\[ Z^\sigma(N) = \{(x, y) \in X^\sigma(N) \times Y^\sigma(N): t_X^\sigma(x) = t_Y^\sigma(y)\}. \]
It follows that if $(x, y) \in Z^\sigma(N)$, $(f_\sigma(x), g_\sigma(y)) \in Z(N)$. Thus $\{(f_\sigma \times g_\sigma)|_{Z^\sigma}: \sigma \in G\}$ is descent data on $Z$. Since $Z$ is an affinoid, it descends to an affinoid over $F$ which represents $X \cap Y$.

The second part of the corollary follows similarly. □

References