On the number of irreducible modular representations of a $P$ and $Q$ polynomial scheme

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Abstract

Whereas results about the number of irreducible modular representations of finite groups are known, those of Bose–Mesner algebras of association schemes are not well understood. In this paper, we consider the number of irreducible modular representations of the adjacency algebra of some $P$ and $Q$ polynomial association schemes. These results should be helpful for investigating the structure of association schemes.

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1. Introduction

The adjacency algebra of an association scheme over a field with positive characteristic is called a modular adjacency algebra. A representation of the algebra is called a modular representation [4]. We want to describe the structure of modular adjacency algebras. In order to do this, it is important to know the number of irreducible modular representations. But the modular representation theory of the algebra is not well understood.

In this paper, we examine some classical $P$ and $Q$ polynomial association schemes which have nice properties. In particular, we discuss the following list [1] of infinite families of $P$ and $Q$ association schemes:
(i) the Johnson scheme $J(v, n)$;
(ii) the Hamming scheme $H(n, q)$;
(iii) the $q$-Johnson scheme $J_q(v, n)$;
(iv) the association scheme of dual polar spaces;
(v) the association scheme of bilinear forms $\text{Bilin}(n, m, q)$;
(vi) the association scheme of hermitian forms $\text{Herm}(n, q)$;
(vii) the association scheme of alternating bilinear forms $\text{Alt}(n, q)$ and;
(viii) the association scheme of quadratic forms $\text{Quad}(n, q)$.

We will determine the number of irreducible modular representations of the above association schemes using the following theorems. (The number of irreducible modular representations of $\mathbb{F}_p X$ is denoted by $k(X)$.)

**Theorem 1.1.** Let $X$ be the $q$-analog of the Johnson scheme. Then

$$k(X) = \left| \left\{ i \in \{0, \ldots, n\} : p \nmid \left( \frac{v - 2i}{v - n - i} \right) \right. \left. \left( q^{i(n-i)} \right) \right\} \right|.$$

**Theorem 1.2.** Let $X$ be the Johnson scheme. Then

$$k(X) = \left| \left\{ i \in \{0, \ldots, n\} : p \nmid \left( \frac{v - 2i}{v - n - i} \right) \right\} \right|.$$

**Theorem 1.3.** Let $X$ be the association scheme of a dual polar space. Then

$$k(X) = \left| \left\{ i \in \{0, \ldots, n\} : p \nmid q^{i(n-i)} \prod_{l=0}^{n-i-1} (1 + q^{e+l-i}) \right\} \right|,$$

where $e$ is 1, 1, 0, 2, $\frac{3}{2}$, $\frac{1}{2}$ in the respective cases $[C_n(q)]$, $[B_n(q)]$, $[D_n(q)]$, $[2D_{n+1}(q)]$, $[2A_{2n}(r)]$, $[2A_{2n-1}(r)]$ and $q = r^2$ for $[2A_{2n}(r)]$, $[2A_{2n-1}(r)]$.

The following theorem deals with all the rest of the list. The result is essentially due to Hanaki, referring to Section 3.

**Theorem 1.4.** Let $X = (X, \{R_i\}_{i=0}^n)$ be the Hamming scheme, the association scheme of bilinear forms, hermitian forms, alternating bilinear forms or quadratic forms. If a prime $p$ divides $|X|$ then $k(X) = 1$; otherwise $k(X) = n + 1$.

In the proofs of Theorems 1.1–1.3, regular semilattice [3] structure associated with these schemes are used. In Section 4, we will give the proofs. This enables the reader to visualise the integral structures of the underlying algebras. Since schemes (i)–(v) are associated with regular semilattices, they are dealt with in a uniform manner.

Yoshikawa [10] described the structure of the Hamming scheme $H(n, q)$ and the number of irreducible modular representations of the Johnson scheme $J(v, n)$ is known [7]. These results are in agreement with the results we are presenting here.
2. Preliminaries

In this section, we assume $X$ to be an association scheme of the list. Let $X = (X, \{R_i\}_{i=0,...,n})$ be an association scheme, and $\{D_i\}_{i=0,...,n}$ its adjacency matrices. Put $n_i$ as the valency of $D_i$ and $m_i$ the multiplicities of the scheme $[1,2]$. Let us denote the adjacency algebra of $X$ over a field $L$ by $LX$. We consider the first eigenmatrix $P$ of the association scheme, in which each entry is an integer. This means that $QX$ is a splitting algebra. So $(Q_p, Z_p, F_p)$ is a splitting $p$-modular system for the adjacency algebra. Let $(\tilde{p})$ be the maximal ideal of $Z_p$. We denote the image of the natural epimorphism $Z_p \rightarrow F_p$ by $\ast$. For a $p$-modular system the reader is referred to [6].

Since $Z_pX/\tilde{p}Z_pX \cong F_pX$ and $\tilde{p}Z_pX \subset J(Z_pX)$, idempotents of $F_pX$ are liftable to idempotents of $Z_pX$. In $F_pX$, 1 is expressed as the summation of central primitive idempotents:

$$1 = f_0 + \cdots + f_s.$$  

Accordingly, the central primitive idempotents decomposition of 1 in $Z_pX$ is

$$1 = e_{B_0} + \cdots + e_{B_s},$$

where $e_{B_i} = f_i$.

Writing $B_i = e_{B_i}Z_pX$ and $B_i^* = e_{B_i}^*F_pX$, we obtain the indecomposable decomposition of $Z_pX$ into two-sided ideals:

$$Z_pX = B_0 \oplus \cdots \oplus B_s.$$  

The corresponding indecomposable decomposition of $F_pX$ into two-sided ideals is

$$F_pX = B_0^* \oplus \cdots \oplus B_s^*.$$  

We call each $B_i$ a block. The number of blocks is equal to the number of irreducible representations of $F_pX$.

3. Self-dual association schemes

First of all, we consider self-dual association schemes.

**Proposition 3.1** (Hanaki [4]). Let $(X, \{R_i\}_{i=0,...,n})$ be an association scheme, and $F$ a field of characteristic $p > 0$. Then $FG$ is semisimple if and only if the Frame number $\mathcal{F}(G)$ is co-prime with $p$.

The Frame number is given as

$$\mathcal{F}(G) = |X|^{n+1} \prod_{i=0}^{n} \frac{n_i}{m_i}.$$
for a commutative association scheme of class $n$. In particular, for a self-dual association scheme, $n_i = m_i$ for all $i \in \{0, \ldots, n\}$, in which case $\mathcal{F}(G) = |X|^{n+1}$. Schemes (v)–(viii) are self-dual and $|X|$ is a prime power [1]; scheme (ii) is self-dual and $|X|$ has a positive integer power. So if $p \nmid q$, then $F\mathfrak{X}$ is semisimple. This means $k(\mathfrak{X}) = n + 1$.

Next we consider the case $p \mid q$.

**Proposition 3.2** (Hanaki [5]). If $|X| = p^r$, then $F\mathfrak{X}$ is a local algebra.

For the above schemes except for the Hamming scheme $H(n, q)$, if $p \mid q$ then $F\mathfrak{X}$ is a local algebra by Proposition 3.2. This means $k(\mathfrak{X}) = 1$. For (ii) with $p \mid q$, $F_p H(n, q) \cong F_p H(n, p)(D_i \mapsto D_i) : F_p$-algebra isomorphism.

Hence, $F_p H(n, q)$ is a local algebra.

4. Regular semilattice

In this section, we consider (i), (iii) and (iv). They are not self-dual. However, they are accompanied by a regular semilattice [2,3,8,9]. We can compute $k(\mathfrak{X})$ using parameters of these posets. We use the same parameters of regular semilattices as in [3].

**Theorem 4.1.** Let $\mathfrak{X}$ be an association scheme associated with a regular semilattice. If the number of classes of $\mathfrak{X}$ and the maximum value of the length function are equal, then

$$k(\mathfrak{X}) = |\{i \in \{0, \ldots, n\} : p \nmid \rho(i, i)\}|,$$

where $\rho(i, i) = \sum_{j=0}^{n} v(i, n - j) P_j(i)$.

**Proof.** Let $\{C_t\}_{t=0,\ldots,n}$ be the Riemann basis with

$$C_t = \sum_{j=0}^{n} v(t, n - j) D_j, \quad t = 0, 1, \ldots, n.$$  

For any $t$, the coefficient of $D_{n-t}$ is 1. This means

$$\bigoplus_{t=0}^{n} F_p D_t = \bigoplus_{t=0}^{n} F_p C_t. \quad (1)$$

We put $I_t := \bigoplus_{j=0}^{t} F_p C_j$; this is an ideal of $F_p \mathfrak{X}$. (Lemma 8 in [3].) Defining $\mathfrak{A}_i := F_p \mathfrak{X}/I_i$ and $I_{-1} := \{0\}$, we get a natural epimorphism:

$$\Pi_i : \mathfrak{A}_{i-1} \longrightarrow \mathfrak{A}_i, \quad i = 0, \ldots, n,$$

with $\text{Ker} \Pi_i = (F_p C_i + I_{i-1})/I_{i-1}$. Let $\text{Rad}(\mathfrak{A}_i)$ be the Jacobson radical of $\mathfrak{A}_i$, $\Pi_i (\text{Rad}(\mathfrak{A}_{i-1})) = \text{Rad}(\mathfrak{A}_i)$. We will write $\text{IRR}(\mathfrak{A}_i)$ for the set of equivalence classes of irreducible $\mathfrak{A}_i$-modules.

Since $\mathfrak{A}_i$ is a commutative algebra,

$$|\text{IRR}(\mathfrak{A}_i)| = \dim_{F_p} \mathfrak{A}_i / \text{Rad}(\mathfrak{A}_i).$$

Since $\dim_{F_p} \text{Ker} \Pi_i = 1$,

$$|\text{IRR}(\mathfrak{A}_{i-1})| = \begin{cases} |\text{IRR}(\mathfrak{A}_i)| & \text{if Ker} \Pi_i \in \text{Rad}(\mathfrak{A}_{i-1}), \\ |\text{IRR}(\mathfrak{A}_i)| + 1 & \text{if Ker} \Pi_i \notin \text{Rad}(\mathfrak{A}_{i-1}). \end{cases}$$
From this, we know that the number of \( i \) such that \( C_i \not\in \text{Rad}(\mathfrak{A}_{l-1}) \) coincides with \([\text{IRR}(\mathbb{F}_p, \mathfrak{X})]\). For \( C_i \in \text{Soc}(\mathfrak{A}_{l-1}) \), \( C_i \in \text{Rad}(\mathfrak{A}_{l-1}) \) if and only if \( C_i^2 \equiv 0 \pmod{I_{l-1}} \). By the congruence equation (9) in [3], we have

\[
C_i^2 \equiv \rho(i, i)C_i \pmod{I_{l-1}}, \quad i = 0, \ldots, n.
\]

Then \( C_i \not\in \text{Rad}(\mathfrak{A}_{l-1}) \) if and only if \( p \nmid \rho(i, i) \). In particular, by Theorem 9 in [3],

\[
\rho(i, i) = \sum_{j=0}^{n} v(i, n - j) P_j(i). \quad \Box
\]

In [3], \( \rho \) is computed for (i)–(iii) and (v). Theorems 1.1 and 1.2 follow from the results on the values of \( \rho \) in [3]. We consider the \( \rho \) for (iv). Let \( q \) be a prime power, and \( V \) be one of the following spaces equipped with a specified form:

- \([C_n(q)] = \mathbb{F}_q^{2n} \) with a nondegenerate symplectic form;
- \([B_n(q)] = \mathbb{F}_q^{2n+1} \) with a nondegenerate quadratic form;
- \([D_n(q)] = \mathbb{F}_q^{2n} \) with a nondegenerate quadratic form of (maximal) Witt index \( n \);
- \( [^2D_{n+1}(q)] = \mathbb{F}_q^{2n+2} \) with a nondegenerate quadratic form of (non-maximal) Witt index \( n \);
- \( [^2A_{2n}(r)] = \mathbb{F}_q^{2n+1} \) with a nondegenerate Hermitian form \( (q = r^2) \);
- \( [^2A_{2n-1}(r)] = \mathbb{F}_q^{2n} \) with a nondegenerate Hermitian form \( (q = r^2) \).

Let \( X \) be the set of maximal totally isotropic subspaces of \( V \). Each element of \( X \) has dimension \( n \). Define the \( i \)th relation \( R_i \) on \( X \) by

\[
(x, y) \in R_i \iff \dim(x \cap y) = n - i.
\]

Then \( \mathcal{X} = (X, [R_i]_{0 \leq i \leq n}) \) is a \( P \) and \( Q \) polynomial scheme [1]. (For details about the association schemes of dual polar spaces, see [9].)

By [8,9], we know that \( v(i, n - j) = \begin{bmatrix} n-j \cr i \end{bmatrix}_q \) and

\[
P_j(i) = \sum_{l=0}^{i} (-1)^{l-j} \frac{(q^n : q)_i}{(q : q)_j} \frac{(n - 1)}{q} \frac{(n - j)}{q} \frac{(j)}{q} q^{(l-j-1)+j(n+1)}.
\]

We need the following equalities:

\[
\begin{bmatrix} a \cr b \end{bmatrix}_q \begin{bmatrix} c \end{bmatrix}_q = \begin{bmatrix} a \cr c \end{bmatrix}_q \begin{bmatrix} a - c \cr b - c \end{bmatrix}_q, \tag{2}
\]

\[
\sum_{i=0}^{a} (-1)^{a-i} q^{\binom{a-i}{2}} \begin{bmatrix} a \cr i \end{bmatrix}_q \begin{bmatrix} i \cr b \end{bmatrix}_q = \delta_{a,b}, \tag{3}
\]

\[
\sum_{i=0}^{a} (-1)^{a-i} q^{\binom{a-i}{2}} \begin{bmatrix} a \cr i \end{bmatrix}_q \begin{bmatrix} i + l \cr b \end{bmatrix}_q = \begin{cases} q^{at}, & \text{if } b = a; \\ 0, & \text{if } b < a. \end{cases} \tag{4}
\]

\[
\sum_{i=0}^{a} q^{\binom{i}{2}} \begin{bmatrix} a \cr i \end{bmatrix}_q z^i = (-z : q)_a, (q\text{-binomial theorem}). \tag{5}
\]
We can prove (4) using (3) and induction on $t$;

\[
\rho(i, i) = \sum_{j=0}^{n} \sum_{l=0}^{j} (-1)^{j-l} \left[ \begin{array}{c} n-j \\ i \end{array} \right]_q (q^{-n} : q)_j \left[ \begin{array}{c} n-1 \\ i-1 \end{array} \right]_q \left[ \begin{array}{c} n-j \\ i-l \end{array} \right]_q \\
\times \left[ \begin{array}{c} j \\ l \end{array} \right]_q q^{(l-j-e)+j(n+e)} \\
\sum_{j=0}^{n} \sum_{l=0}^{j} (-1)^{j} q^{(l-j-e)+j(n+e)-\frac{2(n-j+1)}{2}} \left[ \begin{array}{c} n-j \\ i \end{array} \right]_q \left[ \begin{array}{c} n \\ j \end{array} \right]_q \\
\times \left[ \begin{array}{c} n-i \\ j \end{array} \right]_q \\
\sum_{j=0}^{n} \sum_{l=0}^{j} (-1)^{j} q^{(l-j-e)+j(n+e)-\frac{2(n-j+1)}{2}} \left[ \begin{array}{c} n-j \\ i \end{array} \right]_q \left[ \begin{array}{c} n-1 \\ i-1 \end{array} \right]_q \\
\times \left[ \begin{array}{c} n-l \\ i-l \end{array} \right]_q \left[ \begin{array}{c} n-i \\ j-l \end{array} \right]_q .
\]

We put $h = j - l$;

\[
\rho(i, i) = \sum_{h=0}^{n-i} \left( \sum_{l=0}^{i} (-1)^{l} q^{\frac{h(h-1)}{2} + \frac{l(l-1)}{2} + h} \left[ \begin{array}{c} n-h-l \\ i-l \end{array} \right]_q \left[ \begin{array}{c} n-h-i \\ l \end{array} \right]_q \left[ \begin{array}{c} n-i \\ h \end{array} \right]_q .
\]

By (4),

\[
\rho(i, i) = \sum_{h=0}^{n-i} q^{i(n-i) + \frac{h(h-1)}{2} - hi + eh} \left[ \begin{array}{c} n-i \\ h \end{array} \right]_q \left( q^{e-i} \right)^h.
\]

By (5), if $0 \leq i < n$ then

\[
\rho(i, i) = q^{i(n-i)} \prod_{i=0}^{n-i-1} (1 + q^{l-i+e}),
\]

and $\rho(n, n) = 1$.

**Note 1.** Even if an association scheme has a regular semilattice, (1) does not hold when the number of classes does not coincide with the maximum value of the length function, that is, when $n \neq m$. For (v), this happens when $n = \dim(F) \neq m$ [2].

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References