Invited paper

A tribute to Zadeh's extension principle

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Abstract In this paper, we highlight one of the brightest concepts launched by L. Zadeh, namely, the extension principle. Due to this principle, every one-to-one correspondence and, more generally, every one-to-many correspondence can be extended to fuzzy sets, i.e. sets without precise boundaries.

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1. Introduction

After my graduation in Mathematics, in 1967, from the University of Ghent, I received a grant from the National Institute for research in industry and agriculture (IWONL) for research concerning Low Energy Electron Diffraction (LEED). In this research, classical mathematical techniques, such as the solution of Schrödinger differential and integral equations, using Green functions were sufficient to tackle the LEED problem.

After obtaining my Ph.D. degree in January 1970, I became an assistant in the Department of Mathematical Analysis and began to look for a new research topic. I experienced my first scientific shock. When trying to read papers, I very soon realized that all authors used the language of set theory and modern structures, such as topology, metric spaces, Banach spaces, Hilbert spaces, etc., of which I was completely ignorant. With enthusiasm, I entered the beautiful world of Jean Dieudonné, Henri Cartan, Maurice Fréchet, etc.; all members of the BOURBAKI school. Together with 2 other colleagues, I completely rewrote the basic courses of mathematical analysis, incorporating the continuity and limits of mappings between metric spaces, the derivative as an approximated linear mapping between normed linear spaces, enjoying the beauty and generality of "modern" mathematics.

Indeed, during my bachelor studies, we obtained concepts of the continuity and limit of a function in a point; first for a $\mathbb{R} \rightarrow \mathbb{R}$ function, then for a $\mathbb{R}^2 \rightarrow \mathbb{R}$ function and lastly for a $\mathbb{R}^3 \rightarrow \mathbb{R}$ function, all of which were introduced independently. Fréchet’s metric space unified these 3 concepts and in this way increased insight and avoided redundancy.

A few years later I experienced my second scientific shock while I was reading L. Zadeh’s seminal paper “Fuzzy Sets”. I was immediately attracted by the term itself, which sounds in some ways contradictory. How can a set be fuzzy? So I became curious and since then, ‘fuzzy’ completely determined and filled my scientific life. I guided 29 researchers toward a Ph.D. in fuzzy set theory, co-authored more than 450 papers and have taught, since 1978, 4 courses on this topic for M.S. degrees in mathematics, computer sciences and engineering.

I am honoured by the invitation of Prof. M. Jamshidi to contribute to this special issue dedicated to the work of L. Zadeh, to whom I am extremely grateful. When thinking about the topic of my contribution, I suffered from “embarras du choix”. Zadeh has initiated so many fundamental concepts and contributed so extensively to the development of fuzzy set theory that it is difficult to make a choice. For a long time, I was wavering between the concepts of a linguistic variable and the extension principle. Finally, I selected the extension principle.

2. The extension principle for mappings

Let $f$ be a mapping from an arbitrary set, $X$, into an arbitrary set, $Y$, associating with each object, $x$, belonging to $X$, a unique object, $f(x)$, called the value of $f$ in $x$ of $Y$.

Using the concept of a direct image, such a mapping can be extended to a mapping from the power set $\mathcal{P}(X)$ of $X$ (i.e. the set of all subsets of $X$) to the power set $\mathcal{P}(Y)$ of $Y$ as follows:

$$f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y),$$

$$A \mapsto f(A), \quad \forall A \in \mathcal{P}(X),$$

(1)

with $f(A) = \{y | (\exists x \in A) (y = f(x))\}$ i.e. $f(A)$ consists of all the values of $f$ in points belonging to $A$. 

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It is well-known that to every (crisp) property, $P$ (i.e. a property such that for each object only 2 possibilities occur: the object satisfies the property or does not), there corresponds a subset of $X$, namely, $A = \{x | P\}$, i.e. the subset of $X$ consisting of all objects of $X$ that satisfy the property, $P$. Using classical propositional calculus properties, $P$ may be logically combined, resulting in the conjunction $(P_1 \land P_2)$, disjunction $(P_1 \lor P_2)$, implication (if $P_1$ then $P_2$), and negation (not $P_1$). Every logical combination of properties results in a new subset (using intersection, union, inclusion and complement) from which the direct image can be calculated. In this way, one can extend the point-to-point relationship to a subset-to-subset relationship, obtaining new information. Similarly, using the concept of inverse image with the mapping $f$ a $\mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ mapping, denoted $f^{-1}$ can be associated:

$$f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X),$$

$$B \mapsto f^{-1}(B), \quad \forall B \in \mathcal{P}(Y),$$

with $f^{-1}(B) = \{x | x \in X$ and $f(x) \in B\}$. Similarly, as for the direct image, one can infer subset-to-subset information using the original point-to-point relationship, $f$.

Now, as an example, consider set $X$ of patients in some hospital and set $Y$ of cholesterol levels (or glucose levels, or blood pressures) and the mapping that associates its cholesterol level to each patient. Then due to the above extension from this one-to-one correspondence, one can ask for the cholesterol levels of the subset of all female patients, all patients between 30 and 40 years old, all married patients, all patients that play soccer, etc.

Now we can ask which conclusion can be drawn if instead of crisp subsets of universe $X$, one considers so-called fuzzy sets in $X$ [1], i.e. subsets with imprecise boundaries, such as the subset of old patients, the subset of corpulent patients, the subset of tall patients, etc. It is well-known that such fuzzy sets in $X$ can be mathematically modelled by a mapping from $X$ into the unit interval where the values are called degrees of membership, which reflect gradual membership instead of the classical black-or-white member or non-member.

The answer is given by Zadeh’s extension principle for mappings [2]. So let $f$ be a $X \rightarrow Y$ mapping, $F$ $(X)$ the class of all fuzzy sets in $X$, i.e. the class of all $X \rightarrow [0, 1]$ mappings and $F$ $(Y)$ the class of all fuzzy sets in $Y$. According to Zadeh’s extension principle, $f$ can be extended to a $F$ $(X) \rightarrow F$ $(Y)$ mapping as follows:

$$f : F(X) \rightarrow F(Y),$$

$A \mapsto f(A), \quad \forall A \in F(X),$  

with $f(A)$ given as:

$$f(A) : Y \rightarrow [0, 1],$$

$$y \mapsto 0, \quad \forall y \in Y \setminus \text{rng}(f),$$

$$y \mapsto \sup \{A(x) | x \in X \text{ and } f(x) = y\},$$

$$\forall y \in \text{rng}(f),$$

where $\text{rng}(f)$ denotes the range of $f$.

Similarly, the $X \rightarrow Y$ mapping, $f$, induces a $F$ $(Y) \rightarrow F$ $(X)$ mapping, denoted $f^{-1}$ as follows:

$$f^{-1} : F(Y) \rightarrow F(X),$$

$$B \mapsto f^{-1}(B), \quad \forall B \in F(Y),$$

with $f^{-1}(B)$ given as:

$$f^{-1}(B) : X \rightarrow [0, 1],$$

$$x \mapsto B(f(x)), \quad \forall x \in X.$$  

Coming back to mapping $f$, associating its cholesterol level to each patient, we may now also calculate the fuzzy set of cholesterol levels corresponding to the fuzzy set of old patients, corpulent patients, fond of sports patients, and similarly one may obtain the fuzzy set of patients with a very high cholesterol level.

In the same paper [2], Zadeh also introduced the extension principle for a mapping of several variables, using the extension of the Cartesian product for fuzzy sets.

Let $f$ be a $X_1 \times X_2 \times \cdots \times X_n \rightarrow Y$ mapping and for every $i \in \{1, \ldots, n\}$, $A_i$ a fuzzy set in $X_i$. Then, the direct image of the $n$-tuple $(A_1, A_2, \ldots, A_n)$ under $f$ can be defined as:

$$f(A_1, A_2, \ldots, A_n) : Y \rightarrow [0, 1],$$

$$y \mapsto 0, \quad \forall y \in Y \setminus \text{rng}(f),$$

$$y \mapsto \sup \{\min(A_1(x), A_2(x), \ldots, A_n(x))\},$$

$$(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n,$$

$$f(x_1, x_2, \ldots, x_n) = y, \quad \forall y \in \text{rng}(f).$$

For an extensive list of the properties of these extensions we refer to [3,4].

Due to the extension principle, it is possible to fuzzify any mathematical structure based on set theory and, hence, following the BOURBAKI point of view, any mathematical structure.

The most well-known application is the extension of the arithmetical operations on the real numbers to the calculus of fuzzy numbers or more generally fuzzy quantities. For example, the addition in $\mathbb{R}$ is defined as the $\mathbb{R}^2 \rightarrow \mathbb{R}$ mapping $S$, which associates to every pair $(x, y)$ of real numbers their sum, $x + y$, i.e.:

$$S : \mathbb{R}^2 \rightarrow \mathbb{R},$$

$$(x, y) \mapsto x + y, \quad \forall (x, y) \in \mathbb{R}^2.$$  

Now let $\varphi_1$ and $\varphi_2$ be two fuzzy quantities, i.e. two fuzzy sets in $\mathbb{R}$. Then, applying the extension principle on the mapping $S$, leads to the sum $\varphi_1 + \varphi_2$ of $\varphi_1$ and $\varphi_2$:

$$\varphi_1 + \varphi_2 : \mathbb{R} \rightarrow [0, 1],$$

$$z \mapsto \sup \{\min(\varphi_1(x), \varphi_2(y)) | x + y = z\},$$

$$\forall z \in \mathbb{R}.$$  

In this paper, I will illustrate the power of the extension principle with a fuzzification of classical binary logic, where instead of the two crisp truth values, $T$ (true) and $F$ (false), fuzzy truth values can be considered, introducing, in this way, possible uncertainty about the truth of some propositions.

Let $a$ and $a'$ be two fuzzy truth values, i.e. two fuzzy sets on set $\{T, F\}$ of crisp truth values. More explicitly:

$$a = \{(T, a), (F, \beta)\},$$

$$a' = \{(T, a'), (F, \beta')\},$$

with $\alpha, \beta, \alpha'$ and $\beta'$ belonging to $[0, 1]$. All the classical logical operations, such as negation, conjunction and disjunction, can be extended to fuzzy truth values using the extension principle. Let us illustrate using conjunction $\land$. In the binary case, the conjunction is defined as $\{T, F\} \times \{T, F\} \rightarrow \{T, F\}$ mapping as follows:

$$\land : \{T, F\} \times \{T, F\} \rightarrow \{T, F\},$$

$$(T, T) \mapsto T,$$

$$(\alpha, \beta) \rightarrow F, \quad \text{if } (\alpha, \beta) \neq (T, T).$$
\[ a \land a'(T) = \sup(\min([a(x), a'(y)]))x \land y = T \]
\[ = \min(a(T), a'(T)), \quad (12) \]
\[ a \land a'(F) = \sup(\min([a(x), a'(y)]))x \land y = F \]
\[ = \max(\min(a(T), a'(F)), \quad (13) \]
\[ \min(a(F), a'(T)), \min(a(F), a'(F))). \]

Another important example of the extension principle is the extension of the basic concept of continuity for mappings between fuzzy topological spaces. A fuzzy topological space [5] is an ordered pair \((X, T)\) with \(X\), a non-empty set, and \(T\), a class of fuzzy sets in \(X\), satisfying the following properties:

- \( X \in T \) and \( \emptyset \in T \),
- \( O_1 \in T \) and \( O_2 \in T \implies O_1 \cap O_2 \in T \),
- \((\forall i \in I)(O_i \in T) \implies \bigcup_{i \in I} O_i \in T \),

where \( \cap \) and \( \cup \) are Zadeh’s original intersection and arbitrary union defined by:

\[ O_1 \cap O_2(x) = \min(O_1(x), O_2(x)), \quad \forall x \in X, \]
\[ \cup O_i(x) = \sup[O_i(x)|i \in I], \quad \forall x \in X. \quad (14) \]

Now let \((X_1, T_1)\) and \((X_2, T_2)\) be fuzzy topological spaces and \(f\) a mapping from \(X_1\) into \(X_2\). Then, \(f\) is called fuzzy continuous if:

\[ (\forall O_2 \in T_2)(f^{-1}(O_2) \in T_1), \quad (15) \]

where \(f^{-1}(O_2)\) is defined by the extension principle (see Expression \((6)\)).

Let me give just one illustration of the suitability of the extension principle in the context of fuzzy topology, namely, the fact that the chain rule for fuzzy continuous mappings still holds, i.e. the composition of two fuzzy continuous mappings is also fuzzy continuous. More examples of extensions of logical operations, as well as an overview of what is kept of the classical Boolean structure can be found in [3].

3. The extension principle for relations

It is well-known that most links between classes are not functional, but only of a relational kind. For example, returning to our example in a medical context, let again \(X\) be the set of patients in some hospital and, now, \(Y\) a set of symptoms, and define for \(x \in X\) and \(y \in Y\) a relation \(R\):

\[ (x, y) \in R \iff \text{patient } x \text{ has symptom } y. \quad (16) \]

Then, it is obvious that some patients can have more than one symptom, i.e. relationship \(R\) is not necessarily functional.

First, I will recall the concepts of the direct and inverse image of crisp sets under crisp relations. Let \(R\) be a relation from \(X\) to \(Y\). A subset of \(X\) and \(B\) a subset of \(Y\). Then the direct image \(R(A)\) of \(A\) under \(R\), and the inverse image \(R^{-1}(B)\) of \(B\) under \(R\) are defined as:

\[ R(A) = \{y | (\exists x \in A)((x, y) \in R)\}, \]
\[ R^{-1}(B) = \{x | (\exists y \in B)((x, y) \in R)\}. \quad (17) \]

The extension principle for relations extends these concepts to fuzzy sets and fuzzy relations. So, let \(R\) be a fuzzy relation from \(X\) to \(Y\) (i.e. a fuzzy set in the Cartesian product, \(X \times Y\)), \(A\) a fuzzy set in \(X\), and \(B\) a fuzzy set in \(Y\). Then \(R(A)\) and \(R^{-1}(B)\) are defined as:

\[ R(A) : X \to [0, 1], \]
\[ y \mapsto \sup(\min(a(x), R(x, y)))x \in X, \]
\[ \forall y \in Y, \quad (18) \]

\[ R^{-1}(B) : X \to [0, 1]. \]

Returning to our example of medical diagnostics, let \(R\) be a fuzzy relation between a set of patients, \(X\), and a set of symptoms, \(Y\), where \(R(x, y)\) denotes the degree to which patient \(x\) shows symptom \(y\). Further, let \(A\) be the fuzzy set of corpulent patients and \(B\) the fuzzy set of contagious symptoms. Then, due to the extension principle for relations, we may calculate the fuzzy set of symptoms for the fuzzy set of corpulent patients and the fuzzy set of patients for the fuzzy set of contagious symptoms.

There are numerous domains of application of fuzzy relational calculus, among them: medical diagnostics, information retrieval, relational databases, preference structures, decision making and ordering methods for fuzzy quantities. Its applicability has been substantially increased by the work of Bandler–Kohout and De Baets-Kerre by introducing several refinements of the direct \(R(A)\) and inverse image, \(R^{-1}(B)\), introduced above: subdirect image, superdirect image and square or ultradirect image. For more details see [6,7].

4. Conclusion

In this paper, I have tried to show the huge capabilities of Zadeh’s extension principle in order to fuzzy and colour classical mathematical concepts and structures, and to enrich the extraction of knowledge by supporting information in a natural language.

References