Module Categories with Infinite Radical Cube Zero

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Let \( A \) be an artin algebra and let \( \text{mod } A \) denote the category of finitely generated right \( A \)-modules. We denote by \( \text{rad}(\text{mod } A) \) the ideal in \( \text{mod } A \) generated by all non-invertible morphisms between indecomposable modules in \( \text{mod } A \). The infinite radical \( \text{rad}^\infty(\text{mod } A) \) of \( \text{mod } A \) is the intersection of all powers \( \text{rad}^i(\text{mod } A) \), \( i \geq 1 \), of \( \text{rad}(\text{mod } A) \). It is known that \( A \) is representation-finite if and only if \( \text{rad}^2(\text{mod } A) = 0 \). In this paper we study the representation-infinite algebras \( A \) such that \( \text{rad}^2(\text{mod } A) = 0 \). Examples of such algebras include all representation-infinite tilted algebras of Euclidean type.


Let \( A \) be an artin algebra over a commutative artin ring \( R \). We denote by \( \text{mod } A \) the category of finitely generated right \( A \)-modules and by \( \text{rad}(\text{mod } A) \) the Jacobson radical of \( \text{mod } A \), that is, the ideal in \( \text{mod } A \) generated by all non-invertible morphisms between indecomposable modules in \( \text{mod } A \). The infinite radical \( \text{rad}^\infty(\text{mod } A) \) of \( \text{mod } A \) is the intersection of all powers \( \text{rad}^i(\text{mod } A) \), \( i \geq 1 \), of \( \text{rad}(\text{mod } A) \).

The study of \( \text{rad}^\infty(\text{mod } A) \) has had some importance in describing the category \( \text{mod } A \). We are particularly interested in studying the algebras \( A \)

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whose categories mod $A$ have rad$^s\mod A$ nilpotent. It is known that if $A$
 is a finite dimensional algebra over an algebraically closed field such that
rad$^s\mod A$ is nilpotent, then $A$ is tame. The converse is far from being
true, and we refer to [18] for an example of a domestic algebra whose
module category does not have its infinite radical nilpotent. Recall that an
algebra $A$ is domestic provided there is a positive integer $m$ such that the
indecomposable $A$-modules occur, in each dimension $d$, in a finite number
of discrete and at most $m$ one-parameter families. However, for any tilted
algebra over an algebraically closed field, tameness, domesticity, and the
nilpotency of the infinite radical of its module category are equivalent. It
has been conjectured that if $A$ is such that rad$^s\mod A$ is nilpotent, then
$A$ is domestic. In [18], Kerner and the last named author have shown that
if $A$ is a selfinjective finite dimensional algebra over an algebraically
closed field and admits a simply connected Galois covering, then
rad$^s\mod A$ is nilpotent if and only if $A$ is domestic. Also, if $A$ is a
strongly simply connected finite dimensional algebra over an algebraically
closed field, then rad$^s\mod A$ is nilpotent if and only if $A$ is domestic (see
[29]). We shall show here that if $A$ is an algebra over an algebraically
closed field with $(\text{rad}^s\mod A)^3 = 0$, then $A$ is domestic (Corollary 5.8).

It is known that an artin algebra $A$ is representation-finite if and only if
rad$^s\mod A = 0$, if and only if $(\text{rad}^s\mod A)^2 = 0$ (see [10, 18, 26]). We
shall concentrate our attention on arbitrary representation-infinite artin
algebras $A$ with $(\text{rad}^s\mod A)^3 = 0$. One of our main results is the
following. An artin algebra $A$ is tame concealed if and only if
$(\text{rad}^s\mod A)^3 = 0$ and the Auslander–Reiten quiver of $A$ contains a
faithful regular component. Therefore, in particular, if one starts with any
representation-infinite artin algebra $A$ such that $(\text{rad}^s\mod A)^3 = 0$ and
$\Gamma^*_A$ has a regular component, the above result yields a tame concealed
factor algebra of $A$. We then study how the Auslander–Reiten compo-
nents of such a tame concealed factor algebra of $A$ embed into the
Auslander–Reiten quiver of $A$ (Section 4). Moreover, we show that there
are only finitely many tame concealed factors of $A$ (Theorem 5.5).

We now describe the contents of this paper. Section 1 is devoted to the
preliminaries, and we shall quickly discuss some examples of representa-
tion-infinite artin algebras $A$ with $(\text{rad}^s\mod A)^3 = 0$. In Section 2 we
study the structure of the Auslander–Reiten quiver $\Gamma^*_A$ of such an algebra
$A$ and show that: (1) the regular components of $\Gamma^*_A$ form a family of
pairwise orthogonal generalized standard stable tubes; and (2) at most
finitely many of the $e_r$-orbits of $\Gamma^*_A$ are non-periodic. Section 3 contains
the proof of some of the above-stated results. Finally, in the last two
sections we study algebras $A$ satisfying $(\text{rad}^s\mod A)^3 = 0$ by looking at
those factor algebras $B$ of $A$ which are tame concealed.
The study of the algebras $\mathcal{A}$ with $(\text{rad}^n(\text{mod } \mathcal{A}))^3 = 0$ and such that all the components of $\Gamma_{\mathcal{A}}$ are semiregular is the subject of another paper [11].

1. PRELIMINARIES

1.1. Let $\mathcal{A}$ be an artin algebra over a commutative artin ring $R$; that is, $\mathcal{A}$ is an $R$-algebra which is finitely generated as an $R$-module. By an $\mathcal{A}$-module is meant a finitely generated right $\mathcal{A}$-module. We shall denote by $\text{mod } \mathcal{A}$ the category of all (finitely generated) $\mathcal{A}$-modules, and by $\text{ind } \mathcal{A}$ the full subcategory of $\text{mod } \mathcal{A}$ with one representative of each isomorphism class of indecomposable $\mathcal{A}$-modules. Then $\text{rad mod } \mathcal{A}$ denotes the Jacobson radical of $\text{mod } \mathcal{A}$, that is, the ideal in $\text{mod } \mathcal{A}$ generated by all non-invertible morphisms between indecomposable modules in $\text{mod } \mathcal{A}$. The infinite radical $\text{rad}^\infty(\text{mod } \mathcal{A})$ of $\text{mod } \mathcal{A}$ is defined to be the intersection of all powers $\text{rad}^n(\text{mod } \mathcal{A})$, $i \geq 1$, of $\text{rad( mod } \mathcal{A})$. We denote by $\text{D}$ the standard duality $\text{Hom}_R(-, \tilde{I})$: $\text{mod } \mathcal{A} \to \text{mod } \mathcal{A}^{\text{op}}$, where $\tilde{I}$ is the injective envelope of $R/\text{rad } R$ in $\text{mod } R$. We say that a property holds for almost all indecomposable modules if it holds for, up to isomorphism, all but finitely many of them. Unless otherwise stated all algebras are assumed to be basic and connected.

1.2. We shall denote by $\Gamma_{\mathcal{A}}$ the Auslander–Reiten quiver of $\mathcal{A}$, and by $\tau_{\mathcal{A}} = \text{DTr}$ and $\tau_{\mathcal{A}}^{-1} = \text{TrD}$ the Auslander–Reiten translations in $\Gamma_{\mathcal{A}}$. We shall now agree to identify the vertices of $\Gamma_{\mathcal{A}}$ with the corresponding $\mathcal{A}$-modules in $\text{ind } \mathcal{A}$. By a component of $\Gamma_{\mathcal{A}}$ we mean a connected component of $\Gamma_{\mathcal{A}}$. We observe that a morphism between indecomposable modules lying in different components of $\Gamma_{\mathcal{A}}$ belongs to $\text{rad}^\infty(\text{mod } \mathcal{A})$. We shall use this fact throughout this paper.

Let $\mathcal{C}$ be a component of $\Gamma_{\mathcal{A}}$. Then $\mathcal{C}$ is said to be regular if $\mathcal{C}$ contains neither a projective module nor an injective module, and semi-regular if $\mathcal{C}$ does not contain both a projective and an injective module. A component $\mathcal{C}$ is said to be generalized standard if $\text{rad}^\infty(X, Y) = 0$ for all modules $X$ and $Y$ in $\mathcal{C}$ [28]. Observe that a postprojective component (respectively, a preinjective component) $\mathcal{C}$ is generalized standard because $\text{rad}^\infty(-, X) = 0$ (respectively, $\text{rad}^\infty(X, -) = 0$) for all $X \in \mathcal{C}$. (We shall use here the term postprojective instead of preprojective: we believe it is more suggestive.) Finally, $\mathcal{C}$ is said to be faithful if the intersection $\text{ann } \mathcal{C}$ of the annihilators $\text{ann } X$ of all modules $X$ in $\mathcal{C}$ is zero. We shall denote by $\mathcal{C}_p$ the left stable part of $\mathcal{C}$ obtained from $\mathcal{C}$ by deleting the $\tau_{\mathcal{A}}$-orbits of projective modules, by $\mathcal{C}_r$, the right stable part of $\mathcal{C}$ obtained from $\mathcal{C}$ by deleting the $\tau_{\mathcal{A}}$-orbits of injective modules, and by $\mathcal{C}_s$ the stable part of $\mathcal{C}$ obtained from $\mathcal{C}$ by deleting the $\tau_{\mathcal{A}}$-orbits of both the projective and the injective modules.
1.3. Let \( H \) be a hereditary algebra. It is known that, in this case, we can assume that \( R \) is a field so that \( H \) is a finite dimensional algebra over \( R \). Further, there exists a bilinear form on the Grothendieck group \( K_0(H) \) of \( H \) given by
\[
\langle M, N \rangle = \dim_R \text{Hom}_H(M, N) - \dim_R \text{Ext}_H^1(M, N)
\]
which induces a quadratic form \( q_H \) on \( K_0(H) \otimes_\mathbb{Z} \mathbb{Q} \). It is well-known that \( H \) is representation-finite if and only if \( q_H \) is positive definite. The algebra \( H \) is said to be of tame type if it is not representation-finite and \( q_H \) is positive semidefinite. For the basic representation theory of hereditary algebras we refer the reader, for instance, to [13–15, 24]. Now let \( H \) be a representation-infinite hereditary algebra and let \( n \) denote the rank of \( K_0(H) \). Let \( T \) be a multiplicity-free postprojective tilting \( H \)-module, that is, \( \text{Ext}_H^1(T, T) = 0 \), \( \text{rad}^+(-, T) = 0 \) (which, in this case, is equivalent to \( T \) being postprojective), and \( T \) is a direct sum of \( n \) pairwise non-isomorphic indecomposable \( H \)-modules. The algebra \( B = \text{End}_H(T) \) is called a concealed algebra. If \( H \) is tame hereditary, then \( B \) is called tame concealed, and otherwise wild concealed. It is known that the representation theories of \( B \) and \( H \) are very close to each other. Let \( B \) be a concealed algebra (which does not exclude the possibility of \( B \) being hereditary). Then \( \Gamma_B \) has a postprojective component \( \mathcal{P} \) containing all projectives and a preinjective component \( \mathcal{I} \) containing all injectives. If \( B \) is tame concealed, then the regular components form an infinite family of pairwise orthogonal generalized standard stable tubes \( \mathcal{T}_\rho \), \( \rho \in \Omega \), and all but finitely many of them have rank 1. Moreover, since the family \( \mathcal{T}_\rho \), \( \rho \in \Omega \), separates \( \mathcal{P} \) from \( \mathcal{I} \), all tubes \( \mathcal{T}_\rho \) are faithful, and every map from a module in \( \mathcal{P} \) to a module in \( \mathcal{I} \) belongs to \( (\text{rad}^+(\text{mod } B))^2 \). On the other hand, if \( B \) is wild concealed then the regular components are of the form \( \mathbb{Z} \mathbb{A}_w \) (see [23]). By [25], components of the form \( \mathbb{Z} \mathbb{A}_w \) are not generalized standard.

For unexplained notions in representation theory of artin algebras, we refer the reader to [7, 24].

1.4. We shall finish this section by discussing some examples. The next result gives a criterion for a tilted algebra \( B \) to satisfy the condition \( (\text{rad}^+(\text{mod } B))^3 = 0 \). We refer the reader to [11] for a proof.

**Proposition.** Let \( H \) be a connected representation-infinite hereditary artin algebra, let \( T \) be a tilting \( H \)-module without preinjective direct summands or without postprojective direct summands, and let \( B = \text{End}_H(T) \). Then \( (\text{rad}^+(\text{mod } B))^3 = 0 \) if and only if \( H \) is of Euclidean type.

1.5. Using (1.4), one can characterise the glued algebras \( \mathcal{A} \) (as defined in [1]) which satisfy \( (\text{rad}^+(\text{mod } \mathcal{A}))^3 = 0 \). Namely, let \( \mathcal{A} \) be a representation-infinite finite dimensional algebra over an algebraically closed field. If \( \mathcal{A} \)
is a left glueing of $B_1, \ldots, B_n$ by the non-zero algebra $C$, and if there is no projective module which is a proper successor of the right extremal subsection $\Sigma$ then $(\text{rad}^n(\text{mod } A))^3 = 0$ if and only if $\Sigma$ is a disjoint union of Euclidean graphs. Dual statement can be done for right glued algebras.

We refer the reader to [1] for the definitions and details on this class of algebras.

1.6. In the representation theory of tame simply connected algebras over an algebraically closed field an important role is played by coil enlargements of tame concealed algebras (see [26]), introduced and studied in [2, 3, 6], to which we refer the reader for details. The next result gives a characterisation of coil enlargements $B$ of concealed algebras which satisfies $(\text{rad}^n(\text{mod } B))^3 = 0$.

**Proposition.** Let $B$ be a coil enlargement of a tame concealed algebra $C$. Then the following conditions are equivalent:

(a) $(\text{rad}^n(\text{mod } B))^3 = 0$;
(b) $\text{rad}^n(\text{mod } B)$ is nilpotent;
(c) $B$ is domestic;

**Proof.** We know from [18, (1.7)] that if $B$ is wild or tubular in the sense of [18, (5.1)], then $\text{rad}^n(\text{mod } B)$ is not nilpotent. Then the result is a direct consequence of [6, (4.1) or (4.2)].

2. COMPONENTS OF THE AUSLANDER–REITEN QUIVER

2.1. Let $A$ be an artin algebra such that $(\text{rad}^n(\text{mod } A))^3 = 0$. We shall prove in this section two results concerning the structure of the Auslander–Reiten quiver $\Gamma_A$ of $A$. The first result describes the family of regular components of $\Gamma_A$, while the second shows that all but finitely many $\tau_A$-orbits in $\Gamma_A$ are periodic, or, in other words, that $\Gamma_A$ is a quasi-periodic translation quiver (see [8]).

**Theorem.** Let $A$ be an artin algebra with $(\text{rad}^n(\text{mod } A))^3 = 0$. Let $\Gamma_i$, $i \in I$, denote the family of all regular components of $\Gamma_A$. Then $\Gamma_i$, $i \in I$, are pairwise orthogonal generalized standard stable tubes, and all but finitely many of them have rank 1.

**Proof.** We first show that $\text{rad}^n(X, Y) = 0$ for all $X$ and $Y$ belonging to regular components. Suppose this is not true and let $f: X \to Y$ be a non-zero morphism in $\text{rad}^n(X, Y)$, where $X$ and $Y$ belong to regular components. Consider now the projective cover $\pi: P_A(X) \to X$ of $X$ and the injective envelope $i: Y \to I_A(Y)$ of $Y$. Observe that since $X$ and $Y$
belong to regular components we have that \( \pi \) and \( \iota \) are in \( \text{rad}^n(\text{mod} \ A) \) (1.2). Therefore \( \iota f \pi \) is a non-zero morphism in \( (\text{rad}^n(\text{mod} \ A))^3 \), which contradicts our hypothesis. Hence \( \Gamma_i, i \in I \), are pairwise orthogonal generalized standard components.

**Claim.** Every regular component of \( \Gamma'_A \) contains oriented cycles.

Suppose \( \Gamma \) is a regular component of \( \Gamma'_A \) without oriented cycles. Then, by [30], \( \Gamma \) is of the form \( \mathbb{Z} \Delta \), where \( \Delta \) is a valued quiver without oriented cycles. Since \( \Gamma \) is generalized standard, we have that \( \text{Hom}_A(X, \tau Y) = 0 \) for all \( X \) and \( Y \) in \( \Delta \) and hence, by [25, (2.2)], \( \Delta \) is finite. Moreover, according to [28, (3.3)] it follows that \( B = A/\text{ann} \Gamma \) is a tilted algebra given by a regular tilting module and \( \Gamma \) is a connecting component of \( \Gamma_B \).

Let \( M \) be the sum of all modules in \( \Delta \). Observe that, since \( \Gamma_B \) contains a complete slice in a regular component, each component of \( \Gamma_B \) is semiregular. It follows from tilting theory that \( \Gamma_B \) contains either a stable tube or a (semiregular) component whose stable part contains a component of the form \( \mathbb{Z}A_2 \). Suppose first that \( \Gamma_B \) contains a stable tube \( \mathcal{F} \) and let \( Z \) be in \( \mathcal{F} \). Then \( Z \) is either cogenerated or generated by \( M \). Suppose we have the former (the other situation is dual). Consider the projective cover \( \pi: P_B(Z) \to Z \) of \( Z \), the embedding \( \iota: Z \to M' \) of \( Z \) into some power of \( M \) and the injective envelope \( \epsilon: M' \to I_M(M') \) of \( M' \). It follows that the composition \( \iota \epsilon \pi \) is a non-zero morphism in \( (\text{rad}^n(\text{mod} \ A))^3 \), a contradiction. Suppose now that \( \Gamma_B \) has a semiregular component without injective modules \( \Gamma' \), whose stable part contains a component of the form \( \mathbb{Z}A_2 \). It is known that there exists a sectional path \( X \to \cdots \to Y \) in \( \Gamma' \) such that \( \text{Hom}_B(X, \tau Y) \neq 0 \) [25, (2.2)], and it can be chosen in such a way that there is no projective module which is a predecessor of \( X \) or \( Y \) in \( \Gamma' \). Hence \( \text{rad}^n(X, \tau Y) \) contains a non-zero morphism \( f \), because there are no oriented cycles containing \( X \). Consider the projective cover \( \pi: P_B(X) \to X \) of \( X \) and the injective envelope \( \iota: \tau Y \to I_B(\tau Y) \) of \( \tau Y \). Since the composite \( \iota f \pi \) is non-zero and belongs to \( (\text{rad}^n(\text{mod} \ B))^3 \), which contradicts our hypothesis, the claim is proven.

By [16, 19], a regular component with oriented cycles is a stable tube and so, for each \( i \in I \), \( \Gamma_i \) is a stable tube. Let now \( \text{rk} \Gamma_i \) denote the rank of the tube \( \Gamma_i \). According to [28, (5.10)], we have the inequality

\[
\sum_{i \in I} (\text{rk} \Gamma_i - 1) \leq n - 1,
\]

where \( n \) is the rank of the Grothendieck group of \( A \). Therefore almost all of the stable tubes \( \Gamma_i, i \in I \), have rank 1. This finishes the proof of Theorem 2.1. \( \blacksquare \)
2.2. In order to prove that $\Gamma_d$ is quasi-periodic, we shall need the following general result.

**Proposition.** Let $A$ be an arbitrary artin algebra and $\Gamma$ be a connected stable translation subquiver of $\Gamma_A$ of the form $\mathbb{Z}\Delta$, where $\Delta$ is an infinite locally finite quiver without oriented cycles. Let $M$ be an indecomposable module in $\Gamma$. Then

(a) there exists an infinite sequence of morphisms

$$
\cdots \to M_{r+1} \xrightarrow{f_r} M_r \to \cdots \to M_2 \xrightarrow{f_2} M_1 \xrightarrow{g} M_0 = M
$$

where for each $i$, $M_i \in \Gamma$, $f_i$ is an epimorphism and $gf_1 \cdots f_i$ is a composite of morphisms in a sectional path through $M_{i+1}, \ldots, M_0$.

(b) there exists an infinite sequence of morphisms

$$
M = M_0 \xrightarrow{g'} M'_1 \xrightarrow{f'_1} M'_2 \cdots \to M'_r \xrightarrow{f'_r} M'_{r+1} \to \cdots
$$

where for each $i$, $M_i' \in \Gamma$, $f'_i$ is a monomorphism and $f'_i' \cdots f'_1g'$ is a composite of morphisms in a sectional path through $M_0', \ldots, M'_{i+1}$.

**Proof.** We shall only prove (a), since the proof of (b) is dual.

(a) Let $\Gamma$ and $M$ be as above. Consider first the set $\mathcal{P}$ of all paths

$$
N_k \to \cdots \to N_1 \to N_0 = M
$$

in $\Gamma$ ending at $M$ and such that $\tau^{-1}_dN_k$ is not a predecessor of $M$ in $\Gamma$. Observe that if $N_k \to \cdots \to N_1 \to N_0 = M$ is a path from $\mathcal{P}$, then $\Gamma$ has no paths from $\tau^{-1}_dN_i$ to $N_j$, where $0 \leq i, j \leq k$.

Indeed, if

$$
\tau^{-1}_dN_i = V_0 \to V_1 \to \cdots \to V_p = N_j
$$

is a path in $\Gamma$, then $\Gamma$ admits a path

$$
\tau^{-1}_dN_k \to \tau^{-1}_dN_{k-1} \to \cdots \to \tau^{-1}_dN_i = V_0 \to \cdots \to V_p = N_j \to \cdots \to N_0,
$$

which is a contradiction.

Further, since $\Gamma$ is of the form $\mathbb{Z}\Delta$, with $\Delta$ infinite, $\mathcal{P}$ admits paths of arbitrarily large length. Moreover, all paths from $\mathcal{P}$ are sectional. Indeed, if $N_i \to \cdots \to N_0 = M$ is a path in $\mathcal{P}$ which is not sectional, then there is an $i$, $2 \leq i \leq l$, such that $\tau^{-1}_dM_i = M_{i-2}$ and so a predecessor of $M$,
contradicting our hypothesis on $\mathcal{P}$. Since $\Gamma$ is locally finite, by König's Lemma, there exists an infinite sectional path
\[
\cdots \rightarrow L_r \rightarrow L_{r-1} \rightarrow \cdots \rightarrow L_1 \rightarrow L_0 = M
\]
such that, for each $i \geq 1$, the path
\[
L_i \rightarrow \cdots \rightarrow L_1 \rightarrow L_0 = M
\]
belongs to $\mathcal{P}$. Consider now
\[
\mathcal{M} = \{ L_i : i \in \mathbb{N} \}
\]
\[
\cup \{ X \in \Gamma : \exists \text{ a path } L_r \rightarrow \cdots \rightarrow X \rightarrow \cdots \rightarrow L_s \text{ in } \Gamma, \text{ for some } r,s \}. \]

Now let
\[
L_r = U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_i = L_s \quad (*)
\]
be a path in $\Gamma$. Clearly, $U_i \in \mathcal{M}$ for all $i$. Observe also that $(*)$ is sectional. In fact, if it were not so, then $\tau^{-1}_a U_l = U_{l+2}$ for some $0 \leq l \leq i - 2$, and hence
\[
\tau^{-1}_a L_r = \tau^{-1}_a U_0 \rightarrow \tau^{-1}_a U_1 \rightarrow \cdots \rightarrow \tau^{-1}_a U_l \rightarrow U_{l+3} \rightarrow \cdots \rightarrow U_i = L_s
\]
is a path from $\tau^{-1}_a L_r$ to $L_s$, which contradicts the above claim. Therefore it follows that each path between modules in $\mathcal{M}$ is sectional.

We now define inductively a sequence of modules $M_i$ and subsets $\mathcal{M}_i$ of $\mathcal{M}$. Let first $\mathcal{M}_0 = \mathcal{M}$. Let $M_1$ be a module in $\mathcal{M}_0$ of minimal length and
\[
\mathcal{M}_1 = \{ X \in \mathcal{M} : \text{there is a path from } X \text{ to } M_1 \}.
\]
Suppose now that $M_i$ and $\mathcal{M}_i$ are defined for all $i < j$. We take $M_j$ to be a module in $\mathcal{M}_{j-1} \setminus \{ M_{j-1} \}$ of minimal length and
\[
\mathcal{M}_j = \{ X \in \mathcal{M} : \text{there is a path from } X \text{ to } M_j \}.
\]
Observe that, if $X$ and $Y$ belong to $\mathcal{M}_j$ for some $j$, then any path from $X$ to $Y$ is sectional. Fix a $j \geq 1$ and consider a morphism $h: M_j \rightarrow M_{j-1}$
which is a composite of irreducible morphisms given by a path in $\Gamma$. We will show that $h$ is an epimorphism. Suppose this is not true. Then there are an epimorphism
\[
\alpha = (\alpha_1, \ldots, \alpha_r): M_j \to Y_1 \oplus \cdots \oplus Y_r
\]
and a (proper) monomorphism
\[
\beta = (\beta_1, \ldots, \beta_r): Y_1 \oplus \cdots \oplus Y_r \to M_{j-1}
\]
such that $Y_i \in \text{ind } A$ and
\[
h = \sum_{i=1}^r \beta_i \alpha_i.
\]
Since $\beta$ is a proper monomorphism, then for all $i = 1, \ldots, t, l(Y_i) < l(M_{j-1})$, and hence $Y_i \notin M_{j-1}$. We shall show now that for each $i$ either $\alpha_i$ or $\beta_i$ belongs to rad"(mod $A$). If this is not the case, then for some $i$, there exists a path
\[
M_j \to \cdots \to Y_i \to \cdots \to M_{j-1}
\]
with $Y_i \notin M_{j-1}$, which is a contradiction. Therefore, for each $i$, the morphism $\beta_i \alpha_i$ belongs to rad"($M_j, M_{j-1}$) and so $h = \sum \beta_i \alpha_i \in$ rad"($M_j, M_{j-1}$). The contradiction comes from the fact that $h$ is a composite of irreducible morphisms from a sectional path and hence by [17], it does not belong to rad"(mod $A$). Therefore the morphisms $f_i: M_{i+1} \to M_i$ can now be chosen, for each $i$, to be any composite of irreducible morphisms in a sectional path from $M_{i+1}$ to $M_i$. This finishes the proof. 

2.3. THEOREM. Let $A$ be an artin algebra such that $(\text{rad"}(\text{mod } A))^3 = 0$. Then almost all $\tau_a$-orbits in $\Gamma_A$ are periodic.

Proof. Since all $\tau_a$-orbits in a regular component are periodic (by (2.1)) and since almost all components are regular, it suffices to prove that any non-regular component is quasi-periodic, that is, almost all of its $\tau_a$-orbits are periodic.

Let $\Gamma'$ be a non-regular component of $\Gamma_A$ and suppose it is not quasi-periodic. Then, by duality, we can also assume that there exists a connected component $\Gamma''$ of the right stable part of $\Gamma'$ with infinitely many non-periodic $\tau_a$-orbits. It follows from [21] that $\Gamma''$ is a subquiver of $\mathbb{Z} \Delta'$, closed under successors, where $\Delta'$ is an infinite locally finite quiver without oriented cycles. Then $\Gamma''$ has a (stable) subquiver of the form $\mathbb{Z} \Delta$, where $\Delta$ is an infinite locally finite quiver without oriented cycles. Let $M$
be a module in $\mathbb{Z} \Delta$ such that $\tau_A M$ is not a predecessor of an injective module in $\Gamma$. By (2.2) there exists an infinite sequence of morphisms
\[
\cdots \to M_{r+1} \xrightarrow{f_r} M_r \to \cdots \to M_2 \xrightarrow{f_1} M_1 \xrightarrow{g} M_0 = M, \tag{*}
\]
where for each $i$, $M_i \in \Gamma$, $f_i$ is an epimorphism and $g f_1 \cdots f_i$ is a composite of morphisms in a sectional path through $M_1, \ldots, M_{i+1}$. Since the path (\*) is infinite and consists of pairwise non-isomorphic modules (by [9]), it follows from [25, Lemma 2] that for some $i$ and $j$ there is a non-zero morphism $f: M_i \to \tau_A M_j$, which is in fact in $\text{rad}^\ast(\text{mod}\ A)$ because $\Gamma'$ has no oriented cycles and $M$ is not a predecessor of an injective module in $\Gamma$. Observe that any morphism from a projective to a module $M_i$ is in $\text{rad}^\ast(\text{mod}\ A)$ because it factors through all $M_k$, $k \geq i$. In particular, consider the projective cover $\pi: P_A(M_j) \to M_j$ of $M_j$. Consider also the injective envelope $\iota: \tau_A M_j \to I_A(\tau_A M_j)$ of $\tau_A M_j$, which is also in $\text{rad}^\ast(\text{mod}\ A)$ because $\tau_A M_j$ is not a predecessor of an injective module in $\Gamma$. Now, the composite $\iota f \pi$ is a non-zero morphism and belongs to $(\text{rad}^\ast(\text{mod}\ A))^3 = 0$, in contradiction to the hypothesis.

2.4. COROLLARY. Let $A$ be an artin algebra such that $(\text{rad}^\ast(\text{mod}\ A))^3 = 0$. Then, for each $d \geq 1$, almost all indecomposable $A$-modules of length $d$ are $\tau_A$-invariant.

Proof. It follows from Theorem 2.3 that all components of $\Gamma_A$ are quasi-periodic. Fix now a positive integer $d$. It is known that a quasi-periodic component has at most finitely many indecomposable modules of length $d$ ([8, (5.5)] and [28, (2.6)]). Since there are at most finitely many components of $\Gamma_A$ which are not stable tubes of rank one, we infer that almost all indecomposable modules of length $d$ belong to stable tubes of rank 1 (that is, homogeneous tubes); hence the result.

3. MINIMAL REPRESENTATION-INFINITE ALGEBRAS

3.1. Our main aim in this section is to show that an artin algebra $A$ is tame concealed if and only if $(\text{rad}^\ast(\text{mod}\ A))^3 = 0$ and $\Gamma_A$ contains a faithful regular component. In order to do so we shall first establish a lemma. We recall the following characterisation of concealed algebras proven independently in [1, (3.4)] and [27, (3.3)].

PROPOSITION. The following are equivalent for a representation-infinite artin algebra $A$:

(a) $A$ is concealed;
(b) $pd X = 1$ and $id X = 1$ for almost all modules $X$ in $\text{ind} A$;
(c) $\text{rad}^\ast(-, A) = 0$ and $\text{rad}^\ast(DA, -) = 0$. 

3.2. **Lemma.** A representation-infinite artin algebra \( A \) is tame concealed if and only if \( \text{rad}^*(-,A) = 0, \text{rad}^*(DA,-) = 0 \) and \( (\text{rad}^*(\text{mod} A))^3 = 0 \).

**Proof.** Since the necessity is clear, we shall prove the sufficiency. Suppose that \( A \) is such that (i) \( \text{rad}^*(-,A) = 0 \), (ii) \( \text{rad}^*(DA,-) = 0 \), and (iii) \( (\text{rad}^*(\text{mod} A))^3 = 0 \). Conditions (i) and (ii) imply that \( A \) is concealed by (3.1). Now, if \( A \) is a wild concealed algebra then it contains a component of the form \( ZA_0 \) (1.3), which contradicts (2.1).

3.3. We can now establish our main result of this section.

**Theorem.** Let \( A \) be an artin algebra. Then \( A \) is tame concealed if and only if \( (\text{rad}^*(\text{mod} A))^3 = 0 \) and there exists a faithful regular component.

**Proof.** Since the necessity is clear, we shall only prove the sufficiency. Suppose that \( A \) is such that \( (\text{rad}^*(\text{mod} A))^3 = 0 \) and there exists a faithful regular component. Then there exists a faithful module \( Z \), whose indecomposable summands lie in a regular component \( \mathcal{C} \). If now \( \text{rad}^*(-,A) \neq 0 \), then there exists a non-zero morphism \( f \in \text{rad}^*(P,Q) \) for some indecomposable projective modules \( P \) and \( Q \). Since \( Z \) is faithful, there exists a monomorphism \( \nu: Q \rightarrow Z' \) which belongs to \( \text{rad}^*(\text{mod} A) \) because \( Q \) does not belong to \( \mathcal{C} \). Consider now the injective envelope \( \iota: Z' \rightarrow I(Z') \) of \( Z' \), which is also in \( \text{rad}^*(\text{mod} A) \). The contradiction comes from the fact that \( \nu \iota \tau \) is a non-zero morphism in \( (\text{rad}^*(\text{mod} A))^3 \) and so \( \text{rad}^*(-,A) = 0 \). Dually, one can show that \( \text{rad}^*(DA,-) = 0 \). It follows now from (3.2) that \( A \) is a tame concealed algebra.

3.4. An algebra \( A \) is said to be **minimal representation-infinite** if \( A \) is representation-infinite but any factor algebra \( A/I \), where \( I \) is a non-zero ideal of \( A \), is representation-finite.

**Corollary.** Let \( A \) be a connected finite dimensional algebra over an algebraically closed field. Then \( A \) is tame concealed if and only if \( A \) is minimal representation-infinite and \( (\text{rad}^*(\text{mod} A))^3 = 0 \).

**Proof.** The necessity is well-known. Assume now that \( A \) is minimal representation-infinite and \( (\text{rad}^*(\text{mod} A))^3 = 0 \). The second assumption implies (see [18, (1.7)]) that \( A \) is tame, and then by [12, Corollary F], \( \Gamma_A \) admits a stable tube \( \mathcal{T} \). Clearly, \( \mathcal{T} \) is faithful because \( A \) is minimal representation-infinite. Therefore, by (3.3), \( A \) is tame concealed.
 Examples. (1) Let $A$ be the radical square zero algebra over a field $k$ given by the following quiver

```
\[ \begin{array}{c}
\alpha \\
\cdot \\
\cdot \\
\cdot \\
\end{array} \]
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Then $(\text{rad}^3(\text{mod } A))^3 = 0$ and $A/AeA$ is representation-finite for every non-zero idempotent $e \in A$. However, $A$ is not minimal representation-infinite because if $I$ is generated by the arrow $\alpha$, then $A/I$ is still representation-infinite.

(2) Let $k$ be a field and let $A = k[x, y]/(x, y)^2$. It is known that $A$ is minimal representation-infinite, $(\text{rad}^3(\text{mod } A))^3 \neq 0$ and $(\text{rad}^3(\text{mod } A))^5 = 0$.

4. EMBEDDINGS OF TAME CONCEALED ALGEBRAS

4.1. Let $A$ be an artin algebra with $(\text{rad}^3(\text{mod } A))^3 = 0$. We know from Theorem 3.3 that, for any regular component $C$ in $\Gamma_A$, the factor algebra $A/\text{ann } C$ is tame concealed. In this section we shall study $A$ by looking at those factor algebras of $A$ which are tame concealed. Also, in the next section, we shall prove that for such an algebra $A$ there are only finitely many ideals $I$ such that $A/I$ is tame concealed.

We are particularly interested in the problem of determining which parts of the Auslander–Reiten quiver of a tame concealed factor algebra of $A$ remain a full translation subquiver in $\Gamma_A$. We point out first that the shapes of nonregular components of $A$ with $(\text{rad}^3(\text{mod } A))^3 = 0$ can be complicated. For instance, they can be glueings of coils by directed parts (multicoils) which include the Auslander–Reiten quivers of many representation-finite algebras. We borrow the next example from [3].

Example. Let $A$ be the bound quiver algebra $kQ/I$ over an algebraically closed field $k$ given by the quiver $Q$ of the form

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\[ \begin{array}{c}
\alpha_1 \\
\gamma_1/\beta_1 \\
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and the ideal $I$ in $kQ$ generated by the elements $\sigma_i, \gamma_i, \beta_i, \alpha_i, 1 \leq i \leq m$, and $\epsilon_i, \rho_i, 1 \leq i \leq m - 1$. Then it is easy to see that $(\text{rad}^3(\text{mod} \, A))^3 = 0$ and $\Gamma_A$ consists of postprojective components, preinjective components, tubes, and components of the form

where we identify along the vertical dotted lines.

Clearly, the path algebras $H_i = k\Delta_i, 1 \leq i \leq m$, given by the subquivers

$$\Delta_i : \bullet \leftrightarrow_{\gamma_i}^{\beta_i} \bullet$$

are factor algebras of $A$. Moreover, each of the above glued tubes contains one stable tube of rank 1 of $\Gamma_{H_i}$ and one stable tube of rank 1 of $\Gamma_{H_{i+1}}$ for some $1 \leq i \leq m - 1$. 
Further examples can be found in [3, 6].

4.2. Let \( B \) be a tame concealed factor algebra of \( A \). We shall show first how some components of \( \Gamma_B \) are embedded in \( \Gamma_A \). We start by proving some lemmata.

\textbf{Lemma.} Let \( A \) be an artin algebra with \( (\text{rad} (\text{mod} A))^3 = 0 \). Assume that \( B \) is a tame concealed factor algebra of \( A \) and \( f: M \to N \) is a map in \( \text{ind} B \) which does not belong to \( \text{rad}^2 (\text{mod} B) \). Then \( f \) does not belong to \( \text{rad}^2 (\text{mod} A) \).

\textbf{Proof.} Observe first that \( M \) and \( N \) belong to the same component, say \( C \), of \( \Gamma_B \). Let \( \pi: \underline{P}_B(M) \to M \) be the projective cover of \( M \) and \( \iota: M \to \underline{I}_B(M) \) be the injective envelope in \( \text{mod} B \). Then \( f \pi \neq 0 \), because \( f \neq 0 \). Moreover, \( \iota \in (\text{rad} (\text{mod} B))^2 \) if \( C \) is postprojective, \( \pi \in (\text{rad} (\text{mod} B))^2 \) if \( C \) is preinjective, and \( \iota, \pi \in \text{rad}^2 (\text{mod} B) \) if \( C \) is a stable tube of \( \Gamma_B \). Since \( (\text{rad} (\text{mod} A))^3 = 0 \), we conclude then that \( f \) does not belong to \( \text{rad}^2 (\text{mod} A) \).

\textbf{Corollary.} Let \( A \) be an artin algebra with \( (\text{rad} (\text{mod} A))^3 = 0 \) and let \( B \) be a tame concealed factor algebra of \( A \). Then, for every component \( C \) in \( \Gamma_B \), all modules from \( C \) are contained in one component of \( \Gamma_A \). In particular, if an indecomposable \( B \)-module \( M \) lies on an oriented cycle in \( \Gamma_B \), then it also lies on an oriented cycle in \( \Gamma_A \).

4.3. \textbf{Lemma.} Let \( A \) be an artin algebra with \( (\text{rad} (\text{mod} A))^3 = 0 \) and \( B \) be a tame concealed factor algebra of \( A \). Assume that \( \mathcal{T} \) is a stable tube of \( \Gamma_A \) which contains a \( B \)-module. Then \( \mathcal{T} \) consists entirely of \( B \)-modules.

\textbf{Proof.} We first observe that \( \mathcal{T} \) contains infinitely many \( B \)-modules. In fact, since \( \mathcal{T} \) contains a \( B \)-module \( Y \), then, by Corollary 4.2, it also contains the entire component \( C \) of \( \Gamma_B \) which contains \( Y \), which is infinite. Let now \( X \) be an arbitrary module in \( \mathcal{T} \). Since \( \mathcal{T} \) is a stable tube, there are a path \( X = X_0 \to X_1 \to \cdots \to X_{s} = Y \) of irreducible monomorphisms and a path \( Z = Y_{t} \to Y_{t-1} \to \cdots \to Y_0 = Y \) of irreducible epimorphisms with \( Z \) an indecomposable \( B \)-module. Then \( Y \) and hence also \( X \) are \( B \)-modules. This proves our claim.

4.4. \textbf{Lemma.} Let \( A \) be an artin algebra with \( (\text{rad} (\text{mod} A))^3 = 0 \) and let \( \Gamma \) be a left stable translation subquiver of \( \Gamma_A \) containing no oriented cycles and closed under predecessors in \( \Gamma_A \). Assume that \( B \) is a tame concealed factor algebra of \( A \), and \( M \) is an indecomposable \( B \)-module lying in \( \Gamma \). Then \( M \) is a preinjective \( B \)-module.

\textbf{Proof.} Observe that \( M \) does not belong to stable tubes of \( \Gamma_B \). Indeed, if \( M \) lies in a stable tube, then it would lie on an oriented cycle in \( \Gamma_B \), and consequently, by (4.2), \( M \) lies also on an oriented cycle in \( \Gamma_A \). But this is impossible because \( M \) is in \( \Gamma \). Suppose now that \( M \) is a postprojective
Let \( \pi: P_A(M) \to M \) be the projective cover of \( M \) in \( \text{mod} \ A \) and \( \iota: M \to I_p(M) \) be the injective envelope of \( M \) in \( \text{mod} \ B \). Since \( M \) is a postprojective \( B \)-module, \( \iota \) belongs to \( (\text{rad}^{\Gamma}(\text{mod} \ B))^2 \), and hence also to \( (\text{rad}^{\Gamma}(\text{mod} \ A))^2 \). Moreover, \( \pi \) belongs to \( \text{rad}^{\Gamma}(\text{mod} \ A) \), but the projective modules and it is closed under predecessors in \( \Gamma_A \). Therefore \( \iota \pi \) is non-zero and belongs to \( (\text{rad}^{\Gamma}(\text{mod} \ A))^3 \), a contradiction. Hence \( M \) is preinjective.

4.5. Dually, we have the following.

**Lemma.** Let \( A \) be an artin algebra with \( \text{rad}^{\Gamma}(\text{mod} \ A)^3 = 0 \) and let \( \Gamma \) be a right stable translation subquiver of \( \Gamma_A \) containing no oriented cycles and closed under successors in \( \Gamma_A \). Assume that \( B \) is a tame concealed factor algebra of \( A \), and \( M \) is an indecomposable \( B \)-module lying in \( \Gamma \). Then \( M \) is a postprojective \( B \)-module.

4.6. Let \( A \) be an artin algebra with \( (\text{rad}^{\Gamma}(\text{mod} \ A))^3 = 0 \) and let \( B \) be a tame concealed factor algebra of \( A \). We can now describe the components of \( \Gamma_A \) containing the postprojective and preinjective \( B \)-modules.

**Proposition.** Let \( A \) be an artin algebra with \( (\text{rad}^{\Gamma}(\text{mod} \ A))^3 = 0 \) and let \( B \) be a tame concealed factor algebra of \( A \). Then

(i) There is a left stable connected full translation subquiver \( \mathcal{C} \) of \( \Gamma_A \) containing no oriented cycles and closed under predecessors in \( \Gamma_A \) such that all but finitely many indecomposable preinjective \( B \)-modules lie in \( \mathcal{C} \).

(ii) There is a right stable connected full translation subquiver \( \mathcal{D} \) of \( \Gamma_A \) containing no oriented cycles and closed under successors in \( \Gamma_A \) such that all but finitely many indecomposable postprojective \( B \)-modules lie in \( \mathcal{D} \).

**Proof.** We shall prove only (i), since the proof of (ii) is dual. We know from Lemma 4.2 that all indecomposable preinjective \( B \)-modules belong to one component, say \( \Gamma' \), of \( \Gamma_A \). From Theorem 2.3, \( \Gamma' \) admits at most finitely many non-periodic \( \tau \)-orbits. Further, by Lemma 4.5, we know also that, if \( \mathcal{D} \) is a right stable translation subquiver of \( \Gamma_A \) containing no oriented cycles and closed under successors in \( \Gamma_A \), then \( \mathcal{D} \) does not contain preinjective \( B \)-modules. Observe that there is at most one left stable full translation connected subquiver \( \mathcal{C} \) of \( \Gamma' \) without oriented cycles and closed under predecessors in \( \Gamma' \) containing infinitely many indecomposable preinjective \( B \)-modules. Indeed, suppose that there are two such subquivers, say \( \mathcal{C} \) and \( \mathcal{C}' \), in \( \Gamma' \). Then there are indecomposable preinjective \( B \)-modules \( X \) in \( \mathcal{C} \) and \( X' \) in \( \mathcal{C}' \) such that there is a path from \( X \) to \( X' \) in \( \Gamma_B \). But then, by Lemma 4.2, there is a path from \( X \) to \( X' \) in \( \Gamma_A \), a contradiction because \( \mathcal{C}' \) is closed under predecessors in \( \Gamma_A \).
Suppose now that, for every left stable connected full translation subquiver \( C \) of \( \Gamma_a \) without oriented cycles and closed under predecessors, \( C \) does not contain infinitely many indecomposable preinjective \( B \)-modules. Then, by the above remarks, we infer that either the left stable part \( \Gamma_l \) or the right stable part \( \Gamma_r \) of \( \Gamma \) admits an infinite connected component, say \( \Omega \), with oriented cycles and containing infinitely many indecomposable preinjective \( B \)-modules. Without loss of generality we may assume that \( \Omega \) is a component of \( \Gamma_l \). By [16] and [21] we know that \( \Omega \) is either a stable tube or there is an infinite sectional path

\[
\cdots \rightarrow \tau_a^t X_1 \rightarrow \tau_a^s X_s \rightarrow \cdots \rightarrow \tau_a^t X_2 \rightarrow \tau_a^t X_1 \rightarrow X_s \rightarrow \cdots \rightarrow X_2 \rightarrow X_1
\]

in \( \Omega \) with \( t > s \) such that \( \{X_1, \ldots, X_s\} \) is a complete set of representatives of \( \tau_a \)-orbits in \( \Omega \). In both cases, since \( \Omega \) has infinitely many preinjective \( B \)-modules, there are two sectional paths

\[
M = U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_p = N
\]

and

\[
N = V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_q = M
\]

in \( \Omega \) such that \( M \) is an indecomposable preinjective \( B \)-module and, for any \( 1 \leq j \leq q \), there is an infinite sectional path

\[
\cdots \rightarrow Z^j_2 \rightarrow Z^j_1 \rightarrow Z^j_0 = V_j
\]

in \( \Omega \) with \( Z^j_i \neq V_{j-1} \). Now choose irreducible morphisms \( f_i: U_{i-1} \rightarrow U_i, 1 \leq i \leq p \), and \( g_j: V_{j-1} \rightarrow V_j, 1 \leq j \leq q \). We know from [17] that \( f_p \cdots f_1 \in \text{rad}^s(M, N) \setminus \text{rad}^{s+1}(M, N) \). Moreover, by [19, (1.6)], the maps \( g_j, 1 \leq j \leq q \), are of infinite left degree. Therefore \( g_q \cdots g_1 f_p \cdots f_1 \in \text{rad}^{p+q}(M, M) \setminus \text{rad}^{p+q+1}(M, M) \). In particular, we get that \( \text{rad}(M, M) \neq 0 \). On the other hand, \( \text{End}_a(M) = \text{End}_p(M) \) is a division ring, because \( M \) is an indecomposable preinjective \( B \)-module. Hence we get a contradiction. This finishes our proof of (i).

**4.7.** Let \( A \) be an artin algebra with \( (\text{rad}^n(\text{mod } A))^3 = 0 \) and let \( B \) be a tame concealed factor algebra of \( A \). By (2.1), every regular component of \( \Gamma_a \) is a stable tube. Also, we know from Corollary 4.2 that every component of \( \Gamma_b \) is embedded into a component of \( \Gamma_a \). The next result shows that, in fact, these two components coincide in almost all instances.

**Proposition.** Let \( A \) be an artin algebra with \( (\text{rad}^n(\text{mod } A))^3 = 0 \) and let \( B \) be a tame concealed factor algebra of \( A \). Then all but finitely many stable tubes of \( \Gamma_b \) are full components of \( \Gamma_a \).

**Proof.** By our previous remarks and (4.3), it is enough to show that every non-regular component of \( \Gamma_a \) contains at most finitely many stable
tubes of $\Gamma_B$. Suppose that there is a component $C$ in $\Gamma_A$ containing infinitely many stable tubes, say $T_1, T_2, T_3, \ldots$ of $\Gamma_B$. Observe that $C$ has no sectional path

$$X = Z_0 \to Z_1 \to \cdots \to Z_r = Y$$

with $X \in \mathcal{F}, Y \in \mathcal{F},$ and $i \neq j$. Indeed, if this is the case, and $f_i: Z_{i-1} \to Z_i$, $1 \leq i \leq r$, are irreducible morphisms, then, by [17], $f_1 \cdots f_r$ is non-zero, and hence $\text{Hom}_A(X, Y) = \text{Hom}_A(X, Y) \neq 0$. However, this contradicts the fact that different tubes in $\Gamma_B$ are orthogonal. Hence, if $\Gamma$ is a stable tube in the stable part $\mathcal{C}$ of $\mathcal{C}$, then $\Gamma$ admits modules from at most finitely many tubes $\mathcal{F}$. We know, by (2.3), that all but finitely many $\tau_A$-orbits in $\mathcal{C}$ are periodic. Further, by [16], if $\Gamma'$ is an infinite component of $\mathcal{C}$ containing a periodic module, then $\Gamma'$ is a stable tube. Therefore we conclude that there is a non-periodic $\tau_A$-orbit $D$ in $\mathcal{C}$ containing modules from infinitely many tubes $\mathcal{F}$. We may assume that there is a module $X$ in $\mathcal{C}$ such that $(\tau_A^i X, t \geq 0)$ contains modules from infinitely many tubes $\mathcal{F}$, and that the connected component, say $D$, of $\mathcal{C}$ containing $X$ is non-trivial. Observe that $D$ contains oriented cycles. Indeed, if this is not the case, then there is a connected full translation quiver $E$ of $D$ which is closed under predecessors in $\Gamma$, and contains an indecomposable module from a stable tube of $\Gamma_B$, a contradiction to Corollary 4.2. Therefore $D$ has oriented cycles. Since $D$ is infinite and not a stable tube, by [21], there is an infinite sectional path

$$\cdots \to \tau_A^T X_1 \to \tau_A^T X_s \to \cdots \to \tau_A^T X_2 \to \tau_A^T X_1 \to X_s \to \cdots \to X_2 \to X_1$$

in $D$ with $t > s$ and $\{X_1, \ldots, X_s\}$ is a complete set of representatives of $\tau_A$-orbits in $\mathcal{C}$. But then there is a sectional path

$$\tau_A^{T_{1}}X \to \cdots \to \tau_A^{T_{2}}X$$

in $D$ with $\tau_A^{T_{1}}X$ and $\tau_A^{T_{2}}X$ lying in different tubes of $\Gamma_B$, in contradiction to the fact proven above. This finishes our proof.

5. NUMBER OF IDEALS

5.1. Let $\mathcal{A}$ be an artin algebra. In order to study the complexity of $\mathcal{A}$ it is interesting to look at the number of ideals $I$ such that $\mathcal{A}/I$ is tame concealed. The following example shows that they are not necessarily finite.
Example. Let $\mathcal{A}$ be the wild hereditary algebra over an infinite field given by the following quiver:

\[ \begin{array}{c}
\bullet & \xrightarrow{\alpha} & \bullet \\
\end{array} \]

It is not difficult to see that there are infinitely many ideals $(a)$ generated by different linear combinations of arrows such that the quotients $\mathcal{A}/(a)$ are the Kronecker algebras, thus concealed.

The main purpose of this section is to show that for an artin algebra $\mathcal{A}$ satisfying $(\text{rad}^*(\text{mod} \mathcal{A}))^3 = 0$ the number of ideals $I$ such that $\mathcal{A}/I$ is tame concealed is finite. On the other hand, Example 4.1 shows that the number of such ideals can be arbitrarily large.

5.2. We shall first prove the following general lemma.

Lemma. Let $\mathcal{A}$ be an artin algebra and let $I$ and $J$ be two ideals in $\mathcal{A}$ such that $I \subseteq J$. Assume that $\mathcal{A}/I$ and $\mathcal{A}/J$ are tame concealed algebras. Then $I = J$.

Proof. Observe that $\mathcal{A}/J$ is the quotient of $\mathcal{A}/I$ by $J/I$. Since $\mathcal{A}/I$ is minimal representation-infinite and $\mathcal{A}/J$ is representation-infinite, we get that $J/I = 0$, and so $I = J$.

5.3. Our next immediate concern are the algebras with two simple modules. We first observe that every concealed algebra $\mathcal{B}$ with two simples is hereditary. Indeed, observe that one of the simple $\mathcal{B}$-modules is projective and the other is injective. In order to prove our next result we shall also need the following proposition (see [22]).

Proposition. Let $\mathcal{A}$ be a tame concealed algebra with two simple modules $S_1$ and $S_2$. Then

\[ \mathcal{A} = \begin{pmatrix}
F & F M_G \\
0 & G
\end{pmatrix} \]

where $F = \text{End}_A(S_2)$ and $G = \text{End}_A(S_1)$ are division algebras, and $F M_G$ is an $F$-$G$-bimodule such that

\[ \dim_F(F M_G) \times \dim_G(F M_G) = 4. \]

5.4. Lemma. Let $\mathcal{A}$ be an artin algebra with two simple modules and $(\text{rad}^*(\text{mod} \mathcal{A}))^3 = 0$. Assume that $I$ and $J$ are ideals of $\mathcal{A}$ contained in $\text{rad} \mathcal{A}$ and such that $\mathcal{A}/I$ and $\mathcal{A}/J$ are tame concealed algebras. Then $I = J$. 
Proof. Suppose that \( I \neq J \) and put \( B = A/I, C = A/J \). Replacing, if necessary, \( A \) by \( A/I \cap J \) we may assume, by Lemma 5.2 that \( I \cap J = 0 \).

Let \( S_1 \) and \( S_2 \) be two non-isomorphic simple \( A \)-modules. Since \( I \) and \( J \) are contained in \( \text{rad} \, A \), we get that \( S_1 \) and \( S_2 \) are both \( B \)-modules and \( C \)-modules. We may assume that \( S_1 \) is a simple projective \( B \)-module and \( S_2 \) is a simple injective \( B \)-module. We claim then that \( S_1 \) is also projective in \( \text{mod} \, C \) and \( S_2 \) is also injective in \( \text{mod} \, C \). Indeed, suppose that \( S_1 \) is projective in \( \text{mod} \, C \). Consider the projective cover \( f: P_\infty(S_1) \rightarrow S_1 \) of \( S_1 \) in \( \text{mod} \, B \) and the injective envelope \( g: S_1 \rightarrow I_\infty(S_1) \) of \( S_1 \) in \( \text{mod} \, C \). Then \( f \in (\text{rad}^n(\text{mod} \, B))^2 \) and \( g \in (\text{rad}^n(\text{mod} \, C))^2 \). Hence \( gf \) is a non-zero morphism in \( \text{mod} \, A \) which belongs to \((\text{rad}^n(\text{mod} \, A))^3 \), a contradiction. This proves our claim.

By (5.3) we know that \( B \) and \( C \) are of the form

\[
B = \begin{pmatrix} F & F M_G^* \\ 0 & G \end{pmatrix} \quad C = \begin{pmatrix} F & F M_G^* \\ 0 & G \end{pmatrix}
\]

where \( F = \text{End}_A(S_2) \) and \( G = \text{End}_A(S_1) \) are division algebras, and \( F M_G^* \), \( F M_G^* \) are \( F-G \)-bimodules such that

\[
\dim_F(F M_G^*) \times \dim_G(F M_G^*) = 4 \quad \text{and} \quad \dim_F(F M_G^*) \times \dim_G(F M_G^*) = 4.
\]

Hence, since \( I \cap J = 0 \), we deduce that \( A \) is of the form

\[
A = \begin{pmatrix} F & F M_G \\ 0 & G \end{pmatrix}
\]

where \( F M_G \) is an \( F-G \)-bimodule such that

\[
F M_G^* \cong (F M_G) / I \quad \text{and} \quad F M_G^* \cong (F M_G) / J.
\]

Moreover, we have then the inequality \( \dim_F(F M_G) \times \dim_G(F M_G) \geq 5 \), and so \( A \) is a wild hereditary algebra. In particular, \( \Gamma_A \) admits a component of type \( \mathbb{Z} \mathfrak{A}_n \). But then, by Theorem 2.1, we have a contradiction with our assumption \((\text{rad}^n(\text{mod} \, A))^3 = 0 \). \( \blacksquare \)

5.5. We shall now prove the main result of this section.

**Theorem.** Let \( A \) be an artin algebra with \((\text{rad}^n(\text{mod} \, A))^3 = 0 \). Then there exist at most finitely many ideals \( I \) in \( A \) such that \( A/I \) is tame concealed.

**Proof.** Suppose there exists an infinite sequence of pairwise different ideals \( I_1, I_2, I_3, \ldots \) in \( A \) such that the algebras \( B_j = A/I_j, j \geq 1 \), are tame concealed. Without loss of generality we may assume that \( I_j \subset \text{rad} \, A \) for
any $j \geq 1$. We infer from Lemma 5.2 that $I_j$ is not a subset of $I_i$ and $I_i$ is not a subset of $I_j$, for any $j \neq i$. Moreover, by Lemma 5.4, the rank $n$ of $K_0(A)$ is at least 3. This implies that, for any $i \geq 1$, $\Gamma_B^i$ admits a stable tube of rank at least two [24]. Further, we know from (4.2) that if $\mathcal{F}$ is a stable tube of $\Gamma_B^i$ then either $\mathcal{F}$ is a full component of $\Gamma_A^i$ or all modules of $\mathcal{F}$ belong to one non-regular component of $\Gamma_A^i$.

Let $(\Gamma_A^i)_{\lambda \in \Omega}$ be the family of all regular components of $\Gamma_A^i$. Then, by (2.1), $\Gamma_A^i$, $\lambda \in \Omega$, are pairwise orthogonal generalized standard stable tubes. Denote by $\text{rk} \Gamma_A^i$ the rank of the tube $\Gamma_A^i$. Then, by (28, 5.10), we have the following inequality:

$$\sum_{\lambda \in \Omega} (\text{rk} \Gamma_A^i - 1) \leq n - 1$$

Hence, since $\Gamma_A^i$ admits only finitely many non-regular components, there exists a non-regular component $\mathcal{E}$ in $\Gamma_A^i$ such that for infinitely many $i \geq 1$ there exists a stable tube of rank greater than or equal to 2 in $\Gamma_B^i$ which modules are all contained in $\mathcal{E}$. We know from (4.4) and (4.5) that if $\Gamma''$ is a left (respectively, right) stable translation subquiver of $\Gamma_A^i$ containing no oriented cycles and closed under predecessors (respectively, successors) in $\Gamma_A^i$, then $\Gamma''$ has no module belonging to a stable tube of $\Gamma_B^i$, $i \geq 1$. Further, all but finitely many $\tau_A$-orbits in $\mathcal{E}$ are periodic (2.3). Then there is a connected component $\mathcal{D}$ of either $\mathcal{E}_l$ or $\mathcal{E}_r$ containing oriented cycles, and there is an infinite sequence $1 \leq i_1 < i_2 < i_3 < \cdots$ such that, for each $s \geq 1$, the Auslander–Reiten quiver of $B_i$, admits a stable tube $\mathcal{F}_s$ of rank greater or equal to 2 whose modules all belong to $\mathcal{D}$. By duality, we may assume that $\mathcal{D}$ is a component of $\mathcal{E}_r$. Then either $\mathcal{D}$ is a stable tube, if it contains a periodic module [16], or there exists, by [20, (2.3)], an infinite sectional path

$$\cdots \rightarrow \tau_A^2 X_3 \rightarrow \tau_A X_2 \rightarrow \cdots \rightarrow \tau_A X X_2 \rightarrow \tau_A X X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_2 \rightarrow X_1$$

in $\mathcal{D}$ with $i > s$, where at least one of the modules $X_i$ is not stable, and where $(X_1, \ldots, X_i)$ is a complete set of representatives of $\tau_A$-orbits in $\mathcal{D}$. In both cases, there is an infinite sectional path

$$\cdots \rightarrow Z_3 \rightarrow Z_2 \rightarrow Z_1$$

in $\mathcal{D}$ such that every $Z_i$ has exactly two direct predecessors (respectively, successors) in $\mathcal{E}$, and there are two integers $p \neq q$ such that (\*) contains infinitely many modules from $\mathcal{F}_r$ and infinitely many modules from $\mathcal{F}_q$. We know that all but finitely many modules in $\mathcal{F}_r$ (respectively, $\mathcal{F}_q$) are faithful $B_{ip}$-modules (respectively, $B_{iq}$-modules). Hence there is a subpath

$$M = Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow \cdots \rightarrow Z_m \rightarrow Z_{m+1} \rightarrow \cdots \rightarrow Z_i = N$$
of (*) such that $M$ and $N$ are faithful $B_i$-modules from $\mathcal{T}$, and $Z_m$ is a faithful $B_i$-module from $\mathcal{T}$. Choose irreducible morphisms $f_j: Z_{j-1} \to Z_j$, $1 \leq j \leq l$, in mod $A$, and put $g = f_1 \cdots f_l$. Then, by [17], we get that $g \in \text{rad}(M, N) \setminus \text{rad}^{i+1}(M, N)$. Hence, if some $Z_j$ is a $B_i$-module, then it belongs to $\mathcal{T}_p$. Therefore we may assume that the moduli $Z_1, \ldots, Z_{i-1}$ do not belong to $\mathcal{T}_p$. We claim that $g$ is an irreducible morphism in mod $B_i$. Suppose this is not the case. Then, since $\mathcal{T}_p$ is generalized standard in the Auslander–Reiten quiver of $B_i$, $g$ is a sum of composites of irreducible morphisms between modules from $\mathcal{T}_p$. But then $g \in \text{rad}^{i+1}(M, N)$ because the above path in $\mathcal{D}$ from $M$ to $N$ is a shortest one, a contradiction. This proves that $g$ is irreducible in mod $B_i$, and hence is either a monomorphism or an epimorphism.

Observe that $g$ factors through the faithful $B_i$-module $Z_m$, and consequently $I_y$ is contained in $I_x$ because $\text{ann} M = I_y = \text{ann} N$ and $\text{ann} Z_m = I_x$. Therefore we get a contradiction as required, and this finishes our proof.

5.6. We have the following consequences of Theorem 5.5.

**Corollary.** Let $A$ be an artin algebra with $(\text{rad}(\text{mod } A))^3 = 0$. Then there is a finite number of tame concealed factor algebras $B_1, \ldots, B_r$ of $A$ such that, if $\mathcal{F}$ is a stable tube of $\Gamma_A$, then $\mathcal{F}$ is a component of some $\Gamma_{B_i}$, $1 \leq i \leq r$.

**Proof.** We know that if $\mathcal{F}$ is a stable tube of $\Gamma_A$, then $B = A/\text{ann } \mathcal{F}$ is a tame concealed algebra, and clearly $\mathcal{F}$ is a component of $\Gamma_B$.

5.7. **Corollary.** Let $A$ be an artin algebra with $(\text{rad}(\text{mod } A))^3 = 0$. Then almost all connected components of $\Gamma_A$ are stable tubes of rank one.

5.8. **Corollary.** Let $A$ be a finite dimensional algebra over an algebraically closed field such that $(\text{rad}(\text{mod } A))^3 = 0$. Then $A$ is domestic.

**Proof.** Since $(\text{rad}(\text{mod } A))^3 = 0$, by [18, (1.7)], we infer that $A$ is tame. Then, by [12, Corollary F], for any dimension $d$, all but a finite number of isomorphism classes of indecomposable $A$-modules of dimension $d$ lie in stable tubes of rank 1. From the above corollary we know that the stable tubes of $\Gamma_B$ belong to a finite number of tubular families given by tame concealed algebras. Therefore $A$ is domestic.

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