Explicit constructions of Ramanujan complexes of type $\tilde{A}_d$

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Abstract

In this paper we present for every $d \geq 2$ and every local field $F$ of positive characteristic, explicit constructions of Ramanujan complexes which are quotients of the Bruhat–Tits building $B_d(F)$ associated with $\text{PGL}_d(F)$.

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1. Introduction

In [11] we defined and proved the existence of Ramanujan complexes, see also [2,4] and [8]. The goal of this paper is to present an explicit construction of such complexes.

Our work is based on the lattice constructed by Cartwright and Steger [3]. This remarkable discrete subgroup $\Gamma$ of $\text{PGL}_d(F)$, when $F$ is a local field of positive characteristic, acts simply transitively on the vertices of the Bruhat–Tits building $B_d(F)$, associated with $\text{PGL}_d(F)$. By choosing suitable congruence subgroups of $\Gamma$, we are able to present the 1-skeleton of the corresponding finite quotients of $B_d(F)$ as Cayley graphs of explicit finite groups, with specific sets of generators. The simplicial complex structure is then defined by means of these generators.

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Let \([d^i]_q\) denote the number of subspaces of dimension \(k\) of \(F_d^d\).

**Theorem 1.1.** Let \(q\) be a prime power, \(d \geq 2, e \geq 1\) (\(e > 1\) if \(q = 2\)).

Then, the group \(G = \text{PGL}_d(\mathbb{F}_{q^e})\) has an (explicit) set \(S\) of \([d^1]_q + [d^2]_q + \cdots + [d^{d-1}]_q\) generators, such that the Cayley complex of \(G\) with respect to \(S\) is a Ramanujan complex, covered by \(\mathcal{B}_d(F)\), when \(F = \mathbb{F}_q((y))\).

The Cayley complex of \(G\) with respect to a set of generators \(S\) is the simplicial complex whose 1-skeleton is the Cayley graph \(\text{Cay}(G; S)\), whereas subset of \(i+1\) vertices is an \(i\)-cell iff every two vertices comprise an edge. The generators in Theorem 1.1 are explicitly given in Section 9.

In the case \(d = 2\) there are two types of Ramanujan graphs, bipartite, and non-bipartite [9,10,12]. Here too, given \(r\) dividing \(d\), we produce \(r\)-partite complexes, by proving an analog of Theorem 1.1 for every subgroup of \(\text{PGL}_d(\mathbb{F}_{q^e})\) containing \(\text{PSL}_d(\mathbb{F}_{q^e})\). There is also a version of the theorem for subgroups of \(\text{PGL}_d(L)\) where \(L\) is a finite local ring.

The paper is organized as follows: In Section 2 we describe affine buildings of type \(\tilde{A}_d\) in general, in the language of \(O\)-sublattices of \(F^d\) (where \(O\) is the valuation ring of \(F\)). The Cartwright–Steger group \(\Gamma\) is constructed in Sections 3 and 4. Our construction slightly differs from theirs, but is more convenient for the computations to follow. The simply transitive action of \(\Gamma\) on the building is used in Section 5 to describe the defining relations of \(\Gamma\).

In Section 6 we present and analyze the finite Ramanujan quotients. In particular, we use [11] to show that the complexes constructed here are Ramanujan. We should note that the proof of this result in [11] relies on what is called the global ‘Jacquet–Langlands correspondence’ for function fields (a correspondence between automorphic representations of a division algebra and of \(GL_d\), cf. [6, Theorem VI.1.1] for the characteristic zero case). This correspondence in the function field case is also considered to be true by experts, as the main ingredients of the proof are known; though the task of writing down a complete proof has not been carried out yet.

Originally, \(\Gamma\) is a group of \(d^2 \times d^2\) matrices over a ring \(R\). Section 8 provides an explicit embedding of \(\Gamma\) into \(d \times d\) matrices over a finite extension of \(R\). In Section 9 this embedding is refined to identify finite quotients of \(\Gamma\) with subgroups of \(\text{PGL}_d(L)\) which contain \(\text{PSL}_d(L)\), where \(L\) is a finite local ring; a detailed algorithm is given, with an example in Section 10. In particular the generators of \(\Gamma\) are given as \(d \times d\) matrices over \(\mathbb{F}_q[x]\). For the convenience of the reader, we include a glossary in the final section.

2. Buildings

To every reductive algebraic group over a local field one can associate a building, which is a certain simplicial complex, on which the group acts (see [16]). This complex plays the role of a symmetric space for Lie groups.

Recall that a complex is a structure composed of \(i\)-cells, where the 0-cells are called the vertices, and an \(i\)-cell is a set of \(i+1\) vertices. A complex is simplicial if every subset of a cell is also a cell. The \(i\)-skeleton is the set of all \(i\)-cells in the complex. Buildings are in fact
clique complexes, which means that a set of \( i + 1 \) vertices is a cell if and only if every two vertices form a 1-cell. This property holds for quotient complexes, which will be the subject of Section 6.

We will now describe the affine building associated to \( \text{PGL}_d(F) \), where \( F \) is a local field. These are called ‘buildings of type \( A_{d-1} \)’ because of the Dynkin diagram of the associated Weil group (which is isomorphic to \( S_d \ltimes \mathbb{Z}^{d-1} \)). Let \( \mathcal{O} \) denote the valuation ring of \( F \); choose a uniformizer \( \sigma \) (so for \( F = \mathbb{F}_q((y)), \mathcal{O} = \mathbb{F}_q[[y]] \) and \( \sigma = y \)), and assume \( \mathcal{O}/\sigma \mathcal{O} = \mathbb{F}_q \). Consider the \( \mathcal{O} \)-lattices of full rank in \( F^d \). For every lattice \( L, \sigma L \) is a sublattice, and as \( \mathbb{F}_q \)-vector spaces \( L/\sigma L \cong \mathbb{F}_q^d \). We define an equivalence relation by setting \( L \sim sL \) for every \( s \in F^\times \).

**Remark 2.1.** Let \( \langle \sigma \rangle \) denote the multiplicative subgroup of \( F^\times \) generated by \( \sigma \), and let \( \mathcal{O}^\times \) be the invertible elements of \( \mathcal{O} \). Then \( F^\times = \langle \sigma \rangle : \mathcal{O}^\times \).

Since \( L = sL \) for any element \( s \in \mathcal{O}^\times \), the equivalence classes have the form \( [L] = [\sigma^i L]_{i \in \mathbb{Z}} \). Let \( \mathcal{B}^0 \) be the graph whose vertices are the equivalence classes. There is an edge from \( [L] \) to a class \( x \in \mathcal{B}^0 \), iff there is a representative \( L' \in x \) such that \( \sigma L \subset L' \subset L \). Notice that this is a symmetric relation, since then \( \sigma L' \subset \sigma L \subset L' \).

The vertices of \( \mathcal{B}^0 \) form the 0-skeleton of a complex \( \mathcal{B} \), and the edges are the 1-skeleton, \( \mathcal{B}^1 \). As i-cells of \( \mathcal{B} \) we take the complete subgraphs of size \( i + 1 \) of \( \mathcal{B}^0 \), which correspond to flags

\[
\sigma L_0 \subset L_1 \subset \cdots \subset L_i \subset L_0;
\]

the i-skeleton is denoted \( \mathcal{B}^i \). It immediately follows that \( \mathcal{B} \) has \( (d-1) \)-cells (corresponding to maximal flags in quotients \( L/\sigma L \)). It also follows that there are no higher dimensional cells.

The group \( \text{GL}_d(F) \) acts transitively on lattices by its action on bases. Moreover note that the action preserves inclusion of lattices. We call \( L_0 = \mathcal{O}^d \subset F^d \) the standard lattice. If \( \tau \in \text{GL}_d(F) \) has entries in \( \mathcal{O} \), then \( \tau L_0 = L_0 \). The stabilizer of \( L_0 \) in \( \text{G} \) is thus the maximal compact subgroup \( \text{GL}_d(\mathcal{O}) \).

According to the definition of the equivalence relation, the scalar matrices of \( GL_d(F) \) act trivially on \( \mathcal{B} \), so the action of \( GL_d(F) \) induces a well defined action of \( GL_d(F) \) on (the vertices of) \( \mathcal{B} \), which is easily seen to be an action of an automorphism group. Again, the stabilizer of \( [L_0] \) is the maximal compact subgroup \( GL_d(\mathcal{O}) \). The set of vertices can thus be identified with \( GL_d(F)/GL_d(\mathcal{O}) \).

Since the only ideals of \( \mathcal{O} \) are powers of \( \langle \sigma \rangle \), the Invariant Factor Theorem for \( F \) asserts that any matrix in \( GL_d(F) \) can be decomposed as \( a g a' \) for \( a, a' \in GL_d(\mathcal{O}) \) and \( g = \text{diag}(a_1, \ldots, a_d) \), where \( a_1 \leq \cdots \leq a_d \) are integers. If \( L = a g a'L_0 = a g L_0 \), then \( \sigma^{-i} L = a(\sigma^{-i} g) L_0 \subset L_0 \), and on the other hand \( \sigma^{-i} L = a(\sigma^{-i} g) L_0 \supset L_0 \), so any two lattices of maximal rank are commensurable. Moreover in this case, \( L_0/\sigma^{-i} L \) is annihilated by \( \sigma^{-j} \) and so is a module over \( \mathcal{O}/\sigma^{-j-i} \), a local ring of order \( q^{j-i} \). In particular \( [L_0 : \sigma^{-i} L] \) is a \( q \)-power.

This basic fact allows us to define a color function \( \varphi : \mathcal{B}^0 \to \mathbb{Z}/d \) by \( \varphi(L) = \log_q [L_0 : \sigma^{-j} L] \) for large enough \( i \); this function is well defined since \([\sigma^{-j} L : \sigma^{-j+1} L] = q^d \). Also notice that \( \varphi([\tau L_0]) = v_0(\det(\tau)) \mod d \), when \( v_0 \) is the valuation of \( F \). This shows that \( SL_d(F) \) is color preserving, while on the other hand, \( \tau_\sigma = \text{diag}(\sigma, 1, \ldots, 1) \).
has determinant \( \sigma \), so \( \varrho(\tau_\sigma(L)) = \varrho(L) + 1 \) for every \( L \). It follows that \( \text{GL}_d(F) \) acts transitively on colors.

The color is additive in the sense that if \( L'' \subseteq L' \), then \( \varrho(L'') = \varrho(L') + \log_q [L' : L''] \). Similarly, if \( L \subseteq L_0 \) and \( \tau \in \text{GL}_d(F) \), then \( \varrho(\tau L) = \varrho(\tau L_0) + \varrho(L) \).

The colors provide us with \( d \) Hecke operators, defined on functions of \( B^0 \) by summation over the neighbors of fixed color-shift:

\[
A_k f(x) = \sum_{y \sim x, \varrho(y) - \varrho(x) = k} f(y).
\]

These operators generate the Hecke algebra \( H(\text{PGL}_d(F), \text{PGL}_d(O)) \) (see [11, Section 2] for more details).

For \( 1 \leq t < d \), let \( B[t] \) denote the graph defined on the vertices \( B^0 \), with the edges \( (x, x') \in B^1 \) for which there are \( L \in x \) and \( L' \in x' \) such that \( L' \subseteq L \) and \( [L : L'] = q^t \) (in particular, \( \varrho(x) - \varrho(x') = t \)).

**Remark 2.2.** If \( L' \subset L \) is a sublattice of index \( q \), then \([L], [L']\) are connected in \( B^1 \).

**Proof.** We need to prove that \( \sigma L \subset L' \), but this is obvious since \( L/L' \) is annihilated (as an \( O \)-module) by multiplication by \( \sigma \). \( \square \)

For every \( L' \subset L \) there is a composition series of sublattices \( L' = L^{(m)} \subset \cdots \subset L^{(i)} = q \). It follows that \( B[1] \) is a connected (directed) subgraph of \( B \). In fact, \( B[1] \) determines \( B^1 \), and thus all of \( B \):

**Proposition 2.3.** Vertices \( x, x' \) are connected in \( B^1 \) if and only if there is a chain \( x_0, x_1, \ldots, x_d \in B^0 \) such that \( x_0 = x_d = x, (x_i, x_{i+1}) \in B[1] \) for \( i = 0, \ldots, d - 1 \), and such that \( x' \in \{x_1, \ldots, x_{d-1}\} \).

**Proof.** First assume that such a chain exists, and choose representatives \( L_i \in x_i \) with \( L_d = \sigma L_0 \). By the definition of \( B[1] \), we may assume that \( [L_i : L_{i+1}] = q \) and then \( L_d \subset \cdots \subset L_2 \subset L_1 \subset L_0 \). In particular, \([L_i, [L_0]] \) is a maximal chain of lattices in \( B^1 \) for every \( i \).

On the other hand, if \( \sigma L_0 \subset L' \subset L_0 \) (1) are lattices, we can lift a maximal flag in \( L/\sigma L_0 \cong \mathbb{F}_q^d \) to a maximal chain of lattices refining (1), resulting in a chain \( x_0, \ldots, x_d \). \( \square \)

**Corollary 2.4.** If \((x, x') \in B[t] \), then there is a path of length \( t \) in \( B[1] \) from \( x \) to \( x' \).

Using this criterion, it is easy to see that if the greatest common divisor \( (t', d) \) equals \( t \), and \((x, x') \in B[t'] \), then there is a path from \( x \) to \( x' \) in \( B[t] \). In particular, \( B[t'] \) has the same connected components as \( B[t] \). The final result of this section is not needed in the rest of the paper.

We thank the referee for some simplification in the proof of the following proposition.

**Proposition 2.5.** Let \( t \) be a divisor of \( d \). If \( t \) divides \( \varrho(x') - \varrho(x) \), then there is a path from \( x \) to \( x' \) in the (directed) graph \( B[t] \) defined above.
Proof. We may assume \( x = [L_0], \) and \( x' = [L] \) with \( L = ag'dL_0 = agL_0, \) \( a, \) \( a' \in GL_d(O) \) and \( g = \text{diag}(\sigma^1, \ldots, \sigma^d). \) We may assume \( 0 \leq i_1 \leq \cdots \leq i_d, \) and in particular \( L \subseteq L_0. \) If the claim is true for \( gL_0 \subseteq L_0, \) then acting with \( a \) we get a chain from \( aL_0 = L_0 \) to \( L; \) we can thus assume \( a = 1. \)

Before going on by induction, we multiply \( L \) by a power of \( \sigma \) so that

\[
i_1 \geq \frac{r}{d - t}(i_d - i_1).
\]

Now, by assumption, \( i_1 + \cdots + i_d = rt \) for some \( r \in \mathbb{N}. \)

Case 1. If \( i_{d-t+1} = i_1, \) then we can lower \( t \) of the entries \( i_2, \ldots, i_d, \) each by 1, keeping the increasing order. Let \( i_1', \ldots, i_d' \) denote the resulting values and \( g' = \text{diag}(\sigma^{i_1'}, \ldots, \sigma^{i_d'}). \) Then \( gL_0 \subseteq g'L_0 \subseteq L_0 \) with \( [g' L_0 : gL_0] = q' \) and \( \sigma g'L_0 \subseteq gL_0, \) so \( (g'L_0, gL_0) \in B[t]. \) Since the condition \( i_1 \geq (d/t)(i_d - i_1) \) still holds (as \( i_1 \) was not changed), we are done by induction on \( r. \)

Case 2. Now assume \( i_{d-t+1} = i_1; \) let \( j \) be maximal with \( i_j = i_1. \) If \( j = d \) then \( i_1 = \cdots = i_d, \) and we are done. Therefore assume \( j < d. \) Of course \( j \geq d - t + 1. \) If \( i_1 = 0 \) then \( i_2 = 0 \) too by the assumption (2), so \( L = L_0 \) and again we are done. Assume \( i_1 > 0 \) and \( j < d. \) In the first step we can lower \( i_{j+1}, \ldots, i_d, \) but we also have to lower \( i_1, \ldots, i_{j-(d-j)} \) in order to change exactly \( t \) components. We continue lowering the highest entries, using the remaining \( d-t \) entries \( i_{j-(d-j)+1}, \ldots, i_j \) whenever necessary. In each of the next \( d/t - 1 \) steps, \( i_d \) is lowered by one with \( i_1 \) unharmed—so at the end our condition (2) is still met. The \( d/t \) modules we constructed form a chain in \( L_0, \) climbing from \( L \) using \( B[t] \) steps, and ending with indices which now sum up to \( rt - d, \) so again we are done by induction. \( \square \)

Vertices \( x \) and \( x' \) for which \( (t, d) \) does not divide \( \varphi(x') - \varphi(x) \) cannot be connected in \( B[t], \) so we proved that \( B[t] \) has \( (t, d) \) connected components for every \( 1 \leq t < d. \)

3. The arithmetic lattice

Let \( F_q \) denote the field of order \( q \) (a prime power), and \( F_{q^d} \) the extension of dimension \( d. \) Let \( \phi \) denote a generator of the Galois group \( \text{Gal}(F_{q^d}/F_q) \). Fix a basis \( \zeta_0, \ldots, \zeta_{d-1} \) for \( F_{q^d} \) over \( F_q, \) where \( \zeta_i = \phi^i(\zeta_0). \)

Extend \( \phi \) to an automorphism of the function field \( k_1 = F_{q^d}(y) \) by setting \( \phi(y) = y; \) the fixed subfield is \( k = F_q(y), \) of co-dimension \( d. \) Let \( v_y \) denote the valuation defined by \( v_y(a_my^m + \cdots + a_ny^n) = m(a_m \neq 0, m < n), \) and set \( F = F_q((y)), \) the completion with respect to \( v_y, \) and \( O = F_q[[y]], \) its ring of integers.

Let

\[
R = F_q \left[ y, \frac{1}{y}, \frac{1}{1 + y} \right] \subseteq k,
\]

and let \( R_T \) denote the subring

\[
R_T = F_q \left[ y, \frac{1}{1 + y} \right].
\]

Since \( 1 + y \) is invertible in \( O, \) \( R_T \subseteq O. \)
For a commutative $R_T$-algebra $S$ (namely a commutative ring with unit which is an $R_T$-module, e.g. $k$, $F$, $R$ or $R/I$ for an ideal $I \subset R$), we denote by $y$ the element $y \cdot 1 \in S$. For such $S$ we define an $S$-algebra $A(S)$, by

$$A(S) = \bigoplus_{i,j=0}^{d-1} S \zeta_i z^j,$$

with the relations

$$z \zeta_i = \phi(\zeta_i) z, \quad z^d = 1 + y. \quad (4)$$

The center of $A(S)$ is $S$. We will frequently use the fact that for an $R_T$-algebra $S$, $A(S) = A(R_T) \otimes_{R_T} S$. It is well known that $A(k)$ is a central simple algebra, and so there is a norm map $A(k)^\times \to k^\times$, which induces a norm map $A(S)^\times \to S^\times$ for every $S \subseteq k$. The norm map is a homogeneous form of degree $d$ (in the coefficients of the basis elements $\{z^d\}$), so the norm is also defined for quotients $A(S/I)$. We remark that $F_q^d \otimes_{F_q} S$ is Galois over $S$ and $1 + y$ is invertible in $S$, so $A(S)$ is an Azumaya algebra over $S$ (see [5], where they are called ‘central separable algebras’). This fact will not be used in the rest of the paper.

If $A(S) \cong M_d(S)$, we say that $A(S)$ is split. We need a criterion for this to happen. If $S_1 = F_q^d \otimes_{F_q} S$ is a field (so necessarily $S$ is a subfield), then $A(S)$ is the cyclic algebra $(S_1/S, \phi, 1 + y) = S_1[z]$ with the relations in (4). This is a simple algebra of degree $d$ over its center $S$. Recall Wedderburn’s norm criterion for cyclic algebras [7, Corollary 1.7.5]: the algebra $A(S) = (S_1/S, \phi, 1 + y)$ splits iff $1 + y$ is a norm in the field extension $S_1/S$. More generally, the exponent of $(S_1/S, \phi, 1 + y)$ (i.e. its order in the Brauer group $\text{Br}(S)$) is the minimal $i > 0$ such that $(1 + y)^i$ is a norm. In particular, if this exponent is $d = [S_1 : S]$, $A(S)$ is a division algebra (since the exponent of a central simple algebra is always bounded by the degree of the underlying division algebra).

The algebra $A(R)$ will later be used to construct the desired complexes. As mentioned above, $A(R) \subseteq A(k)$, $k$ being the ring of fractions of $R$; moreover, $A(k)$ is the ring of central fractions of this algebra,

$$A(k) = (R - \{0\})^{-1} A(R).$$

We now consider completions of $A(k)$. The global field $k = F_q(y)$ has the minus degree valuation, defined for $f, g \in F_q[y]$ by $\nu(f/g) = \deg(g) - \deg(f)$. Also recall that the other non-archimedean discrete valuations of $F_q(y)$ are in natural correspondence with the prime polynomials of $F_q[y]$. For a prime polynomial $p \in F_q[y]$, the valuation is defined by $\nu_p(p'f/g) = \nu_p(g) - \nu_p(f)$. The ring of $p$-adic integers in $k$ is $F_p[y]_p = \{f/g : (p, g) = 1\}$. The completion with respect to a valuation $\nu = \nu_p$ is $k_{\nu} = F_p[y]_p((p)) = \sum_{i=-\infty}^{\infty} \alpha_i p^i$ (which we will also denote by $k_p$). The ring of integers of $k_p$ is $O_{k_p} = F_p[y]_p[[p]]$. The notation $v_1/2$ is used for the degree valuation, since the completion of $k$ with respect to this valuation is $F_p((1/y))$; moreover, filtration by the ideal $(1/y)$ of the ring of integers $F_p[[1/y]]$ determines the valuation.

The Albert–Brauer–Hasse–Noether Theorem describes division algebras over $k$ in terms of their local invariants, which translates to an injection $\text{Br}(k) \to \bigoplus \text{Br}(k_p)$. More precisely, the $d$-torsion part of $\text{Br}(k_p)$ is cyclic of order $d$ for every valuation $\nu_p$. 


Fig. 1. Subrings of $F_q((y))$.

Since $k_{1,p}/k_p$ is always unramified (where $k_{1,p}$ is the completion of $k_1$ under $v_p$), these $d$ classes can be written as the cyclic algebras $(k_{1,p}/k_1, \phi, \sigma^i)$ for $i = 0, \ldots, d - 1$, where $\sigma$ is a uniformizer. If $(k_1/k, \phi, c)$ is a cyclic $k$-algebra, the local invariants are determined by the values $\nu(c)$ ([13, Chapters 17–18] is a standard reference, though the focus is on number fields).

There are only two valuations $\nu$ of $k$ for which $\nu(1+y) \neq 0$, namely $\nu_{1+y}$ and $\nu_{1/y}$, for which the values are 1 and $-1$, respectively. We thus have

**Proposition 3.1.** The completions $A(F_q((1/y)))$ and $A(F_q((1+y)))$ of $A(k)$ are division algebras. On the other hand, for any other completion $k_v$ of $k$, $A(k_v)$ splits.

In particular

$$A(R) \subseteq A(k) \otimes_k F = A(F) \cong M_d(F).$$

The same argument embeds

$$A(R_T) \subseteq A(O) \cong M_d(O).$$

We use the algebras $A(S)$ to define algebraic group schemes. For an $R_T$-algebra $S$, let $\tilde{G}'(S) = A(S)^\times$, the invertible elements of $A(S)$, and $G'(S) = A(S)^\times/S^\times$. Recall that for every $R_T$-algebra $S$, one can define a multiplicative function $A(S) \to S^\times$ called the reduced norm (e.g. by taking the determinant in a splitting extension of $S$). In particular, the diagram in Fig. 2 commutes. We can thus define $\tilde{G}'_i(S)$ as the set of elements of $G'(S)$ of norm 1, and $G'_i(S)$ as the image of $\tilde{G}'_i(S)$ under the map $\tilde{G}'(S) \to G'(S)$ (see the square in the middle of Fig. 5).

**Remark 3.2.** The sequence

$$1 \to \mu_d(S) \to \tilde{G}'_i(S) \to G'_i(S) \to 1$$

is exact, where $\mu_d(S)$ is the group of $d$-roots of unity in $S$.

These group schemes are forms of the classical groups $\tilde{G}(S) = \text{GL}_d(S)$, $G(S) = \text{PGL}_d(S)$, $\tilde{G}_1(S) = \text{SL}_d(S)$ and $G_1(S) = \text{PSL}_d(S)$. If $A(S)$ is a matrix ring, we have that $\tilde{G}'(S) = \tilde{G}(S)$ and $G'(S) = G(S)$. 

\[\begin{array}{c}
\mathbb{F}_q((y)) \\
R_T \\
R \\
k = \mathbb{F}_q(y) \\
F = \mathbb{F}_q((y)) \\
k_1 = \mathbb{F}_q^p(y) \end{array}\]
It is useful to have equivalent definitions for the groups $G'(S)$ for various rings $S$. Fix the ordered basis

$$\{\zeta_0, \ldots, \zeta_{d-1}, \xi_0 \zeta, \ldots, \xi_{d-1} \zeta, \ldots, \xi_0 \zeta^{d-1}, \ldots, \xi_{d-1} \zeta^{d-1}\}$$

of $A(S)$ over $S$. Conjugation by an invertible element $a \in A(S)$ is a linear transformation of the algebra. Let $i_S : G'(S) \rightarrow GL_{d^2}(S)$ be the induced embedding. If $S \subseteq S'$, then the diagram

$$
\begin{array}{ccc}
G'(S') & \xrightarrow{i_S} & GL_{d^2}(S') \\
\downarrow & & \downarrow \\
G'(S) & \xrightarrow{i_S} & GL_{d^2}(S)
\end{array}
$$

commutes.

**Proposition 3.3.** Let $R_T \subseteq S \subseteq S'$ be commutative rings, such that $S$ is a Noetherian unique factorization domain. Then

$$i_S G'(S) = i_{S'} G'(S') \cap GL_{d^2}(S),$$

the intersection taken in $GL_{d^2}(S')$.

**Proof.** The inclusion $i_S G'(S) \subseteq i_{S'} G'(S') \cap GL_{d^2}(S)$ is trivial since $A(S) \subseteq A(S')$. Let $\alpha \in i_{S'} G'(S') \cap GL_{d^2}(S)$, then $\alpha$ is an isomorphism of algebras (since it is induced by an element of $A(S')$) and preserves $S$ (as it belongs to $GL_{d^2}(S)$). It is thus an automorphism of $A(S)$, which must be inner [1, Theorem 3.6]. □

The proposition covers, in particular, $S = R_T, R, k, k_v, \mathcal{O}_v$, as well as $\bar{R}_T$, and $\bar{R}$ which are defined in Section 8, with an arbitrary extension $S'$ (usually taken to be from the same list).

**Proposition 3.4.** $G'(R)$ is a discrete subgroup of $G(F)$.

**Proof.** Let $p = 1 + y$. The ring $R = \mathbb{F}_q[1/y, y, 1/(1 + y)]$ embeds (diagonally) as a discrete subgroup of the product

$$F \times k_{1/y} \times k_p.$$ 

This can be seen by letting $a_n = (f_n(y))/(y^h(1 + y)^h)$ be a sequence of non-zero elements in $R$ (with $f_n(\lambda) \in \mathbb{F}_q[\lambda]$ and $i_n, j_n \geq 0$) such that $a_n \rightarrow 0$. We then have that $v_{1/y}(a_n) \rightarrow -\infty$, which implies $i_n = 0$ for $n$ large enough and $v_{1/y}(f_n(y)) \rightarrow -\infty$. Likewise, $v_p(a_n) \rightarrow -\infty$, so $j_n = 0$ for $n$ large enough. This implies that $v_{1/y}(a_n) \rightarrow -\infty$, which is a contradiction.

It follows that the diagonal embedding of $G'(R)$ into

$$G'(F) \times G'(k_{1/y}) \times G'(k_p)$$

is discrete. But an algebraic group over a local field is compact iff it has rank zero [14]. Therefore, by Proposition 3.1, $G'(k_{1/y})$ and $G'(k_p)$ are compact, and $G'(R)$ is discrete in the other component $G'(F) = G(F)$. □
In fact, from general results it follows that $G'(R)$ is a cocompact lattice in $G'(F)$, but we will show it directly when we demonstrate that $G'(R)$ acts transitively on the vertices of the affine building of $G(F) = \text{PGL}_d(F)$.

Consider $G(O) = \text{PGL}_d(O)$, a maximal compact subgroup of $G(F)$, which is equal to $G'(O)$ by Eq. (6). Viewing $K = i_O G'(O)$ and $i_K G'(R)$ as subgroups of $i_F G'(F) \subseteq \text{GL}_d(F)$, the intersection
\[ i_R G'(R) = K \cap i_K G'(R) \tag{7} \]
is finite, being the intersection of discrete and compact subgroups (note that $R \cap O = R_T$).

**Proposition 3.5.** $G'(R_T)$ is a semidirect product of $(z) \cong \mathbb{Z}/d\mathbb{Z}$ acting on $\mathbb{F}_q^\times/\mathbb{F}_q^\times$.

**Proof.** Recall that $R_T = \mathbb{F}_q[y, 1/(1 + y)]$, so that $\mathcal{A}(R_T) = \mathbb{F}_q[y, 1/(1 + y), z]$ with the relations $z\alpha z^{-1} = \phi(\alpha)$ ($\alpha \in \mathbb{F}_q^\times$) and $z^d = 1 + y$. Setting $y = z^d - 1$, we see that $\mathcal{A}(R_T) = \mathbb{F}_q[z, z^{-1}]$ is a skew polynomial ring with one invertible variable over $\mathbb{F}_q$. Every element of $\mathcal{A}(R_T)$ has a monomial $\alpha z^r$ ($\alpha \in \mathbb{F}_q^\times$) with $r$ maximal, called the upper monomial (with respect to $z$), and similarly every element has a lower monomial. The upper monomial of a product $fg$ is equal to the product of the respective upper monomials, and likewise for the lower monomials.

Now let $f, g \in \mathcal{A}(R_T)$ be elements with $fg = 1$, then the product of the upper monomials and that of the lower monomials are both equal to 1, proving that $f$ and $g$ are monomials. Thus, the invertible elements of $\mathcal{A}(R_T)$ are $\tilde{G}(R_T) = \{\alpha z^i : \alpha \in \mathbb{F}_q^\times, i \in \mathbb{Z}\}$. The result is obtained by taking this modulo the center. \[\square\]

4. A simply transitive action on $\mathcal{B}$

We continue with the notation of the last section. The embedding (5) of $\mathcal{A}(R)$ into $\text{M}_d(F)$ induces embeddings $\tilde{G}(R) \hookrightarrow \text{GL}_d(F)$ and $G'(R) \hookrightarrow \text{PGL}_d(F)$, so $G'(R)$ acts on $\mathcal{B}^0$. Notice that $G'(k)$ is dense in $G'(F) = G(F)$, so its action on $\mathcal{B}^0$ is transitive.

In [3], Cartwright and Steger present a subgroup of $G'(k)$, which acts simply transitively on the vertices of the building $\mathcal{B}$. Identifying $\mathcal{B}^0$ with a group, when possible, is an important tool in the description of finite quotients of $\mathcal{B}$. We will use the Cartwright–Steger group, constructed in a different way. In particular we construct the group as a subgroup of $G'(R)$, which is easily shown to be discrete. The proof that it acts transitively relies on the existence of an element of the appropriate norm, and that the action is simple is seen by a relatively easy computation of the group scheme over various rings.

Recall that $\mathcal{A}(R) = R[\zeta_i, z]$ where $z\zeta_i z^{-1} = \phi(\zeta_i)$ and $z^d = 1 + y$, and let
\[ b = 1 - z^{-1} \in \mathcal{A}(R). \]

**Proposition 4.1.** The reduced norm of $b$ in $\mathcal{A}(R)$ is $y/(1 + y)$.

**Proof.** Recall that $\mathcal{A}(R) \subseteq \mathcal{A}(k)$ and that the restriction of the norm function of $\mathcal{A}(k)$ to $\mathcal{A}(R)$ is the norm function of $\mathcal{A}(R)$. As is the case for matrices, if $a \in \mathcal{A}(R)$ generates a subalgebra of dimension $d$, its norm $N(a)$ is the free coefficient in the characteristic polynomial of $a$ multiplied by $(-1)^d$. 
The minimal polynomial of $z$ is $\lambda^{d} - (1 + y)$, so that $N(z) = (-1)^{d-1}(1 + y)$. Likewise, the minimal polynomial of $z - 1$ is $(\lambda + 1)^{d} - (1 + y)$, with the free coefficient $-y$, so that $N(z - 1) = (-1)^{d-1}y$. Finally, since $b = z^{-1}(z - 1), N(b) = N(z - 1)/N(z) = y/(1 + y)$. □

Since $1 + y$ is invertible in $\mathcal{O}$, we claim

**Corollary 4.2.** $N(b) \equiv y \pmod{\mathcal{O}^{\times}}$.

Under the embedding (6), $b \in \mathcal{A}(RT) \subseteq \text{M}_{d}(\mathcal{O})$, so $bL_{0} \subseteq L_{0}$, where $L_{0} = \mathcal{O}^{d}$ is the standard lattice. By the corollary, and since the diagram in Fig. 2 commutes, we have $[L_{0} : bL_{0}] = q$. Note that $b$ is not invertible in $\mathcal{A}(RT)$ (since the norm $y/(1 + y)$ is not invertible in $RT$), but $b$ is invertible in $\mathcal{A}(R)$.

Let $\Omega$ denote the set of neighbors of $[L_{0}]$ in $B$ which have color 1; they correspond to sublattices of index $q$ in $L_{0}$. Consider the element

$$b_{u} = ubu^{-1} = 1 - \frac{u}{\phi(u)}z^{-1}$$

where $u \in \mathbb{F}_{q^{d}}^{\times} \subseteq \mathcal{A}(R)^{\times}$. The embedding $\mathcal{A}(R) \subseteq \text{M}_{d}(F)$ extends the regular embedding $\mathbb{F}_{q^{d}} \hookrightarrow \text{M}_{d}(\mathbb{F}_{q})$ (via the basis $\zeta_{0}, \ldots, \zeta_{d-1}$), and in particular it takes $u$ to a matrix with coefficients in $\mathbb{F}_{q}^{d}$. This proves that $uL_{0} = L_{0}$, and since $\text{GL}_{d}(F)$ preserves the structure of $B$, $u$ permutes the vertices in $\Omega$. Moreover, the sublattices of index $q$ of $L_{0}$ correspond to the points of the projective space $\mathbb{P}_{q^{d}}^{\times}$ via the isomorphism $L_{0}/yL_{0} \cong \mathbb{F}_{q^{d}}^{\times}$. The action of $u$ on the latter space is by multiplication, so $\mathbb{F}_{q^{d}}^{\times} \subseteq \mathcal{A}(R)^{\times}$ acts transitively on $\Omega$.

Finally, we have that $b_{u}(L_{0}) = ubu^{-1}L_{0} = u(bL_{0})$, so we proved:

**Proposition 4.3.** For every lattice $L_{1} \in \Omega$, there is an element $u \in \mathbb{F}_{q^{d}}^{\times}$ (unique up to multiplication by $\mathbb{F}_{q}^{\times}$) such that $b_{u}L_{0} = L_{1}$.

**Definition 4.4.** Let $\tilde{\Gamma}$ denote the subgroup

$$\tilde{\Gamma} = \{b_{u} : u \in \mathbb{F}_{q^{d}}^{\times}/\mathbb{F}_{q}^{\times}\}$$

(8)

of $\tilde{G}'(R) = \mathcal{A}(R)^{\times}$, and let $\Gamma$ be the image of $\tilde{\Gamma}$ under the projection $\tilde{G}'(R) \to G'(R)$.

**Proposition 4.5.** $\Gamma$ acts transitively on $B^{0}$.

**Proof.** We show that $L_{0}$ can be taken to any of its sublattices $L$ by an element of $\Gamma$. The proof is by induction on $[L_{0} : L]$. If $[L_{0} : L] = q$, then $L \in \Omega$ and $b_{u}L_{0} = L$ for some $u$. Assume $[L_{0} : L] > q$, and let $L \subseteq L' \subseteq L_{0}$ be an intermediate sublattice such that $[L' : L] = q$. By the induction hypothesis there is an element $c \in \Gamma$ such that $cL_{0} = L'$. The lattices in $c\Omega$ are the sublattices of index $q$ of $L'$, so for some $u \in \mathbb{F}_{q^{d}}^{\times}$, we have that $c(b_{u}L_{0}) = L$, which proves our claim. □

**Corollary 4.6.** If $L \subseteq L_{0}$ and $[L_{0} : L] = q^{\ell}$, then there are $u_{1}, \ldots, u_{\ell} \in \mathbb{F}_{q^{d}}^{\times}/\mathbb{F}_{q}^{\times}$ such that $[L] = b_{u_{1}} \cdots b_{u_{\ell}}L_{0}$.

Moreover, if $L \subseteq L' \subseteq L_{0}$ where $[L_{0} : L'] = q$ and $[L'] = b_{u_{1}} \cdots b_{u_{k}}L_{0}$, then for suitable $u_{k+1}, \ldots, u_{\ell}$ we have $[L] = b_{u_{1}} \cdots b_{u_{k}}b_{u_{k+1}} \cdots b_{u_{\ell}}L_{0}$.
Let \( R_0 = \mathbb{F}_q[1/y] \). As \( R_0 \) does not contain \( R_T \), \( A(R_0) \) is not defined, so we cannot define \( G'(R_0) \) in the usual way. However, following Proposition 3.3, we set \( G'(R_0) = i_k G'(k) \cap GL_dz(R_0) \).

As mentioned in the Introduction, Cartwright and Steger present in [3] a group \( \Gamma' \) which acts simply transitively on \( B^0 \). Their group is defined as follows:

**Definition 4.7.** The group \( \Gamma' \) is composed of the elements of \( G'(R_0) \) which modulo \( 1/y \) are upper triangular with identity \( d \times d \) blocks on the diagonal.

A careful computation, using \( b_u^{-1} = 1 + (1/y) \sum_{i=0}^{d-1} u \phi^i(u)^{-1} z^i \), reveals that

\[
b_u(\zeta z^k)b_u^{-1} = \zeta z^k + f_{n,k}(\zeta) \sum_{i=0}^{d-1} \frac{u}{\phi^i(u)} z^i + \frac{1}{y} f_{n,k}(\zeta) \sum_{i=0}^{d-1} \frac{u}{\phi^i(u)} z^i
\]

for every \( 0 \leq k < d \) and \( \zeta \in \mathbb{F}_q^* \), where \( f_{u,k}(\zeta) = (\phi^k u/y) \zeta - (\phi^{-1} u/y) \phi^{-1} \zeta \).

It follows that \( b_u \in \Gamma' \), so \( \Gamma' = \langle b_u \rangle \subseteq \Gamma' \). This leads to an easy proof of the main property of \( \Gamma' \):

**Proposition 4.8.** The group \( \Gamma' \) acts simply transitively on \( B^0 \), namely, for \( K = G(O) \),

\[
\Gamma' \cdot K = G(F),
\]

\[
\Gamma' \cap K = 1.
\]

In particular, \( \Gamma' = \Gamma' \).

**Proof.** The first equation, equivalent to \( \Gamma' \) acting transitively on \( B_0 = G(F)/K \), was proved in Proposition 4.5. To compute the intersection, note that \( \Gamma \cap K \subseteq \Gamma' \cap K \subseteq G'(R) \cap K \cap G'(R_T) \) by Eq. (7), so \( \Gamma \cap K \subseteq G'(R_T) \cap \Gamma' \). Let \( \zeta z_j \) be an element of \( G'(R_T) \cap \Gamma' \) for \( \zeta \in \mathbb{F}_q^* \) and \( j = 0, \ldots, d - 1 \) (see Proposition 3.5). By the definition of \( \Gamma' \), the \( d^2 \times d^2 \) matrix representing \( \zeta z_j \) should be upper triangular modulo \( 1/y \). Now, \( (\zeta z_j) z_j (\zeta z_j)^{-1} = (\zeta/\phi(\zeta)) z_j \), so \( \zeta \in \mathbb{F}_q^* \). Furthermore for every \( a \in \mathbb{F}_q^* \),

\[
(\zeta z_j) a (\zeta z_j)^{-1} = \phi^j(a),
\]

proving that \( j = 0 \) and \( \zeta z_j \) is central; thus \( \Gamma' \cap K = 1 \). This proves that \( \Gamma' \) acts simply transitively on \( G(F)/K \), and since \( \Gamma \subseteq \Gamma' \), these groups are equal. \( \square \)

**Proposition 4.9.** The group \( \Gamma \) is a normal subgroup of \( G'(R) \), and \( G'(R) = G'(R_T) \ltimes \Gamma \).

Moreover,

\[
G'(R_0) = G'(R).
\]
Proof. Let $g \in G'(R)$, and write $g = \gamma a$ for $\gamma \in \Gamma \subseteq G'(R)$ and $a \in K$. Then $a = \gamma^{-1}g \in K \cap G'(R) = G'(R_T)$, by Eq. (7), and $G'(R) = \Gamma \cdot G'(R_T)$.

The generators of $\Gamma$ are permuted by $G'(R_T)$: $(\zeta_i z^j)b_u(\zeta_i z^j)^{-1} = b_{\zeta_i \phi(u)}$, so $\Gamma < G'(R)$. Moreover, $\Gamma \cap G'(R_T) \subseteq \Gamma \cap K = 1$, proving the second claim.

Finally, $G'(R_0) \subseteq G'(R)$ but direct inspection using Eq. (9), shows that $\Gamma$ and $G'(R_T)$ are contained in $\text{GL}_d(\mathbb{R}_0)$ (see [3, Theorem 2.6]). □

Since $\Gamma$ acts simply transitively, we can identify $B_0$ with $\Gamma$ by $\gamma \mapsto \gamma[L_0]$. Recall that by Eq. (5), $\Gamma \subseteq \text{PGL}_d(F)$. The determinant (modulo scalars) thus takes $\gamma \mapsto \det(\gamma) \cdot F^\times = F^\times/F^\times d$. Considering this modulo $O^\times$ (which is superfluous if $d$ is prime to $q$), we obtain the cyclic group $F^\times/F^\times d O^\times \cong \langle y \rangle/(y^d)$. Moreover, since $\phi(b_u) = \phi(x) + 1$ and $N(b_u) \equiv y \mod O^\times$ by Corollary 4.2, the diagram in Fig. 3 commutes. We let $\delta$ denote the color map $\Gamma \to \mathbb{Z}/d$. Letting $\Gamma_1 = \text{Ker}(\delta)$, the short sequence

$$1 \longrightarrow \Gamma_1 \longrightarrow \Gamma \longrightarrow \mathbb{Z}/d \mathbb{Z} \longrightarrow 1$$

is exact.

![Fig. 3. Determinant and coloring.](image)

Corollary 4.10. $[\Gamma : \Gamma_1] = d$.

5. The geometry of $\Gamma$

By definition, $\Gamma$ is generated by the elements $b_u = u(1 - z^{-1})u^{-1}$ of $G'(R)$, where $u$ ranges over $\mathbb{F}_q^\times/\mathbb{F}_q^\times$.

Since $b_u = 1 - (u/\phi(u))z^{-1}$, for every $u$, there is a unique $r \in \mathbb{F}_q^\times$ with $N(r) = 1$, such that $b_u = b(r) = 1 - rz^{-1}$, where $N = N_{\mathbb{F}_q^\times/\mathbb{F}_q}$ is the norm map.

Remark 5.1. For every $u \neq v \in \mathbb{F}_q^\times/\mathbb{F}_q^\times$ there are unique $u', v' \in \mathbb{F}_q^\times/\mathbb{F}_q^\times$ such that

$$b_u b_{u'} = b_v b_{v'}. \quad (13)$$

Proof. The conceptual reason is that since $b_u L_0 \neq b_v L_0$, their intersection $L = b_u L_0 \cap b_v L_0$ is a submodule of index $q^2$ of $L_0$ (which contains $yL_0$), and so by Corollary 4.6 there are unique $u', v'$ such that $L = b_u b_{u'} L_0 = b_v b_{v'} L_0$. Since $\Gamma$ acts simply transitively, $b_u b_{u'} = b_v b_{v'}$. 

Computationally, assume first \( d > 2 \). Since \((1 - rz^{-1})(1 - r'z^{-1}) = 1 - (r + r')z^{-1} + r\phi^{-1}(r')z^{-2}\), \( b(r)b(r') = b(s)b(s') \) in \( \Gamma \) iff

\[
r + r' = s + s' \quad \text{and} \quad r\phi^{-1}(r') = s\phi^{-1}(s').
\]

The unique solution is \( r' = (s - r)/(\phi s - \phi r)\phi(s) \) and \( s' = (s - r)/(\phi s - \phi r)\phi(r)\), where \( r' \) and \( s' \) indeed have norm 1.

If \( d = 2 \) then (as elements of \( \Gamma \)) \( b_{(-r)}b_{(r)} = 1 - (r\phi(r)/(1 + y)) \), which is in the center \( R = Z(\mathcal{A}(\Gamma)) \). Therefore in \( \Gamma \), \( b_{(-r)}^{-1} = b_{(-r)} \). Now the unique solution to the above equation is \( b_{(r)}b_{(-r)}^{-1} = 1 = b_{(s)}b_{(-s)}. \)

The following is essentially a reformulation of the simply connectedness of \( \mathcal{B} \).

**Theorem 5.2.** Let \( \mathcal{G}' \) be the abstract group generated by \( \{x_u : u \in \mathbb{Z}_q^\infty/\mathbb{Z}_q^\infty \} \), with the relations \( x_u x_u = x_v x_v \) whenever \( b_u b_u = b_v b_v \) in \( \Gamma \), and a single relation \( x_{u_1} \cdots x_{u_d} = 1 \) for some \( u_1, \ldots, u_d \) such that

\[
r = b_{u_1} \cdots b_{u_d} = 1
\]

in \( \Gamma \). Then \( x_u \mapsto b_u \) is an isomorphism \( \mathcal{G}' \to \Gamma \).

**Proof.** Since \( \Gamma \) acts simply transitively, \( b_{u_1} \cdots b_{u_n} = 1 \) iff \( b_{u_1} \cdots b_{u_n} L = y^1 L \) for some lattice \( L \) and \( i \in \mathbb{Z} \).

We first claim that every equation \( \gamma = b_{u_1} \cdots b_{u_n} = b_{v_1} \cdots b_{v_n} \) (both expressions of the same length) can be obtained from the relations of type (13). Indeed, for \( n \leq 2 \) the claim is obvious.

If \( u_n = v_n \) then we are done by induction, so we assume \( u_n \neq v_n \). By the above remark there are \( u' \) and \( v' \) such that \( \delta = b_{u_1} b_{u'} = b_{v_1} b_{v'} \), and then \( \delta L_0 = b_{u_1} L_0 \cap b_{v_1} L_0 \) is of index \( q^2 \) in \( L_0 \), and contains \( \gamma L_0 \), see Fig. 4. By Corollary 4.6 there are \( u_3, \ldots, u_n \) such that \( \gamma L_0 = b_{u_3} \cdots b_{u_n} L_0 \). By induction the equations \( b_{u_2} \cdots b_{u_n} = b_{v_2} \cdots b_{v_n} \) follow from the relations (13), and so is the case with \( b_{u_1} b_{u_2} \cdots b_{u_n} = b_{v_1} b_{v_2} \cdots b_{v_n} \), which proves our claim. In particular, all the relations of the form (14) are equivalent modulo the relations (13).

Let \( u \in \mathbb{Z}^\infty_q / \mathbb{Z}_q^\infty \). Using Corollary 4.6 we see that \( b_u^{-1} \) is equal to a product of \( d - 1 \) generators. Given a relation, we can thus assume it has the form \( b_{u_1} \cdots b_{u_n} = 1 \). Then \( b_{u_1} \cdots b_{u_n} L_0 = y^1 L_0 \) where \( n = d i \), so \( b_{u_1} \cdots b_{u_n} = r^1 \) is an equality of elements of equal length, which follows from the relations (13) by the above claim.

**Remark 5.3.** If \( d \) is a power of \( \text{char } F \), then Eq. (14) takes a particularly simple form:

\[
b_i^d = (1 - z^{-1})^d = 1 - z^{-d} = y/(1 + y) \in R^\times, \quad \text{so } b_i^d = 1 \text{ in } \Gamma.
\]

From the theorem, together with the case \( d = 2 \) of Remark 5.1, we obtain

**Corollary 5.4.** If \( d = 2 \), then the defining relations of \( \Gamma \) are \( b_{(-r)}^{-1} = b_{(-r)} \). In particular, if \( q \) is odd then \( \Gamma \) is free on \( (q + 1)/2 \) generators, and if \( q \) is even then \( \Gamma \) is a free product of \( q + 1 \) cyclic groups of order 2.
6. Finite quotients of $B$

In this section we specify quotients of $\Gamma$, whose Cayley graphs (with respect to the generators $b_u$) define Ramanujan complexes. These quotients are shown to be subgroups of $\text{PGL}_d$ which contain $\text{PSL}_d$, over a predetermined finite local ring.

Recall the definitions of $\tilde{G}^0$, $\tilde{G}^1$, $G^0$ and $G^1$ from Section 3. Let $S$ be an $R_T$-algebra, and $I \triangleleft S$ an ideal. The map $S \to S/I$ induces an epimorphism of algebras $\mathcal{A}(S) \to \mathcal{A}(S/I)$. It also induces a homomorphism $\tilde{G}(S) \to \tilde{G}(S/I)$ (which, in general, is not onto). The kernel of this map is called the congruence subgroup of $\tilde{G}(S)$ with respect to $I$ and denoted by $\tilde{G}(S, I)$. In a similar manner we have a map $G^0(S) \to G^0(S/I)$ defined by $aS^x \mapsto (a + IA(S))(S/I)^x$, and the kernel is denoted $G^0(S, I)$. Note that $\tilde{G}(S, I)$ is mapped into $G(S, I)$, but this map is not necessarily onto. We set $\tilde{G}^1(S, I) = \tilde{G}^1(S) \cap G^1(S, I)$, and likewise $G^1(S, I) = G^1(S, I) \cap G^1(S)$. (Again, note that $G^1(S, I) = \{aS^x : a \in \tilde{G}^1(S), a \in S^x + IA(S)\}$, while $\tilde{G}^1(S, I) = \{a : a \in \tilde{G}^1(S), a \in 1 + IA(S)\}$, so the map $\tilde{G}^1(S, I) \to G^1(S, I)$ induced by $\tilde{G}(S) \to G^1(S)$ may not be onto). See Fig. 5.
Let $I \triangleleft R$. We define the congruence subgroup of $\Gamma$ to be

$$\Gamma(I) = \Gamma \cap G'(R, I),$$

(15)
a normal subgroup of $G'(R)$ (by Proposition 4.9). Recall that by Proposition 4.8, $\Gamma$ acts
simply transitively on the vertices of $B = G(F)/K$, where $K = G(O)$. The set of vertices in
the quotient complex $B_I = \Gamma(I) \setminus G(F)/K$ is isomorphic to $\Gamma/\Gamma(I)$. To make this an
isomorphism of complexes, we need to define a complex structure on $\Gamma/\Gamma(I)$.

The generators $\{b_\alpha\}$ correspond to the neighbors of color 1 of $[L_0]$. More generally,
by Corollary 4.6 the neighbors of color $k$ correspond to the products $b_{\alpha_1} \cdots b_{\alpha_k}$ which are
headers of a product of $d$ generators which equals 1. The 1g-skeleton of $B_I$ is thus the
Cayley graph of $\Gamma/\Gamma(I)$ with respect to these generators (for $k = 1, \ldots, d - 1$), and there are ‘partial Laplacian operators’ $A_k$ of $\Gamma/\Gamma(I)$, induced from the Hecke operators of $B$.
Frequently, these are the colored Laplacian operators (i.e. they can be defined for $\Gamma/\Gamma(I)$
directly); we address this issue towards the end of this section. The higher dimensional
cells of $\Gamma/\Gamma(I)$ are then defined to make it a clique complex. Thus, $\Gamma(I) \setminus G(F)/K$ and $\Gamma/\Gamma(I)$ become isomorphic complexes.

**Remark 6.1.** If $f(\lambda) \in \mathbb{F}_q[\lambda]$ is divisible by $\lambda$, then $G'(R_0, (f(1/y))) \subseteq G'(R_0, (1/y)) \subseteq \Gamma$, by Definition 4.7 of $\Gamma'$ and Eq. (12). In this case $\Gamma(I) = G'(R_0, I)$ (where $I = (f(1/y))$ so $\Gamma(I) \setminus B \cong \Gamma/\Gamma(I)$.

Let $C = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_i| = 1, z_1 \cdots z_d = 1\}$ and $\sigma : C \to \mathbb{C}^{d-1}$ the map
defined by $(z_1, \ldots, z_d) \mapsto (\lambda_1, \ldots, \lambda_{d-1})$, where

$$\lambda_k = q^{(d-k)/2} \sigma_k(z_1, \ldots, z_d)$$

and $\sigma_k$ are the symmetric functions. Then $\mathbb{S}_d = \sigma(C)$ is the simultaneous spectrum of the Hecke operators $A_1, \ldots, A_{d-1}$ acting on $B$ [11, Theorem 2.9].

Recall [11] that $B_I$ is called a Ramanujan complex if the eigenvalues of every non-trivial simultaneous eigenvector $v \in L^2(B_I), A_k v = \lambda_k v$, satisfy $(\lambda_1, \ldots, \lambda_{d-1}) \in \mathbb{S}_d$. It can be shown that the closure of the union of the sets of simultaneous eigenvalues over any family of quotients of $B$ (with unbounded injective radius) contains $\mathbb{S}_d$.

The following theorem relies on one of the main results of [11], for which we assume
the global Jacquet–Langlands correspondence for function fields. See the Introduction for
more details.

**Theorem 6.2.** For every $d \geq 2$ and every $0 \neq I \triangleleft R$, the Cayley complex of $\Gamma/\Gamma(I)$ is a
Ramanujan complex.

**Proof.** Write $I = R_p$ for some polynomial $p \in \mathbb{F}_q[y]$ which is prime to $y$ and $1 + y$,
then $I \cap R_0$ is generated by $y^{-e} p$ where $e = \deg(p)$. Let $I' = (y^{-e-1} p) \triangleleft R_0$. Set
$N = G'(R_0, I')$, then $N \subseteq \Gamma$ by Remark 6.1, and hence $N \subseteq \Gamma(I)$ (this property is not
satisfied by the more ‘natural’ candidate $G'(R_0, (y^{-e} p))$).

In the notation of [11], $D = \mathcal{A}(k)$ is the division algebra used to define $G' = D^\times/Z^\times$,
$F$ is the completion of $k$ with respect to $v_0 = v_y$, and the ramification places are
of a prime ideal (so \( R \) where the final isomorphism follows from the definition of \( \Gamma \)).

Proof.

\[
T = \{ v_1, v_1 + y \}. \quad \text{The ring } R_0 \text{ is }
\]
\[
R_0 = \bigcap_{v \in \mathcal{V} - \{ v_0 \}} (k \cap \mathcal{O}_v) = \mathbb{F}_q[1/y],
\]
where \( \mathcal{V} \) is the set of valuations of \( k \). Since
\[
R = \bigcap_{v \in \mathcal{V} - \{ v_0 \}} (k \cap \mathcal{O}_v),
\]
every ideal of \( R \) restricts to an ideal of \( R_0 \) which is prime to \( T \).

Note that \( I \) is prime to \( v_1 + y \) (since \( v_1 + y(p) = 0 \)). In [11, Theorem 1.3(a)] we prove that \( X = N \setminus B = N \setminus \Gamma K/K \) is Ramanujan. This complex is isomorphic to \( \Gamma / N \) by \( N \gamma K \mapsto \gamma N \), so \( \Gamma / N \) is Ramanujan. Since \( N \subseteq \Gamma(I) \), \( \Gamma / \Gamma(I) \) is also Ramanujan.

Our next goal is to identify \( \Gamma / \Gamma(I) \) as an abstract group. We assume that \( I \) is a power of a prime ideal (so \( R/I \) is a finite local ring). Let \( (p) = \sqrt{I} \) denote the radical of \( I \), so \( I = (p^s) \) for some \( s \geq 1 \), where \( p \in \mathbb{F}_q[y] \) is prime. By definition of \( \Gamma(I) \), we have that
\[
\Gamma / \Gamma(I) = \Gamma / (\Gamma \cap G'(R, I)) \cong (\Gamma \cdot G'(R, I))/G'(R, I) \leq G'(R) / G'(R, I) \leftrightarrow G'(R/I) \cong \text{PGL}_d(R/I),
\]
where the final isomorphism follows from the definition of \( G'(R/I) \) as \( A(R/I)^\times \) modulo center, and Wedderburn’s theorem which implies that \( A(R/I) \cong M_d(R/I) \).

**Theorem 6.3.** The map \( G'_1(R) \to G'_1(R/I) \) is onto.

Proof. Let \( \mathbb{A} \) denote the ring of Adèles over \( k \), and \( \mathbb{A}_0 = \mathbb{A}/F \). Since \( \tilde{G}'_1 \) is connected, simply connected and simple, and \( G'_1(F) \) is non-compact, the strong approximation theorem ([14, Theorem 7.2], [15]) asserts that \( \tilde{G}'_1(k)G'_1(F) \) is dense in \( \tilde{G}'_1(\mathbb{A}) \) (see [11, Section 3.2]). Therefore, \( \tilde{G}'_1(k) \) is dense in \( \tilde{G}'_1(\mathbb{A}_0) \). Let \( U = \tilde{G}'_1(k_{1/y})\tilde{G}'_1(k_{1+y})\prod_{v \notin v_1, v_1+y, v_5} \tilde{G}'_1(\mathcal{O}_v) \), an open subgroup of \( \tilde{G}'_1(\mathbb{A}_0) \). Then \( \tilde{G}'_1(R) \cap U \) is dense in \( U \).

In particular, \( G'_1(R) \) is dense in \( G'_1(\mathcal{O}_p) \) where \( \mathcal{O}_p \) is the ring of integers in the completion \( k_p \). (Notice that \( R/I \cong \mathcal{O}_p/P_p^s \) where \( P_p \) is the maximal ideal of \( \mathcal{O}_p \). Since \( G'_1(R/I) \cong \text{PSL}_d(R/I) \) which is finite, it is enough to show that the map from \( G'_1(\mathcal{O}_p) \) to \( G'_1(R/I) = G'_1(\mathcal{O}_p/P_p^s) \) is onto.

Let \( c \in G'_1(\mathcal{O}_p/P_p^s) \), so \( c \) is an automorphism of \( A(\mathcal{O}_p/P_p^s) \) which is induced by an element \( \tilde{c} \) of norm one in \( A(\mathcal{O}_p/P_p^s) \). Lift it to an element \( \tilde{a} \) of \( A(\mathcal{O}_p) \), then \( N(\tilde{a}) \in \mathcal{O}_p \) and is equivalent to 1 modulo \( P_p^s \), thus \( \tilde{a} \) is invertible and in \( G'(\mathcal{O}_p) \).

If \( d \) is prime to \( |\mathcal{O}_p/P_p| \), then there is some \( \alpha \in \mathcal{O}_p \) such that \( \alpha^d = N(\tilde{a}) \), so \( N(\alpha^{-1} \tilde{a}) = 1 \). Thus the automorphism induced by \( \alpha^{-1} \tilde{a} \) on \( A(\mathcal{O}_p) \) is in \( G'_1(\mathcal{O}_p) \), call it \( \alpha \). Now \( \alpha \) covers \( c \), and we are done.
When \( d \) is not prime to \(|O_p/P_p|\), we construct an element of \( A(O_p) \) by induction. Let \( v \in \mathbb{F}_{q^d} \) be an element with \( \text{tr} \mathbb{F}_{q^d}/\mathbb{F}_q(v) = 1 \). Take \( \tilde{a}_i = \tilde{a}_i \). Assume \( \text{N}(\tilde{a}_i) \equiv 1 \pmod{p^i} \) for some \( i \geq s \). Write \( \text{N}(\tilde{a}_i) \equiv 1 + gp^{i+1} \pmod{p^{i+1}} \) for some \( g \in R \). Let \( \tilde{a}_{i+1} = (1 - vgp^i)\tilde{a}_i \), so \( \tilde{a}_{i+1} \equiv \tilde{a}_i \pmod{p^i} \). Since \( \text{N}(1 - vgp^i) = \prod_{j=0}^{d-1}(1 - \phi^i(v)g^j) \equiv 1 - gp^i \pmod{p^{i+1}} \) by our choice of \( v \), we have \( \text{N}(\tilde{a}_{i+1}) \equiv (1 - gp^i)(1 + gp^i) \equiv 1 \pmod{p^{i+1}} \). Now let \( \tilde{a}_0 = \lim_{i \to \infty} \tilde{a}_i \). Then \( \text{N}(\tilde{a}_0) = \lim_{i \to \infty} \text{N}(\tilde{a}_i) = 1 \), and \( \tilde{a}_0 \equiv \tilde{a}_i \pmod{p^i} \), proving that the image of \( \tilde{a}_0 \) in \( \tilde{G}'_1(O_p/P_p) \) is \( \tilde{c} \). Consequently, the image of \( \tilde{a}_0 \) in \( G'_1(O_p) \) covers \( c \).

Let \( H = G'(R, I), H_1 = H \cap G'_1(R) = G'_1(R, I) \) and \( \Gamma_1 = \Gamma \cap G'_1(R) \). (Note that the map \( G'_1(R) \to G'(R) \) takes \( \tilde{\Gamma}_1 = \tilde{\Gamma} \cap \tilde{G}'_1(R) \) to \( \{ \gamma R^\times : \gamma \in \tilde{\Gamma}, \text{N}(\gamma) = 1 \} \) which may be a proper subgroup of \( \Gamma_1 = \{ \gamma R^\times : \gamma \in \tilde{\Gamma}, \text{N}(\gamma) \in R^{\times d} \} \).

In order to complete the identification of \( \Gamma / \Gamma'(I) \), we need to compute the index of some subgroups of \( G'(R) \). From the definition of \( R \), it follows that \( R^\times = \mathbb{F}_{q^d}^\times \{ y^i(1 + y)^j : i, j \in \mathbb{Z} \} \). The norm map \( A(R)^\times \to R^\times \) is onto, since \( \text{N}(b_u) = y/(1 + y), \text{N}(z) = 1 + y \), and the norm \( \mathbb{F}_{q^d}^\times \to \mathbb{F}_q^\times \) is onto. Therefore, the map \( G'(R) \to R^\times / R^{\times d} \), defined by \( a R^\times \mapsto \text{N}(a)R^{\times d} \), is also onto. If \( a R^\times \) is in the kernel, then \( \text{N}(a) = t^d \) for some \( t \in R^\times \), and then \( \text{N}(t^{-1}a) = 1 \) and \( a R^\times = t^{-1}a R^\times \in \tilde{G}'_1(R) \). It follows that \( G'(R)/G'_1(R) \equiv R^\times / R^{\times d} \), and

\[
[G'(R) : G'_1(R)] = \left| R^\times / R^{\times d} \right| = d^2(d, q - 1).
\] (16)

On the other hand, from Corollary 4.10, Propositions 3.5 and 4.9, we obtain

\[
[G'(R) : \Gamma_1] = \frac{q^d - 1}{q - 1}d^2.
\] (17)

**Corollary 6.4.** \( \Gamma_1 \) is not contained in any congruence subgroup \( G'_1(R, I) \) for \( 0 \neq I \subset R \) \((I \neq R)\).

**Proof.** We may assume \( I \) is maximal. By Theorem 6.3 the index of \( G'_1(R, I) \) in \( G'_1(R) \) is \( |PSL_d(R/I)| \geq |PSL_d(\mathbb{F}_{q^d})| \geq (q^d - 1)/(q - 1)d(q - 1) \), so that \( |G'(R) : G'_1(R, I)| = d^2(d, q - 1)|G'_1(R) : G'_1(R, I)| > ((q^d - 1)/(q - 1))d^2 = [G'(R) : \Gamma_1] \). \( \square \)

**Corollary 6.5.** For every \( I \subset R \), \( G'_1(R) = \Gamma_1 \cdot H_1 \) where \( H_1 = G'_1(R, I) \).

**Proof.** By Proposition 4.9, \( \Gamma \) and \( H \) are normal in \( G'(R) \), so \( \Gamma_1 H_1 \) is normal in \( G'_1(R) \), and \( \Gamma_1 H_1/H_1 \) is normal in \( G'_1(R) / H_1 \equiv G'_1(R) / I \), which is isomorphic to \( PSL_d(R/I) \).

Recall that \( I = \langle p^s \rangle \), where \( p \) is prime, and \( R/\langle p \rangle = \mathbb{F}_{q^s} \) for an appropriate \( s \). First assume that \( PSL_d(\mathbb{F}_{q^s}) \) is simple, so that \( d > 2 \) or \( q^s > 3 \). The group \( PSL_d(R/I) \) has normal subgroups \( N_j = \text{Ker}(PSL_d(R/I) \to PSL_d(R/\langle p^j \rangle)) \). Now, the composition factors are \( PSL_d(R/I)/N_1 \equiv PSL_d(R/\langle p \rangle) \), and the \( N_j / N_{j+1} \) for \( j = 1, \ldots, s - 1 \), which are all isomorphic to the zero-trace subalgebra of \( \text{M}_{d}(\mathbb{F}_{q^s}) \), and are irreducible \( PSL_d(\mathbb{F}_{q^s}) \)-modules. If \( N \leq PSL_d(R/I) \) is a normal subgroup which is not contained in \( N_1 \), then \( PSL_d(R/I)/N_1 \equiv N/(N \cap N_1) \) is a group of order power of \( q \), which thus has a normal subgroup of index \( q \), producing an impossible composition factor.
This proves $\Gamma_1 H_1 = G_1$, for otherwise $\Gamma_1 H_1 / H_1$ is a proper normal subgroup of $G_1 / H_1$, which is therefore contained in $N_1 / H_1 = G'_1(R, \sqrt{I})$, contradicting the last corollary.

In the special case $d = 2$ and $|R/I| = 3$, $G'_1(R) / H_1 \cong A_4$ and has no non-trivial normal subgroups of index $\leq 2 = [G'_1(R) : \Gamma_1]$. The case $|R/I| = 2$ is impossible since the prime generator $p$ must be different than $y$ and $1 + y$, and so it cannot be linear over $\mathbb{F}_2$. □

There is a delicate point here: the claim that $G'_1(R) = \Gamma_1 \cdot G'_1(R, I)$ holds for every proper ideal of $R$, but cannot be extended to the wider class of ideals of $R_0$ (in the form $G'_1(R) = \Gamma_1 \cdot G'_1(R_0, I)$); indeed, we saw in the proof of Theorem 6.2 that $G'_1(R_0, I) \subseteq \Gamma_1$ for suitable ideals of $R_0$.

From this and the definition of $\Gamma_1$ and $H_1$, it follows that for the subgroups of $G'(R)$ appearing in Fig. 6, the intersections and products can be read from the lattice. The only non-trivial equality is $\Gamma_1 H_1(\Gamma \cap H) = \Gamma_1 H \cap \Gamma H_1$, and this too is easily checked (if $\gamma h = \gamma' h_1$ for $\gamma \in \Gamma_1$, $\gamma \in \Gamma$, $h \in H$ and $h_1 \in H_1$, then $\gamma^{-1} \gamma_1 = h_1 h^{-1} \in \Gamma \cap H$ so $\gamma_1 (h_1 h^{-1})^{-1} h_1 \in \Gamma_1 H_1(\Gamma \cap H)$).

In particular, the quotients of the extensions denoted by double lines are isomorphic to $G_1 / H_1 = G'_1(R) / G'_1(R, I) \cong G'_1(R/I) \cong \text{PSL}_{d}(R/I)$.

Let $L = R/I$. We now observe that

$$\text{PSL}_{d}(L) \cong \Gamma_1 H_1 / H_1 \cong \Gamma \cap H / H \leq \Gamma H / H \leq G'(R) / H \leq \text{PGL}_{d}(L),$$

so we proved:
Theorem 6.6. \(\Gamma_1 = \Gamma / \Gamma(I) \cong \Gamma H / H\) is a subgroup of \(\text{PGL}_d(L)\) which contains \(\text{PSL}_d(L)\).

Proposition 6.7. Let \(r\) denote the order of \(y/(1 + y)\) in \(L^\times / L^\times d\). Then \([\Gamma_1 : \text{PSL}_d(L)] = r\).

Proof. The composition \(\Gamma_1 \cong \Gamma H / H \hookrightarrow \text{PGL}_d(R/I) \xrightarrow{N} L^\times / L^\times d\) carries \(\Gamma_1\) to the subgroup generated by \(N(b) = y/(1 + y)\), and its kernel is \(\Gamma_1 H / H \cong \text{PSL}_d(L)\). \(\square\)

The index of \(\text{PSL}_d(L)\) in \(\Gamma_1\) can be interpreted in another way. As mentioned above, the color map \(\delta : \Gamma \to \mathbb{Z}/d\) of Fig. 3 is in fact an isomorphism \(\Gamma / \Gamma_1 \to \mathbb{Z}/d\), so it induces an isomorphism \(\Gamma H / \Gamma_1 H \cong \Gamma / \Gamma_1 (\Gamma \cap H) \to \mathbb{Z}/r\). In other words, there is an \(r\)-coloring of \(\Gamma H / H\).

A remark about colors is in order. For the building \(\mathcal{B}\), there is a \(d \times d\) matrix \(n_{ij}\) of integers, such that every vertex of color \(i\) has \(n_{ij}\) neighbors of color \(j\) (in fact \(n_{ij} = \left\lfloor \frac{d}{y - y_j} \right\rfloor - \delta_{ij}\), where \(\left\lfloor \frac{d}{y} \right\rfloor\) is the number of subspaces of dimension \(k\) of \(\mathbb{F}_q^d\), and in particular \(n_{ii} = 0\) (so the graph is \(d\)-partite). A quotient complex \(\mathcal{B}_1\) which has a \(d_1\)-coloring still satisfies this property, for the ‘folded’ \(d_1 \times d_1\) matrix whose \((i', j')\) entry is the sum of \(n_{ij}\) over \(i \equiv i' \pmod{d_1}\) and \(j \equiv j' \pmod{d_1}\). Naturally the most interesting case is when the quotient has \(d\) colors, which the above theorem guarantees quite often:

Corollary 6.8. If \(y/(1 + y)\) has order \(d\) in \(L^\times / L^\times d\), then there is a well defined color epimorphism \(\delta_L : \Gamma / \Gamma(I) \to \mathbb{Z}/d\mathbb{Z}\).

7. Construction with given index over \(\text{PSL}_d(L)\)

We combine the results obtained so far to prove Theorem 1.1, in the following, stronger version (the assumption \(q^e > 4d^2 + 1\) is only needed here to control the index \(r\)).

Theorem 7.1. Let \(q\) be a prime power, \(d \geq 2, s \geq 1, e \geq 1\) (\(e > 1\) if \(q = 2\)), and \(r\) a divisor of \((d, q^e - 1)\). Assume \(d\) is prime to \(q\), or \(s = 1\). Also assume that \(q^e > 4d^2 + 1\).

Then, for a suitable prime polynomial \(p \in \mathbb{F}_q[y]\) of degree \(e\) and for \(L = \mathbb{F}_q[y]/(p(y))\), the (unique) subgroup \(G\) of \(\text{PGL}_d(L)\) which has index \(r\) over \(\text{PSL}_d(L)\) has a set of \(\left\lfloor \frac{d}{1} \right\rfloor_q + \left\lfloor \frac{d}{2} \right\rfloor_q + \cdots + \left\lfloor \frac{d}{d - 1} \right\rfloor_q\) generators, such that the Cayley graph of \(G\) with respect to these generators determines a Ramanujan complex.

Let \(J \prec R\) be a prime ideal and \(I = J^s\) for some \(s \geq 1\). Let \(L_0 = R/J \cong \mathbb{F}_{q^e}\), the residue field of \(L = R/I\).

Notice that \(\text{PGL}_d(L)/\text{PSL}_d(L) \cong L^\times / L^\times d\) is usually not a cyclic group, so \(\Gamma_1\) may not be the full group \(\text{PGL}_d(L)\). However, in two important cases, namely if \(d\) is prime to \(q\) or if \(L = L_0\), the quotient \(L^\times / L^\times d\cong L_0^\times / L_0^\times d\) is cyclic.

We want to choose the generator of \(J\) so that \(\Gamma / \Gamma(I)\) has a predetermined index over \(\text{PSL}_d(L)\). Write \(J = (p)\), where \(p\) is a prime polynomial different than \(y\) and \(1 + y\), since \(R = \mathbb{F}_q[y, 1/y, 1/(1 + y)]\). In particular, if \(q = 2\) then we must assume \(e \geq 2\).
It is convenient to have a uniform bound on the number of non-generators of a finite field.

**Lemma 7.2.** Let \( q \) be a prime power and \( e > 1 \), and let \( \mathbb{F}_{q^e} \) be a field of order \( q^e \). There are less than \( 2q^{e/2} \) elements of \( \mathbb{F}_{q^e} \) which do not generate it over \( \mathbb{F}_q \).

**Proof.** We need to bound the size of the union of the maximal subfields of dimensions \( e/p \) for the prime divisors \( p \) of \( e \). If \( e \) is a prime or equals \( 4 \), then the statement is obvious, so we assume \( e \geq 6 \) and is not a prime. Let \( C = 9/q \).

We claim that for every prime \( p \geq 3 \) dividing \( e \), \( q^{e/p} < C p^{-2}q^{e/2} \). Indeed, since \( e \) is not a prime, \( p \leq e/2 \). Now, in the range \( 3 \leq p \leq e/2 \), \( h(p) = 2 \log(p) - (e/2 - e/p) \log(q) \) obtains its maximum on one of the boundaries, and so is bounded by
\[
\max(h(3), h(e/2)) = \max(\log(9) - e \log(q)/6, 2 \log(e/2) - (e/2 - 2) \log(q))
\]
This decreases with \( e \), and is thus bounded by its value at \( e = 6 \), namely \( \log(9)/6 \).

Since \( \sum_{p \geq 3} p^{-2} < 2/9 \) (summing over all the primes), we have that \( \sum_{p \mid e} q^{e/p} < (1 + C \sum_{p \geq 3} p^{-2})q^{e/2} < (1 + 2/q)q^{e/2} \). \( \square \)

**Proposition 7.3.** Let \( q, d, s, r \) and \( e \) be as in Theorem 7.1.

If \( q^e > 4d^2 + 1 \), then there is a polynomial \( p \in \mathbb{F}_q[y] \) of degree \( e \), such that for \( J = (p) \triangleleft R \) and \( I = J^s, L_0 = R/J \) is the field of order \( q^e \), and \( \{I_1 : \text{PSL}_d(L)\} = r \) where \( L = R/I \).

**Proof.** The assumption that \( s = 1 \) or \( d \) is prime to \( q \) guarantees that \( L^x/L^{x/d} \cong L_0^x/L_0^{x/d} \).

Let \( \alpha \) be an element of \( \mathbb{F}_{q^e} \) whose order in \( \mathbb{F}_{q^e}/\mathbb{F}_{q^e}^{d^2} \) is \( r \). Let \( \alpha_1 = \alpha/(1 - \alpha) \). The map \( \alpha \mapsto \alpha_1 \) is one-to-one.

There are \( (q^e - 1)/(d, q^e - 1) \) suitable elements \( \alpha \), and \( \alpha_1 \) belongs to a proper subfield of \( \mathbb{F}_{q^e} \) in less than \( 2q^{e/2} \leq (q^e - 1)/(d, q^e - 1) \) cases. We can thus assume \( \alpha_1 \) generates \( \mathbb{F}_{q^e}/\mathbb{F}_{q^e}^{d^2} \) as a field. Then, the minimal polynomial \( f \in \mathbb{F}_q[y] \) of \( \alpha_1 \) has degree \( e \). Take \( I = (f^e) \). Then \( L_0 = R/(f) \) is isomorphic to \( \mathbb{F}_{q^e} \), and the image of \( y \) in \( L_0 \) satisfies \( y/(1 + y) \mapsto \alpha_1/(1 + \alpha_1) = \alpha \). It has order \( r \) in \( L_0^x/L_0^{x/d} \), and we are done by Proposition 6.7. \( \square \)

Recall that the group \( \Gamma \subseteq G'(R) \) (Definition 4.4) is generated by the \( \left[ \begin{array}{c} d \\ \hline q \end{array} \right] = (q^d - 1)/(q - 1) \) elements \( b_u = u(1 - e^{-1})u^{-1} \), which can be viewed as \( d^2 \times d^2 \) matrices over \( R_0 = \mathbb{F}_q[1/y] \); see Proposition 3.3. The quotient \( G'_1(R)/G'_1(R, I) \) was computed in Section 6. For an explicit identification of this group, it is worth mentioning that \( G'(R)/G'(R, I) \cong i_R G'(R)/i_R G'(R, I) \) \( (i_R \) was defined in Section 3), and likewise for \( G'_1 \). Therefore when dealing with this quotient we may view (the entries of) the generators of \( \Gamma \) as if they were elements of \( R/I \).

Let \( G \) be the group generated by these matrices over \( R_0/(y - e)p^t \). Then \( G \cong \Gamma_I \) and, therefore, has index \( r \) over \( \text{PSL}_d(L) \). For each \( k = 1, \ldots, d - 1 \), the generators of color \( k \) of \( G \) are the products \( b_{u_1} \ldots b_{u_k} \), which can be extended to a product of \( d \) generators which equals \( 1 \) (see Section 5 and Proposition 2.3); in particular, every \( b_u \) is a generator of color \( 1 \). See [3, Section 4] for an alternative description of the generators (as elements of \( A(k) \)).
The description of the complex constructed from the Cayley graph is given at the beginning of Section 6. Theorem 6.2 then completes the proof of Theorem 7.1.

8. Splitting \( A(R) \)

So far \( \Gamma \) is viewed as a group of \( d^2 \times d^2 \) matrices, and the representation of \( \Gamma \) as \( d \times d \) matrices is over the completion \( F \) of \( k \). To facilitate the explicit constructions of finite quotients in the next section, we now present a splitting extension of dimension \( d \) for \( G'(k) \) (and \( G'(R) \)).

Fix an element \( \beta \in F_{q^d} \) such that \( \text{tr}(\beta) \in F_q \) is non-zero. Let \( \tilde{k}_1 = F_{q^d}(x) \), with the obvious action of \( \text{Gal}(F_{q^d}/F_q) \), so the fixed field is \( \tilde{k} = F_q(x) \). Then the norm \( N_{F_{q^d}(x)/F_q(x)}(1 + \beta x) \) is a polynomial of degree \( d \) in \( F_q[x] \), which has the form \( 1 + t_1 x + \cdots + t_d x^d \) for some \( t_i \in F_q \). Then \( t_1 = \text{tr}(\beta) \neq 0 \) by assumption, and \( t_d = N_{F_{q^d}/F_q}(\beta) \neq 0 \), since \( \beta \neq 0 \).

We embed \( k = F_q(y) \) into \( \tilde{k} \) by mapping
\[
y \mapsto t_1 x + \cdots + t_d x^d.
\]
Now \( k \) becomes a subfield of codimension \( d \) of \( F_q(x) \).

**Proposition 8.1.** Taking the completion, \( F_q((x)) = F_q((y)) \), and even \( F_q[[x]] = F_q[[y]] \).

**Proof.** Before giving the proof, we should explain the notation. By \( F_q((x)) \) we mean, as usual, the field of Taylor series \( \{ \sum_{i=-\infty}^{\infty} \alpha_i x^i \} \) with coefficients from \( F_q \); \( F_q[[x]] \) is the subring of elements of the form \( \sum_{i=0}^{\infty} \alpha_i x^i \). As abstract fields, it trivially holds that \( F_q((y)) \cong F_q((x)) \). But now \( F_q((y)) \) is the subfield of \( F_q((x)) \) consisting of Taylor series in the element \( y \) of Eq. (18).

Working in \( F_q((y)) \), Eq. (18) has a solution for \( x \) by Hensel’s lemma (since the derivation at zero, namely \( t_1 \), is non-zero). In fact, the solution has the form \( x = (1/t_1)y - (t_2/t_1^2)y^2 + \cdots \), so that \( x \in F_q[[y]] \). \( \square \)

Tensoring with \( F_q(x) \), we also have that \( F_q((y)) = F_q(x) \). Define \( \tilde{R}_T = F_q[x, 1/(1 + y)] \) and \( \tilde{R} = F_q[x, 1/y, 1/(1 + y)] \), so that \( R_T \subseteq \tilde{R}_T \) and \( R \subseteq \tilde{R} \). Since \( 1 + y \) is invertible in \( F_q[[y]] \), \( \tilde{R}_T \subseteq F_q[[y]] \). The rings mentioned so far appear in Fig. 7.

Notice that \( F_q((x)) \) is a splitting field of \( A(k) \), as \( 1 + y = N(1 + \beta x) \) is a norm in the extension \( F_q(x)/F_q(x) \). We thus obtain the chain of inclusions
\[
A(R) \subseteq A(k) \hookrightarrow A(k) \otimes_k F_q(x) \cong M_d(F_q((x))) = M_d(F_q((y))),
\]
refining Eq. (5). The last equality in this chain follows from Proposition 8.1.

In fact, \( A(\tilde{R}_T) \) is already split, and we now present an explicit isomorphism \( A(\tilde{R}_T) \cong M_d(\tilde{R}_T) \).

**Proposition 8.2.** Let \( M = \tilde{R}_T \otimes F_q^d \), and let \( \rho : M \to \text{End}_{\tilde{R}_T}(M) \) be the regular representation, defined by \( \rho : r \otimes u \mapsto \rho_r \otimes \rho_u \), where \( \rho_r(r') = rr' \) for \( r', r' \in \tilde{R}_T \), and \( \rho_u(u') = uu' \) for \( u, u' \in F_q^d \).
The map $\rho$ is extended by $z \mapsto 1 \otimes \phi + \rho_x \otimes \rho_y \phi$ to an isomorphism

$$A(\bar{R}_T) \xrightarrow{\cong} \text{End}_{\bar{R}_T}(\bar{R}_T \otimes \mathbb{F}_{q^d}').$$

**Proof.** By definition $\rho(z) = \rho(1 + \beta x) \cdot (1 \otimes \phi)$, so that $\rho(z)^{-1} = (1 \otimes \phi)^{-1} \cdot \rho(1 + \beta x)^{-1}$.

We need to check the defining relations of $A(\bar{R}_T)$. Let $r \otimes u, r' \otimes u' \in M$; then

$$\rho(z)\rho(r \otimes u)(r' \otimes u') = \rho(z)(rr' \otimes uu')$$

$$= (1 + \beta x)(rr' \otimes \phi(uu'))$$

$$= (1 + \beta x)(r \otimes \phi(u))(r' \otimes \phi(u'))$$

$$= \rho(r \otimes \phi(u))(1 + \beta x)(r' \otimes \phi(u'))$$

$$= \rho(r \otimes \phi(u))\rho(z)(r' \otimes u'),$$

so that $\rho(z)\rho(r \otimes u)\rho(z)^{-1} = \rho(r \otimes \phi(u))$. Also, $\rho(z)^d = (\rho(1 + \beta x) \cdot (1 \otimes \phi))^d = \rho(N(1 + \beta x)) \cdot (1 \otimes \phi)^d = \rho(1 + y)$.

To check that the extended $\rho$ is an embedding, let $\bar{k} = \mathbb{F}_q(x)$, and recall that $A(\bar{k}) \cong \text{End}_{\bar{k}}(\bar{k} \otimes \mathbb{F}_{q^d})$ by the norm condition. The result then follows from the commutativity of the diagram in Fig. 8, and the fact that $A(\bar{R}_T)$ and $\text{End}_{\bar{R}_T}(M)$ have the same rank $d^2$ over $\bar{R}_T$. □

Since $\bar{R}_T \otimes \mathbb{F}_{q^d}$ is a free module of rank $d$ over $\bar{R}_T$, we obtain an isomorphism $A(\bar{R}_T) \cong M_d(\bar{R}_T)$. 

---

**Fig. 8.** Maps in Proposition 8.2.
Example 8.3. Suppose \( q = 7 \) and \( d = 3 \), and let \( \alpha = \sqrt[3]{3} \) over \( \mathbb{F}_7 \), so that \( \mathbb{F}_{7^3} = \mathbb{F}_7[\alpha] \).

If \( \phi \) is the Frobenius automorphism, we have that \( \phi(\alpha) = \alpha^7 = 4\alpha \). As a normal basis of \( \mathbb{F}_{7^3}/\mathbb{F}_7 \) we take \( \zeta_0 = 1 + \alpha + \alpha^2 \) and \( \zeta_i = \phi^i(\zeta_0) \). We choose \( \beta = \alpha - 2 \) (so that \( \text{tr}(\beta) = 1 \)). Of course \( \{\zeta_i\} \) form a basis of \( \bar{R}_T \otimes \mathbb{F}_{7^3} \) as a free module over \( \bar{R}_T \). Computing directly, we find that \( N(1 + \beta x) = 1 + x - 2x^2 + x^3 \), so that \( y = x - 2x^2 + x^3 \).

In order to present an embedding of \( \mathcal{A}(\bar{R}_T) = \mathbb{F}_{7^3}[x, 1/(1 + y)]/[z] \) into \( \text{M}_3(\mathbb{F}_7[x, 1/(1 + y)]) \), we need to describe the images of \( \mathbb{F}_{7^3}, x, 1/(1 + y) \) and \( z \). For the subfield, check that

\[
\alpha \mapsto \begin{pmatrix} 6 & 3 & 6 \\ 5 & 5 & 6 \\ 5 & 3 & 3 \end{pmatrix}
\]

defines the embedding \( \mathbb{F}_{7^3} \to \text{M}_3(\mathbb{F}_7) \) via the basis we chose. Secondly, \( x \) and \( 1/(1 + y) \) are scalars in \( \bar{R}_T \) and so are mapped to scalar matrices. Finally, since \( \phi(\zeta_i) = \zeta_{i+1} \), by the above proposition we have

\[
z \mapsto \rho(1 + \beta x) \cdot (1 \otimes \phi) = \begin{pmatrix} 1 + 4x & 3x & 6x \\ 5x & 1 + 3x & 6x \\ 5x & 3x & 1 + x \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3x & 6x & 1 + 4x \\ 1 + 3x & 6x & 5x \\ 3x & 1 + x & 5x \end{pmatrix}.
\]

One can check that over \( \bar{R}_T \), \( z^3 \) is the scalar matrix \( 1 + y \).

9. Explicit constructions

In this section we show how to choose the generator \( p^s \in \mathbb{F}_q[y] \) of \( I \), such that \( \Gamma_I = \Gamma/\Gamma(I) \) will have an explicit representation as \( d \times d \) matrices over \( L \). The construction works for arbitrary \( q, d, e = \deg(p) \) and \( s \geq 1 \), but we do not try to control the index \( r \) of \( \Gamma_I \) over \( \text{PSL}_d(L) \) (which can be computed by Proposition 6.7).

Recall that \( H = G'(R, I) \). Our current realization of \( \Gamma H/H \) as a subgroup of \( \text{PGL}_d(R/I) \) is inexplicit, as we use the abstract isomorphism \( G'(R/I) \cong \text{PGL}_d(R/I) \).

We do have an explicit embedding of \( i_R \) of \( G'(R) \) into \( d^2 \times d^2 \) matrices over \( R \), as described before Proposition 3.3, but for practical reasons it is more convenient to have a representation of dimension \( d \).

In Proposition 8.2 (and Example 8.3) we gave an explicit isomorphism of \( \mathcal{A}(\bar{R}_T) \) to matrices over \( \bar{R}_T \), which in particular embeds \( G'(R) \leftrightarrow \text{PGL}_d(\bar{R}) \). Given \( I \lhd R \), there is an ideal \( \bar{I} \lhd \bar{R} \) such that \( I = \bar{I} \cap R \) (since \( \bar{R} = R[x] \) is an integral extension over \( R \) and \( R \) is a principal ideal domain). Let \( \bar{L} = \bar{R}/\bar{I} \), then there is a natural embedding \( L \subseteq \bar{L} \), and we thus have an embedding \( G'(L) \leftrightarrow \text{PGL}_d(\bar{L}) \).

We will show that it is possible to choose \( I \) so that \( \bar{L} \cong L \), where the image of \( x \) in \( L \) (which is what we need for the splitting map of Proposition 8.2) is explicitly identified.

We need some details on the possible choices of \( \bar{I} \). Write \( y(\lambda) = t_1 \lambda + \cdots + t_d \lambda^d \) for the polynomial defined by Eq. (18), so \( y(x) \) is our usual element \( y \) of \( \mathbb{F}_q[x] \). If \( p(\lambda) \in \mathbb{F}_q[\lambda] \) is irreducible and not equal to \( \lambda \) or \( \lambda + 1 \), then \( p(y) \in R \) generates a prime ideal \( J \lhd R \). Since the embedding of \( R \) into \( \bar{R} \) is by sending \( y \) to \( y(x) \), \( J \) lifts to the ideal \( J = \langle p(y(x)) \rangle \) of \( \bar{R} \).
Let \( p(y(\lambda)) = g_1(\lambda) \cdots g_t(\lambda) \) be the decomposition into prime factors over \( \mathbb{F}_q \) and \( \bar{J}_i = \langle g_i(\lambda) \rangle \triangleleft \bar{R} \). Then \( (p(y(\lambda))) = \bar{J}_1 \cdots \bar{J}_t \), where each \( \bar{J}_i \) is a prime ideal of \( \bar{R} \) covering \( J_i \).

If \( \alpha \) is a root of \( g_i \) in some extension, then \( y(\alpha) \in \mathbb{F}_q[\alpha] \) is a root of \( p(\lambda) \). Therefore \( p \) splits in \( \mathbb{F}_q[\alpha] \) and so \( \deg(p(\lambda)) = \deg(g_i) \). This shows once more that \( \bar{L}_0^i = \bar{R}/\bar{J}_i \) is an extension of \( L_0 = R/J \), and in fact \( [\bar{L}_0^i : L_0] = \deg(g_i)/\deg(p) \).

We remark that the degrees of the \( g_i \) are not necessarily equal.

Let \( \gamma_j \) be the roots of \( p(\lambda) \) in \( L_0 \), then \( p(\lambda) = \prod (\lambda - \gamma_j) \) and \( p(y(\lambda)) = \prod (y(\lambda) - \gamma_j) \).

In particular the roots of \( p(y(\lambda)) \) are roots of the factors \( y(\lambda) - \gamma_j \). Let \( \alpha \) be a root of \( p(y(\lambda)) \) coming from an irreducible factor \( g_i(\lambda) \). Then \( \bar{L}_0^i = \mathbb{F}_q[\alpha] = L_0[\alpha] \) (since a generator \( \gamma_j \) of \( L_0 \) equals \( y(\alpha) \)). It follows that \( p(y(\lambda)) \) has a factor \( g_i \) of degree \( \deg(p) \) iff \( \bar{L}_0^i = L_0 \), if \( y(\lambda) - \gamma_j \) has a root in \( L_0 \) for some \( j \) (a condition which is easily seen to be independent of \( j \)).

We now show that \( J = \langle p(y) \rangle \triangleleft R \) can be chosen so that \( \bar{L}_0 = L_0 \) and \( \bar{L} = L \), which provides an explicit realization of Theorem 6.6.

**Theorem 9.1.** Let \( q \) be a prime power and \( d \) an integer. Assume \( q^e \geq 4d^2 \). Then there are irreducible polynomials \( p, g \in \mathbb{F}_q[\lambda] \) of degree \( e \) such that for every \( s \geq 1 \), the embedding \( R \to \bar{R} \) induces an isomorphism \( R/I \cong \bar{R}/\bar{I} \) for \( I = \langle p(y^s) \rangle \triangleleft R \) and \( \bar{I} = \langle g(x^s) \rangle \triangleleft \bar{R} \).

**Proof.** Let \( \mathbb{F}_{q^e} \) be the field of order \( q^e \). For every \( \alpha \in \mathbb{F}_{q^e} \), consider the element \( y(\alpha) \).

Since \( \deg(y(\lambda)) = d \), there are at most \( d \) elements \( \alpha \) with the same \( y(\alpha) \), so in particular (by Lemma 7.2), \( y(\alpha) \) is not a generator of \( \mathbb{F}_{q^e} \) for less than \( 2d q^{e/2} \) choices of \( \alpha \). By the assumption on \( e \), we can choose \( \alpha \) for which \( \gamma = y(\alpha) \) generates the field of \( q^e \) elements (so necessarily \( \alpha \) is also a generator). Let \( p(\lambda) \) and \( g(\lambda) \) be the minimal polynomials of \( \gamma \) and \( \alpha \) over \( \mathbb{F}_q \), respectively. Since \( \alpha \) and \( \gamma \) are generators, \( \deg(p) = \deg(g) = e \).

Now let \( I = \langle p(y^s) \rangle \triangleleft R \) and \( \bar{I} = \langle g(x^s) \rangle \triangleleft \bar{R} \). Since \( g(x) | p(y(x)) \) and \( g(x) \) is prime, \( I = \bar{I} \cap R \), and so \( L = R/I \) embeds into \( \bar{L} = \bar{R}/\bar{I} \). Since \( \deg(p) = \deg(g) \), \( |L| = |\bar{L}| \), and so they are isomorphic. \( \square \)

The same choice of \( p \) and \( g \) works for every \( s \geq 1 \), so the proposition shows that in fact the fraction field \( k = k[x] \) of \( \bar{R} \) is contained in the completion \( k_p \).

We conclude this section with a detailed algorithm giving the generators for a Cayley complex of \( \text{PGL}_d(L) \). Let \( q \) be a given prime power, \( d \geq 2 \), and \( e \geq 1 \). Assume \( q^e \geq 4d^2 \).

**Algorithm 9.2.**

1. (1) In practice, the elements of \( \mathbb{F}_{q^d} \) are polynomials of degree at most \( d \) over \( \mathbb{F}_q \), modulo a fixed irreducible polynomial of degree \( d \). Fix a basis \( \xi_0, \ldots, \xi_{d-1} \) for \( \mathbb{F}_{q^d} \) over \( \mathbb{F}_q \). Let \( \phi \) be any generator of \( \text{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q) \) (namely exponentiation by \( q^d \) where \( d \) is any fixed integer prime to \( d \)). Let \( \beta \in \mathbb{F}_{q^d} \) be an element with \( \beta + \cdots + \phi^{d-1}(\beta) \neq 0 \), let \( t_1, \ldots, t_d \in \mathbb{F}_q \) be defined by \( (1 + \beta \lambda) \cdots (1 + \phi^{d-1}(\beta) \lambda) = 1 + t_1 \lambda + \cdots + t_d \lambda^d \), and set \( y(\lambda) = t_1 \lambda + \cdots + t_d \lambda^d \). For an element \( c \in \mathbb{F}_{q^d} \), \( \rho(c) \in \text{GL}_d(\mathbb{F}_q) \) is the matrix defined by \( c \cdot \xi_j = \sum_{i=0}^{d-1} (\rho(c))_{ij} \xi_i \). Likewise, let \( \varphi \in \text{GL}_d(\mathbb{F}_q) \) be the matrix defined by \( \phi(\xi_j) = \sum_{i=0}^{d-1} \phi_{ij} \xi_i \).

2. Let \( \alpha \in \mathbb{F}_{q^d} \) be an element such that \( \gamma = y(\alpha) \) generates \( \mathbb{F}_{q^d} \) as a field (the existence of which is guaranteed by Theorem 9.1). A quick way to check this property is to verify that \( \gamma^{q^e} \neq \gamma \) for every proper divisor \( e' \) of \( e \). Let \( p(\lambda) \) and \( g(\lambda) \) be the minimal
polynomials (of degree $e$) of $\gamma$ and $\alpha$, respectively, over $\mathbb{F}_q$ (found by Gaussian elimination on $1, \gamma, \ldots, \gamma^e$ or $1, \alpha, \ldots, \alpha^e$).

(3) Fix $s \geq 1$. Let $\tilde{L}$ denote the ring of polynomials of degree $< es$ in the variable $x$, with operations modulo $g(x)^s$. Let $\rho(z) \in \text{GL}_d(L)$ be the matrix $(1 + \rho(\beta)x) \cdot \varphi$, as in Proposition 8.2 and the example following it. Let $b = 1 - \rho(z)^{-1}$ (Gauss elimination can be used to invert $\rho(z)$, where the inverse of $f(x) \in L$ is computed via the extended Euclid’s algorithm for $f(\lambda)$ and $g(\lambda)^s$). For every $u \in \mathbb{F}_d^\times/\mathbb{F}_q^\times$ (one representative from each class), set $b_u = \rho(u) b \rho(u)^{-1}$.

(4) The matrices $[b_u]$ generate a subgroup $G$ of $\text{PGL}_d(L)$, which contains $\text{PSL}_d(L)$. The index $r = [G : \text{PSL}_d(L)]$ can be computed using Proposition 6.7.

(5) Let $P = \{(u_1, \ldots, u_d) : b_{u_1} \cdots b_{u_d} = 1\}$. For every $k = 1, \ldots, d - 1$, let $S_k$ denote the set of products $b_{u_1} \cdots b_{u_k}$ for the headers $(u_1, \ldots, u_k)$ of vectors in $P$. There will be $\left[\frac{d}{k}\right]_q$ different products (up to scalar multiples over $L$), and $S_{d-k}$ is the set of inverses of $S_k$.

(6) The Cayley complex of $G$ with respect to $S_1 \cup \cdots \cup S_{d-1}$ defines a simplicial complex (the clique complex of the Cayley graph). This complex is Ramanujan by Theorem 6.2.

10. An example

We demonstrate the construction of the group $\Gamma$, and the embedding into $d^2 \times d^2$ and $d \times d$ matrices. We choose $q = 2$ and $d = 3$, and write $\mathbb{F}_8 = \mathbb{F}_2[v \mid v^3 = v + 1]$. Fix the basis $\{\xi_0, \xi_1, \xi_2\} = \{1, v, v^2\}$ for $\mathbb{F}_8$ over $\mathbb{F}_2$. Then $\{\xi_0 \xi_j\}$ is a basis for $A(R) = R[v, z \mid v^3 = v + 1, z^3 = 1 + y, zv^{-1} = v^2]$. Over $R = \mathbb{F}_2[y, 1/y, 1/(1 + y)]$. By definition, $\Gamma \subseteq A(R)^\times$ is generated by the conjugates $u(1 - z^{-1})u^{-1}$ for $u \in \mathbb{F}_8^\times/\mathbb{F}_2^\times = \mathbb{F}_8^\times = \{v\}$.

These generators act on $A(R)$ by conjugation, and the corresponding $9 \times 9$ matrices over $R$ are given in the right-hand column of Fig. 9 (they are all of the form $a + by^{-1}$ for $a, b \in \text{GL}_3(\mathbb{F}_2)$).

The relations of $\Gamma$ were discussed in Section 5. If we set $b_i = v^i(1 - z^{-1})v^{-i}$, conjugation by $v$ induces an outer automorphism of order 7 (namely $b_i \mapsto b_{i+1}$, and the defining relations are

\[
\begin{align*}
b_0b_3 &= b_4b_2 = b_6b_5, \\
b_0b_5 &= b_2b_1 = b_3b_6, \\
b_0b_6 &= b_1b_4 = b_5b_3, \\
b_0b_3b_1 &= 1,
\end{align*}
\]

and their $v$-conjugates.

We continue to find a representation of dimension 3 of a finite quotient of $\Gamma$. The first step is to define $R$ as a subring of $\tilde{R} = R[x]$, so we need to choose $\beta$ with $\text{tr}(\beta) \neq 0$; here we take $\beta = 1 + y$, so that $1 + y = N_{\mathbb{F}_8/\mathbb{F}_2}(1 + \beta x) = 1 + x + x^3$ and $y(x) = x + x^3$.

Then $R \hookrightarrow \tilde{R}$ by sending $y \mapsto y(x)$. 


Fig. 9. Generators of $\Gamma$ in the example of Section 10.
Algorithm given in Section 9, we look for an element and onto (since Ramanujan by Theorem 6.2. The Cayley graph with respect to these generators is of Ramanujan complexes.

where $\lambda + x + x^3 = 1 + y(x)$, so $z^{-1} = (1/(1+y(x)))z^2$, and

$$b = 1 - z^{-1} \mapsto \frac{1}{1 + y(x)} \begin{pmatrix} x + x^3 & x^2 & x + x^2 \\ x & x^3 & 1 + x + x^2 \\ x + x^2 & 1 + x^2 & 1 + x^3 \end{pmatrix},$$

where $1 + y(x)$ is invertible in $R$ by definition. We emphasize that this embedding is not onto (since $A(R)$ does not split).

Finally, we present a finite quotient of $\Gamma$. Choose $e = 4$, so $\Gamma'$ will map onto $\text{PGL}_3(L)$, where $L$ is a finite local ring whose residue field is $\mathbb{F}_{16}$ (in general $\Gamma'$ could be mapped onto a subgroup of $\text{PGL}_3(L)$ containing $\text{PSL}_3(\mathbb{F}_3)$). Applying the proof of Theorem 9.1 and the algorithm given in Section 9, we look for an element $\alpha \in \mathbb{F}_{16}$ such that $y(\alpha)$ is a generator.

Arbitrarily choosing an irreducible polynomial of degree 4 over $\mathbb{F}_2$, we set $\mathbb{F}_{16} = \mathbb{F}_2[t]/(t^4 + t + 1 = 0)$. It then turns out that $y(t) = t^3 + t$ is a generator (since $y(t^2) \neq y(t)$), so we choose $\alpha = t$ and $\gamma = y(t)$. The minimal polynomial of $\alpha$ is of course $g(\lambda) = \lambda^4 + \lambda^2 + 1$, and the minimal polynomial of $\gamma$ is computed to be $p(\lambda) = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$. Now $g(\lambda)$ divides $p(y(\lambda)) = p(\lambda^3 + \lambda) = \lambda^{12} + \lambda^9 + \lambda^7 + \lambda^6 + \lambda^5 + \lambda^4 + \lambda^2 + \lambda + 1 = g(\lambda)(\lambda^8 + \lambda^4 + \lambda^3 + \lambda^2 + 1)$. For any $s \geq 1$, when we take $I = \langle p(y^s) \rangle$ and $I' = \langle g(\lambda)^s \rangle$, we have $R \cap I = I$. Since $\deg(p) = \deg(g)$, $R/I \cong \mathbb{F}_2[x]/((x^4 + x + 1)t)$.

The index over $\text{PSL}_d$ is the order of $y/(1+y) = y^3 + y + 1$ in $\mathbb{F}_2[y]/(y^4 + y^3 + y^2 + y + 1)$ modulo its cubes; the order is 3, so $\Gamma'$ maps onto $\text{PGL}_3(L)$. The $3 \times 3$ matrices over $\bar{R}$ corresponding to the generators $b_i$ of $\Gamma'/G' \cap \Gamma = \text{PSL}_3(R/I)$ are given in the intermediate column of Fig. 9. The Cayley graph with respect to these generators is Ramanujan by Theorem 6.2.

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Note added in proof

We recently learned that Alireza Sarveniazi [17] has also given an explicit construction of Ramanujan complexes.
References

[17] A. Sarveniazi, Ramnuajan \((n_1, ..., n_d-1)\)-regular hypergraphs based on Bruhat–Tits buildings of type \( \tilde{A}_{d-1} \), arxiv.org/math.NT/0401181.

Glossary

\( \mathcal{A}(S) \): an \( S \)-algebra for any \( R_T \)-algebra \( S \) with the relations given in Eq. (4) (see Section 3)
\( B = B_d(F) \): Bruhat–Tits building = \( \text{PGL}_d(F) / \text{PGL}_d(O) = G(F)/K \)
\( B^0 \): the set of vertices of \( B \)
\( B_i \): the \( i \)-skeleton of \( B \)
\( B_I = \Gamma(I) \setminus G(F)/K \cong \Gamma / \Gamma(I) \)
\( d \): an integer \( \geq 1 \)
\( F = \mathbb{F}_q((y)) = k_y \) (see Fig. 1)
\( \tilde{G}'(S) = \mathcal{A}(S)^\times \) (in particular \( \tilde{G}'(R) = \mathcal{A}(R)^\times \), and likewise for the other groups defined over \( R \) or other \( R_T \)-algebras)
\( G'(S) = \mathcal{A}(S)^\times / S^\times \).
\( \tilde{G}'_1(S) \): elements of \( \tilde{G}'(S) \) of norm 1
\( G'(S) \): the image of \( \tilde{G}'_1(S) \) under the map \( \tilde{G}'(S) \to G'(S) \) (see Fig. 5)
\( \tilde{G}(S) = \text{GL}_d(S) \)
\( G(S) = \text{PGL}_d(S) \)
\( \tilde{G}_1(S) = \text{SL}_d(S) \)
\( G_1(S) = \text{PSL}_d(S) \)
\( \tilde{G}(S, I) \): congruence subgroup of \( \tilde{G}'(S) \); kernel of \( \tilde{G}'(S) \to \tilde{G}'(S/I) \)
$G'(S, I)$: kernel of $G'(S) \rightarrow \tilde{G}'(S/I)$ (for other congruence subgroups, see Section 6 and in particular Fig. 5)

$\tilde{\Gamma} = \langle b_u : u \in \mathbb{F}_q^\times / \mathbb{F}_q^\times \rangle$ (see Section 4 for definition of $b_u \in \mathcal{A}(R)$)

$\tilde{\Gamma}$: the image of $\Gamma$ under $\tilde{G}'(R) \rightarrow G'(R)$

$\Gamma(I) = \Gamma \cap G'(R, I)$

$\Gamma_1 = \Gamma \cap G'_1(R)$ (see Fig. 6)

$H = G'(R, 1)$

$H_1 = H \cap G'_1(R)$

$I = \langle p^s \rangle \triangleleft R, \ p \in \mathbb{F}_q[y]$ is a prime and $s \geq 1$

$k = \mathbb{F}_q(y)$ (see Fig. 1)

$k_1 = \mathbb{F}_{q^d}(y)$ (see Fig. 1)

$k_p$: the completion of $k$ with respect to $p$

$k = \mathbb{F}_q(x)$

$L = R/I$, a finite local ring

$L_0 = R/J = R/\langle p \rangle$, the residue field of $L$

$L = \bar{R}/I$

$O = \mathbb{F}_q[[y]]$: the ring of integers of $F$

$O_p$: the ring of integers of $k_p$

$q$: a fixed prime power, the order of the finite field $\mathbb{F}_q$

$R_f = \mathbb{F}_q[y, 1/(1+y)] \subseteq O$

$R = \mathbb{F}_q[y, 1/y, 1/(1+y)]$

$R_0 = \mathbb{F}_q[1/y]$

$\bar{R} = \bar{R}[x]$

$\bar{R}_f = \bar{R}_f[x]$

$q : B^0 \rightarrow \mathbb{Z}/d\mathbb{Z}$: the color function of the building

$\nu_p$: valuation of $k$ for the prime polynomial $p \in \mathbb{F}_q[y]$

$\nu_1/y$: the minus degree valuation of $k$

$y$: a transcendental variable over $\mathbb{F}_q$, generates $k$