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LINEAR ALGEBRA AND ITS APPLICATIONS

A characterization and representation of the generalized inverse $A^{(2)}_{T,S}$ and its applications

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Abstract

This paper presents an explicit expression for the generalized inverse $A^{(2)}_{T,S}$. Based on this, we established the characterization, the representation theorem and the limiting process for $A^{(2)}_{T,S}$. As an application, we estimate the error bound of the iterative method for approximating $A^{(2)}_{T,S}$. © 1998 Elsevier Science Inc. All rights reserved.

1. Introduction

It is a well-known fact that the common important six kinds of generalized inverse: the Moore–Penrose inverse A^+, the weighted Moore–Penrose inverse $A^+_{M,N}$, the Drazin inverse A^{D}, the group inverse A_g, the Bott–Duffin inverse $A^{(-1)}_{(L)}$ and the generalized Bott–Duffin inverse $A^{(+)}_{(L)}$ are all generalized inverse $A^{(2)}_{T,S}$, which having the prescribed range T and null space S of [2]-(or outer) inverse of A.

The [2]-inverse has many applications, for example, the application in the iterative methods for solving the nonlinear equations [2,19] and the applications to statistics [10,14,16]. In particular, [2]-inverse play an important role in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverse [17,21].

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This paper presents an explicit expression for the generalized inverse $A_{T,S}^{(2)}$. Based on this, we established the characterization, the representation theorem and the limiting process for $A_{T,S}^{(2)}$. As an application, we estimate the error bound of the iterative method for computing $A_{T,S}^{(2)}$.

Finally, we point out the links between $A_{T,S}^{(2)}$ and the W-weighted Drazin inverse $A_{d,w}$. These results extend the earlier work by various authors [1,5,9,11,13,24,26–28]. As usual, $R(A)$ and $N(A)$ denote the range and null space of A, respectively. The following lemmata are needed in what follows.

Lemma 1.1 ([2], p. 61). *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \leqslant r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m-s$. Then A has a* [2]*-inverse X such that $R(X)=T$ and $N(X)=S$ if and only if*

$$AT \oplus S = \mathbb{C}^m \tag{1.1}$$

in which case X is unique. This X is denoted by $A_{T,S}^{(2)}$.

The next lemma shows that the common six kinds of generalized inverse: $A^+, A_{M,N}^+, A^D, A_g, A_{(L)}^{(-1)}$ and $A_{(L)}^{(+)}$ are all generalized inverse $A_{T,S}^{(2)}$ (for which exists a matrix G such that $R(G)=T$ and $N(G)=S$).

Lemma 1.2. (1) *Let $A \in \mathbb{C}^{m\times n}$. Then, for the Moore–Penrose inverse A^+, the weighted Moore–Penrose inverse $A_{M,N}^+$, one has*

(a) [2] $A^+ = A_{R(A^*),N(A^*)}^{(2)} = (A^*A)^+A^* = A^*(AA^*)^+$,

(b) [20] $A_{M,N}^+ = A_{R(A^\#),N(A^\#)}^{(2)} = (A^\#A)_{N,N}^+A^\# = A^\#(AA^\#)_{M,M}^+$, *where M and N are Hermitian positive definite matrices of order m and n, respectively. In addition, $A^\# = N^{-1}A^*M$.*

(2) *Let $A \in \mathbb{C}^{n\times n}$. Then, for the Drazin inverse A^D, the group inverse A_g, the Bott–Duffin inverse $A_{(L)}^{(-1)}$ and the generalized Bott–Duffin inverse $A_{(L)}^{(+)}$, one has*

(c) [4] $A^D = A_{R(A^K),N(A^K)}^{(2)} = (A^{k+1})_gA^k = A^k(A^{k+1})_g$, *where $k = \mathrm{Ind}(A)$; in particular*, $\mathrm{Ind}(A) = 1$,

$$A_g = A_{R(A),N(A)}^{(2)} = (A^2)_gA = A(A^2)_g.$$

(d) [3,6] $A_{(L)}^{(-1)} = A_{L,L^\perp}^{(2)} = (AP_L + P_{L^\perp})^{-1}$, *where L is a subspace of $\mathbb{C}^n$ and satisfies $AL \oplus L^\perp = \mathbb{C}^n$.*

(e) [6] $A_{(L)}^{(+)} = A_{S,S^\perp}^{(2)} = A_{(S)}^{(-1)}$, *where L is a subspace of $\mathbb{C}^n$, P_L is the orthogonal projector on L, $S = R(P_LA)$, and A is an L-p.s.d. matrix, i.e. A is a Hermitian matrix with the properties: P_LAP_L is nonnegative definite, and $N(P_LAP_L) = N(AP_L)$.*

Lemma 1.3. *Let M be an $2n \times 2n$ matrix partitioned as*

$$M = \begin{bmatrix} A & AQ \\ PA & B \end{bmatrix}.$$

Then

$$\operatorname{rank}(M) = \operatorname{rank}(A) + \operatorname{rank}(B - PAQ).$$

Proof. Immediate from [15], Theorem 19.

2. Main results

In this section, we first give an explicit expression for the generalized inverse $A^{(2)}_{T,S}$, which reduces to the group inverse.

Theorem 2.1. *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \leqslant r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. In addition, suppose $G \in \mathbb{C}^{n\times m}$ such that $R(G) = T$ and $N(G) = S$. If, A has a* [2]*-inverse $A^{(2)}_{T,S}$ then*

$$\operatorname{Ind}(AG) = \operatorname{Ind}(GA) = 1. \tag{2.1}$$

Further, we have

$$A^{(2)}_{T,S} = G(AG)_g = (GA)_g G. \tag{2.2}$$

Proof. It is easy to verify that $R(AG) = AR(G) = AT$,

and

$$S = N(G) \subseteq N(AG).$$

By the assumption of Lemma 1.1, we have $\dim(AT) = m - (m - s) = s$.

Now

$$\dim[R(AG)] + \dim[N(AG)] = m,$$

whence

$$\dim[N(AG)] = m - \dim[R(AG)] = m - s = \dim(S).$$

Thus $N(AG) = S$ so that

$$R(AG) \oplus N(AG) = AT \oplus S = \mathbb{C}^m,$$

i.e.,

$$\operatorname{Ind}(AG) = 1.$$

Let $X = G(AG)_g$. By direct verification, we obtain

$$XAX = G(AG)_g AG(AG)_g = G(AG)_g = X,$$

and

$$R(X) = R[G(AG)_g] \subseteq R(G) = T$$

or

$$N(X) = N[G(AG)_g] \supseteq N[(AG)_g] = N(AG) \supseteq N(G) = S.$$

Obviously, $\operatorname{rank}(X) \leqslant \dim(T)$. On the other hand, it holds

$$\begin{aligned}\operatorname{rank}(X) &= \operatorname{rank}[G(AG)_g] \geqslant \operatorname{rank}[AG(AG)_g] = \operatorname{rank}(AG)\\ &= s = \dim(T).\end{aligned}$$

Thus, $R(X) = T$. In a similar manner giving $N(X) = S$, which is the desired result. It follows similarly that $\operatorname{Ind}(GA) = 1$ and $A^{(2)}_{T,S} = (GA)_g G$. □

From Lemma 1.2 and Theorem 2.1, let G be equal to A^*, $A^{\#}$, A^k, A, P_L and P_S respectively, then $A^{(2)}_{T,S}$ reduces to A^+, $A^+_{M,N}$, A^D, A_g, $A^{(-1)}_{(L)}$ and $A^{(+)}_{(L)}$ correspondingly.

It is a well-known fact that if A is a nonsingular matrix, then the inverse of A, A^{-1}, is the unique matrix X for which

$$\operatorname{rank}\begin{bmatrix} A & I \\ I & X \end{bmatrix} = \operatorname{rank}(A).$$

Next, we present a generalization of this fact to singular matrix A to obtain an analogous result for the generalized inverse $A^{(2)}_{T,S}$ of A.

Theorem 2.2. *Let A, T, S and G be the same as Theorem* 2.1. *Suppose A has a* [2]-*inverse $A^{(2)}_{T,S}$. Then there exist a unique $n \times n$ matrix X such that*

$$GAX = 0, \quad XGA = 0, \quad X^2 = X, \quad \operatorname{rank}(X) = n - s, \tag{2.3}$$

a unique $m \times m$ matrix Y such that

$$YAG = 0, \quad AGY = 0, \quad Y^2 = Y, \quad \operatorname{rank}(Y) = m - s, \tag{2.4}$$

and a unique $n \times m$ matrix Z such that

$$\operatorname{rank}\begin{bmatrix} A & I - Y \\ I - X & Z \end{bmatrix} = \operatorname{rank}(A). \tag{2.5}$$

The matrix Z is the generalized inverse $A^{(2)}_{T,S}$ of A. Further, we have:

$$X = I - A^{(2)}_{T,S} A, \tag{2.6}$$

$$Y = I - A A^{(2)}_{T,S}. \tag{2.7}$$

Proof. To prove the first statement, let U be a nonsingular matrix for which

$$GA = U \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} U^{-1},$$

where J is a nonsingular matrix of order s. It is easy to verify that

$$X = U \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} U^{-1},$$

satisfies the condition (2.3). To show uniqueness, let X_0 be a matrix which satisfies Eq. (2.3). Let $X_1 = U^{-1}X_0U$, and let X_1 be partitioned as

$$X_1 = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

with E being $s \times s$. By Eq. (2.3),

$$\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = 0,$$

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix} \quad \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

so that $E=0$, $F=0$ and $G=0$. It follows that $H=I$, since X_1 again satisfies $X_1^2 = X_1$ and has to have $n-s$. Thus, we obtain $X_0 = X$.

The property (2.4) is proved in a similar manner.

Let $A_{T,S}^{(2)}$ be the [2]-generalized inverse with prescribed range T and null space S. Observe that then Eqs. (2.6) and (2.7) hold. For these X and Y, and

$$\begin{bmatrix} A & I-Y \\ I-X & Z \end{bmatrix} = \begin{bmatrix} A & AA_{T,S}^{(2)} \\ A_{T,S}^{(2)}A & Z \end{bmatrix}.$$

Thus, by Lemma 1.3 and the condition (2.5), we have

$$Z - A_{T,S}^{(2)}AA_{T,S}^{(2)} = 0,$$

implies $Z = A_{T,S}^{(2)}$. This completes the proof of theorem. □

As we know, the important generalized inverses of matrices, for example, $A^+, A_{M,N}^+, A^{\mathrm{D}}$, A_g, $A_{(L)}^{(-1)}$ and $A_{(L)}^{(+)}$ are all [2]-inverse having the corresponding range T and null space S. Therefore the results in Theorem 2.2 are applicable to these generalized inverses.

On the other hand, for the generalised inverse A^+ and A^{D} there are somewhat simpler characterizations.

Corollary 2.1 [9]. *Suppose $A = \mathbb{C}^{m\times n}$ with* rank$(A)=r$. *Then there exist a unique $n\times n$ matrix X such that*

$$AX = 0, \quad X^* = X, \quad X^2 = X, \quad \operatorname{rank}(X) = n-r, \tag{2.8}$$

a unique $m \times m$ matrix Y such that

$$YA = 0, \quad Y^* = Y, \quad Y^2 = Y, \quad \operatorname{rank}(Y) = m-r, \tag{2.9}$$

and a unique $n \times m$ matrix Z such that

$$\operatorname{rank}\begin{bmatrix} A & I-Y \\ I-X & Z \end{bmatrix} = \operatorname{rank}(A). \tag{2.10}$$

The matrix Z is the Moore–Penrose inverse A^+ of A. Further, we have

$$X = I - A^+A, \tag{2.11}$$

and

$$Y = I - AA^+. \tag{2.12}$$

Corollary 2.2 [24]. *Suppose $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(A)=k$ *and* $\mathrm{rank}(A^k) = r$. *Then there exists a unique matrix X such that*

$$A^kX = 0, \quad XA^k = 0, \quad X^2 = X, \quad \mathrm{rank}(X) = n - r, \tag{2.13}$$

and a unique matrix Z such that

$$\mathrm{rank}\begin{bmatrix} A & I - X \\ I - X & Z \end{bmatrix} = \mathrm{rank}(A). \tag{2.14}$$

The matrix Z is the Drazin inverse A^{D} of A. Further, we have

$$X = I - A^{\mathrm{D}}A = I - AA^{\mathrm{D}}. \tag{2.15}$$

Based on Theorem 2.1, we shall next present representation theorem of $A^{(2)}_{T,S}$.

Theorem 2.3. *Under the same hypothesis of Theorem* 2.1. *Then*

$$A^{(2)}_{T,S} = [GA|_{R(G)}]^{-1}G,$$

where $GA|_{R(G)}$ is the restriction of GA to $R(G)$.

Proof. It is a well-known fact that [18], p. 320

$$\begin{aligned}(GA)_g &= [GA|_{R(GA)}]^{-1}[(GA)_gGA] \\ &= [GA|_{R(G)}]^{-1}[(GA)_gGA].\end{aligned}$$

It follows from Theorem 2.1 that

$$\begin{aligned}A^{(2)}_{T,S} &= (GA)_gG = (GA)_gGAA^{(2)}_{T,S} \\ &= [GA|_{R(G)}]^{-1}[(GA)_gGAGA]A^{(2)}_{T,S} \\ &= [GA|_{R(G)}]^{-1}GAA^{(2)}_{T,S} = [GA|_{R(G)}]^{-1}G,\end{aligned}$$

which is the desired result. $\square$

Corollary 2.3 [11]. *Let $A \in \mathbb{C}^{m\times n}$. Then the Moore–Penrose inverse of A can be expressed as*

$$A^+ = [A^*A|_{R(A^*)}]^{-1}A^*.$$

Corollary 2.4 [24]. *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(A) = k$. *Then the Drazin inverse of A can be characterized as*

$$A^{\mathrm{D}} = [A^{k+1}|_{R(A^K)}]^{-1}A^k.$$

For a nonsingular matrix A, A can be characterized in terms of a well-known limit process,

$$A^{-1} = \lim_{\varepsilon\to 0}(A - \varepsilon I)^{-1},$$

where in the limit, as $\varepsilon \to 0$ of the above expression involving $(A - \varepsilon I)^{-1}$, we assume that $\varepsilon \notin \sigma(A)$, the set of all eigenvalues of A. The same assumption will be used in the following.

Based on the above representation Theorem 2.3, we can present an alternative proof for the limiting expression for $A^{(2)}_{T,S}$ [23,26].

Theorem 2.4. *Under the same hypothesis of Theorem* 2.1. *Then*

$$A^{(2)}_{T,S} = \lim_{\varepsilon\to 0}(GA - \varepsilon I)^{-1}G = \lim_{\varepsilon\to 0}G(AG - \varepsilon I)^{-1}. \tag{2.16}$$

Proof. Ben–Israel [1] obtained the limiting process of matrices with index one, i.e.,

$$(GA)_g(GA) = \lim_{\varepsilon\to 0}(GA - \varepsilon I)^{-1}GA.$$

From Theorem 2.1, we have

$$\begin{aligned} A^{(2)}_{T,S} &= (GA)_g G = [(GA)_g GA]A^{(2)}_{T,S} = \lim_{\varepsilon\to 0}(GA - \varepsilon I)^{-1}GAA^{(2)}_{T,S} \\ &= \lim_{\varepsilon\to 0}(GA - \varepsilon I)^{-1}G. \end{aligned}$$

The proof of the remaining part of Eq. (2.16) is similar. □

By Theorem 2.1 and Lemma 1.2, we can arrive at the limiting formulas for $A^{+}, A^{+}_{M,N}, A^{\mathrm{D}}, A_g, A^{(-1)}_{(L)}$ and $A^{(+)}_{(L)}$ immediately (see [2,4,23,26]).

3. Applications

In this section, from Theorem 2.1 and Lemma 1.2, we derive an iterative scheme to compute the generalized inverse $A^{(2)}_{T,S}$, which give a unified treatment of the common important six generalized inverse (see [2,26,27]).

Since $GAA^{(2)}_{T,S} = G$, we have

$$A^{(2)}_{T,S} = A^{(2)}_{T,S} - \beta(GAA^{(2)}_{T,S} - G) = (I - \beta GA)A^{(2)}_{T,S} + \beta G,$$

where β is a relaxation parameter.

Let $P = I - \beta GA$ and $Q = \beta G$, the solution $A^{(2)}_{T,S}$ of matrix equation $X = PX + Q$ can be approximated by the following iterative scheme

$$X_1 = Q, X_{m+1} = PX_m + Q. \tag{3.1}$$

The following theorem presents the sufficient conditions for the iterative algorithm (3.1) to converge to $A^{(2)}_{T,S}$ in terms of the matrix V-norm, V-norm is defined as $\|B\|_V = \|BV\|_2$ for any $B \in \mathbb{C}^{n\times m}$, and V is invertible such that

$$V^{-1}AGV = \begin{bmatrix} J_t & & & & \\ & \ddots & & & \\ & & J_t & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} \text{ is the Jordan form of } AG.$$

Theorem 3.1. *The sequences of approximations*

$$X_{2^m} = \sum_{i=0}^{2^m-1} (I - \beta GA)^i \beta G \tag{3.2}$$

defined by the iterative algorithm (3.1) *converge to the generalized inverse* $A^{(2)}_{T,S}$ *in the matrix form V-norm, if* β *is a fixed real number such that* $\max_{1\leqslant i\leqslant s} |1 - \beta\lambda_i| < 1$ *where* $\lambda_i, i = 1, 2, \ldots, s$ *are the nonzero eigenvalues of* AG.

In the case of convergence we obtain the error estimates

$$\frac{\|A^{(2)}_{T,S} - X_{2^m}\|_V}{\|A^{(2)}_{T,S}\|_V} \leqslant \max_{1\leqslant i\leqslant s} |1 - \beta\lambda_i|^{2^m} + o(\varepsilon) \quad \textit{for } \varepsilon > 0.$$

Proof. We know that

$$A^{(2)}_{T,S}AA^{(2)}_{T,S} = A^{(2)}_{T,S}, \quad A^{(2)}_{T,S}AX_{2^m} = X_{2^m},$$

since

$$A^{(2)}_{T,S}A = P_{T(A^*S^\perp)^\perp} \quad \text{and} \quad X_{2^m} \in R(G) = T.$$

Therefore,

$$\begin{aligned}\|A^{(2)}_{T,S} - X_{2^m}\|_V &= \|[A^{(2)}_{T,S}AA^{(2)}_{T,S} - A^{(2)}_{T,S}AX_{2^m}]V\|_2 \\ &= \|[A^{(2)}_{T,S}VV^{-1}[AA^{(2)}_{T,S} - AX_{2^m}]V\|_2 \\ &\leqslant \|A^{(2)}_{T,S}\|_V\|V^{-1}[AA^{(2)}_{T,S} - AX_{2^m}]V\|_2,\end{aligned}$$

$$
\begin{aligned}
\frac{\|A^{(2)}_{T,S} - X_{2^m}\|_V}{\|A^{(2)}_{T,S}\|_V} &\leqslant \|V^{-1}AA^{(2)}_{T,S}V - V^{-1}AX_{2^m}V\|_2 \\
&\leqslant \|V^{-1}AG(AG)_g V - V^{-1}AX_1 V\|_2^{2^m} \\
&= \left\| \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} - \beta V^{-1}AGV \right\|_2^{2^m} \\
&= \left\| \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} J_1 & & & & \\ & \ddots & & & \\ & & J_t & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix} \right\|_2^{2^m} \\
&\leqslant \left(\max_{1 \leqslant i \leqslant s} |1 - \beta\lambda_i| + \varepsilon \right)^{2^m} = \left(\max_{1 \leqslant i \leqslant s} |1 - \beta\lambda_i| \right)^{2^m} + o(\varepsilon),
\end{aligned}
$$

which concludes the proof. □

4. Concluding remarks

It was shown in [7] that for any complex matrices B and W, m by n and n by m, respectively, $X = B[(WB)^{\mathrm{D}}]^2$ is the unique solution to the equations

$$
\begin{aligned}
&(BW)^k = (BW)^{k+1}XW, \quad X = XWBWX, \\
&BWX = XWB, \quad \text{for some integer } k. \qquad (4.1)
\end{aligned}
$$

The matrix X is called the W-weighted Drazin inverse of B and is written as $X = B_{d,w}$. Interchanging the roles of B and W, then $W_{d,B} = W[(BW)^{\mathrm{D}}]^2$ is the B-weighted Drazin inverse of W. It has shown, moreover, that $X = W_{d,B}$ satisfies $XBX = X$ if and only if

$$
W_{d,W} = W(BW)^{\mathrm{D}}. \qquad (4.2)
$$

The above expression coincides with Theorem 2.1 when $\mathrm{Ind}(BW) = 1$ in Eq. (4.2). For matrices over complex field, therefore, that $A^{\mathrm{D}} = (A^k)_{g,A}$, when A is square, and $A^+ = (A^*)_{g,A}$, by Lemma 1.2.

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