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On a behavior of a slice of the $SL_2(\mathbb{C})$-character variety of a knot group under the connected sum

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ABSTRACT

We observe a behavior of a slice (an algebraic subset) $S_0(K)$ of the $SL_2(\mathbb{C})$-character variety of a knot group under the connected sum of knots. It turns out that the number of 0-dimensional components of $S_0(K)$ is additive under the connected sum of knots.

1. Introduction

In recent years, frameworks of Floer homology have been providing breakthroughs in low-dimensional topology, especially in knot theory. Regarding invariants of knots in 3-sphere S^3, Rasmussen introduced so-called the Rasmussen invariant $s(K)$ [16] by using knot Floer homology. It is very interesting that the Milnor conjecture on the slice genus of torus knots can be shown by using the invariant $s(K)$ (refer to [17]). As regards properties of the invariant $s(K)$, for any alternating knot K', $s(K')$ is exactly half the knot signature $\sigma(K')$, though they are not the same in general. However, $s(K)$ and $\sigma(K)$ have a common property, that is, additivity under the connected sum of knots: $s(K_1 \# K_2) = s(K_1) + s(K_2)$, $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$ for any pair of knots K_1 and K_2. In this paper, we would like to focus on additivity of knot invariants related to $\sigma(K)$.

Concerned with the knot signature $\sigma(K)$, Lin constructed so-called the Casson–Lin invariant $h(K)$ [8] via an algebraic subset of the $SU(2)$-representation space of a knot group. Moreover, he showed that for any knot K the invariant $h(K)$ is half the knot signature $\sigma(K)$. Hence $h(K)$ is also additive under the connected sum: $h(K_1 \# K_2) = h(K_1) + h(K_2)$ for any pair of knots K_1 and K_2.

Based on the Casson–Lin invariant, in the paper [11], the author studied a slice (an algebraic subset) $S_0(K)$ of the $SL_2(\mathbb{C})$-character variety $X(G_K)$ of a knot group G_K. That is a closed algebraic subset of $X(G_K) \subset \mathbb{C}^N$ defined as the intersection of $X(G_K)$ with a certain hyperplane in $\mathbb{C}^N$ (for the precise definition, refer to Section 2). Moreover, in the paper [13], we observed that the slice $S_0(K)$ has a beautiful topological structure; a 2-fold branched/2-sheeted covering space structure. In fact, this structure essentially tells that the slice $S_0(K)$ is a very important object to understand mechanisms of the A-polynomial $A_K(m,l)$ [2] and knot contact homology $HC_*^{ab}(K)$ [14,15] from a representation theoretical viewpoint (for more information, refer to [10–13]).

In these perspectives, we would like to observe whether or not the slice $S_0(K)$ has similar properties to the Casson–Lin invariant $h(K)$ and apply the observation to the studies of the A-polynomial and the knot contact homology. This would give a useful tool to observe the behavior of the invariants $A_K(m,l)$ and $HC_*^{ab}(K)$ under the connected sum of knots. As

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a first step, we look into the behavior of the 0-dimensional components of the slice $S_0(K)$ under the connected sum of knots.

Definition 1.1 *(0-dimensional norm).* For a knot K in S^3, let $\|S_0(K)\|_0$ denote the number of 0-dimensional irreducible components of $S_0(K)$ minus one.

We will mention in Section 4 the reason why we use the terminology “norm” $\|\cdot\|_0$ for the number of 0-dimensional irreducible components. Since for any knot K there exists a unique point p in $S_0(K)$ coming from the abelian representations (see Remark 2.1), the norm $\|S_0(K)\|_0$ actually counts the number of isolated points in $S_0(K)$ except the point p. In particular, $\|S_0(K)\|_0$ is non-negative for any knot K (for more details, refer to the final paragraph of Section 2). In fact, the norm $\|S_0(K)\|_0$ has the following nice property as well as the Casson–Lin invariant $h(K)$ and the Rasmussen invariant $s(K)$.

Theorem 1.2. *For any pair of knots K_1 and K_2 in S^3,*

$$\|S_0(K_1 \# K_2)\|_0 = \|S_0(K_1)\|_0 + \|S_0(K_2)\|_0.$$

We remark that the number of irreducible components of $S_0(K)$ does not have additivity in general (refer to Section 4).

In this paper, we concentrate our interest on a proof of Theorem 1.2 in the following steps. In Section 2, we review the definitions of the $SL_2(\mathbb{C})$-character variety $X(G_K)$ of a knot group and its slice $S_0(K)$. After that, in Section 3, we give a proof of Theorem 1.2. In Section 4, we mention further aspects of this research.

2. $SL_2(\mathbb{C})$-character variety of a knot group and its slice

Let G be a finitely generated and presented group. For a representation $\rho : G \to SL_2(\mathbb{C})$, the character χ_ρ of ρ means a function on G defined by $\chi_\rho(g) := \mathrm{trace}(\rho(g))$, $g \in G$. Note that if representations $\rho_i : G \to SL_2(\mathbb{C})$ $(i = 1, 2)$ are conjugate (namely there exists an element $A \in SL_2(\mathbb{C})$ such that $A^{-1}\rho_2(g)A = \rho_1(g)$ for any $g \in G$), then $\chi_{\rho_1} = \chi_{\rho_2}$. Let $R(G)$ be the set of representations $\rho : G \to SL_2(\mathbb{C})$. $R(G)$ is a non-empty set for any G because of the existence of abelian representations. For each element $g \in G$, we can define a map t_g on $R(G)$ by $t_g(\rho) := \mathrm{trace}(\rho(g)) = \chi_\rho(g)$. Let T denote the ring generated by all the functions t_g, $g \in G$. By Proposition 1.4.1 in [4], the ring T is finitely generated. So we can fix a finite set $g_1, \ldots, g_N$ of G such that $t_{g_1}, \ldots, t_{g_N}$ generate T. Define a map

$$t : R(G) \to \mathbb{C}^N, \quad t(\rho) := \bigl(t_{g_1}(\rho), \ldots, t_{g_N}(\rho)\bigr), \quad \rho \in R(G).$$

Then the image $t(R(G))$ is denoted by $X(G)$ and called the $SL_2(\mathbb{C})$-character variety of G. Indeed, $X(G)$ is a closed (in terms of the Zariski topology) algebraic set (refer to Corollary 1.4.5 in [4]). Since $t_{g_1}(\rho), \ldots, t_{g_N}(\rho)$ generate T, the character χ_ρ of a representation $\rho \in R(G)$ is determined by $t(\rho)$. So there exists a natural one-to-one correspondence between the points of $X(G)$ and the characters of representations in $R(G)$ (i.e., $X(G)$ can be identified with the set of characters of all $\rho \in R(G)$):

$$\{\chi_\rho \mid \rho \in R(G)\} \equiv X(G).$$

For a knot group G_K, we may put $g_1 = \mu$, a meridional element of G_K, in the above setting. Then the intersection of the $SL_2(\mathbb{C})$-character variety $X(G_K)$ and the hyperplane $t_\mu = 0$ becomes a closed algebraic subset of $X(G_K)$. We denote it by $S_0(K)$ and call it a slice of $X(G_K)$.

Here we review some basic properties on the slice $S_0(K)$. For a knot K in S^3, let $R_0(G_K)$ be the set of representations $\rho : G_K \to SL_2(\mathbb{C})$ satisfying $\mathrm{trace}(\rho(\mu)) = \chi_\rho(\mu) = 0$, and $\widehat{R}_0(G_K)$ the set of conjugacy classes of all ρ in $R_0(G_K)$. In fact, the slice $S_0(K)$ can be identified with $\widehat{R}_0(G_K)$:

$$\widehat{R}_0(G_K) \equiv \{\chi_\rho \mid \rho \in R_0(G_K)\} \equiv S_0(K).$$

This can be shown as follows. Let $\Delta_K(t)$ denote the Alexander polynomial of K. Since $\Delta_K(-1) \neq 0$ for any knot K in S^3, by Corollary 4.3 in [6], there do not exist reducible non-abelian representations in $R_0(G_K)$ (see also Burde [1] and de Rham [5]). So $R_0(G_K)$ consists entirely of abelian representations and irreducible representations. Note that any abelian representation in $R_0(G_K)$, which is determined by the image of a meridional element μ, can be conjugate to a representation $\rho_0^{\mathrm{ab}} : G_K \to SL_2(\mathbb{C})$ defined by

$$\rho_0^{\mathrm{ab}}(\mu) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

where $i = \sqrt{-1}$. Thus there exists only one conjugacy class $[\rho_0^{\mathrm{ab}}]$ in $\widehat{R}_0(G_K)$ and only one character $\chi_{\rho_0^{\mathrm{ab}}}$ in the set of characters of all $\rho \in R_0(G_K)$ which come from abelian representations. Moreover, for irreducible representations $\rho_1, \rho_2 \in R_0(G_K)$, ρ_1 and ρ_2 are conjugate if and only if $\chi_{\rho_1} = \chi_{\rho_2}$ (refer to Proposition 1.5.2 in [4]). Therefore, $\widehat{R}_0(G_K)$ can be identified with the set of characters of all $\rho \in R_0(G_K)$, which can be identified with the slice $S_0(K)$.

Let $\hat{t}$ denote the identifying map:

$$\hat{t}: \widehat{R}_0(G_K) \to S_0(K), \quad \hat{t}\big([\rho]\big) := \big(t_{g_1}(\rho), \ldots, t_{g_N}(\rho)\big), \quad \rho \in R_0(G_K).$$

By the above argument, $\widehat{R}_0(G_K)$ is described as a union of $\widehat{R}_0^{\mathrm{ab}}(G_K)$ and $\widehat{R}_0^{\mathrm{irr}}(G_K)$, where

$$\widehat{R}_0^{\mathrm{ab}}(G_K) := \big\{\big[\rho_0^{\mathrm{ab}}\big]\big\},$$
$$\widehat{R}_0^{\mathrm{irr}}(G_K) := \big\{[\rho] \in \widehat{R}_0(G_K) \bigm| \rho\text{: irreducible}\big\}.$$

(Furthermore, this is a disjoint union by Proposition 1.5.2 in [4].) Then $S_0(K)$ is described by

$$S_0(K) = \hat{t}\big(\widehat{R}_0^{\mathrm{ab}}(G_K)\big) \sqcup \hat{t}\big(\widehat{R}_0^{\mathrm{irr}}(G_K)\big).$$

Remark 2.1. $\hat{t}([\rho_0^{\mathrm{ab}}])$ is an isolated point in $S_0(K)$ for any knot K in S^3.

This can be checked as follows. Suppose that $\hat{t}([\rho_0^{\mathrm{ab}}])$ is not isolated in $S_0(K)$. Then there exists an irreducible component C_0 of $S_0(K)$ such that C_0 contains $\hat{t}([\rho_0^{\mathrm{ab}}])$ and $C_0 - \hat{t}([\rho_0^{\mathrm{ab}}])$ is not empty. (Note that $C_0 - \hat{t}([\rho_0^{\mathrm{ab}}])$ consists entirely of (points coming from) irreducible characters by the argument given in the last paragraph.) Since $S_0(K)$ is an algebraic subset of $X(G_K)$, there exists an irreducible component C of $X(G_K)$ that contains C_0. The component C_0 contains (points coming from) irreducible characters, hence C is an irreducible component containing $\hat{t}([\rho_0^{\mathrm{ab}}])$ other than the irreducible component consisting entirely of (points coming from) all abelian characters. This is a contradiction to Theorem 1.2 in [6] with $\Delta_K(-1) \neq 0$ (see also Corollary 4.3 in [6]).

Remark 2.1 shows that $\hat{t}(\widehat{R}_0^{\mathrm{irr}}(G_K))$ is a closed algebraic subset. We denote by $S_0^{\mathrm{irr}}(K)$ the algebraic set $\hat{t}(\widehat{R}_0^{\mathrm{irr}}(G_K))$ and call it the irreducible part of $S_0(K)$. Note that the norm $\|S_0(K)\|_0$ actually counts the number of isolated points in $S_0^{\mathrm{irr}}(K)$. In the next section, we focus on $S_0^{\mathrm{irr}}(K)$.

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is essentially due to Klassen's method used in the proof of Proposition 12 in [7]. Here we show how to calculate the number of 0-dimensional (irreducible) components of the slice $S_0(K)$ by using Klassen's method.

For a knot K_1 (resp. K_2) in S^3, we fix a meridional element μ_1 (resp. μ_2). By the van Kampen theorem, the knot group $G_{K_1\#K_2}$ of the connected sum $K_1 \# K_2$ can be described by the amalgamated free product $G_{K_1} *_f G_{K_2}$ of G_{K_1} and G_{K_2} given by the homomorphism $f: \langle\mu_1\rangle \to \langle\mu_2\rangle$ defined by $f(\mu_1) = \mu_2$. This fact induces the following description of $R_0(G_{K_1\#K_2})$:

$$R_0(G_{K_1\#K_2}) = R_0^{\mathrm{ab}} \cup R_0^1 \cup R_0^2 \cup R_0^{12},$$

where

$$R_0^{\mathrm{ab}} := \big\{\rho_1 * \rho_2 \in R_0(G_{K_1\#K_2}) \bigm| \rho_1\text{: abelian},\ \rho_2\text{: abelian}\big\},$$
$$R_0^1 := \big\{\rho_1 * \rho_2 \in R_0(G_{K_1\#K_2}) \bigm| \rho_1\text{: irreducible},\ \rho_2\text{: abelian}\big\},$$
$$R_0^2 := \big\{\rho_1 * \rho_2 \in R_0(G_{K_1\#K_2}) \bigm| \rho_1\text{: abelian},\ \rho_2\text{: irreducible}\big\},$$
$$R_0^{12} := \big\{\rho_1 * \rho_2 \in R_0(G_{K_1\#K_2}) \bigm| \rho_1\text{: irreducible},\ \rho_2\text{: irreducible}\big\}.$$

R_0^{ab} is the abelian part, the others are irreducible parts. Note that the notation $*$ means the decomposition of a representation derived from the amalgamation $G_{K_1\#K_2} = G_{K_1} *_f G_{K_2}$. More precisely, for any representation $\rho \in R(G_{K_1\#K_2})$ the restriction $\rho_i := \rho|_{G_{K_i}}$ is an element of $R(G_{K_i})$. Then we write $\rho_1 * \rho_2$. Conversely, given representations $\rho_i \in R(G_{K_i})$ $(i = 1, 2)$, we can form $\rho = \rho_1 * \rho_2 \in R(G_{K_1\#K_2})$ if and only if $\rho_1(\mu_1) = \rho_2(\mu_2)$. This description of $R_0(G_{K_1\#K_2})$ naturally induces the following decomposition of $\widehat{R}_0(G_{K_1\#K_2})$:

$$\widehat{R}_0(G_{K_1\#K_2}) = \widehat{R}_0^{\mathrm{ab}} \sqcup \widehat{R}_0^1 \sqcup \widehat{R}_0^2 \sqcup \widehat{R}_0^{12},$$

where

$$\widehat{R}_0^{\mathrm{ab}} := \big\{[\rho_1 * \rho_2] \in \widehat{R}_0(G_{K_1\#K_2}) \bigm| \rho_1\text{: abelian},\ \rho_2\text{: abelian}\big\},$$
$$\widehat{R}_0^1 := \big\{[\rho_1 * \rho_2] \in \widehat{R}_0(G_{K_1\#K_2}) \bigm| \rho_1\text{: irreducible},\ \rho_2\text{: abelian}\big\},$$
$$\widehat{R}_0^2 := \big\{[\rho_1 * \rho_2] \in \widehat{R}_0(G_{K_1\#K_2}) \bigm| \rho_1\text{: abelian},\ \rho_2\text{: irreducible}\big\},$$
$$\widehat{R}_0^{12} := \big\{[\rho_1 * \rho_2] \in \widehat{R}_0(G_{K_1\#K_2}) \bigm| \rho_1\text{: irreducible},\ \rho_2\text{: irreducible}\big\}.$$

Here we can check that $\widehat{R}_0^{\mathrm{ab}} \sqcup \widehat{R}_0^1 \sqcup \widehat{R}_0^2 \sqcup \widehat{R}_0^{12}$ is a disjoint union as follows. It follows from the argument in Section 2 that the component $\widehat{R}_0^{\mathrm{ab}}$ is disjoint from the others. So we focus on the remaining components $\widehat{R}_0^{\mathrm{irr}}(G_{K_1\#K_2}) = \widehat{R}_0^1 \sqcup \widehat{R}_0^2 \sqcup \widehat{R}_0^{12}$. Consider maps $p_i : \widehat{R}_0^{\mathrm{irr}}(G_{K_1\#K_2}) \to \widehat{R}_0(G_{K_i})$ $(i = 1, 2)$ defined by $[\rho_1 * \rho_2] \to [\rho_i]$. Then the images of $\widehat{R}_0^1$ and $\widehat{R}_0^2$ under the map p_1 are $\widehat{R}_0^{\mathrm{irr}}(G_{K_1})$ and $\widehat{R}_0^{\mathrm{ab}}(G_{K_1})$, respectively. Since $\widehat{R}_0^{\mathrm{irr}}(G_{K_1})$ and $\widehat{R}_0^{\mathrm{ab}}(G_{K_1})$ are disjoint from each other in $\widehat{R}_0(G_{K_1})$ (by the argument in Section 2), so are $\widehat{R}_0^1$ and $\widehat{R}_0^2$. Similarly, we can check that $\widehat{R}_0^1$ (resp. $\widehat{R}_0^2$) is disjoint from $\widehat{R}_0^{12}$ by using the maps p_2 (resp. p_1).

By the argument in Section 2, there exists an identifying map

$$\hat{t} : \widehat{R}_0(G_{K_1\#K_2}) \to S_0(K_1 \# K_2).$$

Then we obtain

$$S_0(K_1 \# K_2) = \hat{t}\big(\widehat{R}_0(G_{K_1\#K_2})\big) = \hat{t}\big(\widehat{R}_0^{\mathrm{ab}}\big) \sqcup \hat{t}\big(\widehat{R}_0^1\big) \sqcup \hat{t}\big(\widehat{R}_0^2\big) \sqcup \hat{t}\big(\widehat{R}_0^{12}\big).$$

By Remark 2.1, $\hat{t}(\widehat{R}_0^{\mathrm{ab}})$ consists of an isolated point $\hat{t}([\rho_0^{\mathrm{ab}}])$ in $S_0(K_1 \# K_2)$. The irreducible part $S_0^{\mathrm{irr}}(K_1 \# K_2)$ consists of the others $\hat{t}(\widehat{R}_0^1) \sqcup \hat{t}(\widehat{R}_0^2) \sqcup \hat{t}(\widehat{R}_0^{12})$. It is easy to check that $\hat{t}(\widehat{R}_0^1)$ (resp. $\hat{t}(\widehat{R}_0^2)$) is isomorphic to $S_0^{\mathrm{irr}}(K_1)$ (resp. $S_0^{\mathrm{irr}}(K_2)$) as algebraic set. For example, given an irreducible representation $\rho_1 \in R_0(G_{K_1})$, there exists a unique abelian representation $\rho_2 \in R_0(G_{K_2})$ satisfying $\rho_1(\mu_1) = \rho_2(\mu_2)$. Namely, the map p_1 defined above gives a bijection between $\widehat{R}_0^1$ and $\widehat{R}_0^{\mathrm{irr}}(G_{K_1})$. The bijection naturally induces a bijective polynomial map from $\hat{t}(\widehat{R}_0^1)$ to $\hat{t}(\widehat{R}_0^{\mathrm{irr}}(G_{K_1})) = S_0^{\mathrm{irr}}(G_{K_1})$ so that the following diagram becomes commutative:

$$\begin{array}{ccc}
\widehat{R}_0^1 & \xrightarrow{[\rho_1*\rho_2]\to[\rho_1]} & \widehat{R}_0^{\mathrm{irr}}(G_{K_1}) \\
\downarrow \hat{t} & \circlearrowright & \downarrow \hat{t} \\
\hat{t}(\widehat{R}_0^1) & \xrightarrow[\text{bijective}]{} & \hat{t}(\widehat{R}_0^{\mathrm{irr}}(G_{K_1})).
\end{array}$$

Regarding $\hat{t}(\widehat{R}_0^{12})$, there exists a surjective polynomial map

$$\varphi : \hat{t}\big(\widehat{R}_0^{12}\big) \to \hat{t}\big(\widehat{R}_0^{\mathrm{irr}}(G_{K_1})\big) \times \hat{t}\big(\widehat{R}_0^{\mathrm{irr}}(G_{K_2})\big)$$

induced by a surjection

$$\psi : \widehat{R}_0^{12} \to \widehat{R}_0^{\mathrm{irr}}(G_{K_1}) \times \widehat{R}_0^{\mathrm{irr}}(G_{K_2}), \quad \psi\big([\rho_1 * \rho_2]\big) := \big([\rho_1], [\rho_2]\big).$$

More precisely, the following diagram becomes commutative:

$$\begin{array}{ccc}
\widehat{R}_0^{12} & \xrightarrow{\psi} & \widehat{R}_0^{\mathrm{irr}}(G_{K_1}) \times \widehat{R}_0^{\mathrm{irr}}(G_{K_2}) \\
\downarrow \hat{t} & \circlearrowright & \downarrow \hat{t}\times\hat{t} \\
\hat{t}(\widehat{R}_0^{12}) & \xrightarrow[\varphi]{} & \hat{t}(\widehat{R}_0^{\mathrm{irr}}(G_{K_1})) \times \hat{t}(\widehat{R}_0^{\mathrm{irr}}(G_{K_2})),
\end{array}$$

where $\hat{t} \times \hat{t}([\rho_1], [\rho_2]) := (\hat{t}([\rho_1]), \hat{t}([\rho_2]))$. In the following, we show that for any pair $([\rho_1], [\rho_2]) \in \widehat{R}_0^{\mathrm{irr}}(G_{K_1}) \times \widehat{R}_0^{\mathrm{irr}}(G_{K_2})$, the preimage $\psi^{-1}(([\rho_1], [\rho_2]))$ is a 1-parameter family parametrized by $\lambda \in \mathbb{C} - \{0\}$. Let $([\rho_1], [\rho_2])$ be an element of $\widehat{R}_0(G_{K_1}) \times \widehat{R}_0(G_{K_2})$. We may assume without loss of generality that ρ_1 and ρ_2 satisfy

$$\rho_1(\mu) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \rho_2(\mu).$$

Then the preimage $\psi^{-1}(([\rho_1], [\rho_2]))$ can be described as a set

$$\big\{\big[\rho_1 * A\rho_2 A^{-1}\big] \;\big|\; A \in \mathrm{Stab}\big(\rho_2(\mu)\big)\big\},$$

where $\mathrm{Stab}(\rho_2(\mu))$ is the stabilizer subgroup of $\rho_2(\mu)$ in $SL_2(\mathbb{C})$. Since $\rho_2(\mu)$ is in the maximal abelian subgroup H

$$H := \left\{ A_\lambda := \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \in SL_2(\mathbb{C}) \;\middle|\; \lambda \in \mathbb{C} - \{0\} \right\},$$

$\mathrm{Stab}(\rho_2(\mu))$ is exactly the subgroup H. As ρ_2 is irreducible, $A\rho_2 A^{-1}$ is conjugate to ρ_2 itself if and only if $A = \pm E$, where E is the identity matrix. This shows the desired fact.

By the definition of $\hat{t}$, for any pair $([\rho_1], [\rho_2]) \in \widehat{R}_0^{\mathrm{irr}}(G_{K_1}) \times \widehat{R}_0^{\mathrm{irr}}(G_{K_2})$, the image $\hat{t}(\psi^{-1}(([\rho_1], [\rho_2])))$ has no isolated points (because the image is a 1-parameter family parametrized by λ). Hence $\hat{t}(\widehat{R}_0^{12})$ has no 0-dimensional irreducible components. These results show that the 0-dimensional part of $S_0^{\mathrm{irr}}(K_1 \# K_2)$ is the union of the 0-dimensional parts of $\hat{t}(\widehat{R}_0^1)$ and $\hat{t}(\widehat{R}_0^2)$. This gives the desired equation

$$\big\|S_0(K_1 \# K_2)\big\|_0 = \big\|S_0(K_1)\big\|_0 + \big\|S_0(K_2)\big\|_0.$$

4. Remarks on differences between the 0-dimensional part of $S_0(K)$ and $S_0(K)$ itself

We remark that the equation as in Theorem 1.2 does not hold for the number of irreducible components of $S_0(K)$. Remember the following decomposition given in the proof of Theorem 1.2:

$$S_0(K_1 \# K_2) = \hat{t}\big(\widehat{R}_0(G_{K_1 \# K_2})\big) = \hat{t}\big(\widehat{R}_0^{\mathrm{ab}}\big) \sqcup \hat{t}\big(\widehat{R}_0^{1}\big) \sqcup \hat{t}\big(\widehat{R}_0^{2}\big) \sqcup \hat{t}\big(\widehat{R}_0^{12}\big).$$

Namely, $S_0(K_1 \# K_2)$ consists of irreducible components $\hat{t}(\widehat{R}_0^{\mathrm{ab}})$ (an isolated point), $\hat{t}(\widehat{R}_0^{1}) \sqcup \hat{t}(\widehat{R}_0^{2})$ corresponding to $S_0^{\mathrm{irr}}(K_1) \sqcup S_0^{\mathrm{irr}}(K_2)$ and some extra irreducible components $\hat{t}(\widehat{R}_0^{12})$. If $|\Delta_{K_i}(-1)| > 1$ $(i = 1, 2)$, where $\Delta_K(t)$ is the Alexander polynomial, then the component $\hat{t}(\widehat{R}_0^{12})$ is non-empty. Indeed, if $|\Delta_{K_i}(-1)| > 1$, there exists an irreducible *metabelian* representation in $R_0(G_{K_i})$ (see [11,7,9]). By Klassen's method demonstrated in Section 3 (see also [7]), this gives rise to an irreducible component in $\hat{t}(\widehat{R}_0^{12})$. Note that its dimension is at least one. Here we restate these facts as a remark:

Remark 4.1. For any pair of knots K_1 and K_2 in S^3 satisfying $|\Delta_{K_i}(-1)| > 1$ $(i = 1, 2)$, $\hat{t}(\widehat{R}_0^{12})$ contains an irreducible component of more than 0-dimension. In particular, $\hat{t}(\widehat{R}_0^{12})$ is non-empty with the condition.

We denote by $\|S_0(K)\|$ the number of irreducible components of $S_0^{\mathrm{irr}}(K)$ (including components of more than 0-dimension). Since $S_0^{\mathrm{irr}}(K_1 \# K_2)$ contains irreducible components $\hat{t}(\widehat{R}_0^{1}) \sqcup \hat{t}(\widehat{R}_0^{2})$ corresponding to $S_0^{\mathrm{irr}}(K_1) \sqcup S_0^{\mathrm{irr}}(K_2)$, we obtain the following inequality:

$$\big\|S_0(K_1 \# K_2)\big\| \geqslant \big\|S_0(K_1)\big\| + \big\|S_0(K_2)\big\|.$$

Then Remark 4.1 is showing that for any pair of knots K_i $(i = 1, 2)$ in S^3 satisfying $|\Delta_{K_i}(-1)| > 1$ $(i = 1, 2)$, $\hat{t}(\widehat{R}_0^{1}) \sqcup \hat{t}(\widehat{R}_0^{2})$ is a proper algebraic subset of $S_0^{\mathrm{irr}}(K_1 \# K_2)$. So the following strict inequality holds:

$$\big\|S_0(K_1 \# K_2)\big\| > \big\|S_0(K_1)\big\| + \big\|S_0(K_2)\big\|. \tag{1}$$

Therefore, Theorem 1.2 does not hold for $\|S_0(K)\|$.

As regards knots satisfying the strict inequality (1), for example, we can consider small knots. A knot is said to be small if its exterior contains no closed essential surfaces. For example, any torus knot is small. By Remark 4.1, $\|S_0(K_1 \# K_2)\| > \|S_0(K_1 \# K_2)\|_0$ holds for any pair of knots K_i $(i = 1, 2)$ in S^3 satisfying $|\Delta_{K_i}(-1)| > 1$ $(i = 1, 2)$. In fact, for any small knot K the slice $S_0(K)$ is 0-dimensional and thus $\|S_0(K)\|$ is exactly $\|S_0(K)\|_0$. (Note that $K_1 \# K_2$ is not small for any pair of knots K_1 and K_2.) Combining these facts and Theorem 1.2, we obtain the strict inequality (1) for any pair of small knots K_i $(i = 1, 2)$ satisfying $|\Delta_{K_i}(-1)| > 1$. (For instance, a pair of trefoils satisfies the strict inequality (1).)

The equality $\|S_0(K)\| = \|S_0(K)\|_0$ mentioned above is essentially due to the following facts:

- for any small knot K, the character variety $X(G_K)$ is 1-dimensional (refer to Proposition 2.4 in [2]);
- any small knot is meridionally small (refer to Theorem 2.0.3 in [3]).

In this situation, we can observe a relationship between the maximal degree of the A-polynomial $A_K(m, l)$ in terms of l and the 0-dimensional norm $\|S_0(K)\|_0$ (for 2-bridge knot cases, refer to [11], for more general cases, refer to [12]).

At the end of this paper, we mention a background of the notation $\|S_0(K)\|_0$. Actually, for any small knot K, $\|S_0(K)\|_0$ satisfies properties similar to a norm on a vector space as follows:

(A) $\|S_0(K)\|_0 \geqslant 0$;
(B) $\|S_0(K)\|_0 = 0 \Leftrightarrow K$ is the unknot;
(C) $\|S_0(K_1 \# K_2)\|_0 = \|S_0(K_1)\|_0 + \|S_0(K_2)\|_0$.

(A) was shown in Section 2. (C), which can be thought of as a special case of the triangle inequality, is the consequence of Theorem 1.2. The remaining (B) will be proved in the paper [12]. These are the reason why we use the terminology "norm" $\|\cdot\|_0$ for the number of 0-dimensional irreducible components. Regarding $\|S_0(K)\|$, we remark that (A) clearly holds for any knot, however, (C) does not hold in general as shown in (1). At the present moment, we do not know whether or not (B) for $\|S_0(K)\|$ holds for any knot.

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