On intermediate solutions and the Wronskian for half-linear differential equations

Mariella Cecchi a, Zuzana Došlá b,*,1, Mauro Marini a

a Department of Electronics and Telecommunications, University of Florence, Via S. Marta 3, 50139 Florence, Italy
b Department of Mathematics, Masaryk University, Janáčkovo nám. 2a, 602 00 Brno, Czech Republic

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Abstract
The asymptotic behavior of nonoscillatory solutions of the half-linear differential equation is studied. In particular, two Wronskian-type functions, which have some interesting properties, similar to the one of the Wronskian in the linear case, are given. Using these properties and suitable integral inequalities, the existence of the so-called intermediate solutions is examined and an open problem is solved.

Keywords: Half-linear differential equation; Wronskian; Nonoscillatory solutions; Intermediate solutions

1. Introduction
Consider the half-linear equation

\[(a(t)\Phi(x'))' + b(t)\Phi(x) = 0,\]  \hspace{1cm} (1)

where \(a, b\) are continuous, positive functions for \(t \geq 0\), and \(\Phi(u) = |u|^{p-2}u, p > 1\).

It is well known that (1) exhibits many similarities with the linear equation

\[(a(t)y')' + b(t)y = 0.\] \hspace{1cm} (2)

* Corresponding author.
E-mail addresses: mariella.cecchi@unifi.it (M. Cecchi), dosla@math.muni.cz (Z. Došlá), mauro.marini@unifi.it (M. Marini).

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Nevertheless, in extending to the half-linear equation other typical properties of (2), some problems arise, see [10, §1.3] and references therein. One of them is related to the concept of Wronskian. In [11] it is shown that the Wronskian identity
\[ W(t) \equiv a(t) \left[ y_1(t)y_2'(t) - y_1'(t)y_2(t) \right] = c, \]  
where \( y_i \) are two solutions of (2) and \( c \) is a real constant, does not have analogy in the half-linear case with \( p \neq 2 \). In this paper we present two Wronskian-type functions, which have some remarkable properties, similar to the ones of the function \( W \) in (3) and we apply these results in solving two open problems posed in [13, p. 213], concerning the possible coexistence of solutions of (1) with different asymptotic behavior.

The paper is organized as follows. In Sections 2, 3 some preliminary results, concerning the classification of solutions of (1) and principal solutions, are given. In Section 4 two Wronskian-type functions \( F, G \) are introduced and their monotonicity properties are established. In Section 5 the results on Wronskian-type functions and on the limit characterization of principal solutions are applied to obtain some existence results for the so-called intermediate solutions of (1). These criteria negatively answer to the claimed question in [13]. In Section 6, using results of Section 5, we describe asymptotic properties of functions \( F, G \). Some open problems complete the paper.

2. A classification of solutions

When (1) is nonoscillatory, the asymptotic behavior of its solutions has been considered in many papers. We refer, in particular, to [3,6,9,12–14] and to the monographs [1,10,16]. Here we recall some basic results, which will be useful in the sequel.

Denote
\[ J_a = \int_0^\infty \frac{dt}{\Phi^*(a(t))}, \quad J_b = \int_0^\infty b(t) \, dt, \]
where \( \Phi^* \) is the inverse of the map \( \Phi \), i.e. \( \Phi^*(u) = |u|^{p^* - 2}u, \ p^* = p/(p - 1) \).

Assume that (1) is nonoscillatory. Then any nontrivial solution \( x \) of (1) belongs to one of the following two classes:

\[ M^+_0 = \{ x \text{ solution of (1)}: \exists t_x \geq 0: x(t)x'(t) > 0 \text{ for } t > t_x \}, \]
\[ M^- = \{ x \text{ solution of (1)}: \exists t_x \geq 0: x(t)x'(t) < 0 \text{ for } t > t_x \}, \]
see, e.g., [6] or [10, §4.1.1]. For any nontrivial solution \( x \) of (1), denote by \( x^{[1]} \), \( x^{[1]}(t) = a(t)\Phi(x'(t)) \), the quasiderivative of \( x \). In virtue of the positiveness of the functions \( a, b \), both classes \( M^+, M^- \) can be, a priori, divided into the following subclasses:

\[ M^+_{0,0} = \left\{ x \in M^+: \lim_{t \to \infty} x(t) = c_x, \lim_{t \to \infty} x^{[1]}(t) = 0, \ 0 < |c_x| < \infty \right\}, \]
\[ M^+_{0,0} = \left\{ x \in M^+: \lim_{t \to \infty} \left| x(t) \right| = \infty, \lim_{t \to \infty} x^{[1]}(t) = 0 \right\}, \]
\[ M^+_{\infty,0} = \left\{ x \in M^+: \lim_{t \to \infty} x(t) = 0, \lim_{t \to \infty} x^{[1]}(t) = d_x, \ 0 < |d_x| < \infty \right\}, \]
\[ M^+_{0,\ell} = \left\{ x \in M^+: \lim_{t \to \infty} x(t) = c_x, \lim_{t \to \infty} x^{[1]}(t) = d_x, \ 0 < |c_x|, |d_x| < \infty \right\}, \]
\[ M^-_{\ell,0} = \left\{ x \in M^-: \lim_{t \to \infty} x(t) = 0, \lim_{t \to \infty} x^{[1]}(t) = d_x, \ 0 < |d_x| < \infty \right\}, \]
\[ M_{0,\infty}^- = \left\{ x \in M^- : \lim_{t \to \infty} x(t) = 0, \lim_{t \to \infty} |x^{(1)}(t)| = \infty \right\}, \]
\[ M_{\ell,\infty}^- = \left\{ x \in M^- : \lim_{t \to \infty} x(t) = c_x, \lim_{t \to \infty} |x^{(1)}(t)| = \infty, 0 < |c_x| < \infty \right\}, \]
\[ M_{\ell,\ell}^- = \left\{ x \in M^- : \lim_{t \to \infty} x(t) = c_x, \lim_{t \to \infty} x^{(1)}(t) = d_x, 0 < |c_x|, |d_x| < \infty \right\}. \]

Let \( \mathcal{S} \) be the set of nontrivial solutions of (1). The following holds.

**Lemma 1.**

(i) If \( J_a + J_b = \infty \), then \( M_{\ell,\ell}^+ = \emptyset, M_{\ell,\ell}^- = \emptyset \).

(ii) Assume (1) nonoscillatory. Then
\[ J_a = \infty \iff \mathcal{S} \equiv M^+; \quad J_b = \infty \iff \mathcal{S} \equiv M^- . \]

**Proof.** Claim (i). Let \( J_a = \infty \) and \( x \in M_{\ell,\ell}^+ \). Without loss of generality, suppose \( x(t) > 0 \) for large \( t \). From \( x^{(1)}(t) = a(t)\Phi(x'(t)) \), we have for large \( t \)
\[ x'(t) \sim \frac{1}{\Phi^*(a(t))}, \tag{4} \]
where the symbol \( g_1(t) \sim g_2(t) \) means that \( g_1(t)/g_2(t) \) has a finite nonzero limit, as \( t \to \infty \). Integrating (4) on \((t, \infty)\), we obtain a contradiction. The case \( J_b = \infty \) can be treated in a similar way.

Claim (ii) follows from, e.g., [6, Proposition 1] or [10, Lemmas 4.1.3, 4.1.4]. \( \square \)

Concerning the existence of solutions in the subclasses \( M_{0,\infty}^+, M_{0,\infty}^- \), very few is known in the literature. These solutions are often called intermediate solutions (see, e.g., [13]). Such a terminology is due to the fact that, when \( J_a + J_b = \infty \) and (1) is nonoscillatory, in virtue of Lemma 1, the possible solutions of (1) belong to \( M_{0,0}^+ \cup M_{0,\infty}^+ \cup M_{\ell,\ell}^+ \) (if \( J_a = \infty \)) or to \( M_{0,0}^- \cup M_{0,\infty}^- \cup M_{\ell,\ell}^- \) (if \( J_b = \infty \)) and it results for large \( t \)
\[ |x(t)| < |y(t)| < |z(t)|, \]
for any \( x \in M_{0,0}^+, y \in M_{0,\infty}^+, z \in M_{\ell,\ell}^+ \) or \( x \in M_{0,\ell}^-, y \in M_{0,\infty}^-, z \in M_{\ell,\ell}^- \). These solutions are studied in Section 5, where an answer is given to the question posed in [13] whether intermediate solutions may coexist with solutions in the classes \( M_{\ell,0}^+ \) and \( M_{\ell,\ell}^+ \).

To classify solutions of (1) in the nonoscillatory case, the following integrals play a crucial role:
\[ J_1 = \int_0^\infty \frac{1}{\Phi^*(a(t))} \Phi\left( \int_0^t b(s) \, ds \right) \, dt, \]
\[ J_2 = \int_0^\infty \frac{1}{\Phi^*(a(t))} \Phi\left( \int_t^\infty b(s) \, ds \right) \, dt, \]
\[ Y_1 = \int_0^\infty b(t) \Phi\left( \int_t^\infty \frac{ds}{\Phi^*(a(s))} \right) \, dt, \]
\[ Y_2 = \int_0^\infty b(t) \Phi \left( \int_0^t \frac{ds}{\Phi^*(a(s))} \right) \, dt. \]

It is easy to verify that
\[ J_1 < \infty \implies J_a < \infty; \quad Y_1 < \infty \implies J_a < \infty; \]
\[ J_2 < \infty \implies J_b < \infty; \quad Y_2 < \infty \implies J_b < \infty. \]

Concerning the mutual behavior of integrals \( J_i, Y_i, i = 1, 2 \), the following holds.

**Lemma 2.** ([8, Corollary 1].)

(i1) If \( p \geq 2 \), then
\[ Y_1 = \infty \implies J_1 = \infty; \quad Y_2 = \infty \implies J_2 = \infty. \]

(i2) If \( 1 < p \leq 2 \), then
\[ J_1 = \infty \implies Y_1 = \infty; \quad J_2 = \infty \implies Y_2 = \infty. \]

When \( J_a = J_b = \infty \), then (1) is oscillatory (see, e.g., [1, Theorem 3.8.6] or [10, Theorem 1.2.9]). Consequently, when (1) is nonoscillatory, in view of Lemma 2, it is useful to distinguish the following possible cases:

\( (C_0) \): \( J_a < \infty, \quad J_b < \infty, \quad \text{if } 1 < p; \)
\( (C_1^+) \): \( J_a = \infty, \quad J_b < \infty, \quad J_2 = \infty, \quad Y_2 = \infty, \quad \text{if } 1 < p; \)
\( (C_2^+) \): \( J_a = \infty, \quad J_b < \infty, \quad J_2 = \infty, \quad Y_2 < \infty, \quad \text{if } 2 < p; \)
\( (C_3^+) \): \( J_a = \infty, \quad J_b < \infty, \quad J_2 < \infty, \quad Y_2 = \infty, \quad \text{if } 1 < p < 2; \)
\( (C_4^+) \): \( J_a = \infty, \quad J_b < \infty, \quad J_2 < \infty, \quad Y_2 < \infty, \quad \text{if } 1 < p < 2; \)
\( (C_1^-) \): \( J_a < \infty, \quad J_b = \infty, \quad J_1 = \infty, \quad Y_1 = \infty, \quad \text{if } 1 < p; \)
\( (C_2^-) \): \( J_a < \infty, \quad J_b = \infty, \quad J_1 < \infty, \quad Y_1 = \infty, \quad \text{if } 1 < p < 2; \)
\( (C_3^-) \): \( J_a < \infty, \quad J_b = \infty, \quad J_1 = \infty, \quad Y_1 < \infty, \quad \text{if } 2 < p; \)
\( (C_4^-) \): \( J_a < \infty, \quad J_b = \infty, \quad J_1 < \infty, \quad Y_1 < \infty, \quad \text{if } 1 < p. \)

Observe that all these cases may occur (see, e.g., [8, Examples 1, 2], [13, Example 4.1]). Further observe that if \( (C_0) \) holds, then \( J_i < \infty, \ Y_i < \infty, \ i = 1, 2 \). Since for the linear equation (2) we have \( J_1 = Y_1, \ J_2 = Y_2 \), the cases \( (C_i^\pm), \ i = 2, 3, 4 \), cannot occur in the linear case and so they are, in some sense, typical for the half-linear case with \( p \neq 2 \).

In the cases \( (C_i^+) \) both oscillation and nonoscillation can occur and in the remaining cases \( (C_i^-) \), \( i = 2, 3, 4 \), (1) is nonoscillatory (see, e.g., [1, §3.7] or [10, §3.1]). In the nonoscillatory case, in view of Lemma 1, when any of the cases \( (C_i^+), \ i = 1, \ldots, 4 \) holds, we have \( S \equiv M^+ \) and, similarly, when any of the cases \( (C_i^-), \ i = 1, \ldots, 4 \) holds, we have \( S \equiv M^- \). The upper symbols \( \pm \) for denoting cases \( (C_i^\pm), \ i = 1, \ldots, 4 \), are just suggested by this property.
3. Principal solutions and reciprocity

As it is well known, when (1) is nonoscillatory, the concept of a principal solution has been extended to (1) in [12,16]. More precisely, a nontrivial solution $u$ of (1) is called a principal solution of (1) if for every nontrivial solution $x$ of (1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R}$, we have

\[
\frac{u'(t)}{u(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t. \tag{5}
\]

As in the linear case, the principal solution $u$ exists and is unique up to a constant factor. Any nontrivial solution $x \neq \lambda u$ is called nonprincipal solution. In this section we recall some results in [2,3,6] concerning principal solutions of (1), which will be used later.

When (1) is nonoscillatory, in [9] the question, whether principal solutions are smallest solutions in a neighborhood of infinity also in the half-linear case, has been posed. This problem has been partially solved by the following.

**Theorem 1.** ([3, Theorem 2], [6, Theorem 1].) Assume that (1) is nonoscillatory and any of the following cases occurs:

- $(C_0)$; $(C_i^\pm)$, $i = 2, 3, 4$; $(C_4^+)$, $p \geq 2$; $(C_4^-)$, $1 < p \leq 2$.

Let $u$ be a nontrivial solution of (1). Then $u$ is a principal solution if and only if $\lim_{t \to \infty} u(t)/x(t) = 0$ for any nontrivial solution $x$ of (1) such that $x \neq \lambda u$, $\lambda \in \mathbb{R}$.

**Theorem 2.** ([6, Corollary 1], [2, Theorem 3].)

(i1) The set of principal solutions of (1) is $\mathcal{M}^+_{0,0}$, if any of the cases $(C_3^+)$, $(C_4^+)$ holds.

(i2) The set of principal solutions of (1) is $\mathcal{M}^-_{0,0}$, if any of the cases $(C_0)$, $(C_3^-)$, $(C_4^-)$ holds.

In the study of asymptotic behavior of solutions of (1) the so-called reciprocity principle plays an important role (see, e.g., [10, Chapter 1.2.8]). Recall that the quasiderivative $y = x^{[1]}$ of any solution $x$ of (1) is a solution of

\[
\left(\Phi^*(\frac{1}{b(t)}) \Phi^*(y')\right)' + \Phi^*(\frac{1}{a(t)}) \Phi^*(y) = 0, \tag{6}
\]

which is obtained from (1) by interchanging the function $a$ with $\Phi^*(1/b)$ and $b$ with $\Phi^*(1/a)$. Conversely, the quasiderivative $y^{[1]}(t) = \Phi^*(1/b(t))\Phi^*(y'(t))$ of any solution $y$ of (6) is a solution of (1). For this reason Eq. (6) is called reciprocal equation and, clearly, (1) and (6) have the same character with respect to the oscillation.

Observe that $J_a [J_b]$ for (1) plays the same role as $J_b [J_a]$ for (6) and vice versa. Analogously $J_1 [J_2]$ for (1) plays the same role as $Y_2 [Y_1]$ for (6) and vice versa. Hence, the case $(C_i^+)$ holds for (1) if and only if the case $(C_i^-)$ holds for (6), $i = 1, \ldots, 4$.

In the sequel, we use the reciprocity principle in two contexts: the first one concerns the principal solutions and reads as following.

**Theorem 3.** (See [3, Theorem 1].) Assume (1) nonoscillatory and $J_a + J_b = \infty$. A solution $u$ of (1) is a principal solution if and only if $v = u^{[1]}$ is a principal solution of (6).

The second application of the reciprocity principle concerns the relations between classes $\mathcal{M}^+$ and $\mathcal{M}^-$ and relations between their subclasses for (1) and its reciprocal equation (6). If $x$ is
a solution of (1), \( x \in M^+ \) and \( y = x^{[1]} \), then \( y \) is a solution of (6) and it is straightforward to verify that \( y(t)y'(t) < 0 \) eventually, i.e. \( y \) is a solution of (6) in the class \( M^- \). Similarly, if \( x \in M^- \), then \( y \) is a solution of (6) in the class \( M^+ \). Thus, when any of the cases \( (C^-_i) \), \( i = 1, \ldots, 4 \) occurs, the existence of solutions of (1) in the subclasses of \( M^- \) can be studied by applying the corresponding results obtained for (1) to the reciprocal equation in cases \( (C^+_i) \), \( i = 1, \ldots, 4 \), or vice versa. Such an approach can be also used for studying subclasses of \( M^\pm \) when the case \( (C_0) \) holds. The following result illustrates this application.

**Theorem 4.**

(i) Assume \( J_a = \infty \). Then

\[
M^+_{\ell,0} \neq \emptyset \iff J_2 < \infty, \quad M^+_{\infty,\ell} \neq \emptyset \iff Y_2 < \infty.
\]

(ii) Assume \( J_b = \infty \). Then

\[
M^-_{-\ell,0} \neq \emptyset \iff Y_1 < \infty, \quad M^-_{-\ell,\infty} \neq \emptyset \iff J_1 < \infty.
\]

(iii) Assume the case \( (C_0) \). Then all the solutions of (1), together with their quasiderivatives, are bounded and \( M^+_{\ell,0} \neq \emptyset, M^-_{\ell,\ell} \neq \emptyset \).

**Proof.** Claim (i) follows from [13, Theorems 4.1, 4.2] (see also [6, Theorem A]).

Claim (ii) follows by applying claim (i) to the reciprocal equation (6).

Claim (iii). Let \( x \) be a solution of (1). Applying [2, Theorem 6], the boundedness of \( x \) and \( x^{[1]} \) follows. In view of Theorem 2, the set of principal solutions of (1) coincides with \( M^-_{0,\ell} \).

Because principal solutions are determined up to a constant factor, there exists a unique solution of (1), say \( u \), such that \( u(0) = 1, u \in M^-_{0,\ell} \). Applying the uniqueness result [2, Theorem 1] to the reciprocal equation (6) and taking into account the homogeneity property, i.e. if \( y \) is a solution of (6), then \( \lambda y, \lambda \in \mathbb{R}, \) is a solution too, we obtain that there exists a unique solution of (1), say \( z \), such that \( z(0) = 1, z \in M^+_{\ell,\ell} \). Then each solution \( x \) of (1), such that \( x'(0) \) is different from \( u'(0) \) and \( z'(0) \), belongs to \( M^+_{\ell,\ell} \cup M^-_{\ell,\ell} \). If one of these sets is empty, then, considering the reciprocal equation, the second one would be empty too, which is a contradiction. Thus both classes \( M^+_{\ell,\ell} \), \( M^-_{\ell,\ell} \) must be nonempty.

\[ \square \]

4. An extension of the Wronskian

In this section we introduce two Wronskian-type functions which, in some sense, extend the Wronskian to the half-linear case.

For any two solutions \( x, u \) of (1), consider the functions

\[
F(x, u)(t) = x^{[1]}(t)\Phi(u(t)) - u^{[1]}(t)\Phi(x(t)),
\]

\[
G(x, u)(t) = \Phi^\ast(x^{[1]}(t))u(t) - \Phi^\ast(u^{[1]}(t))x(t).
\]

Clearly, \( F(x, u)(t) = -F(u, x)(t) \) and \( F(x, u)(t) > 0 \) if and only if \( G(x, u)(t) > 0 \). If \( p = 2 \) both functions coincide and read as the Wronskian \( W \) given in (3). In the sequel, we state the main property of the Wronskian-type functions \( F, G \). We start with the following lemma.

**Lemma 3.** Let \( u \) and \( x \) be two solutions of (1). Then the function \( F \) [\( G \)] for (1) is the function \( G \) [\( F \)] for (6).
Proof. Denote the quasiderivative of $u$ and $x$ by $v = u^{[1]}$ and $y = x^{[1]}$, respectively and let $v^{[1]}, y^{[1]}$ be the quasiderivatives of $v$ and $y$, i.e. $v^{[1]} = \Phi^*(v')/\Phi^*(b), y^{[1]} = \Phi^*(y')/\Phi^*(b)$. Then $v, y$ are solutions of (6) and $v^{[1]} = -u, y^{[1]} = -x$. Denoting by $H$ the Wronskian-type function $F$ for (6), we obtain
\[
H(y, v)(t) = y^{[1]}(t)\Phi^*(v(t)) - v^{[1]}(t)\Phi^*(y(t)) = \Phi^*(x^{[1]}(t))u(t) - \Phi^*(u^{[1]}(t))x(t) = G(x, u)(t).
\]
The second assertion follows by using a similar argument. \qed

Theorem 5. Assume (1) nonoscillatory. Let $u$ be a principal solution and $x$ a nonprincipal solution of (1) such that $x(t)u(t) > 0$ eventually. Then the functions $F(x, u), G(x, u)$ are eventually positive. In addition, $F(x, u)$ and $G(x, u)$ are eventually decreasing if
\[
either \quad 1 < p < 2, \ J_a = \infty \quad or \quad p > 2, \ J_b = \infty, \tag{7}
\]
and eventually increasing if
\[
either \quad p > 2, \ J_a = \infty \quad or \quad 1 < p < 2, \ J_b = \infty. \tag{8}
\]

Proof. From (5) we have for large $t$
\[
\frac{x^{[1]}(t)}{\Phi(x(t))} > \frac{u^{[1]}(t)}{\Phi(u(t))}
\]
and so $F(x, u)$ and $G(x, u)$ are eventually positive.
Now we prove the monotonicity of $F$. Assume $J_a = \infty$ and put
\[
\Psi(w) = \frac{d}{dw}\Phi(w).
\]
Taking into account that for $p \neq 1$ and $w, z \neq 0$ it results
\[
\frac{\Phi(w)}{w} = \frac{1}{p - 1}\Psi(w), \quad \Psi(w)\Psi(z) = \Psi(wz),
\]
we have
\[
\frac{d}{dt} F(x, u)(t) = x^{[1]}(t)\Psi(u(t))u'(t) - u^{[1]}(t)\Psi(x(t))x'(t)
\]
\[
= \frac{a(t)u'(t)x'(t)}{p - 1}(\Psi(x'(t))\Psi(u(t)) - \Psi(u'(t))\Psi(x(t)))
\]
\[
= \frac{a(t)u'(t)x'(t)}{p - 1}(\Psi(x'(t)u(t)) - \Psi(u'(t)x(t))).
\]
From (5) we obtain $x'(t)u(t) > u'(t)x(t)$ for large $t$. Since $J_a = \infty$, from Lemma 1 we have $u'(t)x(t) > 0$ eventually. Taking into account that $\Psi(w)$ is increasing for $p > 2$ and $w > 0$ and decreasing for $1 < p < 2$ and $w > 0$, we get the assertion.
In the case $J_b = \infty$, again from Lemma 1, we obtain $x'(t)u(t) < 0$ eventually. Since $\Psi(w)$ is decreasing for $p > 2$ and $w < 0$ and increasing for $1 < p < 2$ and $w < 0$, the assertion again follows.
Finally, the monotonicity of $G$ follows from Theorem 3, Lemma 3 and by applying the first statement to (6). \qed
As already claimed, in the linear case, i.e. for \( p = 2 \), the functions \( F(x, u) \) and \( G(x, u) \) are constant and such a constant is different from zero if \( x \neq \lambda u, \lambda \in \mathbb{R} \). When \( p \neq 2 \) these functions are not constant and the question whether they can approach, as \( t \to \infty \), a finite nonzero limit will be considered in Section 6.

5. Intermediate solutions

In this section, we apply the Wronskian-type functions \( F, G \) and the limit characterization of principal solutions to study the (non)existence of the intermediate solutions of (1). The following result extends [8, Theorem 4, Corollary 5].

**Theorem 6.** Assume (1) is nonoscillatory.

(i) If any of the cases \( (C_i^+) \), \( i = 1, 2, 3 \), holds, then \( M_+^{\infty, 0} \neq \emptyset \).

(ii) If any of the cases \( (C_i^-) \), \( i = 1, 2, 3 \), holds, then \( M_-^{0, \infty} \neq \emptyset \).

**Proof.** Claim (i1). Since (1) is nonoscillatory, in view of Lemma 1 and Theorem 4, in the case \( (C_1^+) \) we have \( M_+^{\infty, 0} \neq \emptyset \). If the case \( (C_2^+) \) holds, again from Theorem 4, \( M_+^{\ell, 0} = \emptyset \) and \( M_+^{\infty, \ell} \neq \emptyset \). From here and Theorem 1 the assertion follows. In the case \( (C_3^+) \) we proceed by using a similar argument.

Claim (i2). The assertion follows by applying claim (i1) to the reciprocal equation (6). \( \square \)

The following result deals with the nonexistence of intermediate solutions and extends to the half-linear case a well-known result for the linear equation (2), see, e.g., [7].

**Theorem 7.** If any of the cases \( (C_0), (C_4^+) \) holds, then \( M_+^{\infty, 0} = M_-^{0, \infty} = \emptyset \).

To prove this result, the following lemma is useful.

**Lemma 4.** Let \( A, B \) two positive functions on \( I = [0, \infty) \) and \( A \in L_{1,\text{loc}}^1(I), B \in L^1(I) \). If \( m \geq 1 \), then for \( t_0 \geq 0 \) the following inequality holds:

\[
\int_{t_0}^{t} A(s) \left( \int_{s}^{\infty} B(u) \, du \right)^m \, ds 
\leq 2^{m-1} \left\{ \left[ \int_{t_0}^{t} B(s) \left( \int_{t_0}^{s} A(u) \, du \right)^{1/m} \, ds \right]^m + \left( \int_{t_0}^{t} B(s) \, ds \right)^m \left( \int_{t_0}^{t} A(s) \, ds \right) \right\}.
\]

**Proof.** Using the generalized Minkowski inequality (see, e.g., (2.59) in [15] with \( \varphi(h, k) = A^{1/m}(h)B(k) \)) we obtain

\[
\int_{t_0}^{t} A(s) \left( \int_{s}^{\infty} B(u) \, du \right)^m \, ds 
\leq \left[ \int_{t_0}^{t} B(s) \left( \int_{t_0}^{s} A(u) \, du \right)^{1/m} \, ds \right]^m + \left( \int_{t_0}^{t} B(s) \, ds \right) \left( \int_{t_0}^{t} A(s) \, ds \right)^m \right]^m.
\]
Applying the inequality $(X + Y)^m \leq 2^{m-1}(X^m + Y^m)$, where $X, Y$ are positive real constants, the assertion follows. □

**Proof of Theorem 7.** Assume the case $(C_0)$. From Theorem 4, each solution of (1) is bounded together with its quasiderivative and so the assertion follows.

Now assume the case $(C_4^+)$. Clearly $J_a = \infty$ and from Lemma 1 we have $M_{0,\infty}^{-} = \emptyset$. Let $M_{\infty,0}^{+} \neq \emptyset$ and, without loss of generality, suppose $x \in M_{\infty,0}^{+}$. $x(t) > 0$, $x'(t) > 0$ for $t > t_0$. Consider the following two cases:

(i) $p \geq 2$,

(ii) $1 < p < 2$.

Assume (i). If $p = 2$, i.e. in the linear case, the set of solutions of (2) is a two-dimensional space and so Theorem 4 gives a contradiction. Now let $p > 2$ and let $u$ be a principal solution of (1), $u(t) > 0$ for $t \geq t_0 \geq 0$. In view of Theorem 2, $u$ is bounded. Taking into account that $x^{[1]}$ and $u^{[1]}$ are eventually positive, we have

$$\lim_{t \to \infty} F(x, u)(t) \leq \lim_{t \to \infty} (x^{[1]}(t) \Phi(u(t))) = 0$$

which gives a contradiction because, in view of Theorem 5, $F(x, u)$ is eventually positive increasing.

Now assume (ii). Integrating (1) on $(t, \infty), t \geq t_0$, we obtain

$$x^{[1]}(t) = \int_{t_0}^{\infty} b(s) \Phi(x(s)) \, ds$$

or

$$x(t) - x(t_0) = \int_{t_0}^{t} \Phi^*(\int_{s}^{\infty} \frac{1}{a(s)} \int_{s}^{r} b(r) \Phi(x(r)) \, dr) \, ds.$$

Putting $p = (1 + m)/m$ and so $p^* = m + 1$, we have

$$x(t) - x(t_0) = \int_{t_0}^{t} \left( \frac{1}{a(s)} \int_{s}^{\infty} b(r) (x(r))^{1/m} \, dr \right)^m \, ds.$$

Since $m > 1$, applying Lemma 4 with $A(s) = a^{-m}(s), B(r) = b(r)x^{1/m}(r)$, we obtain

$$x(t) - x(t_0) \leq 2^{p^*-2} x(t) \Phi^* \left( \int_{t_0}^{t} b(s) \Phi \left( \int_{s}^{\infty} \frac{d\sigma}{\Phi^*(a(\sigma))} \right) \, ds \right)$$

$$+ 2^{p^*-2} \Phi^* \left( \int_{t}^{\infty} b(s) \Phi(x(s)) \, ds \right) \left( \int_{t_0}^{t} \frac{ds}{\Phi^*(a(s))} \right),$$

or, in view of (9),

$$\frac{x(t) - x(t_0)}{x(t)} \leq \gamma \Phi^* \left( \int_{t_0}^{t} b(s) \Phi \left( \int_{s}^{\infty} \frac{d\sigma}{\Phi^*(a(\sigma))} \right) \, ds \right) + \gamma \frac{\Phi^*(x^{[1]}(t))}{x(t)} \int_{t_0}^{t} \frac{ds}{\Phi^*(a(s))}.$$
where $\gamma = 2^{p^* - 2}$. Choosing $t_0$ large so that
\[
\Phi^\ast\left(\int_{t_0}^{\infty} b(s) \Phi\left(\int_{t_0}^{s} \frac{d\sigma}{\Phi^\ast(a(\sigma))}\right) \, ds\right) \leq \frac{1}{2\gamma},
\]
we obtain
\[
\frac{\Phi^\ast(x^{[1]}(t))}{x(t)} \int_{t_0}^{t} \frac{ds}{\Phi^\ast(a(s))} \geq \frac{1}{\gamma} \frac{x(t) - x(t_0)}{x(t)} - \frac{1}{2\gamma}.
\]
Then
\[
\lim_{t \to \infty} \inf \frac{\Phi^\ast(x^{[1]}(t))}{x(t)} \int_{t_0}^{t} \frac{ds}{\Phi^\ast(a(s))} \geq \frac{1}{2\gamma}.
\]
Hence there exist $T > t_0$ such that for any $t \geq T$ it results
\[
x^{[1]}(t) \Phi\left(\int_{t_0}^{t} \frac{ds}{\Phi^\ast(a(s))}\right) \geq \Phi\left(\frac{1}{3\gamma} x(t)\right).
\]
Integrating (1) on $(T, t)$ we obtain
\[
\Phi\left(\frac{1}{3\gamma}\right)(x^{[1]}(T) - x^{[1]}(t)) = \int_{T}^{t} b(s) \Phi\left(\frac{1}{3\gamma} x(s)\right) \, ds
\]
\[
\leq \int_{T}^{t} b(s)x^{[1]}(s) \Phi\left(\int_{t_0}^{s} \frac{d\sigma}{\Phi^\ast(a(\sigma))}\right) \, ds.
\]
Since $x^{[1]}$ is decreasing for $t \geq T$, choosing $T$ large so that
\[
\int_{T}^{\infty} b(s) \Phi\left(\int_{t_0}^{s} \frac{d\sigma}{\Phi^\ast(a(\sigma))}\right) \, ds \leq \Phi\left(\frac{1}{4\gamma}\right),
\]
we obtain
\[
\Phi\left(\frac{1}{3\gamma}\right)(x^{[1]}(T) - x^{[1]}(t)) \leq \Phi\left(\frac{1}{4\gamma}\right)x^{[1]}(T)
\]
or
\[
\left[\Phi\left(\frac{1}{3\gamma}\right) - \Phi\left(\frac{1}{4\gamma}\right)\right]x^{[1]}(T) \leq \Phi\left(\frac{1}{3\gamma}\right)x^{[1]}(t),
\]
which gives a contradiction as $t \to \infty$.

Finally, if the case $(C^-_2)$ occurs, the assertion follows by applying the above argument to the reciprocal equation (6).  \hfill \Box

**Corollary 1.** Assume (1) is nonoscillatory.

If $J_a = \infty$, then $M_{\infty,0}^+ \neq \emptyset$ if and only if $J_2 + Y_2 = \infty$.

If $J_b = \infty$, then $M_{0,\infty}^- \neq \emptyset$ if and only if $J_1 + Y_1 = \infty$.  


Theorems 6, 7, jointly with Theorem 4, give a complete answer to the question posed in [13, p. 213], whether intermediate solutions may coexist with solutions in classes $M_{\ell,0}^+,$ $M_{\infty,0}^+,$ $M_{\ell,0}^\infty,$ $M_{\infty,\ell}^\infty.$ Indeed, for (1) when $J_a = \infty$ at most two of the subclasses $M_{\ell,0}^+,$ $M_{\infty,0}^+,$ $M_{\ell,0}^\infty,$ $M_{\infty,\ell}^\infty$ are nonempty.

6. Asymptotic behavior of functions $F,$ $G$

In this section the limit behavior of the Wronskian-type functions $F,$ $G$ is considered and related with the (non)existence of intermediate solutions.

**Theorem 8.** Let $u$ be a principal solution of (1). The following holds:

(i) If $(C_2^+)\text{ holds, then there exist nonprincipal solutions } x \text{ of } (1) \text{ such that}$

$$\lim_{t \to \infty} F(x,u)(t) = \pm \infty, \quad \lim_{t \to \infty} G(x,u)(t) = \pm \infty,$$

according to whether $x(t)u(t) > 0 \text{ or } x(t)u(t) < 0 \text{ for large } t.$

(ii) If $(C_3^+)\text{ holds, then for all nonprincipal solutions } x \text{ of } (1) \text{ we have}$

$$\lim_{t \to \infty} F(x,u)(t) = 0, \quad \lim_{t \to \infty} G(x,u)(t) = 0.$$

**Proof.** First we prove the statements for the function $F$.

Claim (i). Assume $(C_2^+).$ By Lemma 2, we have $p > 2.$ From Lemma 1, any nontrivial solution of (1) belongs to the class $M^+$ and, in view of Theorems 4, 6, we have $M_{\infty,\ell}^+ \neq \emptyset,$ $M_{\infty,0}^+ \neq \emptyset,$ $M_{\ell,0}^+ = \emptyset.$ By [3, Theorem 2] we have $u \in M_{\infty,0}^+.$ Let $x \in M_{\infty,\ell}^+$ and, without loss of generality, assume $x(t) > 0,$ $u(t) > 0$ for $t \geq T > 0.$ Thus $F(x,u)$ is eventually positive. Assume

$$\lim_{t \to \infty} F(x,u)(t) = c, \quad 0 < c < \infty \text{ and put } \lim_{t \to \infty} x^{[1]}(t) = \ell_x.$$

By using the l’Hôpital rule, we obtain

$$\lim_{t \to \infty} \frac{x(t)}{\int_0^t \Phi^*(1/a(s)) \, ds} = \Phi^*(\ell_x), \quad \lim_{t \to \infty} \frac{u(t)}{\int_0^t \Phi^*(1/a(s)) \, ds} = 0. \quad (11)$$

Hence there exists a constant $k$ such that for $t \geq T$

$$x(t) < k \int_0^t \frac{d\tau}{\Phi^*(a(\tau))}. \quad (12)$$

Since $u$ is unbounded, without loss of generality we can also assume for $t \geq T$

$$\Phi(u(t)) > 3c/\ell_x. \quad (13)$$

Taking into account that $F(x,u)$ is increasing and $x^{[1]}$ is decreasing, from (13) we obtain for $t \geq T$

$$u^{[1]}(t)\Phi(x(t)) \geq -c + x^{[1]}(t)\Phi(u(t)) \geq -c + \ell_x \Phi(u(t)) \geq -c + \left(\frac{\ell_x}{2} + \frac{\ell_x}{3}\right) \Phi(u(t)) \geq \frac{\ell_x}{2} \Phi(u(t)).$$

Then

$$\frac{u^{[1]}(t)}{\Phi(u(t))} \geq \frac{\ell_x}{2} \frac{1}{\Phi(x(t))}$$
or, in view of (12),
\[
\frac{u'(t)}{u(t)} \geq \Phi^*(\ell x/2) \frac{\Phi^*(1/a(t))}{x(t)} \geq h_1 \frac{\Phi^*(1/a(t))}{\int_0^t \Phi^*(1/a(t)) \, dt},
\]
where \( h_1 = k^{-1} \Phi^*(\ell x/2) \). Integrating this inequality on \((T, t)\), we obtain
\[
u(t) \geq h_2 \int_0^t \frac{dt}{\Phi^*(a(t))},
\]
where \( h_2 \) is a suitable positive constant. This inequality yields a contradiction with the second statement in (11). Since \( F(x, u) \) is eventually increasing, the assertion follows. If \((C^-_3)\) holds, from Theorems 1, 4 and 6, by using a similar argument, we obtain the assertion.

Claim (i2). First assume \((C^+_3)\). By Lemma 2 we have \( 1 < p < 2 \). In view of Theorems 2, 4, the set of principal solutions coincides with \( M^+_{\ell,0} \) and the set of nonprincipal solutions with \( M^+_{\infty,0} \).

Let \( x \in M^+_{\infty,0} \) and, without loss of generality, suppose \( x(t) > 0, u(t) > 0 \) for large \( t \). Thus \( F(x, u) \) is eventually positive. From
\[
0 < F(x, u)(t) = x^{[1]}(t) \Phi(u(t)) - u^{[1]}(t) \Phi(x(t)) \leq x^{[1]}(t) \Phi(u(t)),
\]
as \( t \to \infty \) the assertion follows. If \((C^+_3)\) holds, we have \( p > 2 \) and the assertion follows by using the same argument.

Finally the statements for the function \( G \) follow from the above claims, by applying Lemma 3 and Theorem 3. \( \square \)

The following result relates the asymptotic behavior of \( F, G \) with the existence of intermediate solutions.

**Theorem 9.** Assume \((1)\) nonoscillatory and let \( u \) be a principal solution of \((1)\). If \((1)\) does not have intermediate solutions, then the functions \( F(x, u), G(x, u) \) have a finite nonzero limit for each nonprincipal solution \( x \) of \((1)\).

**Proof.** In view of Theorems 6, 7, the possible cases are \((C^+_4)\) and \((C_0)\).

If the case \((C^+_4)\) holds, by Theorems 2, 4 we have \( u \in M^+_{\ell,0}, x \in M^+_{\infty,\ell} \). Without loss of generality, suppose \( u(t) > 0, x(t) > 0 \) for large \( t \). If \( p \geq 2 \), the assertion follows from the inequalities
\[
0 < F(x, u)(t) = x^{[1]}(t) \Phi(u(t)) - u^{[1]}(t) \Phi(x(t)) \leq x^{[1]}(t) \Phi(u(t)),
\]
\[
0 < G(x, u)(t) = \Phi^*(x^{[1]}(t))u(t) - \Phi^*(u^{[1]}(t))x(t) \leq \Phi^*(x^{[1]}(t))u(t),
\]
taking into account that, by Theorem 5, the functions \( F(x, u), G(x, u) \) are eventually increasing. Let \( 1 < p < 2 \). By Theorem 5, the functions \( F(x, u), G(x, u) \) are eventually decreasing. By contradiction, assume that
\[
\lim_{t \to \infty} F(x, u)(t) = 0.
\]
Since \( F(x, cu)(t) = \Phi(c) F(x, u)(t), \) where \( c \neq 0 \), without loss of generality we can assume \( \lim_{t \to \infty} u(t) = 1 \). Let \( \lim_{t \to \infty} x^{[1]}(t) = d_x \). By using the l'Hopital rule, we have
\[
\lim_{t \to \infty} \frac{x(t)}{\int_0^t [\Phi^*(a(s))]^{-1} \, ds} = \Phi^*(d_x), \quad \lim_{t \to \infty} \frac{u^{[1]}(t)}{\int_t^\infty b(s) \, ds} = 1.
\]
From (14) and \( \lim_{t \to \infty} x^{[1]}(t) \Phi(u(t)) = d_x \), we obtain \( \lim_{t \to \infty} u^{[1]}(t) \Phi(x(t)) = dx \) and so
\[
\lim_{t \to \infty} \left( \int_t^\infty b(s) \, ds \right) \Phi\left( \int_0^t \frac{1}{\Phi^*(a(s))} \, ds \right) = 1.
\]
Consequently, for large \( t \)
\[
\frac{1}{\Phi^*(a(t))} \Phi^*\left( \int_t^\infty b(s) \, ds \right) \sim \frac{1}{\Phi^*(a(t))} \int_0^t \frac{1}{\Phi^*(a(s))} \, ds.
\]
Since for \( T > 0 \) we have
\[
J_2 = \int_0^T \frac{1}{\Phi^*(a(t))} \Phi^*\left( \int_t^\infty b(s) \, ds \right) \, dt + \int_T^\infty \frac{1}{\Phi^*(a(t))} \Phi^*\left( \int_t^\infty b(s) \, ds \right) \, dt,
\]
from (15) we obtain \( J_2 = \infty \), i.e. a contradiction. Thus \( F(x, u) \) has a nonzero limit. From here and the existence of \( \lim_{t \to \infty} \Phi^*(x^{[1]}(t))u(t) = \Phi^*(dx) \), \( 0 < dx < \infty \), \( G \) must have a nonzero limit as well.

Assume the case \( (C_{-4}) \). In view of Theorem 3 and Lemma 3, the assertion follows by applying the above argument to the reciprocal equation (6).

Finally, if the case \( (C_0) \) holds, from Theorem 2, we have \( u \in \mathbb{M}_{-0,\ell}^\infty \). Since, by Theorem 4 any nonprincipal solution \( x \) of (1) is bounded jointly with its quasiderivative, we have \( x \in \mathbb{M}_{-0,\ell}^+, \mathbb{M}_{\ell,0}^+ \cup \mathbb{M}_{\ell,\ell}^+ \). From here the assertion follows. \( \square \)

**Remark 1.** If \( (C_1^{\pm}) \) do not occur, then the converse of Theorem 9 holds. Indeed, in these cases, if (1) is nonoscillatory and \( F(x, u), G(x, u) \) have a finite nonzero limit for each nonprincipal solution \( x \) of (1), in virtue of Theorem 8 the possible cases are \( (C_0) \) and \( (C_{4,\ell}^+) \) and so (1) has no intermediate solutions. Observe also that in these cases the asymptotic behavior of \( F(x, u), G(x, u) \) is similar to the one of the Wronskian in the linear case.

From Theorems 6, 7, 9 we obtain the following two consequences.

**Corollary 2.** Let \( u \) be a principal solution of (1) and any of the cases \( (C_0), (C_{4,\ell}^+) \) holds. Then the functions \( F(x, u), G(x, u) \) have a finite nonzero limit for each nonprincipal solution \( x \) of (1).

**Corollary 3.** Assume (1) nonoscillatory and let \( u \) be a principal solution of (1). If (1) does not have intermediate solutions, then the limit \( \lim_{t \to \infty} x(t)/y(t) \) is finite and different from zero for any nonprincipal solution \( x, y \) of (1).

**Proof.** In view of Theorems 6, 7, the possible cases are \( (C_{4,\ell}^+) \) and \( (C_0) \). If the case \( (C_0) \) holds, by Theorems 2, 4, all nonprincipal solutions tend to nonzero limits, which yields the assertion. If any of the cases \( (C_{4,\ell}^+) \) holds, again from Theorems 2, 4, the set of nonprincipal solutions coincides with the class \( \mathbb{M}_{-\ell,\ell}^\infty \) and the assertion follows using the l'Hôpital rule. \( \square \)
Concluding remarks.

(1) If (1) is nonoscillatory and any of the cases \((C_{i}^{\pm})\) occurs, then all nontrivial solutions \(x\) of (1) are intermediate and it remains an open problem the asymptotic behavior of functions \(F, G\).

(2) Does the result in Corollary 3 continue to hold when intermediate solutions exist?

(3) Similarly as in the continuous case, the notion of Casoratian can be extended for the half-linear difference equations. This is treated in [5], where some open problems proposed in [4] are solved.

References