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Convergence of invariant measures for singular stochastic diffusion equations[☆]

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Abstract

It is proved that the solutions to the singular stochastic p-Laplace equation, $p \in (1, 2)$ and the solutions to the stochastic fast diffusion equation with nonlinearity parameter $r \in (0, 1)$ on a bounded open domain $\Lambda \subset \mathbb{R}^d$ with Dirichlet boundary conditions are continuous in mean, uniformly in time, with respect to the parameters p and r respectively (in the Hilbert spaces $L^2(\Lambda)$, $H^{-1}(\Lambda)$ respectively). The highly singular limit case $p = 1$ is treated with the help of stochastic evolution variational inequalities, where $\mathbb{P}$-a.s. convergence, uniformly in time, is established.

It is shown that the associated unique invariant measures of the ergodic semigroups converge in the weak sense (of probability measures).

MSC: 60H15; 35K67; 37L40; 49J45

Keywords: Stochastic evolution equation; Stochastic diffusion equation; p-Laplace equation; 1-Laplace equation; Total variation flow; Fast diffusion equation; Ergodic semigroup; Unique invariant measure; Variational convergence

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1. Introduction

Let $\Lambda \subset \mathbb{R}^d$ be a bounded open domain with Lipschitz boundary $\partial\Lambda$. Let $\{W(t)\}_{t\geqslant 0}$ be a U-valued cylindrical Wiener process on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}(t)\}_{t\geqslant 0}, \mathbb{P})$, where U is a separable Hilbert space.

We are interested in the following two (families of) stochastic diffusion equations, the stochastic p-Laplacian equation, $p \in (1,\infty)$, $B \in L_2(U, L^2(\Lambda))$,

$$(PL_p)\begin{cases} dX_p(t) = \operatorname{div}\left[|\nabla X_p(t)|^{p-2}\nabla X_p(t)\right]dt + B\,dW(t) & \text{in } (0,T)\times\Lambda,\\ X_p(t) = 0 & \text{on } (0,T)\times\partial\Lambda,\\ X_p(0) = x \in L^2(\Lambda) & \text{in } \Lambda. \end{cases}$$

The deterministic p-Laplace equation arises from geometry, quasi-regular mappings, fluid dynamics and plasma physics, see [19,20]. In [27], (PL$_p$) with $B \equiv 0$ is suggested as a model of motion of non-Newtonian fluids. See [28] for the stochastic equation.

We are also interested in the stochastic fast diffusion equation $r \in (0,\infty)$, $B \in L_2(U, H^{-1}(\Lambda))$,

$$(FD_r)\begin{cases} dY_r(t) = \Delta\left(|Y_r(t)|^{r-1}Y_r(t)\right)dt + B\,dW(t), & \text{in } (0,T)\times\Lambda,\\ Y_r(t) = 0, & \text{on } (0,T)\times\partial\Lambda,\\ Y_r(0) = y \in H^{-1}(\Lambda), & \text{in}\Lambda, \end{cases}$$

which models diffusion in plasma physics, curvature flows and self-organized criticality in sandpile models, see e.g. [12,14,36,41] and the references therein.

The above equations considered are called *singular* for $p \in (1,2)$, $r \in (0,1)$ and *degenerate* for $p \in (2,\infty)$, $r \in (1,\infty)$ (porous medium equation). In this paper, we shall investigate the former case.

For $p = 1$, equation (PL$_1$) can be heuristically written as a stochastic evolution inclusion, $B \in L_2(U, H^{-1}(\Lambda))$,

$$(\mathrm{PL}_1)\begin{cases} dX_1(t) \in \operatorname{div}\left[\operatorname{Sgn}(\nabla X_1(t))\right]dt + B\,dW(t) & \text{in } (0,T)\times\Lambda,\\ X_1(t) = 0 & \text{on } (0,T)\times\partial\Lambda,\\ X_1(0) = x \in L^2(\Lambda) & \text{in } \Lambda, \end{cases}$$

where $\operatorname{Sgn}\colon \mathbb{R}^d \to 2^{\mathbb{R}^d}$ is defined by

$$\operatorname{Sgn}(u) := \begin{cases} \dfrac{u}{|u|}, & \text{if } u \in \mathbb{R}^d\setminus\{0\},\\ \left\{v \in \mathbb{R}^d \mid |v| \leqslant 1\right\}, & \text{if } u = 0. \end{cases}$$

A precise characterization of the 1-Laplace operator can be found in [2,3,37]. A typical 2-dimensional example for the so-called total variation flow can be found in image restoration, see [1,3,6] and the references therein.

We shall, however, take use of the stochastic evolution variational inequality-formulation as in [11].

We are particularly interested in continuity of the solutions in the parameters p and r, especially for the case $p \to 1$. Stochastic Trotter-type results in this direction have been obtained by the first named author in [15–17]. However, for the case $p \to 1$, we shall need the theory of

Mosco convergence of convex functionals as in [4], since no strong characterization of the limit is available (which could be treated by Yosida-approximation methods). For $B = 0$ (i.e., the deterministic equation), the convergence of solutions to the evolution problem (PL_p) was proved in [23,40]. See also [39, Ch. 8.3].

With the help of a uniqueness result for invariant measures of the equations considered, obtained by Liu and the second named author [29], we prove tightness and the weak convergence (weak continuity) of invariant measures associated to the ergodic semigroups of the equations (PL_p) and (FD_r). See [9,10,18,22] for other result in this direction.

Organization of the paper

In Section 2, we prove that the solutions to the basic examples are continuous in the parameters p and r resp.

In Section 3, The result of Section 2 is combined with the uniqueness of invariant measures proved in [29] in order to obtain the weak continuity of invariant measures in the parameters p and r resp.

In Section 4, we prove a convergence result for the stochastic p-Laplace equation as $p \to 1$, using another notion of a solution. For the limit $p = 1$, however, uniqueness of the invariant measure is an open question. The matter is further investigated in [22].

The Appendix collects some well-known results on Mosco (variational) convergence and Mosco convergence in L^p-spaces, needed for the proof in Section 4.

2. Convergence of solutions

Compare with [16, Theorem 2].

Theorem 2.1. *Let* $\{p_n\} \subset \left(1 \vee \frac{2d}{2+d}, 2\right]$, $n \in \mathbb{N}$, $p_0 \in \left(1 \vee \frac{2d}{2+d}, 2\right]$ *such that* $p_n \to p_0$. *Let* $X_n := X_{p_n}$, $n \in \mathbb{N}$, $X_0 := X_{p_0}$ *be the solutions to* (PL_{p_n}), $n \in \mathbb{N}$, (PL_{p_0}) *resp. Then for* $x \in L^2(\Lambda)$.

$$\lim_n \mathbb{E}\left[\sup_{t\in[0,T]} \|X_n(t) - X_0(t)\|^2_{L^2(\Lambda)}\right] = 0.$$

Proof. For $p \in (1,\infty)$, define $a_p\colon \mathbb{R}^d \to \mathbb{R}^d$ by $a_p(x) := |x|^{p-2}x$. Furthermore, let $A_p\colon W_0^{1,p}(\Lambda) \to (W_0^{1,p})^*(\Lambda)$ be defined by $A_p(y) := -\mathrm{div}\left[a_p(\nabla y)\right]$, where $y \in W_0^{1,p}(\Lambda)$. To be more specific,

$$_{(W^{1,p})^*}\left\langle A_p(y), z\right\rangle_{(W^{1,p})} = \int_\Lambda \left\langle a_p(\nabla y), \nabla z\right\rangle d\xi, \quad \forall z \in W_0^{1,p}(\Lambda).$$

We first consider the following approximating equations for (PL_p)

$$\begin{cases} dX_p^\varepsilon(t) + A_p^\varepsilon\left(X_p^\varepsilon(t)\right)dt = B\,dW(t) \\ X_p^\varepsilon(0) = x \end{cases} \tag{2.1}$$

where for any $u \in L^2(\Lambda)$,

$$A_p^\varepsilon(u) = -(1-\varepsilon\Delta)^{-1}\,\mathrm{div}\left[a_p^\varepsilon\left(\nabla(1-\varepsilon\Delta)^{-1}u\right)\right]$$

and a_p^ε is the Yosida approximation of a_p i.e., for any $r \in \mathbb{R}^d$,

$$a_p^\varepsilon(r) = \frac{1}{\varepsilon}\left(1 - \left(1 + \varepsilon a_p\right)^{-1}(r)\right).$$

In particular, for $u, v \in L^2(\Lambda)$,

$$\left(A_p^\varepsilon(u), v\right)_{L^2(\Lambda)} = \int_\Lambda \left\langle a_p^\varepsilon(\nabla R_\varepsilon u), \nabla R_\varepsilon(v)\right\rangle d\xi,$$

where $R_\varepsilon := (1 - \varepsilon\Delta)^{-1}$ is the resolvent of the Dirichlet Laplacian.

We shall use the following strategy ($\mathbb{P}$-a.s.)

$$\begin{aligned}
&\|X_n(t) - X_0(t)\|_{L^2(\Lambda)}^2 \\
&\quad \leqslant 3\left\|X_n(t) - X_n^\varepsilon(t)\right\|_{L^2(\Lambda)}^2 + 3\left\|X_n^\varepsilon(t) - X_0^\varepsilon(t)\right\|_{L^2(\Lambda)}^2 + 3\left\|X_0^\varepsilon(t) - X_0(t)\right\|_{L^2(\Lambda)}^2 \\
&\quad =: I_1(n,\varepsilon) + I_2(n,\varepsilon) + I_3(\varepsilon)
\end{aligned}$$

uniformly in $t \in [0, T]$.

At this point we need to prove the following lemma. We introduce the notation $r_\varepsilon^p(r) := \left(1 + \varepsilon a_p\right)^{-1}(r)$. □

Lemma 2.2. *Under our assumptions, if we let X_p^ε be the solution to (2.1) and $\tilde{X}_p^\varepsilon := (1 - \varepsilon\Delta)^{-1}X_p^\varepsilon$, we have that*

$$\mathbb{E}\int_0^t \int_\Lambda \left|r_\varepsilon^p\left(\nabla \tilde{X}_p^\varepsilon(s)\right)\right|^p d\xi\, ds \leqslant C_t\left(\|x\|_{L^2(\Lambda)}^2 + \|B\|_{HS}^2\right), \tag{2.2}$$

for all $t \in [0, T]$.

Proof. We know by the definition of a_p that

$$\left\langle a_p(r), r\right\rangle \geqslant |r|^p.$$

On the other hand we have by Itō's formula, applied to the function $u \mapsto \|u\|_{L^2(\Lambda)}^2$, that

$$\begin{aligned}
&\mathbb{E}\left\|X_p^\varepsilon(t)\right\|_{L^2(\Lambda)}^2 + 2\mathbb{E}\int_0^t \int_\Lambda \left\langle a_p^\varepsilon\left(\nabla \tilde{X}_p^\varepsilon(s)\right), \nabla \tilde{X}_p^\varepsilon(s)\right\rangle d\xi\, ds \\
&\quad \leqslant C_t\left(\|x\|_{L^2(\Lambda)}^2 + \|B\|_{HS}^2\right).
\end{aligned} \tag{2.3}$$

By the definition of the Yosida approximation we have that

$$a_p^\varepsilon(r) = a_p\left(r_\varepsilon^p(r)\right)$$

and

$$\left\langle a_p^\varepsilon(r), r\right\rangle = \left\langle a_p^\varepsilon\left(r_\varepsilon^p(r)\right), r_\varepsilon^p(r)\right\rangle + \frac{1}{\varepsilon}\left|r - r_\varepsilon^p(r)\right|^2.$$

We rewrite as follows

$$\begin{aligned}&\mathbb{E}\int_0^t\int_\Lambda\left\langle a_p^\varepsilon\left(\nabla\tilde{X}_p^\varepsilon(s)\right),\nabla\tilde{X}_p^\varepsilon(s)\right\rangle d\xi\,ds\\&\quad\geqslant\mathbb{E}\int_0^t\int_\Lambda\left\langle a_p\left(r_\varepsilon^p\left(\nabla\tilde{X}_p^\varepsilon(s)\right)\right),r_\varepsilon^p\left(\nabla\tilde{X}_p^\varepsilon(s)\right)\right\rangle d\xi\,ds\\&\quad\geqslant\mathbb{E}\int_0^t\int_\Lambda\left|r_\varepsilon^p\left(\nabla\tilde{X}_p^\varepsilon(s)\right)\right|^p d\xi\,ds.\end{aligned}$$

Plugging into (2.3) proves (2.2). □

We shall prove now that $\mathbb{P}$-a.s.

$$\lim_{\varepsilon\to0}\sup_{t\in[0,T]}\left\|X_p(t)-X_p^\varepsilon(t)\right\|_{L^2(\Lambda)}^2=0,\quad\text{uniformly in }p\in\left(\frac{2d}{d+2},2\right).$$

We set $\tilde{X}_p^\varepsilon=(1-\varepsilon\Delta)^{-1}X_p^\varepsilon$ and $\tilde{X}_p^\lambda=(1-\lambda\Delta)^{-1}X_p^\lambda$. Then by (2.1), we have that

$$\begin{aligned}&\frac{1}{2}\left\|X_p^\varepsilon(t)-X_p^\lambda(t)\right\|_{L^2(\Lambda)}^2\\&\qquad+\int_0^t\int_\Lambda\left\langle a_p^\varepsilon\left(\nabla\tilde{X}_p^\varepsilon(s)\right)-a_p^\lambda\left(\nabla\tilde{X}_p^\lambda(s)\right),\nabla\tilde{X}_p^\varepsilon(s)-\nabla\tilde{X}_p^\lambda(s)\right\rangle d\xi\,ds=0\quad\mathbb{P}\text{-a.s.}\end{aligned}$$

Setting $\nabla\tilde{X}_p^\varepsilon(s)=u^\varepsilon$ and $\nabla\tilde{X}_p^\lambda(s)=u^\lambda$ and using

$$a_p^\varepsilon(u)\in a_p\left(\left(1+\varepsilon a_p\right)^{-1}(u)\right),$$

we get by the monotonicity of a_p that

$$\left\langle a_p^\varepsilon\left(u^\varepsilon\right)-a_p^\lambda\left(u^\lambda\right),u^\varepsilon-u^\lambda\right\rangle\geqslant\left\langle a_p^\varepsilon\left(u^\varepsilon\right)-a_p^\lambda\left(u^\lambda\right),\varepsilon a_p^\varepsilon\left(u^\varepsilon\right)-\lambda a_p^\lambda\left(u^\lambda\right)\right\rangle.$$

This leads to

$$\begin{aligned}&\frac{1}{2}\left\|X_p^\varepsilon(t)-X_p^\lambda(t)\right\|_{L^2(\Lambda)}^2\\&\quad\leqslant\int_0^t\int_\Lambda\left(\varepsilon\left|a_p^\varepsilon\left(\nabla\tilde{X}_p^\varepsilon(s)\right)\right|^2+\lambda\left|a_p^\lambda\left(\nabla\tilde{X}_p^\lambda(s)\right)\right|^2\right)d\xi\,ds\quad\mathbb{P}\text{-a.s.}\end{aligned}\tag{2.4}$$

We can now prove that $\mathbb{P}$-a.s.

$$\int_0^t\int_\Lambda\left|a_p^\varepsilon\left(\nabla\tilde{X}_p^\varepsilon(s)\right)\right|^2 d\xi\,ds\leqslant C_t\tag{2.5}$$

for some C_t independent of p and ε.

Using Jensen's inequality (for $t\mapsto t^{p/(2p-2)}$) and taking into account that $\left|a_p(r)\right|\leqslant|r|^{p-1}$, we obtain

$$\begin{aligned}&\int_0^t\int_\Lambda\left|a_p^\varepsilon\left(\nabla\tilde{X}_p^\varepsilon(s)\right)\right|^2 d\xi\,ds\\&\quad\leqslant(t\,|\Lambda|)^{1-((2p-2)/p)}\left(\int_0^t\int_\Lambda\left|a_p\left(r_\varepsilon^p\left(\nabla\tilde{X}_p^\varepsilon(s)\right)\right)\right|^{p/(p-1)}d\xi\,ds\right)^{(2p-2)/p}\end{aligned}$$

$$\leqslant (1+t\,|\Lambda|)\left(\int_0^t\int_\Lambda \left|r_\varepsilon^p\left(\nabla\tilde{X}_p^\varepsilon(s)\right)\right|^p d\xi\, ds\right)^{(2p-2)/p}$$
$$\leqslant C_t + C_t\left(\int_0^t\int_\Lambda \left|r_\varepsilon^p\left(\nabla\tilde{X}_p^\varepsilon(s)\right)\right|^p d\xi\, ds\right), \tag{2.6}$$

where $|\Lambda| = \int_\Lambda d\xi$.

Now by Lemma 2.2 we have (2.5) for a constant C_t independent of p and ε, and passing to the limit for $\varepsilon, \lambda \to 0$ in (2.4) we get that $\mathbb{P}$-a.s.

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\left\|X_p(t) - X_p^\varepsilon(t)\right\|_{L^2(\Lambda)}^2 = 0, \quad \text{uniformly in } p \in \left(1\vee\frac{2d}{d+2}, 2\right).$$

As a consequence, $I_1(n,\varepsilon)$ and $I_3(\varepsilon)$ tend to zero as $\varepsilon \downarrow 0$, uniformly in n. For $I_2(n,\varepsilon)$, using the monotonicity of $a_{p_n}^\varepsilon$ we have

$$\frac{1}{2}\left\|X_{p_n}^\varepsilon(t) - X_{p_0}^\varepsilon(t)\right\|_{L^2(\Lambda)}^2 + \int_0^t\int_\Lambda \left\langle a_{p_n}^\varepsilon\left(\nabla\tilde{X}_{p_0}^\varepsilon(s)\right) - a_{p_0}^\epsilon\left(\nabla\tilde{X}_{p_0}^\varepsilon(s)\right), \nabla\tilde{X}_{p_n}^\varepsilon(s) - \nabla\tilde{X}_{p_0}^\varepsilon(s)\right\rangle_d d\xi ds \leqslant 0.$$

Since

$$\begin{aligned}
&\frac{1}{2}\left\|X_{p_n}^\varepsilon(t) - X_{p_0}^\varepsilon(t)\right\|_{L^2(\Lambda)}^2 \\
&\quad\leqslant \int_0^t\int_\Lambda \Big[(1-\varepsilon\Delta)^{-1}\operatorname{div}\Big(a_{p_n}^\varepsilon\left(\nabla\tilde{X}_{p_0}^\varepsilon(s)\right) \\
&\qquad - a_{p_0}^\epsilon\left(\nabla\tilde{X}_{p_0}^\varepsilon(s)\right)\Big)\Big]\left[X_{p_n}^\varepsilon(s) - X_{p_0}^\varepsilon(s)\right] d\xi ds \\
&\quad\leqslant \Bigg(\int_0^t\int_\Lambda \Big((1-\varepsilon\Delta)^{-1}\operatorname{div} a_{p_n}^\varepsilon\left(\nabla\tilde{X}_{p_0}^\varepsilon(s)\right) \\
&\qquad - (1-\varepsilon\Delta)^{-1}\operatorname{div} a_{p_0}^\epsilon\left(\nabla\tilde{X}_{p_0}^\varepsilon(s)\right)\Big)^2 d\xi ds\Bigg)^{1/2} \\
&\qquad \times\left(\int_0^t\int_\Lambda \left(X_{p_n}^\varepsilon(s) - X_{p_0}^\varepsilon(s)\right)^2 d\xi ds\right)^{1/2}.
\end{aligned}$$

We only need to prove that

$$\begin{aligned}
&\Bigg(\int_0^t\int_\Lambda \Big((1-\varepsilon\Delta)^{-1}\operatorname{div} a_{p_n}^\varepsilon\left(\nabla\tilde{X}_{p_0}^\varepsilon(s)\right) \\
&\qquad - (1-\varepsilon\Delta)^{-1}\operatorname{div} a_{p_0}^\epsilon\left(\nabla\tilde{X}_{p_0}^\varepsilon(s)\right)\Big)^2 d\xi ds\Bigg)^{1/2} \to 0
\end{aligned}$$

and that follows from

$$A_{p_n}^\varepsilon(u) \to A_{p_0}^\varepsilon(u), \quad \text{strongly in } L^2((0,T)\times\Lambda), \tag{2.7}$$

where $A_{p_n}^\varepsilon(u) = (1-\varepsilon\Delta)^{-1}\operatorname{div} a_{p_n}^\varepsilon(u)$ (as in (2.1)).

Indeed, we obtain (2.7) by the following arguments:

Since $a^{\varepsilon}_{p_n}(u) \to a^{\epsilon}_{p_0}(u)$ pointwise, which follows from Lemma Appendix A.5 and [4, Proposition 3.29], and since $\left\{a^{\varepsilon}_{p_n}(u)\right\}_n$ is bounded $a.e.$ on $(0,T)\times\Lambda$ we get by Lebesgue's dominated convergence theorem

$$\left\langle \operatorname{div} a^{\varepsilon}_{p_n}(u) - \operatorname{div} a^{\epsilon}_{p_0}(u), v\right\rangle_{L^2((0,T)\times\Lambda)} = \left\langle a^{\varepsilon}_{p_n}(u) - a^{\epsilon}_{p_0}(u), \nabla v\right\rangle_{L^2((0,T)\times\Lambda)} \overset{n}{\to} 0,$$

for all $v \in L^2((0,T)\times\Lambda)$.

That means

$$\operatorname{div} a^{\varepsilon}_{p_n}(u) \to \operatorname{div} a^{\epsilon}_{p_0}(u), \quad \text{weakly in } L^2((0,T)\times\Lambda)$$

and this leads to

$$(1-\varepsilon\Delta)^{-1}\operatorname{div} a^{\varepsilon}_{p_n}(u) \to (1-\varepsilon\Delta)^{-1}\operatorname{div} a^{\varepsilon}_{p_0}(u), \quad \text{strongly in } L^2((0,T)\times\Lambda),$$

which is (2.7).

We have proved that

$$\lim_n \sup_{t\in[0,T]} \|X_n(t) - X_0(t)\|_{L^2(\Lambda)} = 0 \quad \mathbb{P}\text{-a.s.}.$$

The convergence

$$\lim_n \mathbb{E}\left[\sup_{t\in[0,T]} \|X_n(t) - X_0(t)\|^2_{L^2(\Lambda)}\right] = 0$$

is established by Lebesgue's dominated convergence theorem and [28, Eq. (1.3)], where the constant can be controlled uniformly in p by Itō's formula, Poincaré inequality and Grönwall's lemma. We refer to [38] for the p-dependence of Poincaré constants. □

Theorem 2.3. *Let* $\{r_n\} \subset \left(0 \vee \frac{d-2}{d+2}, 1\right]$, $n \in \mathbb{N}$, $r_0 \in \left(0 \vee \frac{d-2}{d+2}, 1\right]$ *such that* $r_n \to r_0$. *Let* $Y_n := Y_{r_n}$, $n \in \mathbb{N}$, $Y_0 := Y_{r_0}$ *be the solutions to* (FD_{r_n}), $n \in \mathbb{N}$, (FD_{r_0}) *resp. Then for* $y \in H^{-1}(\Lambda)$,

$$\lim_n \mathbb{E}\left[\sup_{t\in[0,T]} \|Y_n(t) - Y_0(t)\|^2_{H^{-1}(\Lambda)}\right] = 0.$$

Proof. We need to show that

$$\lim_n \mathbb{E}\left[\sup_{t\in[0,T]} \|Y_n(t) - Y_0(t)\|^2_{H^{-1}(\Lambda)}\right] = 0.$$

Using the same approximation as in [10] consider

$$\begin{aligned} &\|Y_n(t) - Y_0(t)\|_{H^{-1}(\Lambda)} \\ &\quad\leqslant \left\|Y_n(t) - Y^{\varepsilon}_n(t)\right\|_{H^{-1}(\Lambda)} + \left\|Y^{\varepsilon}_n(t) - Y^{\varepsilon}_0(t)\right\|_{H^{-1}(\Lambda)} + \left\|Y^{\varepsilon}_0(t) - Y_0(t)\right\|_{H^{-1}(\Lambda)} \\ &\quad= I_1 + I_2 + I_3. \end{aligned}$$

For I_1 and I_3 we have the convergence uniformly in r_n for $r_n > 1/2$, arguing as in [10], Proposition 2.6 and using at the end Jensen's inequality for $L^2(\Lambda) \subset L^{2r_n}(\Lambda)$.

For I_2 note that the pointwise convergence of $\Psi_{r_n}(x) = |x|^{r_n-1}x$ to $\Psi_{r_0}(x) = |x|^{r_0-1}x$ imply the convergence of the resolvent in $\mathbb{R}$ and then we get the result arguing as in [15]. □

3. Convergence of invariant measures

In this section, we shall present a result on convergence of invariant measures associated to equations (PL_p), (FD_r) respectively.

Let $\{X_p^x(t)\}_{t\geqslant 0}$ be the variational solution associated to equation (PL_p) starting at $x \in L^2(\Lambda)$. Similarly, let $\{Y_r^y(t)\}_{t\geqslant 0}$ be the variational solution associated to equation (FD_r) starting at $y \in H^{-1}(\Lambda)$.

Let

$$P_t^p F(x) := \mathbb{E}\left[F(X_p^x(t))\right], \quad F \in C_b(L^2(\Lambda)), \quad t \geqslant 0,$$

be the semigroup associated to equation (PL_p).

Let

$$Q_t^r G(y) := \mathbb{E}\left[G(Y_r^y(t))\right], \quad G \in C_b(H^{-1}(\Lambda)), \quad t \geqslant 0,$$

be the semigroup associated to equation (FD_r).

Recently, Liu and the second named author obtained the following result:

Proposition 3.1. *Suppose that $p \in \left(1 \vee \frac{2d}{2+d}, 2\right]$, $r \in \left(0 \vee \frac{d-2}{d+2}, 1\right]$. Then $\{P_t^p\}$ and $\{Q_t^r\}$ are ergodic and admit unique invariant measures μ_p, ν_r respectively. It holds that μ_p is supported by $W_0^{1,p}(\Lambda)$ and ν_r is supported by $L^{r+1}(\Lambda)$. Also*

$$\int_{L^2(\Lambda)} \|x\|_{1,p}^p \, \mu_p(dx) < +\infty, \tag{3.1}$$

and

$$\int_{H^{-1}(\Lambda)} \|y\|_{r+1}^{r+1} \, \nu_r(dy) < +\infty. \tag{3.2}$$

Proof. See [29, Propositions 3.2 and 3.4]. □

Theorem 3.2. (i) *Let $\{p_n\} \subset \left(1 \vee \frac{2d}{2+d}, 2\right]$, $n \in \mathbb{N}$, $p_0 \in \left(1 \vee \frac{2d}{2+d}, 2\right]$ such that $p_n \to p_0$. Set $P_t^n := P_t^{p_n}$, $P_t^0 := P_t^{p_0}$.*

Then the unique invariant measures μ_n, $n \in \mathbb{N}$, μ_0 resp. associated to $\{P_t^n\}$, $n \in \mathbb{N}$, $\{P_t^0\}$ converge in the weak sense, i.e.

$$\lim_n \int_{L^2(\Lambda)} F(x)\, \mu_n(dx) = \int_{L^2(\Lambda)} F(x)\, \mu_0(dx) \quad \forall F \in C_b(L^2(\Lambda)).$$

(ii) *Let $\{r_n\} \subset \left(0 \vee \frac{d-2}{d+2}, 1\right]$, $n \in \mathbb{N}$, $r_0 \in \left(0 \vee \frac{d-2}{d+2}, 1\right]$ such that $r_n \to r_0$. Set $Q_t^n := Q_t^{r_n}$, $Q_t^0 := Q_t^{r_0}$.*

Then the unique invariant measures ν_n, $n \in \mathbb{N}$, ν_0 resp. associated to $\{Q_t^n\}$, $n \in \mathbb{N}$, $\{Q_t^0\}$ converge in the weak sense, i.e.

$$\lim_n \int_{L^2(\Lambda)} F(x)\, \nu_n(dx) = \int_{L^2(\Lambda)} F(x)\, \nu_0(dx) \quad \forall F \in C_b(L^2(\Lambda)).$$

Proof. Let us prove (i) first. By Proposition 3.1, we see that $\{P_t^n\}$, $n \in \mathbb{N}$, $\{P_t^0\}$ admit unique invariant measures μ_n, $n \in \mathbb{N}$, μ_0 resp. Let $p_1 := \inf_n p_n$. By the convergence $p_n \to p_0$, $p_1 \in \left(1 \vee \frac{2d}{2+d}, 2\right]$ and the embedding $W_0^{1,p_1}(\Lambda) \subset L^2(\Lambda)$ is compact.

Let $\theta > 0$. Set

$$K_\theta := \left\{ x \in L^2(\Lambda) \,\middle|\, \|x\|_{1,p_1}^{p_1} \leqslant \theta^{-1} + |\Lambda| \right\}.$$

Clearly, K_θ is compact in $L^2(\Lambda)$. Now by (3.1),

$$\mu_n\{K_\theta^{c}\} = \mu_n \left\{ \|\cdot\|_{1,p_1}^{p_1} - |\Lambda| \geqslant \theta^{-1} \right\} \leqslant \theta \int_{L^2(\Lambda)} \|x\|_{1,p_n}^{p_n} \, \mu_n(dx) \leqslant \theta \|B\|_{HS}^2.$$

Hence the family of measures $\{\mu_n\}_{n\in\mathbb{N}}$ is tight and has a weak accumulation point $\tilde{\mu}$, i.e. $\mu_{n_k} \to \tilde{\mu}$ weakly. By the Krylov–Bogoliubov theorem, for $F \in C_b(L^2(\Lambda))$,

$$\begin{aligned} \int_{L^2(\Lambda)} F(x)\, \mu_{n_k}(dx) &= \lim_{T\to+\infty} \frac{1}{T} \int_0^T P_t^{n_k} F(x)\, dt \\ &= \lim_{T\to+\infty} \frac{1}{T} \int_0^T (P_t^{n_k} F(x) - P_t^0 F(x))\, dt \\ &\quad + \lim_{T\to+\infty} \frac{1}{T} \int_0^T P_t^0 F(x)\, dt \\ &=: \varepsilon_k + \int_{L^2(\Lambda)} F(x)\, \mu_0(dx). \end{aligned}$$

By Theorem 2.1 and dominated convergence, $\varepsilon_k \to 0$ as $k \to +\infty$ and hence

$$\int_{L^2(\Lambda)} F(x)\, \tilde{\mu}(dx) = \int_{L^2(\Lambda)} F(x)\, \mu_0(dx).$$

As a consequence, for the whole sequence, $\mu_n \to \mu_0$ weakly.

The proof for (ii) can be carried out by similar arguments. □

4. The case $p = 1$

For $p = 1$, the situation is more complicated. We would like to find a convex functional Φ^1 such that the stochastic 1-Laplace equation

$$(\mathrm{PL}_1) \begin{cases} dX_1(t) = \operatorname{div}\left[\dfrac{\nabla X_1(t)}{|\nabla X_1(t)|}\right] dt + B\, , dW(t) & \text{in}(0,T)\times\Lambda, \\ X_1(t) = 0 & \text{on } (0,T)\times\partial\Lambda, \\ X_1(0) = x & \text{in } \Lambda, \end{cases}$$

can be written as

$$\begin{cases} dX_1(t) \in -\partial\Phi^1(X_1(t))\, dt + B\, dW(t) & \text{in}(0,T), \\ X_1(0) = x, \end{cases} \tag{4.1}$$

where $\partial\Phi^1$ is the subdifferential of Φ^1.

We shall need the spaces $BV(\Lambda)$ and $BV(\mathbb{R}^d)$. For $f \in L^1_{\mathrm{loc}}(\Lambda)$, define the *total variation*

$$\|\mathrm{D}f\|(\Lambda) = \sup\left\{ \int_\Lambda f \operatorname{div}\psi\, d\xi \,\middle|\, \psi \in C_0^\infty\left(\Lambda; \mathbb{R}^d\right),\ |\psi| \leqslant 1 \right\}$$

$BV(\Lambda)$ is defined to be equal to $\{f \in L^1(\Lambda) \mid \|\mathrm{D}f\|(\Lambda) < \infty\}$. Denote the $d-1$-dimensional Hausdorff measure on $\partial\Lambda$ by $\mathscr{H}^{d-1}$. For $f \in BV(\Lambda)$ there is an element $f^\Lambda \in L^1(\partial\Lambda, d\mathscr{H}^{d-1})$ called the *trace* such that

$$\int_\Lambda f \operatorname{div}\psi \, d\xi = -\int_\Lambda \langle \psi, d[\mathrm{D}f]\rangle + \int_{\partial\Lambda} \langle \psi, \nu\rangle f^\Lambda \, d\mathscr{H}^{d-1} \quad \forall \psi \in C^1(\overline{\Lambda}; \mathbb{R}^d),$$

where $[\mathrm{D}f]$ denotes the distributional gradient of f on Λ (which is a $\mathbb{R}^d$-valued Radon measure here) and ν denotes the outer unit normal on $\partial\Lambda$. $BV(\mathbb{R}^d)$ is defined similarly by setting $\Lambda = \mathbb{R}^d$. Define also $\|\mathrm{D}f\|(\mathbb{R}^d)$ in the above manner. Note that for $f \in BV(\Lambda)$ (extended by zero outside Λ) it holds that $f \in BV(\mathbb{R}^d)$ and that

$$\|\mathrm{D}f\|(\mathbb{R}^d) = \|\mathrm{D}f\|(\Lambda) + \int_{\partial\Lambda} \left|f^\Lambda\right| d\mathscr{H}^{d-1}, \tag{4.2}$$

cf. [1, Theorem 3.87].

Remark 4.1. By Ambrosio et al. [1, Corollary 3.49], if $d \in \{1, 2\}$, then

$$W_0^{1,1}(\Lambda) \subset BV(\Lambda) \subset L^2(\Lambda)$$

continuously. If $d = 1$, then

$$BV(\Lambda) \subset\subset L^2(\Lambda)$$

compactly.

For further results in spaces of functions of bounded variation, we refer to [1, Ch. 3].

We shall return to Eq. (4.1). Recall that the subdifferential $\partial\Phi^1$ in $L^2(\Lambda)$ is defined by $\eta \in \partial\Phi^1(x)$ iff

$$\Phi^1(x) - \Phi^1(y) \leqslant \int_\Lambda \eta\,(x-y)\, d\xi, \quad \forall y \in \operatorname{dom} \Phi^1. \tag{4.3}$$

One possible choice for Φ^1 is the (homogeneous) energy

$$\tilde{\Phi}(u) := \begin{cases} \displaystyle\int_\Lambda |\nabla u|\, d\xi, & \text{if } u \in W_0^{1,1}(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus W_0^{1,1}(\Lambda). \end{cases}$$

In this case, if $u \in W_0^{1,1}(\Lambda)$, and if $U :=\subset L^2(\Lambda)$, then we have that $u \in \operatorname{dom} \partial\tilde{\Phi}$ and $U = \partial\tilde{\Phi}(u)$.

However, $\tilde{\Phi}$ fails to be lower semi-continuous in $L^2(\Lambda)$ which is a necessary ingredient for the theory. Therefore, it is convenient to consider its relaxed functional in $L^2(\Lambda)$, which is equal to

$$\Phi^1(u) := \begin{cases} \|\mathrm{D}u\|(\mathbb{R}^d), & \text{if } u \in BV(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus BV(\Lambda), \end{cases}$$

see Eq. (4.2) above. Φ^1 is proper, convex and lower semi-continuous in $L^2(\Lambda)$ and an extension of $\tilde{\Phi}$ in the sense that $\operatorname{dom} \Phi^1 \supset \operatorname{dom} \tilde{\Phi}$ and $\Phi^1 \leq \tilde{\Phi}$. Compare with [3,24,37,40].

Following the approach of Barbu et al. [11], we shall give the definition of a solution for equations (PL_p), $p \in [1, 2]$.

Definition 4.2. Set $V_p := W_0^{1,p}(\Lambda)$, $p \in (1,2]$, $V_1 := BV(\Lambda)$. Let Φ^1 be defined as above. For $p \in (1,2]$, let

$$\Phi^p(x) := \begin{cases} \frac{1}{p}\int_\Lambda |\nabla x|^p \, d\xi, & \text{if } u \in W_0^{1,p}(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus W_0^{1,p}(\Lambda). \end{cases}$$

A stochastic process $X = X^x$ with $\mathbb{P}$-a.s. continuous sample paths in $H := L^2(\Lambda)$ is said to be a *solution* to equation (PL_p), $p \in [1,2]$ if

$$X \in C_W([0,T];H) \cap L^p\left((0,T)\times\Omega, V_p\right), \qquad X(0) = x \in H$$

and

$$\frac{1}{2}\|X(t) - Y(t)\|_{L^2(\Lambda)}^2 + \int_0^t \left(\Phi^p(X(s)) - \Phi^p(Y(s))\right) ds$$
$$\leqslant \frac{1}{2}\|x - Y(0)\|_{L^2(\Lambda)}^2 + \int_0^t (G(s), X(s) - Y(s))_{L^2(\Lambda)} \, ds, \quad t \in [0,T],$$

for all $G \in L_W^2(0,T;H)$ and $Y \in C_W([0,T];H) \cap L^p\left((0,T)\times\Omega; V_p\right)$ satisfying the equation

$$dY(t) + G(t)\, dt = B\, dW(t), \quad t \in [0,T]. \tag{4.4}$$

Suppose for a while that $1 < p < 2$, $d = 1, 2$. Arguing as in [31, Example 4.1.9, Theorem 4.2.4], we can easily prove existence and uniqueness of the solution X_p for equation (PL_p), in the usual (strong) variational sense, as in Krylov and Rozovskiĭ [26], Pardoux [30]. We shall refer to Prévôt, Röckner [31, Definition 4.2.1]. By Itō's formula, we see that X_p is also a solution in the sense of the definition above.

Here, $W(t)$ is a cylindrical Wiener process on $L^2(\Lambda)$ of the form

$$W(t) = \sum_{n=1}^\infty \gamma_n(t) e_n, \quad t \geq 0,$$

where $\{\gamma_n\}$ is a sequence of mutually independent real Brownian motions on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ and $\{e_n\}$ is an orthonormal basis of $L^2(\Lambda)$. We shall make further specifications. BB^* is assumed to be a linear, continuous, non-negative, symmetric operator on $L^2(\Lambda)$ with eigenbasis $\{e_n\}$ and corresponding sequence of eigenvalues $\{\lambda_n\}$. Let $(-\Delta, \mathrm{dom}(-\Delta))$ be the Dirichlet Laplacian in $L^2(\Lambda)$, in particular, $\mathrm{dom}(-\Delta) = H^2(\Lambda) \cap H_0^1(\Lambda)$. Assume for simplicity that $\{e_n\}$ is an eigenbasis of $-\Delta$ with corresponding sequence of eigenvalues $\{\mu_n\}$. We shall assume that

$$\sum_{n=1}^\infty \lambda_n^{1+\kappa}\mu_n < \infty \tag{4.5}$$

for some $\kappa > 0$. For the situation considered in this paper, it is enough to set $Q := (-\Delta)^{-1-\delta}$ with $\delta > \frac{1}{2} + \kappa$ for $d = 1$ and $\delta > 1 + \kappa$ for $d = 2$.

Regarding equation (PL_1), well-posedness of the problem as well as existence and uniqueness of the solution were proved by Barbu et al. in [11].

Remark 4.3. Note that in [11], the space $BV_0(\Lambda)$ is introduced, consisting of $BV(\Lambda)$-functions with zero trace. They claim, however, that the energy

$$\Psi(u) := \begin{cases} \|\mathrm{D}u\|(\Lambda), & \text{if } u \in BV_0(\Lambda), \\ +\infty, & \text{if } u \in L^2(\Lambda) \setminus BV_0(\Lambda). \end{cases}$$

is lower semi-continuous which is not the case. Consider, for example, a sequence u_n of trace zero Lipschitz functions on Λ with $\|\mathrm{D}u_n\|(\Lambda) = 1$ converging in $L^2(\Lambda)$ to $\mathbb{1}_\Lambda$. Then

$$\varliminf_n \Psi(u_n) = 1 < +\infty = \Psi(\mathbb{1}_\Lambda).$$

Fortunately, all results of [11] remain true, if one replaces Ψ (denoted by Φ in their paper) by Φ^1. We do not repeat the steps taken in the proof of [11] here, but note that for their existence and uniqueness result relies on an approximation $\{\Psi^\varepsilon\}$ of Ψ which "does not see" the trace-term in (4.2), i.e. maps $L^2(\Lambda)$ functions on a joint subspace of $BV_0(\Lambda)$ and $\mathrm{dom}(\Phi^1)$. In fact, $\{\Psi^\varepsilon\}$ is defined similarly to (4.6).

Other results of stochastic evolution variational inequalities can be found in [8,13,32–34].

We are now able to formulate the main result of this section.

Theorem 4.4. *Let $d \in \{1, 2\}$. The sequence of solutions $\{X_p\}_p$ to equations* (PL$_p$) *is convergent for $p \to 1$ to the solution X_1 of equation* (PL$_1$)*, strongly in $L^2(\Lambda)$, uniformly in $[0, T]$, $\mathbb{P}-a.s.$, i.e.,*

$$\lim_{p\to 1} \sup_{t\in[0,T]} \left\| X_p(t) - X_1(t) \right\|_{L^2(\Lambda)} = 0, \quad \mathbb{P} - a.s.$$

There is some evidence that the following conjecture is true, see [21,22,25].

Conjecture 4.5. *Let $d \in \{1, 2\}$. Then the semigroup*

$$P_t^1 F(x) := \mathbb{E}\left[F\left(X_1(t, x)\right)\right], \quad F \in C_b(L^2(\Lambda)),$$

admits a unique invariant measure μ_1.

Theorem 4.6. *Let $d = 1$. Suppose that Conjecture 4.5 is true. Let $X_p = X_p(t, x)$ be the solution to equation* (PL$_p$)*, $p \in [1, 2]$. Let $\{p_n\} \subset (1, 2]$ such that $\lim_n p_n = 1$. Let*

$$P_t^p F(x) := \mathbb{E}\left[F\left(X_p^x(t)\right)\right], \quad \varphi \in C_b(L^2(\Lambda)),$$

be the semigroup associated to equation (PL$_p$)*. Let μ_{p_n}, $n \in \mathbb{N}$, μ_1 be the associated unique invariant measures on $L^2(\Lambda)$. Then*

$$\mu_{p_n} \to \mu_1 \quad \text{in the weak sense.}$$

Proof. Note that by Remark 4.1, the embedding $BV(\Lambda) \subset L^2(\Lambda)$ is compact. The proof is similar to that of Theorem 3.2, $W_0^{1,p_1}(\Lambda)$ therein replaced by $BV(\Lambda)$. □

Proof of Theorem 4.4. For each $\varepsilon > 0$, let $R_\varepsilon := (1 - \varepsilon\Delta)^{-1}$ be the resolvent of the (Dirichlet) Laplace operator $(-\Delta, \mathrm{dom}(-\Delta))$, where $\mathrm{dom}(-\Delta) = H_0^1(\Lambda) \cap H^2(\Lambda)$. For $p \in [1, 2]$, $\varepsilon > 0$, let

$$\Phi_\varepsilon^p(u) := \int_\Lambda j_\varepsilon^p(\nabla R_\varepsilon u)\, d\xi, \quad u \in L^2(\Lambda). \quad \square \tag{4.6}$$

Lemma 4.7. *Let $\{p_n\} \subset [1, 2]$ such that $\lim_n p_n = 1$. Let $\varepsilon > 0$. Then for $u \in L^2(\Lambda)$, we have that*

$$\lim_n \Phi_\varepsilon^{p_n}(u) = \Phi_\varepsilon^1(u). \tag{4.7}$$

Furthermore, if $u_n \rightharpoonup u$ converges weakly in $L^2(\Lambda)$, we have that

$$\lim_n \Phi_\varepsilon^{p_n}(u_n) \geq \Phi_\varepsilon^1(u). \tag{4.8}$$

Also, each Φ_ε^p, $p \in [1, 2]$, $\varepsilon > 0$, is continuous w.r.t. the weak topology of $L^2(\Lambda)$.

Proof. Since R_ε maps to $\operatorname{dom}(-\Delta) \subset H_0^1(\Lambda)$, it is clear that $\nabla R_\varepsilon u \in L^2(\Lambda; \mathbb{R}^d)$ and hence (4.7) follows from (A.2).

Let $u_n \in L^2(\Lambda)$, $n \in \mathbb{N}$, $u \in L^2(\Lambda)$, such that $u_n \rightharpoonup u$ weakly in $L^2(\Lambda)$. If we can proof that $\nabla R_\varepsilon u_n \rightharpoonup \nabla R_\varepsilon u$ weakly in $L^2(\Lambda; \mathbb{R}^d)$, we can apply (A.3) and Lemma 4.7 follows. Indeed, we even have that $\nabla R_\varepsilon u_n \to \nabla R_\varepsilon u$ strongly in $L^2(\Lambda; \mathbb{R}^d)$.

The last part follows by repeating the compactness argument above and the strong $L^2(\Lambda; \mathbb{R}^d)$-continuity of the Ψ_ε^pś. □

We first consider the following approximating equations for (PL_p)

$$\begin{cases} dX_p^\varepsilon(t) + A_p^\varepsilon\left(X_p^\varepsilon\right) dt = B\, dW(t) \\ X_p^\varepsilon(0) = x \end{cases} \tag{4.9}$$

where for any $u \in L^2(\Lambda)$,

$$A_p^\varepsilon(u) = -(1 - \varepsilon\Delta)^{-1} \operatorname{div}\left[a_p^\varepsilon\left(\nabla(1 - \varepsilon\Delta)^{-1} u\right)\right]$$

and a_p^ε is the Yosida approximation of a_p i.e., for any $r \in \mathbb{R}^d$,

$$a_p^\varepsilon(r) = \frac{1}{\varepsilon}\left(1 - \left(1 + \varepsilon a_p\right)^{-1}(r)\right).$$

In particular, for $u, v \in L^2(\Lambda)$,

$$\left(A_p^\varepsilon(u), v\right)_{L^2(\Lambda)} = \int_\Lambda \left\langle a_p^\varepsilon(\nabla R_\varepsilon u), \nabla R_\varepsilon(v)\right\rangle d\xi.$$

We shall consider a similar approximation for equation (PL_1)

$$\begin{cases} dX_1^\varepsilon(t) + A^\varepsilon\left(X_1^\varepsilon\right) dt = B\, dW(t) \\ X_1^\varepsilon(0) = x \end{cases} \tag{4.10}$$

where for any $u \in L^2(\Lambda)$,

$$A^\varepsilon(u) = -(1 - \varepsilon\Delta)^{-1} \operatorname{div}\left[\beta^\varepsilon\left(\nabla(1 - \varepsilon\Delta)^{-1} u\right)\right]$$

with

$$\beta^\varepsilon(r) = \begin{cases} \dfrac{r}{\varepsilon}, & \text{if } |r| \leqslant \varepsilon, \\ \dfrac{r}{|r|}, & \text{if } |r| > \varepsilon. \end{cases}$$

In particular, for $u, v \in L^2(\Lambda)$,

$$\left(A^{\varepsilon}(u), v\right)_{L^2(\Lambda)} = \int_{\Lambda} \left\langle \beta^{\varepsilon}(\nabla R_{\varepsilon} u), \nabla R_{\varepsilon}(v) \right\rangle d\xi.$$

Note that β^{ε} is the Yosida approximation of the sign function, i.e., for any $r \in \mathbb{R}^d$,

$$\beta^{\varepsilon}(r) = \frac{1}{\varepsilon}\left(1 - (1 + \varepsilon \operatorname{sgn})^{-1}(r)\right).$$

In particular, $\beta^{\varepsilon} = \nabla j^{\varepsilon}$, where j_{ε} is the convex function defined by

$$j^{\varepsilon}(r) = \begin{cases} \dfrac{|r|^2}{2\varepsilon}, & \text{if } |r| \leqslant \varepsilon, \\ |r| - \dfrac{\varepsilon}{2}, & \text{if } |r| > \varepsilon. \end{cases}$$

We shall use the following strategy to prove the main result

$$\begin{aligned} &\left\|X_p(t) - X_1(t)\right\|_{L^2(\Lambda)} \\ &\quad \leqslant \left\|X_p(t) - X_p^{\varepsilon}(t)\right\|_{L^2(\Lambda)} + \left\|X_p^{\varepsilon}(t) - X_1^{\varepsilon}(t)\right\|_{L^2(\Lambda)} + \left\|X_1^{\varepsilon}(t) - X_1(t)\right\|_{L^2(\Lambda)} \end{aligned}$$

$\mathbb{P}$-a.s. and uniformly in $t \in [0, T]$.

Step 1

We note that, taking Remark 4.3 into account, the result of [11, Eq. (4.8)] remains valid in our case. Hence,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \left\|X_1^{\varepsilon}(t) - X_1(t)\right\|_{L^2(\Lambda)} = 0, \quad \mathbb{P}\text{-a.s.}$$

Step 2

Note that we have proved above (proof of Theorem 2.1) that

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \left\|X_p(t) - X_p^{\varepsilon}(t)\right\|_{L^2(\Lambda)} = 0, \quad \mathbb{P}\text{-a.s. uniformly in } p \in (1, 2).$$

Step 3

In order to complete the proof we still need to show that for all $\varepsilon > 0$ fixed we have

$$\lim_{p \to 1} \sup_{t \in [0,T]} \left\|X_p^{\varepsilon}(t) - X_1^{\varepsilon}(t)\right\|_{L^2(\Lambda)} = 0, \quad \mathbb{P}\text{-a.s.}$$

To this aim, we consider the definition of the solution for equations

$$\begin{cases} dX_p^{\varepsilon}(t) + A_p^{\varepsilon}\left(X_p^{\varepsilon}\right) dt = B\, dW(t) \\ X_p^{\varepsilon}(0) = x \end{cases}$$

as

$$\begin{aligned} &\frac{1}{2}\left\|X_p^{\varepsilon}(t) - Y(t)\right\|_{L^2(\Lambda)}^2 + \int_0^t \left(\Phi_{\varepsilon}^p\left(X_p^{\varepsilon}(s)\right) - \Phi_{\varepsilon}^p(Y(s))\right) ds \\ &\quad \leqslant \frac{1}{2}\|x - Y(0)\|_{L^2(\Lambda)}^2 + \int_0^t \left(G(s), X_p^{\varepsilon}(s) - Y(s)\right)_{L^2(\Lambda)} ds, \\ &\quad \text{for all } t \in [0, T], \mathbb{P}\text{-a.s.} \end{aligned}$$

We take $Y = X_1^\varepsilon$, the solution of equation

$$\begin{cases} dX_1^\varepsilon(t) + A^\varepsilon\left(X_1^\varepsilon\right) dt = B\,dW(t) \\ X_1^\varepsilon(0) = x. \end{cases}$$

and using the definition of the subdifferential we get that

$$\frac{1}{2}\left\|X_p^\varepsilon(t) - X_1^\varepsilon(t)\right\|_{L^2(\Lambda)}^2 + \int_0^t \left(\Phi_\varepsilon^p\left(X_p^\varepsilon(s)\right) - \Phi_\varepsilon^p\left(X_1^\varepsilon(s)\right) + \Phi_\varepsilon^1\left(X_1^\varepsilon(s)\right)\right.$$
$$\left. - \; \Phi_\varepsilon^1\left(X_p^\varepsilon(s)\right)\right) ds \leqslant \frac{1}{2}\left\|x - X_1^\varepsilon(0)\right\|_{L^2(\Lambda)}^2 = 0, \tag{4.11}$$

for $t \in [0, T]$ and $\mathbb{P}$-a.s.. By estimate (2.3), we can extract a subsequence $\{p_n\}$ with $\lim_n p_n = 1$ such that for $X_n^\varepsilon := X_{p_n}^\varepsilon$ we have that for dt-a.a. $t \in [0, T]$, $X_n^\varepsilon(t) \rightharpoonup Z^\varepsilon(t)$ weakly in $L^2(\Lambda)$, $\mathbb{P}$-a.s. for some $dt \otimes \mathbb{P}$-measurable Z^ε that satisfies

$$\sup_{t\in[0,T]} \|Z^\varepsilon(t)\|_{L^2(\Lambda)} \le \varliminf_n \sup_{t\in[0,T]} \|X_n(t)\|_{L^2(\Lambda)} \quad \mathbb{P}\text{-a.s.}$$

We shall need following lemma. Set $\Phi_\varepsilon^n := \Phi_\varepsilon^{p_n}$.

Lemma 4.8.

$$\Phi_\varepsilon^n(X_1^\varepsilon(\cdot)) - \Phi_\varepsilon^n(X_n^\varepsilon(\cdot)) + \Phi_\varepsilon^1(X_n^\varepsilon(\cdot)) - \Phi_\varepsilon^1(X_1^\varepsilon(\cdot))$$

is $\mathbb{P}$-a.s. bounded above by a function in $L^\infty(0, T)$.

Proof. Set $u := X_n^\varepsilon(\cdot)$, $v := X_1^\varepsilon(\cdot)$. Recall that in our notation, $R_\varepsilon := (1 - \varepsilon\Delta)^{-1}$.

Let us treat the term $\Phi_\varepsilon^1(u) - \Phi_\varepsilon^1(v)$ first. By the definition of the subgradient it is bounded by $(\nabla\Phi_\varepsilon^1(u), u - v)_{L^2(\Lambda)}$. But this term is equal to

$$\int_\Lambda \left\langle\beta^\varepsilon(\nabla R_\varepsilon(u)), \nabla R_\varepsilon(u - v)\right\rangle d\xi.$$

Since $|\beta^\varepsilon| \le 1$, we get that the latter is bounded by $\|\nabla R_\varepsilon(u - v)\|_{L^2(\Lambda;\mathbb{R}^d)}$. By the proof of Lemma 4.7, ∇R_ε is a bounded operator from $L^2(\Lambda)$ to $L^2(\Lambda; \mathbb{R}^d)$.

We get that

$$\Phi_\varepsilon^1(X_n^\varepsilon(\cdot)) - \Phi_\varepsilon^1(X_1^\varepsilon(\cdot)) \leqslant C \sup_n \|X_n^\varepsilon(\cdot)\|_{L^2(\Lambda)} + C\|X_1^\varepsilon(\cdot)\|_{L^2(\Lambda)}$$

which is $\mathbb{P}$-a.s. in $L^\infty(0, T)$ again by estimate (2.3).

We continue with the term $\Phi_\varepsilon^n(v) - \Phi_\varepsilon^n(u)$. By the definition of the subgradient it is bounded by $(\nabla\Phi_\varepsilon^n(v), v - u)_{L^2(\Lambda)}$, which is equal to

$$\int_\Lambda \left\langle a_p^\varepsilon(\nabla R_\varepsilon(v)), \nabla R_\varepsilon(v - u)\right\rangle d\xi.$$

Noticing that r_ε^p is a contraction on $\mathbb{R}^d$, we can use a similar estimate as in (2.6) to get that the latter is bounded by

$$C + C\|\nabla R_\varepsilon(v)\|_{L^2(\Lambda;\mathbb{R}^d)}\|\nabla R_\varepsilon(v - u)\|_{L^2(\Lambda;\mathbb{R}^d)}.$$

Arguing as above, we see that this term is bounded by

$$C + C \sup_n \|X_n^\varepsilon(\cdot)\|_{L^2(\Lambda)}\|X_1^\varepsilon(\cdot)\|_{L^2(\Lambda)} + C\|X_1^\varepsilon(\cdot)\|_{L^2(\Lambda)}^2,$$

which is $\mathbb{P}$-a.s. in $L^\infty(0, T)$ by estimate (2.3). $\square$

We take the limit superior in (4.11) and continue investigating

$$\overline{\lim_n} \int_0^t \left[\Phi_\varepsilon^n(X_1^\varepsilon(s)) - \Phi_\varepsilon^n(X_n^\varepsilon(s)) + \Phi_\varepsilon^1(X_n^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s)) \right] ds.$$

By Lemma 4.8, we can apply (reverse) Fatou's lemma such that it is sufficient to prove that

$$\overline{\lim_n} \left[\Phi_\varepsilon^n(X_1^\varepsilon(s)) - \Phi_\varepsilon^n(X_n^\varepsilon(s)) + \Phi_\varepsilon^1(X_n^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s)) \right] \leq 0.$$

$\mathbb{P}$-a.s. and for ds-a.e. $s \in [0, T]$. At this point, we apply Lemma 4.7 and get that

$$\begin{aligned} &\overline{\lim_n} \left[\Phi_\varepsilon^n(X_1^\varepsilon(s)) - \Phi_\varepsilon^n(X_n^\varepsilon(s)) + \Phi_\varepsilon^1(X_n^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s)) \right] \\ &\quad \leqslant \overline{\lim_n}\, \Phi_\varepsilon^n(X_1^\varepsilon(s)) - \underline{\lim_n}\, \Phi_\varepsilon^n(X_n^\varepsilon(s)) + \overline{\lim_n}\, \Phi_\varepsilon^1(X_n^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s)) \\ &\quad \leqslant \Phi_\varepsilon^1(X_1^\varepsilon(s)) - \Phi_\varepsilon^1(Z^\varepsilon(s)) + \Phi_\varepsilon^1(Z^\varepsilon(s)) - \Phi_\varepsilon^1(X_1^\varepsilon(s)) \\ &\quad = 0. \end{aligned}$$

$\mathbb{P}$-a.s. and for ds-a.e. $s \in [0, T]$.

Final step. Going back to

$$\begin{aligned} &\left\| X_p(t) - X_1(t) \right\|_{L^2(\Lambda)} \\ &\quad \leqslant \left\| X_p(t) - X_p^\varepsilon(t) \right\|_{L^2(\Lambda)} + \left\| X_p^\varepsilon(t) - X_1^\varepsilon(t) \right\|_{L^2(\Lambda)} + \left\| X_1^\varepsilon(t) - X_1(t) \right\|_{L^2(\Lambda)}. \end{aligned}$$

$\mathbb{P}$-a.s. and uniformly in $t \in [0, T]$, we can complete the proof using Steps I–III as follows. Let $\delta > 0$. Pick $\varepsilon_0 > 0$, independent of p, such that the first and the third term are less than $\delta/3$. Having fixed ε_0 in such a way, we can pick p such that the second term is less than $\delta/3$. □

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Appendix A. Some results on variational convergence

Let H be a separable Hilbert space. For a proper, convex functional $\Phi\colon H \to (-\infty, +\infty]$, the *Legendre transform* Φ^* is defined by

$$\Phi^*(y) := \sup_{x \in H} \left[(x, y)_H - \Phi(x) \right], \quad y \in H.$$

For two functionals $F, G\colon H \to (-\infty, +\infty]$ the *infimal convolution* $F\#G$ is defined by

$$(F\#G)(y) := \inf_{x \in H} \left[F(x) + G(y - x) \right], \quad y \in H.$$

For a proper, convex, l.s.c. functional $\Phi\colon H \to (-\infty, +\infty]$, for each $\varepsilon > 0$, define the *Moreau–Yosida regularization*

$$\Phi_\varepsilon := \Phi\#\frac{1}{2\varepsilon}\|\cdot\|_H^2.$$

Φ_ε is a continuous convex function. Also, $\lim_{\varepsilon \searrow 0} \Phi_\varepsilon = \Phi$ pointwise.

It holds that

$$(\Phi_\varepsilon)^* = \Phi^* + \frac{\varepsilon}{2}\|\cdot\|_H^2. \tag{A.1}$$

See e.g. [7, Section 2.2] and [4, Chapter. 3].

Recall following definition.

Definition Appendix A.1 (*Mosco Convergence*)**.** Let $\Phi^n\colon H \to (-\infty, +\infty]$, $n \in \mathbb{N}$, $\Phi\colon H \to (-\infty, +\infty]$ be proper, convex, l.s.c. functionals. We say that $\Phi^n \xrightarrow{M} \Phi$ in the *Mosco sense* if

$$\forall x \in H\ \forall x_n \in H,\ n \in \mathbb{N},\quad x_n \rightharpoonup x \text{ weakly in } H\colon \varliminf_n \Phi^n(x_n) \geqslant \Phi(x). \tag{M1}$$

$$\forall y \in H\ \exists y_n \in H,\ n \in \mathbb{N},\quad y_n \to y \text{ strongly in } H\colon \varlimsup_n \Phi^n(y_n) \leqslant \Phi(y). \tag{M2}$$

We shall need following theorem.

Theorem Appendix A.2. *Let $\Phi^n\colon H \to (-\infty, +\infty]$, $n \in \mathbb{N}$, $\Phi\colon H \to (-\infty, +\infty]$ be proper, convex, l.s.c. functionals. Then the following conditions are equivalent.*

(i) $\Phi^n \xrightarrow{M} \Phi$.

(ii) $(\Phi^n)^* \xrightarrow{M} \Phi^*$.

(iii) $\forall \varepsilon > 0$, $\forall x \in H$: $\lim_n \Phi_\varepsilon^n(x) = \Phi_\varepsilon(x)$.

Proof. See [4, Theorems 3.18 and 3.26]. □

Corollary Appendix A.3. *Suppose that $\Phi^n \xrightarrow{M} \Phi$. Then for each $\varepsilon > 0$, $\Phi_\varepsilon^n \xrightarrow{M} \Phi_\varepsilon$, too.*

Proof. Suppose that $\Phi^n \xrightarrow{M} \Phi$. By Theorem Appendix A.2, $(\Phi^n)^* \xrightarrow{M} \Phi^*$, too.

If we can prove for each $\varepsilon > 0$ that $(\Phi_\varepsilon^n)^* \xrightarrow{M} (\Phi_\varepsilon)^*$, we are done by Theorem Appendix A.2. (M2) in Definition Appendix A.1 follows easily, using (A.1) and (M2) for $\{(\Phi_n)^*\}$ and Φ^*.

Let $x_n \in H$, $n \in \mathbb{N}$, $x \in H$ such that $x_n \rightharpoonup x$ weakly in H. By (A.1), weak lower semi-continuity of the norm and (M1) in Definition Appendix A.1 for $\{(\Phi_n)^*\}$ and Φ^* we get that

$$\begin{aligned}\varliminf_n (\Phi_\varepsilon^n)^*(x_n) &= \varliminf_n \left[(\Phi^n)^*(x_n) + \frac{\varepsilon}{2}\|x_n\|_H^2\right]\\ &\geqslant \varliminf_n (\Phi^n)^*(x_n) \varliminf_n \frac{\varepsilon}{2}\|x_n\|_H^2 \geqslant \Phi^*(x) + \frac{\varepsilon}{2}\|x\|_H^2 = (\Phi_\varepsilon)^*(x). \quad \square\end{aligned}$$

A.1. The L^p-case

Let $p \in [1, 2]$. We define $j^p\colon \mathbb{R}^d \to \mathbb{R}$ by $j^p(x) := \frac{1}{p}|x|^p$. Obviously, if $p > 1$, each j^p is a convex C^1-function. For $\varepsilon > 0$, let

$$j_\varepsilon^p(x) := \inf_{y\in\mathbb{R}^d}\left[j^p(y) + \frac{1}{2\varepsilon}|x-y|^2\right]$$

be its regularization. For $u \in L^2(\Lambda; \mathbb{R}^d)$, set

$$\Psi^p(u) := \int_\Lambda j^p(u)\,d\xi.$$

Ψ^p is a continuous convex functional on $L^2(\Lambda; \mathbb{R}^d)$ for each $p \in [1, 2]$.

Lemma Appendix A.4. *For $\varepsilon > 0$, let Ψ^p_ε be the Moreau–Yosida regularization of Ψ^p in $L^2(\Lambda; \mathbb{R}^d)$. Then*

$$\Psi^p_\varepsilon(v) = \int_\Lambda j^p_\varepsilon(v)\, d\xi \quad \forall v \in L^2(\Lambda; \mathbb{R}^d).$$

Proof. Straightforward from [35, Theorem 14.60]. □

We would like to prove a convergence result, which shall be useful later. See the Appendix for the terminology. Compare also with [5].

Lemma Appendix A.5. *Let $\{p_n\} \subset [1, 2]$, $p_0 \in [1, 2]$ such that $\lim_n p_n = p_0$. Then*

$$\Psi^{p_n} \xrightarrow{M} \Psi^{p_0} \quad \textit{in the Mosco sense in } L^2(\Lambda; \mathbb{R}^d).$$

Proof. Let us prove (M1) in Definition Appendix A.1 first. Let $u_n \in L^2(\Lambda; \mathbb{R}^d)$, $n \in \mathbb{N}$, $u \in L^2(\Lambda; \mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $L^2(\Lambda; \mathbb{R}^d)$. W.l.o.g. $\underline{\lim}_n \Psi^{p_n}(u_n) < +\infty$. Extract a subsequence (also denoted by $\{u_n\}$) such that

$$\underline{\lim}_n \Psi^{p_n}(u_n) = \lim_n \Psi^{p_n}(u_n).$$

Let $v \in L^\infty(\Lambda; \mathbb{R}^d)$. Clearly,

$$\lim_n \int_\Lambda \langle u_n, v\rangle\, d\xi = \int_\Lambda \langle u, v\rangle\, d\xi.$$

Also, by Hölder's inequality,

$$\frac{1}{p_n}\left|\int_\Lambda \langle u_n, v\rangle\, d\xi\right|^{p_n} \leqslant \Psi^{p_n}(u_n) \times \begin{cases} |\Lambda|^{p_n-1}\|v\|^{p_n}_{L^\infty(\Lambda;\mathbb{R}^d)}, & \text{if } p_0 = 1,\\ \left(\int_\Lambda |v|^{p_n/(p_n-1)}\, d\xi\right)^{p_n-1}, & \text{if } p_0 > 1,\end{cases}$$

(here $|\Lambda| = \int_\Lambda d\xi$). Upon taking the limit $n \to \infty$, we get that

$$\frac{1}{p_0}\left|\int_\Lambda \langle u, v\rangle\, d\xi\right|^{p_0} \leqslant \underline{\lim}_n \Psi^{p_n}(u_n) \times \begin{cases} \|v\|_{L^\infty(\Lambda;\mathbb{R}^d)} & \text{if } p_0 = 1,\\ \left(\int_\Lambda |v|^{p_0/(p_0-1)}\, d\xi\right)^{p_0-1}, & \text{if } p_0 > 1.\end{cases}$$

Taking the supremum over all $v \in L^\infty(\Lambda; \mathbb{R}^d)$ with $\|v\|^{p_0/(p_0-1)}_{L^\infty(\Lambda;\mathbb{R}^d)} \leqslant 1$ and using the l.s.c. property of the supremum, we get that

$$\Psi^{p_0}(u) = \frac{1}{p_0}\int_\Lambda |u|^{p_0}\, d\xi \leqslant \underline{\lim}_n \Psi^{p_n}(u_n).$$

Since the same argument works for any subsequence of $\{u_n\}$, we have proved (M1).

We are left to prove (M2) in Definition Appendix A.1. Let $u \in L^2(\Lambda; \mathbb{R}^d)$. Clearly for a.e. $\xi \in \Lambda$

$$\lim_n \frac{1}{p_n}|u(\xi)|^{p_n} = \frac{1}{p_0}|u(\xi)|^{p_0}.$$

But for all $p \in [1, 2]$,

$$\frac{1}{p}|u|^p \leqslant 1_\Lambda + |u|^2 \in L^1(\Lambda).$$

Hence an application of Lebesgue's dominated convergence theorem yields

$$\lim_n \Psi^{p_n}(u) = \Psi^{p_0}(u).$$

(M2) is proved. □

Theorem Appendix A.2, Corollary Appendix A.3 and Lemmas Appendices A.4 and A.5 together give:

Corollary Appendix A.6. *Let $\{p_n\} \subset [1, 2]$ such that $\lim_n p_n = 1$. Let $\varepsilon > 0$. Then for $u \in L^2(\Lambda; \mathbb{R}^d)$, we have that*

$$\lim_n \int_\Lambda j_\varepsilon^{p_n}(u)\, d\xi = \int_\Lambda j_\varepsilon^1(u)\, d\xi. \tag{A.2}$$

Furthermore, if $u_n \rightharpoonup u$ converges weakly in $L^2(\Lambda; \mathbb{R}^d)$, we have that

$$\underline{\lim}_n \int_\Lambda j_\varepsilon^{p_n}(u_n)\, d\xi \geqslant \int_\Lambda j_\varepsilon^1(u)\, d\xi. \tag{A.3}$$

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