Analogous results to two classical characterizations of covering properties by products

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Received 6 February 1995; revised 1 September 1995

Abstract

First, as an analogue of Dowker’s theorem for countable paracompactness, we prove a characterization of countable metacompactness in terms of subnormality of products. Second, as an analogue of Tamano’s theorem for paracompactness, we give a characterization of Lindelöfness in terms of normality of products. © 1998 Elsevier Science B.V.

Keywords: Countably paracompact; Countably metacompact; Subnormal; Normal; Lindelöf

AMS classification: 54B10; 54D18; 54D20

Introduction

Dowker’s theorem in [4] is the first result which indicated an important implication between covering properties and products. This results has had great influence upon the study of covering properties, and several analogous results have been obtained. Recently, Good and Tree [6] gave a list of such analogues, and asked whether there is a product theorem for subnormality in the same vein. For this question, they stated without proof that subnormality of \( X \times [0, 1] \) implies countable metacompactness of \( X \). However, Good and Tree kindly informed the author that their statement has not been proved yet. In Section 1, we prove it in a slightly generalized form. This immediately yields another generalization of Dowker’s theorem. We also point out that our result is essentially different from all the analogues in the list in [6].

After Dowker’s theorem, Tamano [11] characterized paracompactness in terms of product. This result is known as Tamano’s theorem. In Section 2, as an analogue of this theorem, we characterize Lindelöfness in terms of normality of products.

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Throughout this paper, all spaces are assumed to be $T_1$-spaces.

1. Analogue of Dowker’s theorem

Recall that a space $X$ is countably paracompact (countably metacompact) if every countable open cover of $X$ has a locally finite (point-finite) open refinement.

Let us begin with a classical result of Dowker [4].

Dowker’s theorem. For a normal space $X$, the following are equivalent.
(a) $X$ is countably paracompact.
(b) $X \times (\omega + 1)$ is normal.
(c) $X \times [0, 1]$ is normal.

Recall that a space $X$ is subnormal [3,9] (normal) if for any disjoint closed sets $A$ and $B$ in $X$, there are disjoint $G_\delta$-sets (open sets) $G$ and $H$ such that $A \subset G$ and $B \subset H$. Note that a space $X$ is subnormal (normal) if and only if every binary open cover of $X$ has a countable (finite) closed refinement.

A space $X$ is countably subparacompact [9] if every countable open cover of $X$ has a countable closed refinement. Note that countably subparacompact spaces are, equivalently, countably metacompact and subnormal (see [9, Theorem 2.5]).

A list of analogues of Dowker’s theorem was given in [6, p. 118]. Here we can add another analogue as follows.

Theorem 1.1. For a subnormal space $X$, the following are equivalent.
(a) $X$ is countably metacompact.
(b) $X$ is countably subparacompact.
(c) $X \times 2^\omega$ is subnormal.
(d) $X \times [0, 1]$ is subnormal.

The equivalence of (a) and (d) in Theorem 1.1 was stated in [6, p. 127] without proof. Subsequently, Good and Tree stated that their (unpublished) proof of this equivalence was erroneous (see Note added in proof). Nevertheless, as we give a proof of it below, the equivalence holds true. Thus Theorem 1.1 is an answer to the question raised in [6, p. 127].

Proof of Theorem 1.1. (a) $\Leftrightarrow$ (b) As stated above, see [9, Theorem 2.5].
(b) $\Rightarrow$ (d) It is easy to check that $X \times [0, 1]$ is countably subparacompact (cf. [13, Lemma 2.2] for the case of $\kappa = \omega$). Hence it is subnormal.
(d) $\Rightarrow$ (c) Since $2^\omega$ can be considered as a closed subspace of $[0, 1]$, this is obvious.
(c) $\Rightarrow$ (a) Let $\{U_i: i \in \omega\}$ be a countable increasing open cover of $X$. Let $G_j = \bigcup_{i \in \omega} (U_i \times \pi_i^{-1}(j))$ for each $j \in 2$, where each $\pi_i$ denotes the projection of $2^\omega$ onto the $i$th coordinate. Then $\{G_0, G_1\}$ is a binary open cover of $X \times 2^\omega$. There is a countable closed cover $\{K_{n,j}: n \in \omega, j \in 2\}$ of $X \times 2^\omega$ such that $K_{n,j} \subset G_j$ and $K_{n,j} \subset K_{n+1,j}$.
for each \( n \in \omega \) and \( j \in 2 \). For each \( s = (k_0, \ldots, k_{n-1}) \in 2^n \), \( n \in \omega \), and \( j \in 2 \), take the point \( a_{s,j} \subset 2^\omega \) defined by \( \pi_i(a_{s,j}) = k_i \) if \( i < n \), and \( \pi_i(a_{s,j}) = j \) if \( i \geq n \). For each \( n \in \omega \), let

\[
F_n = \{ x \in X : (x, a_{s,1-j}) \in K_{n,j}, \ s \in 2^n \text{ and } j \in 2 \}.
\]

Then each \( F_n \) is a closed set in \( X \). Pick \( n \in \omega \) and \( x \in F_n \). There are some \( s \in 2^n \) and \( j \in 2 \) with \( (x, a_{s,1-j}) \in K_{n,j} \). Since \( K_{n,j} \subset G_j \) and \( a_{s,1-j} \notin \pi_i^{-1}(j) \) if \( i \geq n \), there is \( k < n \) with \( (x, a_{s,1-j}) \in U_k \times \pi_k^{-1}(j) \). So we have \( x \in U_k \subset U_n \). Hence \( F_n \subset U_n \) for each \( n \in \omega \). Next, pick any \( x \in X \). By the Baire category theorem, there are some \( m \in \omega \) and \( \ell \in 2 \) such that \( K_{m,\ell} \cap (\{x\} \times 2^\omega) \) has the nonempty interior in \( \{x\} \times 2^\omega \). There are some \( n \geq m \) and \( s = (k_0, \ldots, k_{n-1}) \in 2^n \) such that \( \{x\} \times (\bigcap_{i<n} \pi_i^{-1}(k_i)) \subset K_{m,\ell} \). Then it follows that \( (x, a_{s,1-\ell}) \in K_{m,\ell} \subset K_{n,\ell} \). So we have \( x \in F_n \). Hence \( \{F_n : n \in \omega\} \) is a countable closed cover of \( X \) such that \( F_n \subset U_n \) for each \( n \in \omega \). This implies from [7, Theorem] that \( X \) is a countably metacompact. ☐

Theorem 1.1 immediately yields a generalization of Dowker’s theorem.

Corollary 1.2. For a normal space \( X \), the following are equivalent.

(a) \( X \) is countably paracompact.
(b) \( X \times (\omega + 1) \) is normal.
(c) \( X \times [0,1] \) is subnormal.

Remark. It should be noticed that Theorem 1.1 and Corollary 1.2 are essentially different from all the analogues in the list of [6, p. 118]. Because we can replace \([0,1]\) with \( \omega + 1 \) in all of them, but we cannot do in Theorem 1.1 and Corollary 1.2. In fact, consider a Dowker space \( Y \), whose existence was assured by Rudin [10]. Since the product of a subnormal space and a countable space is subnormal, it follows that \( Y \times (\omega + 1) \) is subnormal. On the other hand, \( Y \) is normal, but not countably metacompact.

2. Analogue of Tamano’s theorem

For a Tychonoff space \( X \), we denote by \( \beta X \) the Stone–Čech compactification of \( X \). Let us restate Tamano’s characterization of paracompactness in [11,12].

Tamano’s theorem. For a Tychonoff space \( X \), the following are equivalent.

(a) \( X \) is paracompact.
(b) \( X \times \beta X \) is normal.
(c) \( X \times \gamma X \) is normal for some compactification \( \gamma X \) of \( X \).

Recall that a regular space \( X \) is Lindelöf if every open cover of \( X \) has a countable subcover. A space \( X \) is \( \omega_1 \)-compact if every closed discrete subset in \( X \) is at most countable. Note that Lindelöf spaces are \( \omega_1 \)-compact.
**Lemma 2.1.** Let $D$ be an uncountable discrete space and $A(D)$ the one-point compactification of $D$ with the nonisolated point $p$. Then $A(D)^2\backslash\{(p,p)\}$ is not normal.

This seems to be known. In fact, the proof is actually done in that of [5, Lemma 2.5].

**Lemma 2.2.** Let $C$ be a countably compact space and $X$ a subspace of $C$. If the subspace $(X \times C) \cup (C \times X)$ of the square $C^2$ is normal, then $X$ is $\omega_1$-compact.

**Proof.** Let $Z = (X \times C) \cup (C \times X)$. Notice that $Z = C^2\backslash(C\backslash C)^2$. Assume that $Z$ is normal, but $X$ is not $\omega_1$-compact. There is an uncountable closed discrete subset $D$ in $X$. Let $D^*$ be the set of all accumulation points of $D$ in $C$. Let $A(D)$ and $p$ be as in Lemma 2.1. Let $E = (D \cup D^*)^2\backslash(D^*)^2$ and $F = A(D)^2\backslash\{(p,p)\}$. Note that $D^*$ is open discrete in $E$ and $F$. Consider the function $f: E \to F$ defined by $f(x, x') = (x, x'), f(x, y) = (x, p)$ and $f(y, x) = (p, x)$ for each $x, x' \in D$ and each $y \in D^*$. Then $f$ is continuous. Pick $x \in D$. Let $G$ be an open set in $E$ with $f^{-1}(p, x) \subset G$. Let $U$ be an open set in $C$ such that $D^* \subset U$ and $U \times \{x\} \subset G$. Since $C$ is countably compact, $D^* \cap U$ must be finite. Then $W = (U \cap D) \cup \{p\}$ is an open neighborhood of $p$ in $A(D)$. Moreover, we have

$$f^{-1}(W \times \{x\}) = U \times \{x\} \subset G.$$  

Thus $f$ is a closed map from $E$ onto $F$. Since $E$ is closed in $Z$, it is normal. Since normality is preserved under closed maps, $F$ is normal. However, by Lemma 2.1, $F$ is not normal. This is a contradiction. \[ \square \]

Now, we can obtain a characterization of Lindelöf spaces analogous to Tamano’s theorem.

**Theorem 2.3.** For a Tychonoff space $X$, the following are equivalent.

(a) $X$ is Lindelöf.

(b) The subspace $(X \times \beta X) \cup (\beta X \times X)$ of the square $(\beta X)^2$ is normal.

(c) The subspace $(X \times \gamma X) \cup (\gamma X \times X)$ of the square $(\gamma X)^2$ is normal for some compactification $\gamma X$ of $X$.

**Proof.** (a) $\Rightarrow$ (b) Since $X$ is Lindelöf, so are both $X \times \beta X$ and $\beta X \times X$. Hence $(X \times \beta X) \cup (\beta X \times X)$ is Lindelöf, so that it is normal.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) Let $Z = (X \times \gamma X) \cup (\gamma X \times X)$. Let $K$ be a compact subset of $\gamma X \backslash X$. Then $\Delta = \{(x, x) : x \in X\}$ and $X \times K$ are disjoint closed sets not only in $X \times \gamma X$ but also in $Z$. So $\Delta$ and $X \times K$ are completely separated in $Z$, hence in $X \times \gamma X$. It follows from [12, Theorem 3.1] that $X$ is paracompact. Recall that a $\omega_1$-compact, paracompact

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Lemma 2.2 was previously included in another paper, which was not published. We wish to thank the referee of that paper for suggesting this simpler proof than our original one.
space is Lindelöf (see [2, Corollary 1] which is essentially due to [1]). Hence it follows
from Lemma 2.2 that $X$ is Lindelöf. □

Theorem 2.3 immediately yields Katětov's old result in [8].

**Corollary 2.4.** Let $K$ be a compact space. If $K^2$ is hereditarily normal, then $K$ is
hereditarily Lindelöf.

**Note added in proof.** Good and Tree have recently commented on an erratum to [6] in

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