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Phase semantics and decidability of elementary affine logic

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Abstract

Light, elementary and soft linear logics are formal systems derived from Linear Logic, enjoying remarkable normalization properties. In this paper, we prove decidability of Elementary Affine Logic, **EAL**. The result is obtained by semantical means, first defining a class of phase models for **EAL** and then proving soundness and (strong) completeness, following Okada's technique. Phase models for Light Affine Logic and Soft Linear Logic are also defined and shown complete.

Keywords: Linear logic; Light linear logic; Soft linear logic; Optimal reduction

1. Introduction

The logical characterization of computational complexity classes has a long tradition. The most followed path has been to extensionally characterize complexity classes as the models for certain logical theories. Logical systems, however, have a built-in computational mechanism—normalization. The definition of logical systems which could be normalized inside an interesting class, and which, at the same time, could give extensional characterization of that same class, is a much more recent research direction. The first interesting logical system to have a polytime reduction strategy was Bounded Linear Logic [11]. In this system, however, the bound on the resources is explicitly present in the syntax, as a polynomial indexing the modality. A better system is Light Linear Logic (**LLL**) [10], where the introduction of three modalities (and a

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suitable management of contexts) allows for a polynomial reduction strategy (in the proof-net notation, and once the box-nesting depth is fixed); moreover, any polynomial time computable function can be defined inside **LLL**. In between **LLL** and full Linear Logic, the same paper [10] introduced Elementary Linear Logic (**ELL**), a system with a (Kalmar) elementary time normalization and defining all elementary time computable functions (see [8] for an in-depth study of **ELL**'s expressiveness and normalization). All these systems derive from Linear Logic—they limit the computational explosion of normalization by controlling weakening and contraction via modalities (called *exponentials* in the linear logic jargoon). It was first observed by Asperti that these systems maintain their main computational and expressive properties even in the presence of *full weakening*. The resulting polynomial system, Light Affine Logic (**LAL**) was introduced in [1] and studied in depth in [4]. From the same papers it is clear how to define the affine version of the elementary logic, **EAL**. One last system to appear on the scene is Lafont's Soft Linear Logic (**SLL**) [15], a system with a simple syntax, still enjoying polynomial normalization at fixed depth. All these logics have been introduced and justified as mere formal systems, without any reference to an intended or implied logical semantics. Okada, Kanovich, Scedrov and, later, Terui, introduced and investigated notions of models for **LLL** [12] and intuitionistic **LAL** [21]. The present paper builds on this previous work, defining classes of phase models for **EAL** and **SLL**. Following Lafont [14]—who proved that the addition of full weakening to Linear Logic yields the finite model property, and hence decidability—we prove that **EAL** is decidable, by showing it enjoys the finite model property. This same technique, on the other hand, cannot be applied to **SLL**, since it is not an affine logic and it can be easily proved to be undecidable. We show, however, that even the multiplicative fragment of **SLL** does not enjoy the finite model property.

We proceed in an incremental way, by first introducing a notion of a phase model for **LAL** (a variation on the one in [21]) and then showing how the same technique can be applied to build models for **EAL**. After having obtained our main result, we apply our schema to **SLL**, for which no semantics have been introduced so far. Our notions are simple—and, we believe, natural—extensions of the usual definition used for linear logic (they are "elementary" definitions, if a pun is allowed, etc.). Since our interest here is mainly in using semantics to derive syntactical properties, we looked for the simplest notion.

1.1. Related and previous work

The use of phase spaces as models of linear logic dates way back to the origins [9]. Lafont [14] used phase semantics to show that free weakening turns full linear logic (which is undecidable [16]) into a decidable affine logic **LLW**. Noticeably, the same result was previously obtained by Kopylov [13] with different tools. The use of phase semantics to deduce cut-elimination from completeness (strong completeness) is due to Okada [17–19]. In these papers, the technique is applied and generalized to a large number of logics. Okada and Terui [20] attack the decidability of affine variants of the same systems, extending Lafont's approach to various intuitionistic fragments of Linear Logic, including some substructural ones. In all the logics of [17–20], however, the exponential is always introduced in the standard linear logic fashion— the modality "!" is governed by an S4-like rule. This makes it difficult to directly apply the results of these papers to logics with restricted exponentials, like **EAL**, **LAL**, etc., where "!" is only a functor.

The semantics for systems with restricted exponentials is studied in [12,21]. Kanovich et al. [12] introduced the notion of phase model for **LLL**, extending the usual notion of phase semantics for linear logic by the use of fibrations. They showed that their class of models is complete for **LLL**, obtaining also a strong completeness result, à la Okada. While fibred models provide a deep insight on **LLL**, they are not handy to be used as a tool to prove decidability. Indeed, the quotient of a fibred model modulo a logical congruence (see Section 3) is not directly a (fibred) model. Finally, Terui [21] gives classes of phase models for intuitionistic **LAL**, for which he proves the finite model property and, hence, decidability. To obtain this result, though, the original notion of a model has to be generalized, in order to allow the result of a quotient to be a (generalized) model. See Section 4.3 for a detailed comparison.

1.2. Motivations

At the end of this introduction, it is time to mention the beginning of all of this. The interest of **EAL** is not only (in our eyes: is not much) in its role in the description of complexity classes. The properties ensuring its elementary normalization (that is, the box depth of a link never changes during proof-net normalization) have a remarkable interpretation in the context of the optimal reduction of λ-terms à la Lévy (see [3] as general reference). Lamping's approach to optimal reduction of λ-terms is a graph rewriting algorithm that can be thought of as composed of two parts. The first part—the *abstract* algorithm—is responsible for optimal beta-reduction and incremental duplication; the second part—the *oracle*—allows for the presence in the graph of enough distributed information to make the abstract algorithm correct with respect to the usual notion of reduction. While the abstract algorithm is simple, clear and compelling, the oracle is complex, heavy and, to a certain extent, debatable. There are λ-terms, however, for which the oracle is not needed, resulting in a much simpler (and more efficient!) graph rewriting reduction. λ-terms which are typeable inside **EAL** form a large class of terms with this property. This is the starting point of our interest in **EAL**. In [2] we used **EAL** as a tool to prove a complexity result on optimal reduction (and as a by-product we showed that **EAL**-typeable terms form a large and expressive class). Then, we investigated [6] the possibility to automatically infer **EAL**-typeability. Finally, Coppola and Ronchi della Rocca [7] prove the existence of principal types for **EAL**. The present paper completes the picture with the semantical perspective.

1.3. Outline of the paper

The structure of the paper is the following. Section 2 introduces formal systems for **LAL**, **EAL** and **SLL**. Section 3 recalls the standard notion of a phase space, states some relevant properties and gives some results that are used throughout the paper. Phase models for **LAL** are defined in Section 4; we prove strong completeness (that is

Identity and Cut.

$$\frac{}{\vdash A, A^{\perp}}\ I_{\mathcal{GMALL}} \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ U_{\mathcal{GMALL}}$$

Logical Rules.

$$\frac{}{\vdash \perp^{\perp}}\ \perp^{\perp}_{\mathcal{GMALL}} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp_{\mathcal{GMALL}} \qquad \frac{}{\vdash \Gamma, \top}\ \top_{\mathcal{GMALL}}$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes_{\mathcal{GMALL}} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A ⅋ B}\ ⅋_{\mathcal{GMALL}}$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus 1_{\mathcal{GMALL}} \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus 2_{\mathcal{GMALL}} \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ \&_{\mathcal{GMALL}}$$

Fig. 1. Multiplicative and Additive Linear Logic, **MALL**.

soundness, completeness, and cut-elimination) in Section 4.1; in Section 4.2 we show that the finite model property holds for **LAL**. The definition and properties of the phase models for **EAL** and **SLL** are the subject of Sections 5 and 6, respectively; decidability of **EAL** is in Section 5.2.

2. Systems

All the systems we will describe make use of subsets of the logical language generated by the grammar

$$\begin{aligned} A, B ::= & \perp \mid \perp^{\perp} \mid \top \mid \top^{\perp} \mid \alpha \mid \alpha^{\perp} \mid A ⅋ B \mid A \otimes B \\ & A \& B \mid A \oplus B \mid !A \mid ?A \mid \S A \mid \overline{\S} A \end{aligned}$$

where α ranges over a set $\mathcal{L}$ of *atoms*. The unary operator $\perp$ is extended to the whole language in the usual De Morgan style; in particular, the following *syntactical* equivalences hold on exponential formulae:

$$?A^{\perp} \equiv (!A)^{\perp} \tag{1}$$

$$!A^{\perp} \equiv (?A)^{\perp} \tag{2}$$

$$\S A^{\perp} \equiv (\overline{\S} A)^{\perp} \tag{3}$$

$$\overline{\S} A^{\perp} \equiv (\S A)^{\perp} \tag{4}$$

The logics we are interested in are obtained from the core of Multiplicative and Additive Linear Logic (**MALL**, Fig. 1) by adding suitable rules for the exponential connectives.

LAL is a logical system characterizing polynomial time. The rules of the sequent calculus $\mathcal{GLAL}$ for **LAL** are summarized in Fig. 2. Notice that we do not suppose

Structural Rules.

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}\, W_{\mathcal{GLAL}} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\, C_{\mathcal{GLAL}}$$

Exponential Rule.

$$\frac{\vdash A, B}{\vdash ?A, !B}\, S_{\mathcal{GLAL}} \quad \frac{\vdash A}{\vdash !A}\, \overline{S}_{\mathcal{GLAL}} \quad \frac{\vdash \Gamma, \Delta, A}{\vdash ?\Gamma, \overline{\S}\Delta, \S A}\, P_{\mathcal{GLAL}}$$

Fig. 2. Light Affine Logic, **LAL**.

the connective § to be self-dual. We call $\mathbf{LAL}_{\mathcal{SD}}$ the variant of **LAL** where § *is* self-dual—the connective $\overline{\S}$ is omitted from the language and the equivalence

$$\S A^{\perp} \equiv (\S A)^{\perp}$$

takes the place of both (3) and (4). Moreover, the rule $P_{\mathcal{GLAL}}$ becomes

$$\frac{\vdash \Gamma, \Delta, A}{\vdash ?\Gamma, \S\Delta, \S A}\, P_{\mathcal{GLAL}_{\mathcal{SD}}}$$

in the underlying sequent calculus $\mathcal{GLAL}_{\mathcal{SD}}$. It is well known that in this system the cut rule is not eliminable, as shown by the proof:

$$\cfrac{\cfrac{\cfrac{}{\vdash A, A^{\perp}}}{\vdash ?A, \S A^{\perp}} \quad \cfrac{\cfrac{}{\vdash A, A^{\perp}}}{\vdash \S A, ?A^{\perp}}}{\vdash ?A, ?A^{\perp}}$$

This makes $\mathcal{GLAL}_{\mathcal{SD}}$ not suitable to be studied with phase semantics, since we want to derive cut-elimination from completeness.

If we add to **MALL** a functorial exponential rule with restricted contraction and weakening, we get Elementary Linear Logic (**ELL**), sketched first in [10] and studied in depth in [8] (although with a slightly different syntax). The key feature of **ELL** is its elementary time normalization, a property which is maintained by adding a full weakening rule. The resulting logic, Elementary Affine Logic (**EAL**), is obtained from **MALL** by adding the rules in Fig. 3. We already discussed in the introduction its relevance for optimal reduction.

The last system we will consider is Soft Linear Logic [15], a system in which, at fixed box depth, proof-nets have a polynomial reduction in their size. It is obtained from **MALL** by adding the same exponential rule we used in **ELL**, but with a strong restriction on contraction and weakening. The system **SLL** is obtained by adding to **MALL** the rules of Fig. 4, where $A^{(n)}$ stands for

$$\underbrace{A, \ldots, A}_{n\ times}.$$

Structural Rules.

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}\ W_{\mathcal{GEAL}} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ C_{\mathcal{GEAL}}$$

Exponential Rule.

$$\frac{\vdash \Gamma, A}{\vdash ?\Gamma, !A}\ S_{\mathcal{GEAL}}$$

Fig. 3. Elementary Affine Logic, **EAL**.

Exponential Rules.

$$\frac{\vdash \Gamma, A^{(n)} \quad n \geqslant 0}{\vdash \Gamma, ?A}\ M_{\mathcal{GSLL}} \qquad \frac{\vdash \Gamma, A}{\vdash ?\Gamma, !A}\ S_{\mathcal{GSLL}}$$

Fig. 4. Soft Linear Logic, **SLL**.

Given a formal system $\mathcal{F}$ we will use the following abbreviations:

- $\Xi_{\mathcal{F}}$ will be the set of well formed formulae which are *provable* in $\mathcal{F}$;
- $\partial(\mathcal{F})$ will be the *decision problem* of provability on $\mathcal{F}$, which could be seen as a language over the set of $\mathcal{F}$ well formed formulae.

3. Preliminaries on phase semantics

We recall in this section the basic definitions and properties of phase semantics for Linear Logic, see [14]. A *phase space* is a pair $(M, \bot)$ where M is a commutative monoid and $\bot$ is a subset of M. If $(M, \bot)$ is a phase space and $X, Y \subseteq M$, we will use the following notations:

$$XY = \{xy \mid x \in X, y \in Y\},$$
$$X \multimap Y = \{z \in M \mid \forall x \in Xxz \in Y\}.$$

We write $X^\bot$ for $X \multimap \bot$. The following lemma establishes some basic results and can be easily proved.

Lemma 3.1. *If* $(M, \bot)$ *is a phase space and* $X, Y \subseteq M$, *then*:
(i) $X \subseteq X^{\bot\bot}$;
(ii) $Y^\bot \subseteq X^\bot$ *whenever* $X \subseteq Y$;
(iii) $X \multimap Y^\bot = (X\,Y)^\bot$;
(iv) $X^\bot \cap Y^\bot = (X \cup Y)^\bot$;
(v) $(X^{\bot\bot} Y^{\bot\bot})^\bot = (XY)^\bot$.

If $(M, \perp)$ is a phase space and X is a subset of M such that $X = X^{\perp\perp}$, we say that X is a *fact*. The following technical lemma is handy in proving that sets are facts.

Lemma 3.2. *If $(M, \perp)$ is a phase space and $X, Y \subseteq M$*, then:
(i) *$\perp$ is a fact*;
(ii) *X is a fact if and only if $X = Z^{\perp}$ for some $Z \subseteq M$*;
(iii) *If Y is a fact, then $X \multimap Y$ is a fact*;
(iv) *If X and Y are facts, then $X \cap Y$ is a fact*;
(v) *If X is a fact, then $\perp^{\perp} \multimap X = X$.*

Useful properties can be extended from individual elements of a phase space to whole subsets of the underlying monoid:

Lemma 3.3. *If $(M, \perp)$ is a phase space and X is a subset of M such that $x \in \{x^2\}^{\perp\perp}$ for every $x \in X$, then $X \subseteq (X^2)^{\perp\perp}$.*

If $(M, \perp)$ is a phase space, then the set $\{x \in M \mid x \in \{x^2\}^{\perp\perp}\}$ will be denoted as $J(M)$.

Lemma 3.4. *If $(M, \perp)$ is a phase space, then $J(M)$ is a submonoid of M.*

Let $(M, \perp)$ be a phase space and let $f, g \subseteq M \times M$ be binary relations. If, for every fact $X \subseteq M$, $f(X) \subseteq g(X)^{\perp\perp}$, then we say that f is *bounded* by g.

Lemma 3.5. *Let $(M, \perp)$ be a phase space and let $f, g \subseteq M \times M$ be binary relations. If $f(x) \subseteq g(x)^{\perp\perp}$ whenever $x \in M$, then f is bounded by g.*

Proof. Suppose $X \subseteq M$ and let $y \in f(X)$. Then $y \in f(x)$, where $x \in X$ and, by hypothesis, $f(x) \subseteq g(x)^{\perp\perp}$. But, clearly, $g(x)^{\perp\perp} \subseteq g(X)^{\perp\perp}$, meaning that $y \in g(X)^{\perp\perp}$. $\square$

If $(M, \perp)$ is a phase space, *a relational monoid homomorphism* is a binary relation $f \subseteq M \times M$ such that $1_M \in f(1_M)$ and $f(x)f(y) \subseteq f(xy)$ for every x and y in M.

Lemma 3.6. *If $(M, \perp)$ is a phase space, $f \subseteq M \times M$ is a relational monoid homomorphism and $A, B \subseteq M$, then $f(A \multimap B) \subseteq f(A) \multimap f(B)$.*

Proof. If $x \in f(A \multimap B)$, then $x \in f(y)$, where $y \in A \multimap B$. Now, if $z \in f(A)$, then $z \in f(w)$ where $w \in A$, and then $xz \in f(y)f(w) \subseteq f(yw) \subseteq f(B)$, because $yw \in B$. $\square$

If $(M, \perp)$ is a phase space, a *logical congruence* on $(M, \perp)$ is an equivalence relation $\sim$ on M such that:
(i) $xz \sim yw$ whenever $x \sim y$ and $z \sim w$;
(ii) $\perp$ is closed with respect to $\sim$.
If M is a set and $\sim$ is an equivalence on M, then $\pi : M \rightarrow M/{\sim}$ is the $\sim$ *canonical map*, that is the function which maps every element of M to its equivalence class; moreover,

$[x]_\sim$ will be the equivalence class modulo $\sim$ containing x. A logical congruence $\sim$ is said to have *finite index* if $M/\!\sim$ is a finite set.

Lemma 3.7. *If* $(M, \perp)$ *is a phase space*, $\sim$ *is a logical congruence on* M *and* $A \subseteq M$ *is a fact*, *then* A *is closed under* $\sim$.

Lemma 3.8. *If* $(M, \perp)$ *is a phase space and* $\sim$ *is a logical congruence on* $(M, \perp)$, *then* $J(M)$ *is closed under* $\sim$.

Proof. If $x \in J(M)$ and $x \sim y$, then $\{x\}^\perp = \{y\}^\perp$. But

$$\begin{aligned}\{x\}^\perp = \{y\}^\perp &\Rightarrow \{x\}^{\perp\perp} = \{y\}^{\perp\perp}\\ &\Rightarrow (\{x\}^{\perp\perp}\{x\}^{\perp\perp})^{\perp\perp} = (\{y\}^{\perp\perp}\{y\}^{\perp\perp})^{\perp\perp}\\ &\Rightarrow (\{x\}\{x\})^{\perp\perp} = (\{y\}\{y\})^{\perp\perp}\\ &\Rightarrow \{x^2\}^{\perp\perp} = \{y^2\}^{\perp\perp}.\end{aligned}$$

By Lemma 3.7, y must be in $\{x^2\}^{\perp\perp}$, and the thesis easily follows. □

Lemma 3.9. *If* $(M, \perp)$ *is a phase space and* $\sim$ *is a logical congruence on* $(M, \perp)$, *then* $\pi(XY) = \pi(X)\pi(Y)$ *and* $\pi(X \multimap Y) = \pi(X) \multimap \pi(Y)$.

If $(M, \perp)$ is a phase space, $f \subseteq M \times M$ is a binary relation, and $\sim$ is a logical congruence, then $f^\sim \subseteq (M/\!\sim) \times (M/\!\sim)$ is defined by letting $([x]_\sim, [y]_\sim) \in f^\sim$ whenever $(x, y) \in f$.

Lemma 3.10. *If* $(M, \perp)$ *is a phase space*, $\sim$ *is a logical congruence on* $(M, \perp)$, $f \subseteq M \times M$ *is a binary relation and* $X \subseteq M$, *then*:
(i) $\pi(f(X)) \subseteq f^\sim(\pi(X))$;
(ii) $\pi(f(X)) = f^\sim(\pi(X))$ *whenever* X *is closed on* $\sim$;
(iii) $\pi(f(\pi^{-1}(X))) = f^\sim(X)$ *whenever* X *is closed* $\sim$.

Proof. Easy, from the definition of $f^\sim$. □

If M is monoid, an *ideal* for M is a set $X \subseteq M$ such that $XM \subseteq X$.

Lemma 3.11. *Let* M *be a commutative monoid. Then*:
(i) *If* $Y \subseteq M$ *is an ideal*, *then* $X \multimap Y$ *is an ideal*;
(ii) *Every finite union of ideals is an ideal.*

Proof. (i) If Y is an ideal, then $YM = Y$ because, obviously, $Y \subseteq YM$. Let then x be an element of $(X \multimap Y)M$; this means that $x = yz$ with $y \in X \multimap Y$ and $z \in M$. Let then w be an element of X; it is clear that $xw = (yz)w = (yw)z \in Y$ and that $x \in X \multimap Y$.

(ii) It suffices to observe that if $X_1,\ldots,X_n \subseteq M$ are ideals, then

$$(X_1 \cup \cdots \cup X_n)M = X_1M \cup \cdots \cup X_nM = X_1 \cup \ldots \cup X_n$$

which is the thesis. □

An ideal of M is a *principal ideal* if it can be written as xM, for x in M. An ideal is said to have *finite type* if it is a finite union of principal ideals. A monoid M is *noetherian* if all its ideals have finite type. The following classical result will be useful later.

Lemma 3.12. *A free, finitely generated, commutative monoid is noetherian.*

If $(M,\bot)$ is a phase space, we denote as $\equiv$ the logical congruence defined by letting $x \equiv y$ iff $\{x\}^\bot = \{y\}^\bot$.

Lemma 3.13. *Let $(M,\bot)$ be a phase space, where M is a free and finitely generated monoid such that every fact is an ideal. Then $M/\equiv$ is finite.*

Proof. We can assume, without losing generality, that M is $\mathbb{N}^k$ and that the operation which makes M a monoid is the addition. From the hypothesis, $\bot$ is an ideal and, by Lemma 3.12, we can conclude that

$$\bot = \bigcup_{i=1}^{n} u_i\mathbb{N}^k = \bigcup_{i=1}^{n} \{x \in \mathbb{N}^k \mid x \geqslant u_i\}$$

for $u_1,\ldots,u_n \in \mathbb{N}^k$. But it is now clear that there can be only finitely many subsets of $\mathbb{N}^k$ in the form $\{x\}^\bot$, simply because $\{x\}^\bot = \{inf\{x, sup\{u_1,\ldots,u_n\}\}\}^\bot$ for every x. This, by definition of $\equiv$, yields the thesis. □

4. Light affine logic

A *light affine phase space* is a quintuple $(M,\bot,\phi,\xi)$ where:

- $(M,\bot)$ is a phase space;
- $\xi \subseteq M \times M$ is a relational monoid homomorphism;
- $\phi \subseteq M \times M$ is bounded by ξ, includes $(1_M, 1_M)$ and $\phi(M) \subseteq J(M)$;
- $\bot \subseteq M^\bot$.

Light affine phase spaces form the algebraic structure of the models we are proposing. They can be seen as ordinary phase spaces with the additional structure (namely, ξ and ϕ) needed to model exponentials.

Proposition 4.1. *If $(M,\bot,\phi,\xi)$ is a light affine phase space and $X \subseteq M$ is a fact, then $\bot \subseteq X$.*

Proof. Since X is a fact, by 3.2 $X = Y^\bot$ for some $Y \subseteq M$. By Lemma 3.1, $M^\bot \subseteq Y^\bot = X$, which yields the thesis. □

A *phase model for* **LAL** is a light affine phase space, enriched with an interpretation for atoms, that is a tuple $(M, \perp, \phi, \xi, \sigma)$ where $(M, \perp, \phi, \xi)$ is a light affine phase space and $\sigma : \mathcal{L} \to \mathcal{P}(M)$ maps every atom to a fact.

Given a phase model $\mathcal{M}$ for **LAL**, we can associate a fact $[\![A]\!]_{\mathcal{M}}$ to every formula A in the usual way. Nonexponential formulae can be treated as in **MALL**:

$$
\begin{aligned}
[\![\perp]\!]_{\mathcal{M}} &= \perp\\
[\![\perp^{\perp}]\!]_{\mathcal{M}} &= \perp^{\perp}\\
[\![\top]\!]_{\mathcal{M}} &= M\\
[\![\top^{\perp}]\!]_{\mathcal{M}} &= M^{\perp}\\
[\![\alpha]\!]_{\mathcal{M}} &= \sigma(\alpha)\\
[\![\alpha^{\perp}]\!]_{\mathcal{M}} &= \sigma(\alpha)^{\perp}\\
[\![A \parr B]\!]_{\mathcal{M}} &= ([\![A]\!]_{\mathcal{M}}^{\perp}[\![B]\!]_{\mathcal{M}}^{\perp})^{\perp}\\
[\![A \otimes B]\!]_{\mathcal{M}} &= ([\![A]\!]_{\mathcal{M}}[\![B]\!]_{\mathcal{M}})^{\perp\perp}\\
[\![A \& B]\!]_{\mathcal{M}} &= [\![A]\!]_{\mathcal{M}} \cap [\![B]\!]_{\mathcal{M}}\\
[\![A \oplus B]\!]_{\mathcal{M}} &= ([\![A]\!]_{\mathcal{M}} \cup [\![B]\!]_{\mathcal{M}})^{\perp\perp}
\end{aligned}
$$

The semantics for the exponentials is the following:

$$
\begin{aligned}
[\![!A]\!]_{\mathcal{M}} &= (\phi([\![A]\!]_{\mathcal{M}}))^{\perp\perp}\\
[\![?A]\!]_{\mathcal{M}} &= (\phi([\![A]\!]_{\mathcal{M}}^{\perp}))^{\perp}\\
[\![\S A]\!]_{\mathcal{M}} &= (\xi([\![A]\!]_{\mathcal{M}}))^{\perp\perp}\\
[\![\overline{\S} A]\!]_{\mathcal{M}} &= (\xi([\![A]\!]_{\mathcal{M}}^{\perp}))^{\perp}
\end{aligned}
$$

This definition can be easily extended to sequents, allowing to define that a $\mathcal{GLAL}$ sequent $\vdash \Gamma$ is *verified* in a phase model $\mathcal{M}$ for **LAL** if and only if $1 \in [\![\vdash \Gamma]\!]_{\mathcal{M}}$.

Proposition 4.2. *If* $(M, \perp, \phi, \xi)$ *is a light affine phase space and* $X \subseteq M$ *is a fact, then* $\perp \subseteq X$.

Proof. Since X is a fact, by 3.2 $X = Y^{\perp}$ for some $Y \subseteq M$. By Lemma 3.1, $M^{\perp} \subseteq Y^{\perp} = X$, which yields the thesis. □

4.1. Strong completeness

In proving strong completeness for the class of models we are proposing, we follow the usual methodology first introduced in [17]. The *syntactical model for* **LAL** is the quintuple $\mathcal{M}^{\mathcal{L}} = (M^{\mathcal{L}}, \perp^{\mathcal{L}}, \phi^{\mathcal{L}}, \xi^{\mathcal{L}}, \sigma^{\mathcal{L}})$ defined as follows:

- $M^{\mathcal{L}}$ is the commutative monoid generated by all formulae of **LAL**; this structure is isomorphic to the set of all $\mathcal{GLAL}$ sequents (endowed with sequent juxtaposition);

- $\bot^{\mathcal{L}}$ is the set of all cut-free provable sequents in $\mathcal{GLAL}$;
- $\xi^{\mathcal{L}}$ is defined by $\xi^{\mathcal{L}}(A_1 \ldots A_n) = \{\overline{\S}A_1 \ldots \overline{\S}A_n\}$ for every sequence $A_1 \ldots A_n$ of **LAL** formulae and $n \geqslant 0$;
- $\phi^{\mathcal{L}}$ is defined by

$$\phi^{\mathcal{L}}(A_1 \ldots A_n) = \begin{cases} \{1_{M^{\mathcal{L}}}\} & \text{if } n = 0, \\ \{?A_1\} & \text{if } n = 1, \\ \emptyset & \text{otherwise;} \end{cases}$$

- $\sigma^{\mathcal{L}}$ is defined by $\sigma^{\mathcal{L}}(\alpha) = \{\alpha\}^{\bot}$ for every $a \in \mathcal{L}$.

Lemma 4.3. *$\mathcal{M}^{\mathcal{L}}$ is a phase model for* **LAL**.

Proof. The only interesting properties to be verified are the ones on $\phi^{\mathcal{L}}$. First of all, we show that $\phi^{\mathcal{L}}$ is bounded by $\xi^{\mathcal{L}}$. Now, if $A_1 \ldots A_n \in M^{\mathcal{L}}$, then:
- If $n = 0$, then $\phi^{\mathcal{L}}(1) = \{1\} \subseteq \{1\}^{\bot\bot} = \{\xi^{\mathcal{L}}(1)\}^{\bot\bot}$;
- If $n = 1$, then $\phi^{\mathcal{L}}(A_1) = \{?A_1\} \subseteq \{\overline{\S}A_1\}^{\bot\bot} = \{\xi^{\mathcal{L}}(A_1)\}^{\bot\bot}$, because, for every Γ

$$\Gamma, \overline{\S}A_1 \in \bot \Rightarrow \Gamma, ?A_1 \in \bot$$

  as it can be proved by an easy induction on the structures of $\mathcal{GLAL}$ proofs;
- If $n > 1$, then $\phi^{\mathcal{L}}(A_1 \ldots A_n) = \emptyset$.

This means that $\phi^{\mathcal{L}}(A_1 \ldots A_n) \subseteq \{\xi^{\mathcal{L}}(A_1 \ldots A_n)\}^{\bot\bot}$, which yields, by Lemma 3.5, $\phi^{\mathcal{L}}(X) \subseteq \{\xi^{\mathcal{L}}(X)\}^{\bot\bot}$ for every $X \subseteq M^{\mathcal{L}}$. If $\Gamma, ?A, ?A$ is cut-free provable, then $\Gamma, ?A$ is cut-free provable, too. As a consequence, $\phi^{\mathcal{L}}$ ranges over $J(M)$. □

Lemma 4.4 (Okada). *For every formula A in* **LAL**, *we have that $[\![A]\!] \subseteq \{A\}^{\bot}$ in $\mathcal{M}^{\mathcal{L}}$.*

Proof. We can prove this by a structural induction on A. The only interesting inductive cases are the following:
- $A = !B$; by inductive hypothesis, $[\![B]\!] \subseteq \{B\}^{\bot}$ and so $\phi([\![B]\!]) \subseteq \phi(\{B\}^{\bot})$. Now, if $CB \in \bot$, then, by rule $S_{\mathcal{GLAL}}$, $?C!B \in \bot$; moreover, if $B \in \bot$ then, by rule $\overline{S}_{\mathcal{GLAL}}$, $!B \in \bot$. Then, we can conclude that $\phi([\![B]\!])^{\bot\bot} \subseteq \{!B\}^{\bot}$, proving the inclusion $[\![A]\!] \subseteq \{A\}^{\bot}$;
- $A = ?B$; by inductive hypothesis, $[\![B]\!] \subseteq \{B\}^{\bot}$ and so $[\![B]\!]^{\bot} \supseteq \{B\}^{\bot\bot}$; obviously, $B \in \{B\}^{\bot\bot}$, and so $\phi([\![B]\!]^{\bot}) \supseteq \{?B\}$, which yields $[\![A]\!] = \phi([\![B]\!]^{\bot})^{\bot} \subseteq \{?B\}^{\bot} = \{A\}^{\bot}$;
- $A = \S B$; by inductive hypothesis, $[\![B]\!] \subseteq \{B\}^{\bot}$ and so $\xi([\![B]\!]) \subseteq \xi(\{B\}^{\bot})$. Now, if $\Gamma B \in \bot$, then, by rule $P_{\mathcal{GLAL}}$, $\overline{\S}\Gamma\, \S B \in \bot$; this means that $\xi(\{B\}^{\bot}) \subseteq \{\S B\}^{\bot}$. Then, we can conclude that $\xi([\![B]\!])^{\bot\bot} \subseteq \{\S B\}^{\bot}$, proving the inclusion $[\![A]\!] \subseteq \{A\}^{\bot}$;
- $A = \overline{\S}B$; by inductive hypothesis, $[\![B]\!] \subseteq \{B\}^{\bot}$ and so $[\![B]\!]^{\bot} \supseteq \{B\}^{\bot\bot}$; obviously, $B \in \{B\}^{\bot\bot}$, and so $\xi([\![B]\!]^{\bot}) \supseteq \{\overline{\S}B\}$, which yields $[\![A]\!] = \xi([\![B]\!]^{\bot})^{\bot} \subseteq \{\overline{\S}B\}^{\bot} = \{A\}^{\bot}$. □

Lemma 4.5. *If $\vdash \Gamma$ is provable in* **LAL**, *then $\vdash \Gamma$ is verified in all phase models for* **LAL**.

Proof. We proceed by induction on the structure of the proof π of $\vdash \Gamma$:

- If the last $\mathcal{GLAL}$ rule used to build π is $W_{\mathcal{GLAL}}$ Γ can be written as Δ, A and the immediate premise of $\vdash \Gamma$ in π will be $\vdash \Delta$. It is now easy to realize that

$$1 \in [\![\vdash \Delta, A]\!] \quad \Leftrightarrow \quad [\![\vdash \Delta]\!]^{\perp} [\![A]\!]^{\perp} \subseteq \perp$$
$$1 \in [\![\vdash \Delta]\!] \quad \Leftrightarrow \quad [\![\vdash \Delta]\!]^{\perp} \subseteq \perp$$

  Lemma 4.2 implies $[\![A]\!]^{\perp} \subseteq \perp^{\perp}$ while, by inductive hypothesis, $[\![\vdash \Delta]\!]^{\perp} \subseteq \perp$. As a consequence, $[\![\vdash \Delta]\!]^{\perp} [\![A]\!]^{\perp} \subseteq \perp$ and thus $1 \in [\![\vdash \Delta, A]\!]$, that is the thesis; here, we have used the condition $\perp \subseteq M^{\perp}$;
- If the last $\mathcal{GLAL}$ rule used to build π is $C_{\mathcal{GLAL}}$, then Γ can be written as $\Delta, ?A$ and the premise of $\vdash \Gamma$ in π can itself be written as $\vdash \Delta, ?A, ?A$. Now we can write

$$1 \in [\![\vdash \Delta, ?A]\!] \Leftrightarrow [\![\vdash \Delta]\!]^{\perp} [\![?A]\!]^{\perp} \subseteq \perp$$
$$1 \in [\![\vdash \Delta, ?A, ?A]\!] \Leftrightarrow [\![\vdash \Delta]\!]^{\perp} ([\![?A]\!]^{\perp} [\![?A]\!]^{\perp})^{\perp\perp} \subseteq \perp$$

  By Lemma 3.3, it follows that $\phi([\![A]\!]^{\perp}) \subseteq (\phi([\![A]\!]^{\perp})\phi([\![A]\!]^{\perp}))^{\perp\perp}$; closing the subsets we obtain

$$\phi([\![A]\!]^{\perp})^{\perp\perp} \subseteq ((\phi([\![A]\!]^{\perp})\phi([\![A]\!]^{\perp}))^{\perp\perp})^{\perp\perp} = (\phi([\![A]\!]^{\perp})^{\perp\perp}\phi([\![A]\!]^{\perp})^{\perp\perp})^{\perp\perp};$$

  this yields, in particular

$$[\![?A]\!]^{\perp} \subseteq ([\![?A]\!]^{\perp}[\![?A]\!]^{\perp})^{\perp\perp};$$

  as a consequence,

$$[\![\vdash \Delta]\!]^{\perp}[\![?A]\!]^{\perp} \subseteq [\![\vdash \Delta]\!]^{\perp}([\![?A]\!]^{\perp}[\![?A]\!]^{\perp})^{\perp\perp}$$

  from which the thesis can be easily obtained; here we have used the conditions on ϕ;
- If the last rule applied in π is $S_{\mathcal{GLAL}}$, then it suffices to notice that the following chain of implications holds:

$$\begin{aligned} 1 \in [\![\vdash A, B]\!] &\Rightarrow 1 \in ([\![A]\!]^{\perp}[\![B]\!]^{\perp})^{\perp} \\ &\Rightarrow 1 \in [\![A]\!]^{\perp} \multimap [\![B]\!] \\ &\Rightarrow [\![A]\!]^{\perp} \subseteq [\![B]\!] \\ &\Rightarrow \phi([\![A]\!]^{\perp}) \subseteq \phi([\![B]\!]) \\ &\Rightarrow 1 \in (\phi([\![A]\!]^{\perp})\phi([\![B]\!])^{\perp})^{\perp} \\ &\Rightarrow 1 \in (\phi([\![A]\!]^{\perp})^{\perp\perp}\phi([\![B]\!])^{\perp\perp\perp})^{\perp} \\ &\Rightarrow 1 \in [\![\vdash ?A, !B]\!]; \end{aligned}$$

- If the last rule applied in π is $\overline{S}_{\mathcal{GLAL}}$, then it suffices to notice that the following chain of implications holds:

$$1 \in [\![\vdash A]\!] \Rightarrow 1 \in \phi([\![A]\!])$$

$$\Rightarrow 1 \in \phi([\![A]\!])^{\perp\perp}$$
$$\Rightarrow 1 \in [\![!A]\!];$$

- If the last rule applied in π is $P_{\mathcal{GLAL}}$, then it suffices to notice that the following chain of implications holds:

$$
\begin{aligned}
&1 \in [\![\vdash A_1,\dots,A_n,B_1,\dots,B_m,C]\!]\\
&\Rightarrow 1 \in ([\![A_1]\!]^{\perp}\dots[\![A_n]\!]^{\perp}[\![B_1]\!]^{\perp}\dots[\![B_m]\!]^{\perp}[\![C]\!]^{\perp})^{\perp}\\
&\Rightarrow 1 \in [\![A_1]\!]^{\perp}\dots[\![A_n]\!]^{\perp}[\![B_1]\!]^{\perp}\dots[\![B_m]\!]^{\perp} \multimap [\![C]\!]\\
&\Rightarrow [\![A_1]\!]^{\perp}\dots[\![A_n]\!]^{\perp}[\![B_1]\!]^{\perp}\dots[\![B_m]\!]^{\perp} \subseteq [\![C]\!]\\
&\Rightarrow \xi([\![A_1]\!]^{\perp})\dots\xi([\![A_n]\!]^{\perp})\xi([\![B_1]\!]^{\perp})\dots\xi([\![B_m]\!]^{\perp}) \subseteq \xi([\![C]\!])\\
&\Rightarrow (\xi([\![A_1]\!]^{\perp})^{\perp\perp}\dots\xi([\![A_n]\!]^{\perp})^{\perp\perp}\xi([\![B_1]\!]^{\perp})\dots\xi([\![B_m]\!]^{\perp}))^{\perp\perp} \subseteq \xi([\![C]\!])^{\perp\perp}\\
&\Rightarrow (\phi([\![A_1]\!]^{\perp})\dots\phi([\![A_n]\!]^{\perp})\xi([\![B_1]\!]^{\perp})\dots\xi([\![B_m]\!]^{\perp}))^{\perp\perp} \subseteq \xi([\![C]\!])^{\perp\perp}\\
&\Rightarrow 1 \in ((\phi([\![A_1]\!]^{\perp})\dots\phi([\![A_n]\!]^{\perp})\xi([\![B_1]\!]^{\perp})\dots\xi([\![B_m]\!]^{\perp}))^{\perp\perp}\xi([\![C]\!])^{\perp})^{\perp}\\
&\Rightarrow 1 \in ((\phi([\![A_1]\!]^{\perp})^{\perp\perp}\dots\phi([\![A_n]\!]^{\perp})^{\perp\perp}\xi([\![B_1]\!]^{\perp})^{\perp\perp}\dots\xi([\![B_m]\!]^{\perp})^{\perp\perp})^{\perp\perp}\xi([\![C]\!])^{\perp})^{\perp}\\
&\Rightarrow 1 \in ([\![\vdash ?A_1,\dots,?A_n,\overline{\S}B_1,\dots,\overline{\S}B_M]\!]^{\perp}[\![\S C]\!]^{\perp})^{\perp}\\
&\Rightarrow 1 \in [\![\vdash ?A_1,\dots,?A_n,\overline{\S}B_1,\dots,\overline{\S}B_m,\S C]\!];
\end{aligned}
$$

  we used the boundness condition on ϕ and ξ.

This concludes the proof. □

Theorem 4.6 (Strong completeness). *Let A be a formula. The following four conditions are then equivalent*:
(i) $\vdash A$ *is provable in* **LAL**;
(ii) *Every phase model for* **LAL** *verifies* A;
(iii) $\mathcal{M}^{\mathcal{L}}$ *verifies* A;
(iv) $\vdash A$ *is cut-free provable in* **LAL**.

4.2. Decidability

To prove that **LAL** is decidable, we will prove it enjoys the finite model property. Following [14], we will iteratively reduce the size of the monoid underlying our syntactical model until we reach a finite monoid; during this process, we will maintain, as an invariant, the fact that the model we are dealing with is itself complete with respect to $\mathcal{GLAL}$.

4.2.1. Excluding useless elements from the model

If A is a **LAL** formula and $\vdash A$ is provable in $\mathcal{GLAL}$ with proof π, then π could, in general, contain formulae that are not subformulae of A; due to cut elimination, however, we can state that $\vdash A$ is provable if and only if there is a proof π for $\vdash A$ which contains only subformulae of $\vdash A$. This simple observation, known as the *subformula property*, can be exploited in the context of phase spaces to drastically reduce the size of $\mathcal{M}^{\mathcal{L}}$.

If A is a formula, we will denote as $\mathbf{LAL}_A$ the logic obtained by restricting our logical language to subformulae of A; similarly we can denote the restriction of $\mathcal{GLAL}$ to subformulae of A as $\mathcal{GLAL}_A$. The notions of a phase model and of verifiability by a phase model can be easily extended to $\mathbf{LAL}_A$. The *syntactic model for* $\mathcal{GLAL}_A$ is the quintuple $\mathcal{M}_A^{\mathcal{L}} = (M_A^{\mathcal{L}}, \bot_A^{\mathcal{L}}, \phi_A^{\mathcal{L}}, \xi_A^{\mathcal{L}}, \sigma_A^{\mathcal{L}})$ where:

- $M_A^{\mathcal{L}}$ is the free monoid generated by all the subformulae of A;
- $\bot_A^{\mathcal{L}}$ is the set of all cut-free provable $\mathcal{GLAL}_A$ sequents;
- $\phi_A^{\mathcal{L}}$ is defined as follows:

$$\phi_A^{\mathcal{L}}(A_1 \dots A_n) = \begin{cases} \{1_{M_A^{\mathcal{L}}}\} & \text{if } n = 0 \\ \{?A_1\} & \text{if } n = 1 \text{ and } ?A_n \text{ is a subformula for } A \\ \emptyset & \text{otherwise;} \end{cases}$$

- $\xi_A^{\mathcal{L}}$ is defined as follows:

$$\xi_A^{\mathcal{L}}(A_1 \dots A_n) = \begin{cases} \{\bar{\S}A_1 \dots \bar{\S}A_n\} & \text{if all the } \bar{\S}A_i \text{ are subformulae of } A \\ \emptyset & \text{otherwise;} \end{cases}$$

- $\sigma_A^{\mathcal{L}}$ is defined as follows:

$$\sigma_A^{\mathcal{L}}(\alpha) = \begin{cases} \{\alpha\}^{\bot} & \text{if } \alpha \text{ is a subformula of } A \\ \{\alpha^{\bot}\}^{\bot\bot} & \text{if } \alpha^{\bot} \text{ is a subformula of } A \\ & \text{but } \alpha \text{ is not a subformula of } A. \end{cases}$$

Lemma 4.7. *For every formula A, we have that $[\![A]\!] \subseteq \{A\}^{\bot}$ in $\mathcal{M}_A^{\mathcal{L}}$.*

Lemma 4.8. *If $\Gamma = A_1, \dots, A_n$, where all the A_i are subformulae of A and $\vdash \Gamma$ is provable in $\mathcal{GLAL}$, then all the phase models for $\mathbf{LAL}_A$ verify $\vdash \Gamma$.*

Proof. Assume, by way of contradiction, that a phase model for $\mathcal{GLAL}_A$ exists that does not verify $\vdash \Gamma$. Then, we could easily obtain a phase model for $\mathcal{GLAL}$ that does not verify $\vdash \Gamma$, too: it suffice to extend σ to atoms not in A in arbitrary way. By strong completeness, however, $\vdash \Gamma$ could not be provable in $\mathcal{GLAL}$, and this clearly does not agree with the hypothesis, because if $\vdash \Gamma \in \Xi_{\mathcal{GLAL}_A}$ then $\vdash \Gamma \in \Xi_{\mathcal{GLAL}}$. □

We can then give a result that strongly links $\mathcal{GLAL}_A$ to $\mathcal{GLAL}$:

Theorem 4.9. *If A is a formula, then the following three conditions are equivalent*:
(i) *$\vdash A$ is provable in $\mathcal{GLAL}_A$;*
(ii) *$\vdash A$ is provable in $\mathcal{GLAL}$;*
(iii) *$\mathcal{M}_A^{\mathcal{L}}$ verifies A.*

4.2.2. Exploiting logical congruences

At this point, we need to analyse how our phase models for $\mathbf{LAL}_A$ behave with respect to *logical congruences* on the underlying phase space.

Given a logical congruence $\sim$ on the phase space $(M, \bot)$ and a phase model $\mathcal{M} = (M, \bot, \phi, \xi, \sigma)$ for $\mathbf{LAL}_A$, the quintuple $\mathcal{M}/\sim = (M^{\sim}, \bot^{\sim}, \phi^{\sim}, \xi^{\sim}, \sigma^{\sim})$ is defined as follows:

- $M^{\sim}$ is the *quotient* monoid of M with respect to $\sim$;
- $\bot^{\sim}$ is the subset $\pi(\bot)$ of $M^{\sim}$;
- $\sigma^{\sim}$ is defined from σ by letting $\sigma^{\sim}(\alpha) = \pi(\sigma(\alpha))$.

Lemma 4.10. *If $\sim$ is a logical congruence, $\mathcal{M}$ is a phase model for $\mathbf{LAL}_A$, then $\mathcal{M}/\sim$ is a phase model, too.*

Proof. The only interesting facts to be verified are the properties of $\xi^{\sim}$ and $\phi^{\sim}$. Now:

$$\begin{aligned}
\xi^{\sim}([x]_\sim)\xi^{\sim}([y]_\sim) &= \{\pi(z)\pi(w) \mid z \in \xi([x]_\sim), w \in \xi([y]_\sim)\}\\
&= \{\pi(v) \mid v \in \xi([x]_\sim)\xi([y]_\sim)\}\\
&\subseteq \{\pi(v) \mid v \in \xi([x]_\sim[y]_\sim)\}\\
&= \xi^{\sim}([x]_\sim[y]_\sim).
\end{aligned}$$

Moreover, if $x \in M$, then

$$\begin{aligned}
\phi^{\sim}([x]_\sim) &= \phi(\pi(\pi^{-1}([x]_\sim)))\\
&= \pi(\phi(\pi^{-1}([x]_\sim)))\\
&\subseteq \pi(\xi(\pi^{-1}([x]_\sim))^{\bot\bot})\\
&= \pi(\xi(\pi^{-1}([x]_\sim)))^{\bot\bot}\\
&= \xi^{\sim}([x]_\sim)^{\bot\bot}.
\end{aligned}$$

This, by Lemma 3.5, implies that $\phi^{\sim}$ is bounded by $\xi^{\sim}$. Now, notice that

$$\begin{aligned}
x \in \{x^2\}^{\bot\bot} &\Rightarrow [x]_\sim \in \pi(\{x^2\}^{\bot\bot})\\
&\Rightarrow [x]_\sim \in \{\pi(x^2)\}^{\bot\bot}\\
&\Rightarrow [x]_\sim \in \{[x]_\sim[x]_\sim\}^{\bot\bot}.
\end{aligned}$$

This means that $\phi^{\sim}$ ranges over $J(M^{\sim})$. All the other conditions can be trivially verified. □

We can now give an essential result.

Proposition 4.11. *If A is a formula in $\mathbf{LAL}_B$ (where A is a subformula of B), $\sim$ is a logical congruence on $(M, \bot)$ and $\mathcal{M} = (M, \bot, \phi, \xi, \sigma)$ is a phase model for $\mathbf{LAL}_B$, then*:

(i) $\pi([\![A]\!]_{\mathcal{M}}) = [\![A]\!]_{\mathcal{M}/\sim}$;
(ii) $1_M \in [\![A]\!]_{\mathcal{M}} \Leftrightarrow 1_{M^\sim} \in [\![A]\!]_{\mathcal{M}/\sim}$;
(iii) *$\mathcal{M}$ verifies A iff $\mathcal{M}/\sim$ verifies A.*

Proof. If we prove the first of the three claims, the proof is finished, because the other two can be easily deduced from the first. We can proceed by induction on the structure of A, and the only interesting inductive cases are those involving ϕ and ξ. But the results on Lemmas 3.7, 3.9 and 3.10 give us exactly what we need. □

At this point, we can give an essential result.

Theorem 4.12. *Let $\sim$ be a logical congruence of finite index on $(M_A^{\mathcal{L}}, \bot_A^{\mathcal{L}})$. Then the following five conditions are equivalent*:
(i) $\vdash A$ *is provable in* **LAL**.;
(ii) *All finite phase models for* **LAL** *verify* A;
(iii) $\mathcal{M}_A^{\mathcal{L}}/\sim$ *verifies* A;
(iv) $\mathcal{M}_A^{\mathcal{L}}$ *verifies* A;
(v) $\vdash A$ *is cut-free provable in* **LAL**.

Proving that **LAL** enjoys the finite model property involves finding a logical congruence that fits the conditions of the previous theorem. But, by results on Section 4.2.1, $\equiv$ is a good candidate:

Proposition 4.13. *If $(M_A^{\mathcal{L}}, \bot_A^{\mathcal{L}}, \phi_A^{\mathcal{L}}, \sigma_A^{\mathcal{L}})$ is the syntactical model for* $\mathbf{LAL}_A$, *then every fact $X \subseteq M_A^{\mathcal{L}}$ is an ideal.*

Proof. Let us suppose that $X \subseteq M_A^{\mathcal{L}}$ is a fact. Then $X = Y^\bot$ for Y a subset of $M_A^{\mathcal{L}}$. Let now x be an element of $XM_A^{\mathcal{L}}$; it is obvious that $x = yz$, where $y \in X = Y^\bot$ and $z \in M_A$. At this point, let w be an element of Y and let us show that $xw \in \bot$. By definition, $yw \in \bot$ and then, by the definition of a light affine phase space, $yw \in (M_A^{\mathcal{L}})^\bot$. We can then conclude that $xw = (yz)w = (yw)z \in \bot$. □

Theorem 4.14. *The problem $\partial(\mathcal{GLAL})$ is decidable.*

Proof. **LAL** enjoys the subformula property, in view of its cut-elimination. From this, we obtain a semidecision procedure for $\partial(\mathcal{GLAL})$. On the other hand, from the finite model property for **LAL** we obtain a semidecision procedure for the complement of this set. □

4.3. Comparison with previous work

The idea of using some kind of morphism (on the underlying monoid) to capture the semantic of modalities goes back to [12], where it has been used to model **LLL** through *fibred phase spaces*; note that § is not a self-dual connective in [12], too. Our definition is simpler than the notion of fibred phase spaces, in that we let the sequence of phase spaces in a fibred phase space collapse to just one. This choice could be regarded as a useless restriction, but it makes easier to reach the goal of proving decidability of **LAL**. From another perspective, we get more general structures than fibred phase spaces, since we shift from single-valued morphisms (functions) to multi-value morphism (binary relations). Other constraints (such as the *intermediate value property*) are not needed in our definition, reflecting the simpler structure of **LAL** with respect to **LLL** [1]. The strong constraint $\bot \subseteq M^\bot$ models affinity, as in [14] for **LLW**.

Our models are similar to those proposed by Terui [21] in the context of **ILAL**, the intuitionistic variant of Light Affine Logic. Terui proposed two classes of models. The first one is slightly less general than ours, exponential connectives being interpreted by way of usual morphisms on the underlying monoid. This class can be used to prove strong completeness, but is not closed under the quotient model construction. In particular $x \sim y$ does not necessarily imply $\xi(x) \sim \xi(y)$, meaning that $\xi^\sim$ can be multi-valued even if ξ is a function. This problem cannot be circumvented by merely adding additional conditions on the definition of a model. Indeed, even the syntactical model $\mathcal{M}^{\mathcal{L}}$ exhibits this behaviour; as an example, $A, B \equiv A \parr B$, while $\overline{\S}A, \overline{\S}B \not\equiv \overline{\S}(A \parr B)$.

To prove decidability of **ILAL**, Terui introduces *generalized models*, where exponential connectives are interpreted by functions on $\mathcal{P}(M)$ (M being the underlying monoid); these objects must satisfy a number of remarkable conditions, monotonicity *in primis*. Every phase model for **LAL**, as we have defined it, is actually a generalized model in the sense of Terui, once relations are interpreted as powerset functions. The converse does not hold, since one can easily build a generalized model where $\overline{\S}$ is interpreted by way of a function $\Theta : \mathcal{P}(M) \to \mathcal{P}(M)$ where $\Theta(\emptyset) \neq \emptyset$. This function is not induced by any binary relation on M.

Our class of models is closed under the quotient construction and, at the same time, does not involve higher-order structures, such as powerdomains. Moreover, this definition smoothly scales to **EAL** and **SLL**, as we will see in the next two sections.

5. Elementary affine logic

An *elementary affine phase space* is a triple $(M, \bot, \phi)$ where:

- $(M, \bot)$ is a phase space;
- $\phi \subseteq M \times M$ is a relational monoid homomorphism such that $\phi(M) \subseteq J(M)$;
- $\bot \subseteq M^\bot$.

Proposition 5.1. *If* $(M, \bot, \phi)$ *is an affine elementary phase space and* $X \subseteq M$ *is a fact, then* $\bot \subseteq X$.

A *phase model for* **EAL** is a quadruple $(M, \bot, \phi, \sigma)$, where $(M, \bot, \phi)$ is an elementary affine phase space and the *interpretation* σ maps every atom α in $\mathcal{L}$ to a fact $\sigma(\alpha) \subseteq M$. The interpretation of non-exponential formulae (with respect to a phase model) is defined as for **MALL**; for exponential formulae we have:

$$[\![!A]\!]_{\mathcal{M}} = (\phi([\![A]\!]_{\mathcal{M}}))^{\bot\bot}$$
$$[\![?A]\!]_{\mathcal{M}} = (\phi([\![A]\!]_{\mathcal{M}}^{\bot}))^{\bot}.$$

5.1. Strong completeness

The *syntactical model for* **EAL** is the quadruple $\mathcal{M}^{\mathcal{A}} = (M^{\mathcal{A}}, \perp^{\mathcal{A}}, \phi^{\mathcal{A}}, \sigma^{\mathcal{A}})$ where:

- $M^{\mathcal{A}}$ is the commutative monoid generated by all formulae of **EAL**; this structure is isomorphic to the set of $\mathcal{GEAL}$ sequents (with juxtaposition);
- $\perp^{\mathcal{A}}$ is the set of all cut-free provable sequents in $\mathcal{GEAL}$;
- $\phi^{\mathcal{A}}$ is defined by imposing $\phi^{\mathcal{A}}(A_1 \dots A_n) = \{?A_1 \dots ?A_n\}$ for every sequence $A_1, \dots, A_n$ of **EAL** formulae and $n \geqslant 0$;
- $\sigma^{\mathcal{A}}$ is defined by $\sigma^{\mathcal{A}}(a) = \{a\}^{\perp}$ for any $a \in \mathcal{L}$.

Lemma 5.2. *$\mathcal{M}^{\mathcal{A}}$ is a phase model for* **EAL**.

Proof. We only check that $\perp \subseteq M^{\perp}$. If $\Gamma \in \perp$ then, by definition, $\vdash \Gamma$ is cut-free provable in $\mathcal{GEAL}$. If, now, $\vdash \Delta$ is an arbitrary sequent in M, it is clear that $\vdash \Gamma, \Delta$ is cut-free provable in $\mathcal{GEAL}$, because the following is a valid $\mathcal{GEAL}$ deduction

$$\dfrac{\overline{\vdash \Gamma}}{\vdash \Gamma, \Delta}\ W_{\mathcal{GEAL}}$$

As an immediate consequence, $\perp \subseteq M^{\perp}$. □

On this class of phase models, we can give soundness and completeness results in the same way as we did for **LAL**.

Lemma 5.3 (Okada). *For every* **EAL** *formula A, we have that $[\![A]\!] \subseteq \{A\}^{\perp}$ in $\mathcal{M}^{\mathcal{A}}$.*

Proof. We can prove this by a structural induction on A. The only two interesting inductive cases are the following:

- If $A = !B$ then, by inductive hypothesis, $[\![B]\!] \subseteq \{B\}^{\perp}$ and so $\phi([\![B]\!]) \subseteq \phi(\{B\}^{\perp})$. Now, if $\Gamma B \in \perp$, then, by rule $S_{\mathcal{GEAL}}$, $?\Gamma !B \in \perp$; this means that $\phi(\{B\}^{\perp}) \subseteq \{!B\}^{\perp}$. Then, we can conclude that $\phi([\![B]\!])^{\perp\perp} \subseteq \{!B\}^{\perp}$, proving the inclusion $[\![A]\!] \subseteq \{A\}^{\perp}$.
- If $A = ?B$, we can observe that, by inductive hypothesis, $[\![B]\!] \subseteq \{B\}^{\perp}$ and then $[\![B]\!]^{\perp} \supseteq \{B\}^{\perp\perp}$; this, in particular, yields $\phi([\![B]\!]^{\perp}) \supseteq \phi(\{B\}^{\perp\perp})$. Obviously $B \in \{B\}^{\perp\perp}$ and then $\phi(\{B\}^{\perp\perp}) \supseteq \{?B\}$. We can then infer the inclusion $\phi([\![A]\!]^{\perp})^{\perp} \subseteq \{A\}^{\perp}$.

All other cases can be proved exactly as for **MALL** (see, for example, [14]). □

Lemma 5.4. *If $\vdash \Gamma$ is provable in* **EAL**, *then $\vdash \Gamma$ is verified in all phase models for* **EAL**.

Proof. We can proceed exactly as for $\mathcal{GLAL}$ (see Lemma 4.5). If π is a proof of $\vdash \Gamma$ (which must exist by hypothesis), we proceed by induction on the structure of π. But the only inductive case which asks for an argument different from the ones used for $\mathcal{GLAL}$ is the one involving the $P_{\mathcal{GEAL}}$ rule; in this case:

$$\begin{aligned} 1 \in [\![\vdash A_1 \dots A_n, B]\!] &\Rightarrow 1 \in ([\![A_1]\!]^{\perp} \dots [\![A_n]\!]^{\perp} [\![B]\!]^{\perp})^{\perp} \\ &\Rightarrow 1 \in [\![A_1]\!]^{\perp} \dots [\![A_n]\!]^{\perp} \multimap [\![B]\!] \end{aligned}$$

$$
\begin{aligned}
&\Rightarrow [\![A_1]\!]^\perp \dots [\![A_n]\!]^\perp \subseteq [\![B]\!]\\
&\Rightarrow \phi([\![A_1]\!]^\perp)\dots\phi([\![A_n]\!]^\perp) \subseteq \phi([\![B]\!])\\
&\Rightarrow \phi([\![A_1]\!]^\perp)\dots\phi([\![A_n]\!]^\perp) \subseteq \phi([\![B]\!])^{\perp\perp}\\
&\Rightarrow 1 \in (\phi([\![A_1]\!]^\perp)\dots\phi([\![A_n]\!]^\perp)\phi([\![B]\!])^\perp)^\perp\\
&\Rightarrow 1 \in (\phi([\![A_1]\!]^\perp)^{\perp\perp}\dots\phi([\![A_n]\!]^\perp)^{\perp\perp}\phi([\![B]\!])^{\perp\perp\perp})^\perp\\
&\Rightarrow 1 \in ([\![?A_1]\!]^\perp\dots[\![?A_n]\!]^\perp[\![!B]\!]^\perp)^\perp\\
&\Rightarrow 1 \in [\![\vdash ?A_1,\dots,?A_n,!B]\!].
\end{aligned}
$$

This concludes the proof. □

Theorem 5.5 (Strong completeness). *Let A be a formula. The following four conditions are then equivalent*:
(i) *$\vdash A$ is provable in* **EAL**;
(ii) *Every phase model for* **EAL** *verifies A*;
(iii) *$\mathcal{M}^{\mathcal{A}}$ verifies A*;
(iv) *$\vdash A$ is cut-free provable in* **EAL**.

5.2. Decidability

Like **LAL**, **EAL** enjoys the finite model property. In this section, we will prove that **EAL** provability is decidable, following again [14] in building a phase model whose underlying monoid is finitely generated.

Remark 5.6. One referee suggested that the decidability of **EAL** could be obtained by reduction to the decidability of **LAL**. Indeed, let $[\cdot]$ be the translation from **EAL** formulae to **LAL** formulae defined as $[!A] = \S A$ and leaving all other connectives (? included) unchanged. It is easy to see that a formula A is cut-free **EAL**-provable iff $[A]$ is cut-free **LAL**-provable, see [5]. The cut-elimination theorem (which we derived in 5.5) allows to conclude.

If A is a formula, we will denote as $\mathbf{EAL}_A$ the logic obtained by restricting our logical language to subformulae of A; similarly we can denote the restriction of $\mathcal{GEAL}$ to subformulae of A as $\mathcal{GEAL}_A$. The notions of a phase model and of verifiability by a phase model can be easily extended to $\mathbf{EAL}_A$.

The *syntactic model for $\mathcal{GEAL}_A$* is the quadruple $\mathcal{M}_A^{\mathcal{A}} = (M_A^{\mathcal{A}}, \perp_A^{\mathcal{A}}, \phi_A^{\mathcal{A}}, \sigma_A^{\mathcal{A}})$, where:
- $M_A^{\mathcal{A}}$ is the free monoid generated by all the subformulae of A;
- $\perp_A^{\mathcal{A}}$ is the set of all cut-free provable $\mathcal{GEAL}_A$ sequents;
- $\phi_A^{\mathcal{A}}$ is defined as follows:

$$
\phi_A^{\mathcal{A}}(A_1 \dots A_n) = \begin{cases} \{?A_1 \dots ?A_n\} & \text{if all the } ?A_i \text{ are subformulae of } A\\ \emptyset & \text{otherwise;}\end{cases}
$$

- $\sigma_A^{\mathcal{A}}$ is defined as follows:

$$\sigma_A^{\mathcal{A}}(\alpha) = \begin{cases} \{\alpha\}^{\perp} & \text{if } \alpha \text{ is a subformula of } A \\ \{\alpha^{\perp}\}^{\perp\perp} & \text{if } \alpha^{\perp} \text{ is a subformula of } A \\ & \text{but } \alpha \text{ is not a subformula of } A. \end{cases}$$

Lemma 5.7. *For every formula A, we have that $[\![A]\!] \subseteq \{A\}^{\perp}$ in $\mathcal{M}_A^{\mathcal{A}}$.*

Lemma 5.8. *If $\Gamma = A_1,\ldots,A_n$, where all the A_i are subformulae of A and $\vdash\Gamma$ is provable in $\mathcal{GEAL}$, then all the phase models for $\mathbf{EAL}_A$ verify $\vdash\Gamma$.*

Proof. Assume, by way of contradiction, that a phase model for $\mathcal{GEAL}_A$ exists that does not verify $\vdash\Gamma$. Then, we could easily obtain a phase model for $\mathcal{GEAL}$ that does not verify $\vdash\Gamma$, too. By Theorem 5.5, $\vdash\Gamma$ could not be provable in $\mathcal{GEAL}$, but this clearly does not agree with the hypothesis, because if $\vdash\Gamma \in \Xi_{\mathcal{GEAL}_A}$ then $\vdash\Gamma \in \Xi_{\mathcal{GEAL}}$. □

We can then give a result that strongly links $\mathcal{GEAL}$ and $\mathcal{GEAL}_A$:

Theorem 5.9. *If A is a formula, then the following three conditions are equivalent*:
(i) *$\vdash A$ is provable in $\mathcal{GEAL}_A$*;
(ii) *$\vdash A$ is provable in $\mathcal{GEAL}$*;
(iii) *$\mathcal{M}_A^{\mathcal{A}}$ verifies A.*

Given a logical congruence $\sim$ on the phase space $(M,\perp)$ and a phase model $\mathcal{M} = (M,\perp,\phi,\sigma)$ for $\mathbf{EAL}_A$, the quadruple $\mathcal{M}/\!\sim\; = (M^{\sim},\perp^{\sim},\phi^{\sim},\sigma^{\sim})$ is defined as follows:
- $M^{\sim}$ is the *quotient* monoid of M with respect to $\sim$;
- $\perp^{\sim}$ is the subset $\pi(\perp)$ of $M^{\sim}$;
- $\sigma^{\sim}$ is defined from σ by letting $\sigma^{\sim}(a) = \pi(\sigma(s))$.

Lemma 5.10. *If $\mathcal{M} = (M,\perp,\phi,\sigma)$ is a phase model for $\mathbf{EAL}_A$, then $\mathcal{M}/\!\sim$ is a phase model for $\mathbf{EAL}_A$.*

Lemma 5.11. *If A is a formula in $\mathbf{EAL}_B$ (where A is a subformula of B), $\sim$ is a logical congruence on $(M,\perp)$ and $\mathcal{M} = (M,\perp,\phi,\sigma)$ is a phase model for $\mathbf{EAL}_B$, then*:
(i) $\pi([\![A]\!]_{\mathcal{M}}) = [\![A]\!]_{\mathcal{M}/\sim}$;
(ii) $1_M \in [\![A]\!]_{\mathcal{M}} \Leftrightarrow 1_{M^{\sim}} \in [\![A]\!]_{\mathcal{M}/\sim}$;
(iii) *$\mathcal{M}$ verifies A iff $\mathcal{M}/\!\sim$ verifies A.*

At this point, we can give a result similar to Theorem 4.12.

Theorem 5.12. *Let $\sim$ be a logical congruence of finite index on $(M_A^{\mathcal{A}},\perp_A^{\mathcal{A}})$, such that $\mathcal{M}_A^{\mathcal{A}} = (M_A^{\mathcal{A}},\perp_A^{\mathcal{A}},\phi_A^{\mathcal{A}},\sigma_A^{\mathcal{A}})$ respects $\sim$. Then the following five conditions are*

equivalent:
(i) $\vdash A$ *is provable in* **EAL**.;
(ii) *All finite phase models for* **EAL** *verify* A;
(iii) $\mathcal{M}_A^{\mathcal{A}}/\sim$ *verifies* A;
(iv) $\mathcal{M}_A^{\mathcal{A}}$ *verifies* A;
(v) $\vdash A$ *is cut-free provable in* **EAL**.

Proving that **EAL** enjoys the finite model property involves finding a logical congruence that fits the conditions of the previous theorem. Once again, $\equiv$ can be fruitfully used in this context:

Proposition 5.13. *If* $(M_A^{\mathcal{A}}, \perp_A^{\mathcal{A}}, \phi_A^{\mathcal{A}}, \sigma_A^{\mathcal{A}})$ *is the syntactical model for* $\mathbf{EAL}_A$, *then every fact* $X \subseteq M_A^{\mathcal{A}}$ *is an ideal.*

Finally, we can give the result we have anticipated at the beginning of this section.

Theorem 5.14. *The problem* $\partial(\mathcal{GEAL})$ *is decidable.*

6. Soft linear logic

6.1. Phase semantics

A *soft phase space* is a triple $(M, \perp, \phi)$ such that
- $(M, \perp)$ is a phase space;
- $\phi \subseteq M \times M$ is a relational monoid homomorphism such that $(X^n)^\perp \subseteq \phi(X)^\perp$ for every $X \subseteq M$ and every natural number n.

A *phase model for* **SLL** is a quadruple $(M, \perp, \phi, \sigma)$ where $(M, \perp, \phi)$ is a soft phase space and the *intepretation* σ maps every atom α of $\mathcal{L}$ to a fact $\sigma(\alpha) \subseteq M$. Given a phase model $\mathcal{M} = (M, \perp, \phi, \sigma)$ for **SLL**, we can associate to every formula A in **SLL** and to every sequent $\vdash \Gamma$ in $\mathcal{GSLL}$ a fact as we have previously done for **EAL**, with the same interpretation for exponential formulae:

$$[\![!A]\!]_{\mathcal{M}} = (\phi([\![A]\!]_{\mathcal{M}}))^{\perp\perp}$$
$$[\![?A]\!]_{\mathcal{M}} = (\phi([\![A]\!]_{\mathcal{M}}^{\perp}))^{\perp}.$$

The *syntactical model for* **SLL** is the quadruple $\mathcal{M}^{\mathcal{S}} = (M^{\mathcal{S}}, \perp^{\mathcal{S}}, \phi^{\mathcal{S}}, \sigma^{\mathcal{S}})$ defined as follows:
- $M^{\mathcal{S}}$ is the commutative monoid generated by all formulae of **SLL**; this structure is isomorphic to the set of all $\mathcal{GSLL}$ sequents (with juxtaposition);
- $\perp^{\mathcal{S}}$ is the set of all cut-free provable sequents in **SLL**;
- $\phi^{\mathcal{S}}$ is defined by letting $\phi^{\mathcal{S}}(A_1 \dots A_n) = \{?A_1 \dots ?A_n\}$ for every sequence $A_1 \dots A_n$ of formulae in **SLL** and $n \geqslant 0$;
- $\sigma^{\mathcal{S}}$ is defined, as usual, by putting $\sigma^{\mathcal{S}}(\alpha) = \{\alpha\}^\perp$ for every $\alpha \in \mathcal{L}$.

Lemma 6.1. $\mathcal{M}^{\mathcal{S}}$ *is a phase model for* $\mathcal{GSLL}$.

Proof. The only interesting thing to prove is the condition on $\phi^{\mathcal{S}}$. But let $A \subseteq M^{\mathcal{S}}$ and let $\Gamma \in (A^n)^{\perp}$, $\Delta \in \phi(A)$; this means, in particular, that $\Delta = ?\Lambda$, for Λ being an element of A; by definition, $\Gamma, \Lambda^n \in \perp$ and, by rule $M_{\mathcal{GSLL}}$, $\Gamma, \Delta \in \perp$. This concludes the proof. □

Lemma 6.2 (Okada). *For every formula A in* **SLL**, *we have that* $[\![A]\!] \subseteq \{A\}^{\perp}$ *in* $\mathcal{M}^{\mathcal{S}}$.

Proof. We can proceed exactly as for Lemma 5.3. □

Lemma 6.3. *If* $\vdash \Gamma$ *is provable in* **SLL**, *then every phase model for* **SLL** *verifies* $\vdash \Gamma$.

Proof. We can proceed, exactly as for **EAL**, by induction on the structure of π. The only interesting inductive cases are the following two:

- If the last rule applied is $M_{\mathcal{GSLL}}$, we can write

$$\begin{aligned} 1 \in [\![\vdash A^{(n)}, B]\!] &\Rightarrow 1 \in (([\![A]\!]^{\perp})^n [\![B]\!]^{\perp})^{\perp} \\ &\Rightarrow 1 \in ((([\![A]\!]^{\perp})^n)^{\perp\perp} [\![B]\!]^{\perp})^{\perp} \\ &\Rightarrow 1 \in (\phi([\![A]\!]^{\perp})^{\perp\perp} [\![B]\!]^{\perp})^{\perp} \\ &\Rightarrow 1 \in ([\![?A]\!]^{\perp} [\![B]\!]^{\perp})^{\perp} \\ &\Rightarrow 1 \in [\![\vdash ?A, B]\!]. \end{aligned}$$

- If the last rule applied is $S_{\mathcal{GSLL}}$, then it is sufficient to observe that

$$\begin{aligned} 1 \in [\![\vdash A_1 \ldots A_n, B]\!] &\Rightarrow 1 \in ([\![A_1]\!]^{\perp} \ldots [\![A_n]\!]^{\perp} [\![B]\!]^{\perp})^{\perp} \\ &\Rightarrow 1 \in [\![A_1]\!]^{\perp} \ldots [\![A_n]\!]^{\perp} \multimap [\![B]\!] \\ &\Rightarrow [\![A_1]\!]^{\perp} \ldots [\![A_n]\!]^{\perp} \subseteq [\![B]\!] \\ &\Rightarrow \phi([\![A_1]\!]^{\perp}) \ldots \phi([\![A_n]\!]^{\perp}) \subseteq \phi([\![B]\!]) \\ &\Rightarrow \phi([\![A_1]\!]^{\perp}) \ldots \phi([\![A_n]\!]^{\perp}) \subseteq \phi([\![B]\!])^{\perp\perp} \\ &\Rightarrow 1 \in (\phi([\![A_1]\!]^{\perp}) \ldots \phi([\![A_n]\!]^{\perp}) \phi([\![B]\!])^{\perp})^{\perp} \\ &\Rightarrow 1 \in (\phi([\![A_1]\!]^{\perp})^{\perp\perp} \ldots \phi([\![A_n]\!]^{\perp})^{\perp\perp} \phi([\![B]\!])^{\perp\perp\perp})^{\perp} \\ &\Rightarrow 1 \in ([\![?A_1]\!]^{\perp} \ldots [\![?A_n]\!]^{\perp} [\![!B]\!]^{\perp})^{\perp} \\ &\Rightarrow 1 \in [\![\vdash ?A_1, \ldots, ?A_n, !B]\!]. \end{aligned}$$

All the other cases can be proved exactly as we have done for **EAL**. □

Theorem 6.4 (Strong completeness). *Let A be an* **SLL** *formula. The following four conditions are then equivalent*:

(i) $\vdash A$ *is provable in* **SLL**;
(ii) *Every phase model for* **SLL** *verifies* A.
(iii) $\mathcal{M}^{\mathcal{S}}$ *verifies* A;
(iv) $\vdash A$ *is cut-free provable in* **SLL**;

6.2. Finite model property

SLL does not admit full weakening and, as a consequence, Lafont's argument cannot be applied to this logic. In this section, we will show that even the multiplicative fragment of **SLL** (which we call **MSLL**) does not enjoy the finite model property. Indeed, the counter example given in [14] for **MELL** works also for **SLL**. We will not give the definitions of phase models for this fragment, because they can be easily inferred from the ones for full **SLL**.

Proposition 6.5. *If* $\alpha, \beta \in \mathcal{L}$, *then the formula*

$$A = ?\alpha^{\perp} ⅋ ?(\alpha^{\perp} ⅋ \beta^{\perp}) ⅋ ?(\alpha \otimes \beta \otimes \perp) ⅋ \beta$$

is verified by every finite phase model for **SLL** *but is not provable.*

Proof. We can adapt the counterexample on the finite model property for **MELL** [14] with only minor variations. In the following, we will write B^n for

$$\underbrace{B \otimes \ldots \otimes B}_{n\ times}$$

If $(M, \perp, \phi, \sigma)$ is an arbitrary finite phase model, then there will be a finite number of subsets of M and, in particular, a finite number of facts; this naturally yields the existence of two natural numbers p and q with $p < q$, such that $[\![\beta^p]\!] = [\![\beta^q]\!]$. Let now B be the formula $(\beta^p)^{\perp} ⅋ \beta^q$; then

$$\begin{aligned} [\![B^{\perp}]\!] &= [\![((\beta^p)^{\perp} ⅋ \beta^q)^{\perp}]\!] = [\![\beta^p \otimes (\beta^q)^{\perp}]\!] \\ &= ([\![\beta^p]\!][\![(\beta^q)^{\perp}]\!])^{\perp\perp} \\ &= ([\![\beta^p]\!][\![\beta^q]\!]^{\perp})^{\perp\perp} \\ &\subseteq \perp. \end{aligned}$$

But $1 \in [\![?B^{\perp}]\!]^{\perp}$, because

$$\begin{aligned} [\![B^{\perp}]\!] \subseteq \perp &\Rightarrow [\![B^{\perp}]\!]^{\perp} \supseteq \perp^{\perp} \\ &\Rightarrow \phi([\![B^{\perp}]\!]^{\perp}) \supseteq \phi(\perp^{\perp}) \\ &\Rightarrow \phi([\![B^{\perp}]\!]^{\perp})^{\perp\perp} \supseteq \phi(\perp^{\perp})^{\perp\perp} \\ &\Leftrightarrow [\![?B^{\perp}]\!]^{\perp} \supseteq \phi(\perp^{\perp})^{\perp\perp} \end{aligned}$$

and, obviously,

$$\begin{aligned} 1 \in \phi(1) \subseteq \phi(\perp^{\perp}) &\Rightarrow \{1\}^{\perp\perp} \subseteq \phi(\perp^{\perp})^{\perp\perp} \\ &\Rightarrow 1 \in \phi(\perp^{\perp})^{\perp\perp}. \end{aligned}$$

On the other hand, it is clear that $\vdash ?B^{\perp} ⅋ A \in \Xi_{\mathcal{GELL}}$, because

$$?B^{\perp} ⅋ A \equiv ?(\beta^p \otimes (\beta^q)^{\perp}) ⅋ ?\alpha^{\perp} ⅋ ?(\alpha^{\perp} ⅋ \beta^{\perp}) ⅋ ?(\alpha \otimes \beta \otimes \perp) ⅋ \beta$$

and in $\mathcal{GSLL}$ the following three deductions are valid

$$\dfrac{}{\vdash \underbrace{\beta \otimes \cdots \otimes \beta}_{p\ times}, \underbrace{\beta^\perp, \ldots, \beta^\perp}_{p\ times}}$$

$$\dfrac{}{\vdash \underbrace{\beta^\perp, \ldots, \beta^\perp}_{q\ times}, \underbrace{\alpha^\perp, \ldots, \alpha^\perp}_{q-1\ times}, \underbrace{\alpha \otimes \beta \otimes \bot, \ldots, \alpha \otimes \beta \otimes \bot}_{q-1\ times}, \beta}$$

$$\dfrac{\vdash \beta^p \otimes (\overbrace{\beta^\perp ⅋ \ldots ⅋ \beta^\perp}^{q\ times}), \overbrace{\alpha^\perp, \ldots, \alpha^\perp}^{q-1\ times}, \overbrace{\beta^\perp, \ldots, \beta^\perp}^{p\ times}, \overbrace{\alpha \otimes \beta \otimes \bot, \ldots, \alpha \otimes \beta \otimes \bot}^{q-1\ times}, \beta}{\vdash ?(\beta^p \otimes (\beta^q)^\perp) ⅋ ?\alpha^\perp ⅋ ?(\alpha^\perp ⅋ \beta^\perp) ⅋ ?(\alpha \otimes \beta \otimes \bot) ⅋ \beta}$$

By Theorem 6.4, M verifies $\vdash ?B^\perp ⅋ A$ and so even $\vdash A$, because $[\![\vdash ?B^\perp ⅋ A]\!]$ and $[\![?B^\perp]\!]^\perp$ both contain 1_M and, moreover, $[\![\vdash ?B^\perp ⅋ A]\!] = ([\![?B^\perp]\!]^\perp [\![A]\!]^\perp)^\perp = [\![?B^\perp]\!]^\perp \multimap [\![A]\!]$. The only thing that remains to be proved is the fact that $\vdash A$ is not provable in $\mathcal{GSLL}$. But if $\vdash A$ were provable, then it would we provable in the sequent calculus for **MELL**. By a result given in [14], however, $\vdash A$ is not provable in **MELL**. □

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References

[1] A. Asperti, Light affine logic, in: Proc. 13th Annual IEEE Symp. on Logic in Computer Science, 1998, pp. 300–308.
[2] A. Asperti, P. Coppola, S. Martini, (Optimal) duplication is not elementary recursive, in: 27th ACM Conf. on Principles of Programming Languages, ACM Press, New York, 2000, pp. 96–107.
[3] A. Asperti, S. Guerrini, Optimal Implementation of Functional Programming Languages, Cambridge University Press, Cambridge, 1998.
[4] A. Asperti, L. Roversi, Intuitionistic light affine logic, ACM Trans. Comput. Logic 3 (1) (2002) 137–175.
[5] P. Baillot, Checking polynomial time complexity with types, in: Proc. 2nd IFIP Internat. Conf. on Theoretical Computer Science, 2000, pp. 370–382.
[6] P. Coppola, S. Martini, Typing lambda terms in elementary logic with linear constraints, in: Proc. 6th Internat. Conf. on Typed Lambda-Calculus and Appl., 2001, pp. 76–90.
[7] P. Coppola, S. Ronchi della Rocca, Principal typing in elementary affine logic, in: Proc. 7th Internat. Conf. on Typed Lambda-Calculus and Applications 2003, pp. 90–104.
[8] V. Danos, J.-B. Joinet, Linear logic & elementary time, Inform. and Comput. 183 (1) (2003) 123–137.
[9] J.-Y. Girard, Linear logic, Theoret. Comput. Sci. 50 (1987) 1–102.
[10] J.-Y. Girard, Light linear logic, Inform. and Comput. 143 (1) (1998) 175–204.
[11] J.-Y. Girard, A. Scedrov, P.J. Scott, Bounded linear logic: a modular approach to polynomial time computability, Theoret. Comput. Sci. 97 (1992) 1–66.
[12] M.I. Kanovich, M. Okada, A. Scedrov, Phase semantics for light linear logic, Theoret. Comput. Sci. 294 (2003) 525–549.

[13] A.P. Kopylov, Decidability of linear affine logic, Inform. and Comput. 164 (2001) 173–198, Expanded version of a paper in the Proc. 10th Annual Symp. on Logic in Computer Science, 1995, pp. 496–504.
[14] Y. Lafont, The finite model property for various fragments of linear logic, J. Symbolic Logic 62 (1997) 1202–1208.
[15] Y. Lafont, Soft linear logic and polynomial time, Theoret. Comput. Sci. (2004), in press.
[16] P. Lincoln, J. Mitchell, A. Scedrov, N. Shankar, Decision problems for propositional linear logic, Ann. Pure Appl. Logic 56 (1992) 239–311.
[17] M. Okada, Phase semantics for higher order completeness, cut-elimination and normalization proofs, Electron. Notes in Theoret. Comput. Sci. 3 (1996), Special Issue on Linear Logic 96, Tokyo Meeting.
[18] M. Okada, Phase semantics cut-elimination and normalization proofs of first and higher order linear logic, Theoret. Comput. Sci. 227 (1999) 333–396.
[19] M. Okada, A uniform semantic proof for cut-elimination and completeness of various first and higher order logics, Theoret. Comput. Sci. 281 (2002) 471–498.
[20] M. Okada, K. Terui, The finite model property for various fragments of intuitionistic linear logic, J. Symbolic Logic 64 (2) (1999) 790–802.
[21] K. Terui, Light Logic and Polynomial Time Computation, Ph.D. Thesis, Faculty of Letters, Keio University, 2002.