Discrete Applied Mathematics 17 (1987) 255-261
North-Holland

BOUNDS FOR THE SOLUTION SET OF LINEAR COMPLEMENTARITY PROBLEMS*

Panos M. PARDALOS
Computer Science Department, The Pennsylvania State University, University Park, PA 16802, USA

J.B. ROSEN
Computer Science Department, University of Minnesota, Minneapolis, MN 55455, USA

Received 13 January 1986
Revised 2 May 1986

We give here bounds for the feasible domain and the solution norm of Linear Complementarity Problems (LCP). These bounds are motivated by formulating the LCP as a global quadratic optimization problem and are characterized by the eigenstructure of the corresponding matrix. We prove boundedness of the feasible domain when the quadratic problem is concave, and give easily computable bounds for the solution norm for the convex case. We also obtain lower and upper bounds for the solution norm of the general nonconvex problem.

Keywords. Linear complementarity problem, quadratic programming, bounds.

Introduction

We are concerned here with some properties of the linear complementarity problem, that has the following form:

Find $x \in \mathbb{R}^n$ such that

$$Mx+q \geq 0, \qquad x \geq 0, \qquad x^{\mathrm{T}}Mx+q^{\mathrm{T}}x=0 \tag{LCP}$$

(or prove that such an x does not exist) where $M_{n\times n}$ is a real matrix and $q \in \mathbb{R}^n$. For given M and q the problem is generally denoted by LCP(M, q).

This problem has many important applications in science and technology including fluid flow, economic equilibrium analysis, and numerical solutions of differential equations. For a review and many references regarding these applications see [1] and [7].

If the LCP is solvable, then it has a solution that occurs at some vertex of the associated polyhedral set $S=\{x: Mx+q\geq 0, x\geq 0\} \subseteq \mathbb{R}^n$. An important question of both practical and theoretical interest is the boundedness of the solution set or of

*This research was supported in part by the Division of Computer Research, National Science Foundation under Research Grant DC8405489.

the feasible domain S. Attempts to answer this question can be found in [5], [11]. More recently, Mangasarian [8] obtained some easily computable bounds for the positive semidefinite LCP. For problems with unbounded feasible domain S or unbounded solution set see [3] and [4].

In this note we present simple numerical bounds for the cases where the matrix $\bar{M}=M+M^{\mathrm{T}}$ is positive or negative definite, then generalize for the indefinite case. In particular we show that when $\bar{M}$ is negative definite, then the corresponding feasible domain S is always bounded, and when $\bar{M}$ is positive definite, then the solution set is also contained in an easily computed rectangle (that may be tight in some cases). Most of the results are motivated by formulating LCP as a global optimization problem [10].

Global optimization formulation

The possibility of formulating the LCP as an equivalent constrained quadratic problem, was first discussed in [2]. This quadratic problem has the form:

$$\operatorname*{global\,min}_{x\in S} \phi(x)=\tfrac{1}{2}x^{\mathrm{T}}(M+M^{\mathrm{T}})x+q^{\mathrm{T}}x \tag{GP}$$

and $S=\{x\colon x\in\mathbb{R}^n,\ Mx+q\geq 0,\ x\geq 0\}$.

In this formulation a lower bound is known a priori, that is $\phi(x)\geq 0$. If the LCP has a finite number of solutions, then all solutions are vertices of S. If there are an infinite number of solutions, one occurs at some vertex of S. If $v\in S$ is such a solution vertex, then

$$\operatorname*{global\,min}_{x\in S} \phi(x)=\phi(v)=0.$$

Because of this requirement, the LCP may have no solution even if the feasible domain is nonempty.

Example. Let

$$M=\begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}, \qquad q^{\mathrm{T}}=(-1,\ -1).$$

Then LCP(M,q) has a nonempty feasible domain S, but has no solution since the global minimum occurs at the vertex $v=(0,1)$ with $\phi(v)=1>0$. The problem of characterizing LCPs with a nonempty feasible domain that have a solution remains a difficult one.

The set S may also be unbounded ([3], [4], [6]). Since the solutions occur at vertices of S and since we may have only finitely many of them, the set of vertices is contained in some bounded rectangle [9, p., 30]. We need numerical bounds that are easily computable and practical for computational considerations.

Boundedness of feasible domain and solution norm of LCP

The matrix $\bar{M}$ is symmetric and therefore all its eigenvalues are real numbers. In what follows we assume $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, and use the euclidean norm.

Theorem 1. *If $\bar{M}$ is negative definite, then the feasible domain of the corresponding LCP is bounded. In fact, it is contained in the rectangle*

$$\{x: 0 \le x_i \le 2\|q\|/|\lambda_n|,\ i=1,\dots,n\}.$$

Proof. We have

$$\phi(x) = \tfrac{1}{2}x^{\mathrm{T}}\bar{M}x + q^{\mathrm{T}}x \le \tfrac{1}{2}\lambda_n\|x\|^2 + \|q\| \cdot \|x\| = (\tfrac{1}{2}\lambda_n\|x\| + \|q\|)\|x\|.$$

Since feasibility requires $\phi(x) \ge 0$, x is infeasible if $\phi(x) < 0$, that is if $\frac{1}{2}\lambda_n\|x\| + \|q\| < 0$. Therefore x is infeasible if $\|x\| > 2\|q\|/|\lambda_n|$, so that the feasible domain must be contained in the rectangle $\{x: 0 \le x_i \le 2\|q\|/|\lambda_n|\}$.

Remarks. When $\bar{M}$ is negative semidefinite (has at least one zero eigenvalue), the corresponding feasible domain S may be bounded or unbounded, as shown by the examples:

(a) The feasible domain S of the LCP with the following data is unbounded.

$$M = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad q^{\mathrm{T}} = (-1, 2).$$

In this example, the matrix $\bar{M}$ has eigenvalues $\lambda_2 = -2$, $\lambda_1 = 0$

(b) The next example shows that although $\bar{M}$ may have zero eigenvalues, S can be bounded.

$$M = \begin{bmatrix} -1 & 0 \\ 4 & -4 \end{bmatrix}, \qquad q^{\mathrm{T}} = (2, -2).$$

Later we give a different proof of a similar result about the boundedness of S when $\bar{M}$ is negative definite matrix.

In the case where $\bar{M}$ is positive definite the corresponding feasible set S is always unbounded. However, in that case we may obtain bounds on the (unique) solution of LCP. Different bounds which also depend on the eigenvalues of $\bar{M}$ are given in [8].

Theorem 2. *If $\bar{M}$ is positive definite, then the vertex that solves the LCP is contained in the rectangle*

$$\{x: 0 \le x_i \le 2\|q\|/\lambda_1,\ i=1,\dots,n\}.$$

Proof. When $\bar{M}$ is positive definite, we have

$$\phi(x) = \tfrac{1}{2}x^T \bar{M}x + q^T x \geq \tfrac{1}{2}\lambda_1 \|x\|^2 - \|q\| \cdot \|x\| \geq (\tfrac{1}{2}\lambda_1 \|x\| - \|q\|)\|x\|.$$

If $\|x\| > 2\|q\|/\lambda_1$, then $\phi(x) > 0$ and therefore the vertex v such that $\phi(v) = 0$ belongs to the rectangle $\{x: 0 \leq x_i \leq 2\|q\|/\lambda_1\}$.

We consider now a more general approach, where we assume only that the matrix $\bar{M}$ is nonsingular. We first state and prove a simple but very useful lemma.

Lemma 1. *If $x \in \mathbb{R}^n$ is a solution to* LCP(M, q) *and $\bar{x} = -\bar{M}^{-1}q$ (generally $\bar{x}$ is a stationary point of $\phi(x)$), then we have*

$$(x-\bar{x})^T \bar{M}(x-\bar{x}) = q^T \bar{M}^{-1} q. \tag{1}$$

Proof. If x is a solution of the LCP and z any vector in $\mathbb{R}^n$, then it is easy to see that

$$\phi(z) = \tfrac{1}{2}(z-x)^T \bar{M}(z-x) + z^T \bar{M}x + q^T x + q^T z. \tag{2}$$

Now let $z = \bar{x}$. Using the fact that $\bar{x} = -\bar{M}^{-1}q$ and $\phi(\bar{x}) = -\tfrac{1}{2}q^T \bar{M}^{-1} q$ substituting in the above relation (2) we prove that

$$(x-\bar{x})^T \bar{M}(x-\bar{x}) = -q^T \bar{x} = q^T \bar{M}^{-1} q.$$

Using Lemma 1, we are going to prove a number of results which depend on the sign of smallest and largest eigenvalues λ_1 and λ_n of $\bar{M}$.

Theorem 3. *Assume that $\lambda_n > 0$. Then any solution x of* LCP(M, q) *satisfies*

$$\|x-\bar{x}\| \geq \|q\|/\lambda_n.$$

Proof. From (1), using $\lambda_1 \|x\|^2 \leq x^T \bar{M}x \leq \lambda_n \|x\|^2$ for any x, we have

$$\lambda_n \|x-\bar{x}\|^2 \geq \|q\|^2/\lambda_n.$$

Since $\lambda_n > 0$,

$$\|x-\bar{x}\|^2 \geq \left(\frac{\|q\|}{\lambda_n}\right)^2 \quad \text{or} \quad \|x-\bar{x}\| \geq \frac{\|q\|}{\lambda_n}.$$

Corollary 1. *If $\bar{M}$ is positive definite and x is the unique solution of* LCP(M, q), *then*

$$\frac{\|q\|}{\lambda_n} \leq \|x-\bar{x}\| \leq \frac{\|q\|}{\lambda_1}.$$

Example. Note that the above bounds may be tight as shown by the next simple example:

$$M=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad q^{\mathrm{T}}=(-1,0)$$

The unique solution of the corresponding LCP, is the vertex $x^{\mathrm{T}}=(1,0)$. Using the above bounds we obtain $\frac{1}{2}\le\|x-\bar{x}\|\le\frac{1}{2}$, that is, $\|x-\bar{x}\|=\frac{1}{2}$.

Theorem 4. *If $\lambda_1<0$ and x is a solution of* LCP(M, q), *then*

$$\|x-\bar{x}\|\ge\|q\|/|\lambda_1|.$$

Proof. From (1), $\lambda_1\|x-\bar{x}\|^2\le\|q\|^2/\lambda_1$, and since $\lambda_1<0$ we have

$$\|x-\bar{x}\|^2\ge\left(\frac{\|q\|}{\lambda_1}\right)^2 \quad\text{or}\quad \|x-\bar{x}\|\ge\frac{\|q\|}{|\lambda_1|}.$$

Corollary 2. *If $\lambda_1<0$ and $\lambda_n>0$, then the solution x of* LCP(M, q) *satisfies*

$$\|x-\bar{x}\|\ge\lambda\|q\|, \quad \lambda=\max\left\{\frac{1}{|\lambda_1|},\frac{1}{\lambda_n}\right\}.$$

Corollary 3. *If $\bar{M}$ is negative definite and x is a solution of* LCP(M, q), *then*

$$\frac{\|q\|}{|\lambda_1|}\le\|x-\bar{x}\|\le\frac{\|q\|}{|\lambda_n|}.$$

In fact, the upper bounds obtained above for the negative definite case are special cases of a more general theorem.

Theorem 5. *If $\bar{M}$ is negative definite, then the feasible domain S is contained in the rectangle*

$$\{x\colon 0\le x_i\le\bar{x}_i+\|q\|/|\lambda_n|,\ i=1,\dots n\}.$$

Proof. Let $z=x-\bar{x}$ or $x=z+\bar{x}$. Then

$$\begin{aligned}\phi(x)&=\tfrac{1}{2}z^{\mathrm{T}}\bar{M}z+q^{\mathrm{T}}z+\phi(\bar{x})+z^{\mathrm{T}}M\bar{x}=\tfrac{1}{2}z^{\mathrm{T}}\bar{M}z+q^{\mathrm{T}}z-\tfrac{1}{2}q^{\mathrm{T}}\bar{M}^{-1}q\\&=\tfrac{1}{2}z^{\mathrm{T}}\bar{M}z-\tfrac{1}{2}q^{\mathrm{T}}\bar{M}^{-1}q=\tfrac{1}{2}(z^{\mathrm{T}}\bar{M}z-q^{\mathrm{T}}\bar{M}^{-1}q).\end{aligned}$$

Then we have that $\phi(x)\le\frac{1}{2}(\lambda_n\|z\|^2+(1/\lambda_n)\|q\|^2)$. Then x is infeasible if $\phi(x)<0$ or is infeasible when $\|z\|>\|q\|/|\lambda_n|=\beta$. Therefore $\|x-\bar{x}\|\le\beta$ and

$$S\subseteq\{x\colon 0\le x_i\le\bar{x}_i+\beta,\ i=1,\dots,n\}.$$

Example. Consider the following LCP(M, q):

$$M=\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad q^{\mathrm{T}}=(-1,2).$$

Then $\bar{M}$ has eigenvalues $\lambda_1=\lambda_2=-2$ and $\bar{x}=(-.5, 1)$. Applying the theorem we see that, for any feasible point x we have $\|x-\bar{x}\|\leq\sqrt{5}/2$. Note that for this example the bound is tight if $x=(0, 2)$ (the solution of the LCP(M, q)).

In case the matrix M itself is symmetric and nonsingular with eigenvalues $\lambda_1\leq\cdots\leq\lambda_n$, then the following is true:

Theorem 6. *Suppose that $\lambda_n>0$. If x is a solution vector of* LCP(M, q)*, then we have*

$$\|x\|\leq\lambda_n\|\bar{x}\|\cdot\|M^{-1}\|.$$

Proof. Consider the following LCP(M^{-1}, $-M^{-1}q$):

Find $v\in\mathbb{R}^n$ satisfying

$$v^{\mathrm{T}}M^{-1}v-(M^{-1}q)^{\mathrm{T}}v=0,$$

$$M^{-1}v-M^{-1}q\geq 0, \qquad v\geq 0.$$

It is easy to see that if v is solution of LCP(M^{-1}, $-M^{-1}q$), then $x=M^{-1}(v-q)$ is a solution of LCP(M, q). Using the same techniques as in Lemma 1 for the above LCP we prove that

$$(v-q)^{\mathrm{T}}M^{-1}(v-q)=-q^{\mathrm{T}}M^{-1}(v-q).$$

Then

$$\frac{1}{\lambda_n}\|v-q\|^2\leq\|-q^{\mathrm{T}}M^{-1}\|\cdot\|v-q\|=\|\bar{x}\|\cdot\|v-q\|$$

which implies:

$$\|v-q\|\leq\lambda_n\|\bar{x}\|.$$

But $x=M^{-1}(v-q)$, so from the above relation we obtain:

$$\|x\|\leq\|M^{-1}\|\cdot\|v-q\|\leq\lambda_n\|\bar{x}\|\cdot\|M^{-1}\|.$$

Concluding remarks

Bounds for the solution norm and the feasible domain of the Linear Complementarity Problem were derived. Since the bounds involve the eigenvalues of the corresponding symmetric matrix $\bar{M}$ in the quadratic formulation, this indicates the significance of the eigenstructure in proving certain results.

Such bounds may be useful in obtaining information on where solutions lie without actually solving the problem. This is important when the problem is non-convex and therefore difficult to solve. Finally, these bounds can be useful in enumerative and global optimization methods of solution of LCPs (see for example [10]).

References

[1] Y.M. Bershchanskii and M.V. Meerov, The complementarity problem: theory and methods of solution, Automation and Remote Control 44 (6) (1983) 687–710.

[2] R.W. Cottle, Note on a fundamental theorem in quadratic programming, SIAM J. Appl. Math. 12 (3) (1964) 663–665.

[3] R.W. Cottle, The principal pivoting method of quadratic programming, in: G.B. Dantzig and A.F. Veinott, eds., Mathematics of the Decision Sciences, Part I, Lectures in Applied Mathematics, Vol. 11 (A.M.S., Providence, R.I., 1968) 144–162.

[4] R.W. Cottle, Solution rays for a class of complementarity problems, Math. Programming Study 1 (1974) 59–70.

[5] R. Doverspike, Some perturbation results for the linear complementarity problem, Math. Programming 23 (1982) 181–192.

[6] C.E. Lemke, Recent results on complementarity problems, in: J.B. Rosen, O.L. Mangasarian and K. Ritter, eds., Nonlinear Programming (Academic Press, New York, 1970) 349–384.

[7] C.E. Lemke, A survey of complementarity theory, in: R.W. Cottle, F. Giannessi and J.I. Lions, eds., Variational Inequalities and Complementarity Problems: Theory and Applications (Wiley, New York, 1980) 213–239.

[8] O.L. Mangasarian, Simple computable bounds for solutions of linear complementarity problems and linear programs, Math. Programming Study 25 (1985) 1–12.

[9] C.H. Papadimitriou and K. Steiglitz, Combinatorial Optimization: Algorithms and Complexity (Prentice-Hall, Englewood Cliffs, NJ, 1983).

[10] P.M. Pardalos and J.B. Rosen, Global optimization approach to the linear complementarity problem, Technical Report UMSI 85/10, Supercomputer Institute, University of Minnesota (1985).

[11] S.M. Robinson, Bounds for error in the solution of a perturbed linear program, Linear Algebra Appl. 6 (1973) 69–81.