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The coarse geometric Novikov conjecture and uniform convexity

Gennadi Kasparov, Guoliang Yu*, [1]

Department of Mathematics, 1326 Stevenson Center, Vanderbilt University, Nashville, TN 37240, USA

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Abstract

The coarse geometric Novikov conjecture provides an algorithm to determine when the higher index of an elliptic operator on a noncompact space is nonzero. The purpose of this paper is to prove the coarse geometric Novikov conjecture for spaces which admit a (coarse) uniform embedding into a uniformly convex Banach space.

Keywords: Higher index; Elliptic operator; Positive scalar curvature; Novikov conjecture; Uniform convexity; (Coarse) Uniform embedding

1. Introduction

The classic Atiyah–Singer index theory of elliptic operators on compact manifolds has been vastly generalized to higher index theories of elliptic operators on noncompact spaces in the framework of noncommutative geometry [4] by Connes–Moscovici for covering spaces [8], Baum–Connes for spaces with proper and cocompact discrete group actions [2], Connes–Skandalis for foliated manifolds [9], and Roe for noncompact complete Riemannian manifolds [33]. These higher index theories have important applications to geometry and topology.

* Corresponding author. Fax: +1 615 343 0215.
E-mail addresses: kasparov@math.vanderbilt.edu (G. Kasparov), gyu@math.vanderbilt.edu (G. Yu).
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In the case of a noncompact complete Riemannian manifold, the coarse geometric Novikov conjecture provides an algorithm to determine when the higher index of an elliptic operator on the noncompact complete Riemannian manifold is nonzero. The purpose of this paper is to prove the coarse geometric Novikov conjecture under a certain mild geometric condition suggested by Gromov [16].

Let Γ be a metric space; let X be a Banach space. A map $f : \Gamma \to X$ is said to be a (coarse) uniform embedding [15] if there exist non-decreasing functions ρ_1 and ρ_2 from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}$ such that

(1) $\rho_1(d(x, y)) \leqslant \|f(x) - f(y)\| \leqslant \rho_2(d(x, y))$ for all $x, y \in \Gamma$;
(2) $\lim_{r\to+\infty} \rho_i(r) = +\infty$ for $i = 1, 2$.

In this paper, we prove the following result:

Theorem 1.1. *Let Γ be a discrete metric space with bounded geometry. If Γ is uniformly embeddable into a uniformly convex Banach space, then the coarse geometric Novikov conjecture holds for Γ, i.e., the index map from $\lim_{d\to\infty} K_*(P_d(\Gamma))$ to $K_*(C^*(\Gamma))$ is injective, where $P_d(\Gamma)$ is the Rips complex of Γ and $C^*(\Gamma)$ is the Roe algebra associated to Γ.*

Recall that a discrete metric space Γ is said to have bounded geometry if $\forall\, r > 0$, $\exists\, N(r) > 0$ such that the number of elements in $B(x, r)$ is at most $N(r)$ for all $x \in \Gamma$, where $B(x, r) = \{y \in \Gamma : d(y, x) \leqslant r\}$. A Banach space X is called uniformly convex if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if $x, y \in S(X)$ and $\|x - y\| \geqslant \varepsilon$, then $\left\|\frac{x+y}{2}\right\| < 1 - \delta$, where $S(X) = \{x \in X,\ \|x\| = 1\}$.

The coarse geometric Novikov conjecture implies that the higher index of the Dirac operator on a uniformly contractible Riemannian manifold is nonzero (recall that a Riemannian manifold is said to be uniformly contractible if for every $r > 0$, there exists $R \geqslant r$ such that every ball with radius r can be contracted to a point in a ball with radius R). By Proposition 4.33 of [33] and the Lichnerowicz argument, Theorem 1.1 implies the following result:

Corollary 1.2. *Let M be a Riemannian manifold with bounded geometry. If M admits a uniform embedding into a uniformly convex Banach space and is uniformly contractible, then M cannot have uniformly positive scalar curvature.*

In general, Gromov conjectures that a uniformly contractible Riemannian manifold with bounded geometry cannot have uniformly positive scalar curvature [16].

The possibility of using a uniform embedding into a uniformly convex Banach space in order to study the Novikov conjecture was suggested by Gromov [16]. The main new ideas in the proof of Theorem 1.1 consist of a construction of a family of uniformly almost flat Bott vector bundles over the uniformly convex Banach space and a K-theoretic finitization technique. The uniform convexity condition is used in a crucial way to construct this family of uniformly almost flat Bott vector bundles.

The coarse geometric Novikov conjecture is false if the bounded geometry condition is removed [39]. The coarse geometric Novikov conjecture for bounded geometry spaces uniformly embeddable into Hilbert space was proved in [40]. The proof of the Hilbert space case makes the use of an algebra of the Hilbert space introduced in [21–23].

W. B. Johnson and N. L. Randrianarivony showed that l_p ($p > 2$) does not admit a (coarse) uniform embedding into a Hilbert space [27]. More recently, M. Mendel and A. Naor proved that l_p does not admit a (coarse) uniform embedding into l_q if $p > q \geqslant 2$ [31]. N. Brown and E. Guentner proved that every bounded geometry space admits a (coarse) uniform embedding into a strictly convex and reflexive Banach space [3] (recall that a Banach space X is said to be strictly convex if $\left\| \frac{x+y}{2} \right\| < 1$ for any two distinct unit vectors x and y in X). It is an open question whether every separable metric space admits a (coarse) uniform embedding into some uniformly convex Banach space. N. Ozawa proved that expanders do not admit a (coarse) uniform embedding into a uniformly convex Banach space with an unconditional basis [32]. We also would like to mention the conjecture that if M is a compact smooth manifold, then any countable subgroup of the diffeomorphism group $Diff(M)$ of M admits a (coarse) uniform embedding into C_p for some $p > 1$, where C_p is the Banach space of all Schatten-p class operators on a Hilbert space (recall that C_p is uniformly convex for all $p > 1$).

We remark that the K-theory for complex Banach algebras throughout this paper is the 2-periodic complex topological K-theory.

This paper is organized as follows: In Section 2, we collect a few facts about uniform convexity which will be used later in this paper. In Section 3, we introduce (Banach) Clifford algebras over a Banach space. In the case of a uniformly convex Banach space, we use the (Banach) Clifford algebras to construct a certain Bott vector bundles over the Banach space and show that the Bott vector bundles are uniformly almost flat in a certain Banach sense. As suggested by Misha Gromov, Section 3 might be of independent interest to experts in Banach space theory. In Section 4, we briefly recall the coarse geometric Novikov conjecture and the K-theoretic localization technique. Section 5, we introduce a K-theoretic finitization technique. In Section 6, we use the Bott vector bundles to construct Bott maps in K-theory. Finally in Section 7, we prove the main result of this paper.

In a separate paper, we will show how uniform convexity can be used to study K-theory for C^*-algebras associated to discrete groups. In particular, we shall prove the Novikov conjecture for groups uniformly embeddable into uniformly convex Banach spaces.

2. Uniform convexity of Banach spaces

In this section, we collect a few facts about uniformly convex Banach spaces which will be used in this paper. A beautiful account of the theory of uniformly convex Banach space can be found in Diestel's book [10].

For convenience of the readers, we give a proof of the following classic result in the theory of Banach spaces.

Proposition 2.1. *Let X be a Banach space (over $\mathbb{R}$). Assume that X^* is uniformly convex.*

(1) $\forall\, x \in S(X)$, $\exists$ *a unique* $x^* \in S(X^*)$ *such that* $x^*(x) = 1$.
(2) *The map*: $x \to x^*$, *from* $S(X)$ *to* $S(X^*)$ *is uniformly continuous.*

Proof. Let us first prove part (1) of Proposition 2.1 . Existence of x^* follows from the Hahn–Banach theorem. Let g be another element in $S(X^*)$ satisfying $g(x) = 1$. We have $\frac{x^*+g}{2}(x) = 1$. This implies that $\|\frac{x^*+g}{2}\| = 1$. Uniform convexity of X^* implies $g = x^*$.

Next we shall prove part (2) of the proposition. Given $\varepsilon > 0$, let $\delta > 0$ be as in the definition of uniform convexity of X^*. Assume that a pair of vectors x and $y \in S(X)$ satisfies $\|x - y\| < \delta$. We have $\frac{x^*+y^*}{2}(x) > 1 - \delta$. This implies that $\|\frac{x^*+y^*}{2}\| > 1 - \delta$. By uniform convexity of X^*, we have $\|x^* - y^*\| < \varepsilon$. $\square$

We remark that part (1) of Proposition 2.1 is still true under the weaker condition that X^* is strictly convex.

For any $x \in X$, we define $x^* \in X^*$ by

$$x^* = \begin{cases} \|x\| \left(\frac{x}{\|x\|}\right)^* & \text{if } x \neq 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The following result is a consequence of Enflo's theorem [13] and Asplund's averaging technique [1]. (See also [10, p. 87].) It plays an important role in the proof of the main result of this paper.

Theorem 2.2. *Let* Γ *be a discrete metric space. If* Γ *admits a uniform embedding into a uniformly convex Banach space, then* Γ *is uniformly embeddable into a uniformly convex Banach space* X *such that its dual space* X^* *is also uniformly convex.*

3. (Banach) Clifford algebras and Bott vector bundles over Banach spaces

In this section, we shall first introduce (Banach) Clifford algebras over a Banach space. In the case of a uniformly convex Banach space, we use the (Banach) Clifford algebras to construct Bott vector bundles over the Banach space and show that the Bott vector bundles are uniformly almost flat in a certain Banach sense.

Let X be a Banach space over $\mathbb{R}$. Let V be a finite-dimensional subspace of X and V^* its dual space. We define a pairing: $\left(V \oplus V^*\right) \times \left(V \oplus V^*\right) \to \mathbb{R}$ by

$$q(x \oplus g),\ y \oplus h = h(x) + g(y)$$

for all $x \oplus g,\ \ y \oplus h \in V \oplus V^*$.

Let $W = V \oplus V^*$ be the given norm;

$$\|x \oplus g\| = \sqrt{\|x\|^2 + \|g\|^2}$$

for all $x \oplus g \in W = V \oplus V^*$.

Let $\otimes^n W = \overbrace{W \otimes \cdots \otimes W}^{n}$ for $n \geqslant 1$ and $\otimes^0 W = \mathbb{R}$.

Endow $\otimes^0 W$ with the standard norm. For $n \geqslant 1$, endow $\otimes^n W$ with the following norm:

$$\|u\| = \sup\{(\phi_1 \otimes \cdots \otimes \phi_n)(u) : \phi_k \in W^*, \|\phi_k\| \leqslant 1, \ 1 \leqslant k \leqslant n\}$$

for all $u \in \otimes^n W$, where W^* is the dual (Banach) space of W.

Let

$$T(W) = \left\{ \bigoplus_{n=0}^{\infty} a_n \ : \ a_n \in \otimes^n W \, , \ \sum_{n=0}^{\infty} \|a_n\| < +\infty \right\}$$

be the tensor algebra.

Observe that $T(W)$ is a Banach algebra over $\mathbb{R}$ with the following norm:

$$\left\| \bigoplus_{n=1}^{\infty} a_n \right\| = \sum_{n=0}^{\infty} \|a_n\|.$$

Let $T_{\mathbb{C}}(W)$ be the complexification of the Banach algebra $T(W)$. $T_{\mathbb{C}}(W)$ is a Banach algebra over $\mathbb{C}$.

Let $I_{\mathbb{C}}(W)$ be the closed two-sided ideal in $T_{\mathbb{C}}(W)$ generated by all elements of the form

$$w_1 \otimes w_2 + w_2 \otimes w_1 + 2q(w_1, w_2), \quad w_1, w_2 \in W.$$

The Clifford algebra $Cl(W)$ is defined as the quotient Banach algebra:

$$Cl(W) = T_{\mathbb{C}}(W)/I_{\mathbb{C}}(W).$$

$Cl(W)$ is a finite-dimensional complex Banach algebra with the natural quotient norm.

Let $C_{\mathrm{b}}(W, Cl(W))$ be the Banach algebra of all bounded continuous functions on W with values in $Cl(W)$, where the norm of each element $f \in C_{\mathrm{b}}(W, Cl(W))$ is defined by

$$\|f\| = \sup_{w \in W} \|f(w)\|.$$

Let $C_0(W, Cl(W))$ be the Banach algebra of all continuous functions on W with values in $Cl(W)$, vanishing at ∞, where the norm on $C_0(W, Cl(W))$ is inherited from the norm on $C_{\mathrm{b}}(W, Cl(W))$.

Throughout the rest of the paper, let X be a uniformly convex Banach space over $\mathbb{R}$ such that its dual space X^* is also uniformly convex. We shall next construct a family of uniformly almost flat representatives of the Bott generators in the K-group $K_0(C_0(W, Cl(W)))$ for all finite-dimensional subspaces $V \subseteq X$.

We need to recall the concept of the index of a generalized Fredholm operator in the context of Banach algebras.

Let B be a ($\mathbb{Z}_2$-)graded unital complex Banach algebra and let A be a ($\mathbb{Z}_2$-)graded ideal in B. Assume that the grading on B is induced by a grading operator ε in B satisfying $\varepsilon^2 = 1$, $\|\varepsilon\| = 1$. We have $B^{(0)} = \{b \in B : \varepsilon^{-1} b \varepsilon = b\}$ and $B^{(1)} = \{b \in B : \varepsilon^{-1} b \varepsilon = -b\}$.

Let $F \in B$ be an element of degree one such that $F^2 - 1 \in A$.

Write

$$\alpha = \left(\frac{1+\varepsilon}{2}\right) F \left(\frac{1-\varepsilon}{2}\right),$$

$$\alpha' = \left(\frac{1-\varepsilon}{2}\right) F \left(\frac{1+\varepsilon}{2}\right).$$

Let

$$\omega = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha' & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$\omega = \begin{pmatrix} \alpha + (1-\alpha\alpha')\alpha & -(1-\alpha\alpha') \\ 1-\alpha'\alpha & \alpha' \end{pmatrix},$$

$$\omega^{-1} = \begin{pmatrix} \alpha' & 1-\alpha'\alpha \\ -(1-\alpha\alpha') & \alpha + \alpha(1-\alpha'\alpha) \end{pmatrix}.$$

Define

$$\mathit{index}(F) = \omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \omega^{-1}.$$

Clearly, $\mathit{index}(F)$ is an idempotent in $M_2(A)$.

We have

$$\mathit{index}(F) = \begin{pmatrix} \alpha\alpha' + (1-\alpha\alpha')\alpha\alpha' & \alpha(1-\alpha'\alpha) + (1-\alpha\alpha')\alpha(1-\alpha'\alpha) \\ (1-\alpha'\alpha)\alpha' & (1-\alpha'\alpha)^2 \end{pmatrix}.$$

We define

$$Index(F) = [index(F)] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(A)\,,$$

where $K_0(A)$ is the K-group of A considered as a Banach algebra without grading.

Notice that $Index(F)$ is an obstruction to the invertibility of F.

Let φ be a continuous function on $\mathbb{R}$ such that $0 \leqslant \varphi(t) \leqslant 1$, $\exists\, 0 < c_1 < c_2$ satisfying $\varphi(t) = 0$ if $t \leqslant c_1$ and $\varphi(t) = 1$ if $t \geqslant c_2$.

Note that the uniform convexity of X^* is equivalent to the uniform smoothness of X (see [10, p. 36]), therefore, all linear subspaces V of X have the property that both V and V^* are uniformly convex. Let V be a finite-dimensional subspace of X. For any $x \in V$, let $x^* \in X^*$ be defined as above. The restriction of x^* to V is still denoted by x^*. Since V is naturally isometric to $(V^*)^*$ (with its natural norm), $(V^*)^*$ is uniformly convex. For any $h \in V^*$, we identify $h^* \in (V^*)^*$ with an element (still denoted by h^*) in V.

Let $F_{V,\varphi}$, $\in C_b(W, Cl(W))$ be defined by

$$F_{V,\varphi}(0 \oplus 0) = 0,$$

$$F_{V,\varphi}(x \oplus h) = \frac{\varphi\left(\sqrt{\|x\|^2 + \|h\|^2}\right)}{\sqrt{\|x\|^2 + \|h\|^2 + i(h(x) - x^*(h^*))}} \left(\frac{h^* \oplus x^* - x \oplus h}{2} + i\,\frac{x \oplus h + h^* \oplus x^*}{2}\right),$$

for all nonzero $x \oplus h \in W = V \oplus V^*$.

Note that $\sqrt{\|x\|^2 + \|h\|^2 + i(h(x) - x^*(h^*))}$ is well defined as a continuous complex-valued function of $x \oplus h$ since $h(x) - x^*(h^*)$ are real numbers.

Endow $C_b(W, Cl(W))$ with the grading induced by the natural grading operator ε of $Cl(W)$ (considered as a constant function on W). It is not difficult to see that the norm of the grading operator is 1.

Clearly $F_{V,\varphi}$ has degree one. It is also straightforward to verify that

$$F_{V,\varphi}^2 - 1 \in C_0(W, Cl(W)),$$

where 1 is the identity element of $C_b(W, Cl(W))$.

Lemma 3.1. *Index* $(F_{V,\varphi})$ *is a generator for* $K_0(C_0(W, Cl(W)))$.

Proof. Let $\|\cdot\|_0$ be the Euclidean norm on V. We define a homotopy of norms on V and V^* by

$$\|\cdot\|_t = \sqrt{t\|\cdot\|^2 + (1-t)\|\cdot\|_0^2}$$

for $t \in [0, 1]$.

For each $x \in V$, let $x^{*,0} \in V^*$ be given by $x^{*,0}(y) =< y, x >$ for all $y \in V$, where $<, >$ is the inner product on V corresponding to the Euclidean structure. We define $x^{*,t} \in V^*$ by

$$x^{*,t} = tx^* + (1-t)x^{*,0}$$

for $t \in [0, 1]$. For each $h \in V^*$, let $h^{*,0} \in V$ be given by $h(x) =< x, h^{*,0} >$ for all $x \in V$. We define $h^{*,t} \in V$ by

$$h^{*,t} = th^* + (1-t)h^{*,0}$$

for $t \in [0, 1]$.

We can define a homotopy $F_{V,\varphi}(t)$ by, respectively, replacing x^*, h^* and $\| \cdot \|$ with $x^{*,t}$, $h^{*,t}$, and $\| \cdot \|_t$ in the definition of $F_{V,\varphi}$. We have

$$\mathit{Index}(F_{V,\varphi}) = \mathit{Index}(F_{V,\varphi}(1)) = \mathit{Index}(F_{V,\varphi}(0))$$

in $K_0(C_0(W, Cl(W)))$. But it is straightforward to verify that $\mathit{Ind}(F_{V,\varphi}(0))$ is the Bott generator for $K_0(C_0(W, Cl(W)))$. $\square$

We have the following proposition.

Proposition 3.2. *Let*

$$\mathit{index}(F_{V,\varphi}) = a_{V,\varphi} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

for some $a_{V,\varphi} \in M_2(C_0(W, Cl(W)))$.

(1) *Given* φ, *there exists* $R > 0$ *such that* $\mathit{supp}(a_{V,\varphi}) \subseteq B_W(0, R)$ *for any finite-dimensional subspace* $V \subseteq X$, *where* $\mathit{supp}(a_{V,\varphi}) = \left\{\xi \in W : a_{V,\varphi}(\xi) \neq 0\right\}$ ($a_{V,\varphi}$ is *identified as an element of* $C_0(W, M_2(Cl(W)))$, *and* $B_W(0, R) = \{\xi \in W : \|\xi\| < R\}$, $(\|\xi\| = \sqrt{\|x\|^2 + \|h\|^2}$ *if* $\xi = x \oplus h \in W = V \oplus V^*)$.
(2) *There exists* $C > 0$ *such that* $\|\mathit{index}(F_{V,\varphi})\| \leqslant C$ *for any finite dimensional subspace* $V \subseteq X$, *where the norm of each element* $f \in C_b(W, Cl(W))$ *is defined by* $\|f\| = \sup_{w \in W} \|f(w)\|$, *and* $M_2(C_b(W, Cl(W))) = C_b(W, Cl(W)) \otimes M_2(\mathbb{C})$ *is endowed with a Banach algebra tensor product norm* (*for example*, *the projective tensor product norm*).

Proof. We have the following identity:

$$\left(\frac{h^* \oplus x^* - x \oplus h}{2} + i\frac{x \oplus h + h^* \oplus x^*}{2}\right)^2 = \|x\|^2 + \|h\|^2 + i(h(x) - x^*(h^*))$$

for all $x \oplus h \in W = V \oplus V^*$.

Now part (1) of Proposition 3.2 follows from the definition of $index(F_{V,\varphi})$, the definition of φ, and the above identity.

Recall that V and V^* are real Banach spaces. Hence we have

$$|\sqrt{\|x\|^2 + \|h\|^2 + i(h(x) - x^*(h^*))}\ | \geqslant \sqrt{\|x\|^2 + \|h\|^2}$$

for all $x \oplus h \in W = V \oplus V^*$.

This inequality, together the definitions of $F_{V,\varphi}$ and φ and the norm on $V \oplus V^*$, implies that

$$\|F_{V,\varphi}\| \leqslant 2.$$

Now part (2) of Proposition 3.2 follows from the above inequality, the definition of $index(F_{V,\varphi})$, and the fact that the norm of the grading operator in $C_{\mathrm{b}}(W, Cl(W))$ is 1. $\square$

The concept of almost flat bundles has been successfully used to study the Novikov conjecture and positive scalar curvature problem in [6,7,17,19,26]. We shall introduce a slight variation of almost flatness suitable for the purpose of this paper.

Given a natural number k, real numbers $r > 0$, $\varepsilon > 0$ and a subspace U of W, an idempotent p in $M_k(C_{\mathrm{b}}(W, Cl(W)))$ is said to be (r, ε)-flat relative to U if

$$\|p(u_1) - p(u_2)\| < \varepsilon$$

for any u_1 and u_2 in W satisfying $u_1 - u_2 \in U$ and $\|u_1 - u_2\| \leqslant r$, where p is identified with an element in $C_{\mathrm{b}}(W, M_k(Cl(W)))$.

The following result says that the family of idempotents $\{index(F_{V,\varphi})\}_V$ is uniformly almost flat.

Proposition 3.3. $\forall r > 0$, $\varepsilon > 0$, *there exists* φ *such that* $index(F_{V,\varphi})$ *is* (r, ε)*-flat relative to* $V \oplus 0 \subseteq W$ *for any finite dimensional subspace* $V \subseteq X$, *where* $W = V \oplus V^*$.

Proof. Given $N > 0$, $\delta > 0$, let φ be a smooth function on $\mathbb{R}$ satisfying

$$0 \leqslant \varphi(t) \leqslant 1, \quad |\varphi'(t)| < \delta$$

for all $t \in \mathbb{R}$, $\varphi(t) = 0$ if $t \leqslant N$, and there exists $N' > N$ such that $\varphi(t) = 1$ if $t > N'$.

By Proposition 2.1, given $\varepsilon' > 0$, there exists $N > 0$ (independent of V) such that

$$\left\| \frac{x_1 \oplus h}{\sqrt{\|x_1\|^2 + \|h\|^2}} - \frac{x_2 \oplus h}{\sqrt{\|x_2\|^2 + \|h\|^2}} \right\| < \varepsilon',$$

$$\left\| \frac{h^* \oplus x_1^*}{\sqrt{\|x_1\|^2 + \|h\|^2}} - \frac{h^* \oplus x_2^*}{\sqrt{\|x_2\|^2 + \|h\|^2}} \right\| < \varepsilon'$$

for all $x_1, x_2 \in V$, and $h \in V^*$ satisfying $\|x_1 - x_2\| < r$, $\|(x_1, h)\| \geqslant N$.

Write

$$x' = \frac{x}{\sqrt{\|x\|^2 + \|h\|^2}},$$

$$h' = \frac{h}{\sqrt{\|x\|^2 + \|h\|^2}}$$

for all nonzero $x \oplus h \in W = V \oplus V^*$.

We have

$$\begin{aligned} &F_{V,\varphi}(x \oplus h) \\ &= \frac{\varphi\left(\sqrt{\|x\|^2 + \|h\|^2}\right)}{\sqrt{1 + i(h'(x') - (x')^*((h')^*))}} \left(\frac{((h')^* \oplus (x')^*) - x' \oplus h'}{2} \right. \\ &\quad \left. + i\,\frac{x' \oplus h' + ((h')^* \oplus (x')^*)}{2} \right), \end{aligned}$$

for all nonzero $x \oplus h \in W = V \oplus V^*$.

By choosing ε' and δ small enough, Proposition 3.3 follows from the above facts, part (2) of Proposition 2.1, part (3) of Proposition 2.3, the definitions of φ, $F_{V,\varphi}$, *index*$(F_{V,\varphi})$, and straightforward estimates. $\square$

4. The coarse geometric Novikov conjecture and localization

In this section, we shall briefly recall the coarse geometric Novikov conjecture and the localization technique.

Let M be a locally compact metric space. Let H_M be a separable Hilbert space equipped with a faithful and nondegenerate $*$-representation of $C_0(M)$ whose range contains no nonzero compact operator.

Definition 4.1 (*Roe [33]*). (1) The support of a bounded linear operator $T: H_M \to H_M$, denoted by *supp*(T), is the complement of the set of points $(m, m') \in M \times M$ for which there exist ϕ and ψ in $C_0(M)$ such that

$$\psi T \phi = 0, \quad \phi(m) \neq 0 \quad \text{and} \quad \psi(m') \neq 0.$$

(2) The propagation of a bounded operator $T: H_M \to H_M$, denoted by propagation (T), is defined to be

$$\sup\{d(m, m') : (m, m') \in \; supp\,(T)\};$$

(3) A bounded operator $T : H_M \to H_M$ is locally compact if the operators ϕT and $T\phi$ are compact for all $\phi \in C_0(M)$.

Definition 4.2 (*Roe [33]*). The Roe algebra $C^*(M)$ is the operator norm closure of the $*$-algebra of all locally compact, finite propagation operators acting on H_M.

It should be pointed out that $C^*(M)$ is independent of the choice of H_M up to a $*$-isomorphism (cf. [25]). Throughout the rest of this paper, we choose H_M in the definition of $C^*(M)$ to be $l^2(Z) \otimes H$, where Z is a countable dense subset of M, H is a separable and infinite-dimensional Hilbert space, and $C_0(M)$ acts on H_M by: $\phi(g \otimes h) = (\phi g) \otimes h$ for all $\phi \in C_0(M), g \in l^2(Z), h \in H$ (ϕ acts on $l^2(Z)$ by pointwise multiplication).

Let Γ be a locally finite discrete metric space (a metric space is called locally finite if every ball contains finitely many elements).

Definition 4.3. For each $d \geqslant 0$, the Rips complex $P_d(\Gamma)$ is the simplicial polyhedron where the set of all vertices is Γ, and a finite subset $\{\gamma_0, \ldots, \gamma_n\} \subseteq \Gamma$ spans a simplex iff $d(\gamma_i, \gamma_j) \leqslant d$ for all $0 \leqslant i, j \leqslant n$.

Endow $P_d(\Gamma)$ with the spherical metric. Recall that the spherical metric is the maximal metric whose restriction to each simplex is the metric obtained by identifying the simplex with part of a unit sphere endowed with the standard Riemannian metric. The distance of a pair of points in different connected components of $P_d(\Gamma)$ is defined to be infinity. The use of spherical metric is necessary to avoid certain pathological phenomena when Γ does not have bounded geometry. If Γ has bounded geometry, one can instead use the Euclidean/simplicial metric on $P_d(\Gamma)$.

Conjecture 4.1 (*The coarse geometric Novikov conjecture*). *If Γ is a discrete metric space with bounded geometry, then the index map Ind from* $\lim_{d\to\infty} K_*(P_d(\Gamma))$ *to* $\lim_{d\to\infty} K_*(C^*(P_d(\Gamma)))$ *is injective, where* $K_*(P_d(\Gamma)) = KK_*(C_0(P_d(\Gamma), \mathbb{C})$ *is the K-homology group of the locally compact space* $P_d(\Gamma)$.

It should be pointed out that $\lim_{d\to\infty} K_*(C^*(P_d(\Gamma)))$ is isomorphic to $K_*(C^*(\Gamma))$.

Conjecture is false if the bounded geometry condition is dropped [39]. By Proposition 4.33 in [33], the coarse geometric Novikov conjecture implies Gromov's conjecture that a uniformly contractible Riemannian manifold with bounded geometry cannot have uniformly positive scalar curvature and the zero-in-the spectrum conjecture stating that the Laplace operator acting on the space of all L^2-forms of a uniformly contractible Riemannian manifold has zero in its spectrum.

The localization algebra introduced in [38] will play an important role in the proof of our main result. For the convenience of the readers, we shall briefly recall its definition and its relation with K-homology.

Definition 4.4. Let M be a locally compact metric space. The localization algebra $C_L^*(M)$ is the norm-closure of the algebra of all uniformly bounded and uniformly norm-continuous functions

$$a : [0, \infty) \to C^*(M)$$

satisfying

$$\textit{propagation}\,(a(t)) \to 0$$

as $t \to \infty$.

There exists a local index map [38]

$$Ind_L : K_*(M) \to K_*(C_L^*(M)).$$

Theorem 4.5 (*Yu [38]*). *If P is a locally compact and finite-dimensional simplicial polyhedron endowed with the spherical metric, then the local index map $Ind_L : K_*(P) \to K_*(C_L^*(P))$ is an isomorphism.*

For the convenience of readers, we give an overview of the proof of the above theorem given in [38]. Given a locally compact and finite-dimensional simplicial polyhedron P endowed with the spherical metric, let P_1 and P_2 be two simplicial sub-polyhedrons of P. Endow P_1, P_2, $P_1 \cup P_2$ and $P_1 \cap P_2$ with metrics inherited from P. The local nature of the localization algebra can be used to prove a Mayer–Vietoris for the K-groups of the localization algebras for $P_1 \cup P_2$, P_1, P_2 and $P_1 \cap P_2$, and a certain (strong) Lipschitz homotopy invariance of the K-theory of the localization algebra. Now Theorem 4.5 follows from an induction argument on the dimension of skeletons of P using the Mayer–Vietoris sequence and the (strong) Lipschitz homotopy invariance for K-theory of localization algebras.

The evaluation homomorphism e from $C_L^*(M)$ to $C^*(M)$ is defined by

$$e(a) = a(0)$$

for all $a \in C_L^*(M)$.

In the definitions of $C^*(P_d(\Gamma))$ and $C_L^*(P_d(\Gamma))$, we choose a countable dense subset Γ_d of $P_d(\Gamma)$ in such a way that if $d' > d$, then $\Gamma_d \subseteq \Gamma_{d'}$. Hence there are natural inclusion homomorphisms from $C^*(P_d(\Gamma))$ to $C^*(P_{d'}(\Gamma))$ and from $C_L^*(P_d(\Gamma))$ to $C_L^*(P_{d'}(\Gamma))$ when $d' > d$.

If Γ is a locally finite discrete metric space, we have the following commuting diagram:

$$\begin{tikzcd} & \lim_{d\to\infty} K_*(P_d(\Gamma)) \arrow[dl, "\mathrm{Ind}_L"'] \arrow[dr, "\mathrm{Ind}"] & \\ \lim_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) \arrow[rr, "e_*"] & & \lim_{d\to\infty} K_*(C^*(P_d(\Gamma))). \end{tikzcd}$$

Theorem 4.5 implies that in order to prove the coarse geometric Novikov conjecture, it is enough to show that

$$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) \to \lim_{d\to\infty} K_*(C^*(P_d(\Gamma)))$$

is injective.

Let P be a locally compact and finite-dimensional simplicial polyhedron endowed with the spherical metric. Let Q be a simplicial sub-polyhedron of P with the metric inherited from P.

Let $P\backslash Q = \{x \in P : x \notin Q\}$. We define the relative K-homology group of (P, Q) by

$$K_*(P, Q) = K_*(P\backslash Q) = KK_*(C_0(P\backslash Q), \mathbb{C}).$$

Endow $P\backslash Q$ with the metric inherited from P. We define $C_L^*(\partial(P\backslash Q))$ to be the closed subalgebra of $C_L^*(P\backslash Q)$ generated by elements $a \in C_L^*(P\backslash Q)$ such that there exists $c_t > 0$ ($t \in [0,\infty)$) satisfying $\lim_{t\to\infty} c_t = 0$, and $supp(a(t)) \subseteq \{(x, y) \in (P\backslash Q) \times (P\backslash Q) : d((x, y), Q \times Q) < c_t\}$ for all $t \in [0,\infty)$, where c_t depends on a. Notice that $C_L^*(\partial(P\backslash Q))$ is the closed two sided ideal of $C_L^*(P\backslash Q)$.

Let $C_L^*(Q; P)$ be the closed subalgebra of $C_L^*(P)$ generated by all elements a in $C_L^*(P)$ such that there exists $c_t > 0$ ($t \in [0,\infty)$) satisfying $\lim_{t\to\infty} c_t = 0$, and $supp(a(t)) \subseteq \{(x, y) \in P \times P : d((x, y), Q \times Q) < c_t\}$ for all $t \in [0,\infty)$, where c_t depends on a. It is not difficult to see that $C_L^*(Q; P)$ is a closed two-sided ideal of $C_L^*(P)$.

Observe that $C_L^*(P)/C_L^*(Q, P)$ is naturally isomorphic to $C_L^*(P\backslash Q)/C_L^*(\partial(Q\backslash P))$.

Next we shall define a local index map

$$Ind_L : \; K_*(P, Q) \to K_*(C_L^*(P)/C_L^*(Q, P)) \cong K_*(C_L^*(P\backslash Q)/C_L^*(\partial(Q\backslash P))).$$

Recall that if M is a locally compact topological space, the K-homology groups $K_i(M) = KK_i(C_0(M), \mathbb{C})$ $(i = 0, 1)$ are generated by the following cycles modulo a certain equivalence relation [28]:

(1) a cycle for $K_0(M)$ is a pair (H_M, F), where H_M is a Hilbert space with a $*$-representation of $C_0(M)$ and F is a bounded linear operator acting on H_M such that $F^*F - I$ and $FF^* - I$ are locally compact, and $\phi F - F\phi$ is compact for all $\phi \in C_0(M)$;

(2) a cycle for $K_1(M)$ is a pair (H_M, F), where H_M is a Hilbert space with a $*$-representation of $C_0(M)$ and F is a self-adjoint operator acting on H_M such that $F^2 - I$ is locally compact, and $\phi F - F\phi$ is compact for all $\phi \in C_0(M)$.

Let $(H_{P\backslash Q}, F)$ be a cycle for $K_0(P; Q) = K_0(C_0(P\backslash Q))$. Without loss of generality, we can assume that $H_{P\backslash Q}$ is a nondegenerate $*$-representation of $C_0(P\backslash Q)$ whose range contains no nonzero compact operator.

For each positive integer n, there exists a locally finite open cover $\{U_{n,i}\}_i$ for $P\backslash Q$ such that $diameter(U_{n,i}) < 1/n$ for all i. Let $\{\phi_{n,i}\}_i$ be a continuous partition of unity subordinate to $\{U_{n,i}\}_i$. Define a family of operators $F(t)$ $(t \in [0, \infty))$ acting on $H_{P\backslash Q}$ by

$$F(t) = \sum_i ((n-t)\phi_{n,i}^{\frac{1}{2}} F \phi_{n,i}^{\frac{1}{2}} + (t-n+1)\phi_{n+1,i}^{\frac{1}{2}} F \phi_{n+1,i}^{\frac{1}{2}})$$

for all positive integer n and $t \in [n-1, n]$, where the infinite sum converges in strong topology. Lemma 2.6 of [39] implies that $F(t)$ is a bounded and uniformly norm-continuous function from $[0, \infty)$ to the C^*-algebra of all bounded operators acting on $H_{P\backslash Q}$. Note that

$$propagation(F(t)) \to 0 \quad as \quad t \to \infty.$$

Using the above facts it is not difficult to see that $F(t)$ is a multiplier of $C_L^*(P\backslash Q)/C_L^*(\partial(Q\backslash P))$ and $F(t)$ is a unitary modulo $C_L^*(P\backslash Q)/C_L^*(\partial(Q\backslash P))$. Hence $F(t)$ gives rise to an element

$$[F(t)] \in K_0(C_L^*(P)/C_L^*(Q, P)) \cong K_0(C_L^*(P\backslash Q)/C_L^*(\partial(Q\backslash P))).$$

We define the local index of the cycle $(H_{P\backslash Q}, F)$ to be $[F(t)]$.

Similarly we can define the local index map

$$Ind_L : \ K_1(P; Q) \to K_1(C_L^*(P)/C_L^*(Q, P)) \cong K_1(C_L^*(P\backslash Q)/C_L^*(\partial(Q\backslash P))).$$

Proposition 4.6. *Ind_L is an isomorphism from $K_*(P, Q)$ to $K_*(C_L^*(P)/C_L^*(Q; P))$.*

Proof. We have the following commutative diagram:

$$
\begin{array}{ccccccc}
\to & K_*(Q) & \to & K_*(P) & \to & K_*(P,Q) & \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & K_*(C_L^*(Q;P)) & \to & K_*(C_L^*(P)) & \to & K_*(C_L^*(P)/C_L^*(Q;P)) & \to \\
\to & K_{*+1}(Q) & \to & K_{*+1}(Q) & \to & \cdots & \\
 & \downarrow & & \downarrow & & & \\
\to & K_{*+1}(C_L^*(Q;P)) & \to & K_{*+1}(C_L^*(P)) & \to & \cdots, &
\end{array}
$$

where the vertical map from $K_*(Q)$ to $K_*(C_L^*(Q;P))$ is the composition of the local index map with the homomorphism from $K_*(C_L^*(Q))$ to $K_*(C_L^*(Q;P))$ induced by the inclusion homomorphism from $C_L^*(Q)$ to $C_L^*(Q;P)$.

Now Proposition 4.6 follows from the five lemma, Theorem 4.5, and the fact that the inclusion homomorphism from $K_*(C_L^*(Q))$ to $K_*(C_L^*(Q;P))$ is an isomorphism (cf. Lemma 3.10 in [38]). $\square$

5. Finitization of K-theory

In this section, we introduce algebras associated to a sequence of finite metric spaces. The new algebras allow us to localize K-theory of the Roe algebra associated to an infinite metric space to its finite metric subspaces (we call this process finitization).

Let Γ be a discrete metric space with bounded geometry, let $\{F_n\}_{n=1}^{\infty}$ be a sequence of finite metric subspaces of Γ.

We define $C^*_{\text{alg}}(\{F_n\}_n)$ to be the algebra

$$\left\{ \bigoplus_{n=1}^{\infty} a_n \,:\, a_n \in C^*(F_n),\ \sup_n \|a_n\| < +\infty,\ \sup_n (\mathit{propagation}(a_n)) < +\infty \right\}.$$

Endow $C^*_{\text{alg}}(\{F_n\}_n)$ with the following norm:

$$\left\| \bigoplus_{n=1}^{\infty} a_n \right\| = \sup_n \|a_n\|.$$

Definition 5.1. The C^*-algebra $C^*(\{F_n\}_n)$ is defined to be the norm completion of $C^*_{\text{alg}}(\{F_n\}_n)$.

Definition 5.2. The C^*-algebra $C^*(\{\partial_\Gamma F_n\}_n)$ is defined to be the closed subalgebra of $C^*(\{F_n\}_n)$ generated by elements $\bigoplus_{n=1}^{\infty} a_n$ such that there exists $r > 0$ satisfying

$$\mathit{supp}(a_n) \subseteq (F_n \cap B_\Gamma(\Gamma - F_n, r)) \times (F_n \cap B_\Gamma(\Gamma - F_n, r))$$

for all n, where $B_\Gamma(\Gamma - F_n, r) = \{x \in \Gamma : d(x, \Gamma - F_n) < r\}$ if $\Gamma - F_n \neq \emptyset$, and $B_\Gamma(\Gamma - F_n, r) = \emptyset$ if $\Gamma - F_n = \emptyset$.

It is easy to see that $C^*(\{\partial_\Gamma F_n\}_n)$ is a two-sided ideal in $C^*(\{F_n\}_n)$.

Definition 5.3. We define the C^*-algebra $C^*(\left\{F_n, \partial_\Gamma F_n\right\}_n)$ to be the quotient algebra $C^*\left(\{F_n\}_n\right)/C^*(\left\{\partial_\Gamma F_n\right\}_n)$.

Throughout the rest of this paper, let $\chi_n \in l^\infty(\Gamma)$ be the characteristic function of F_n.

Define a homomorphism

$$S : C^*\left(\left\{F_n, \partial_\Gamma F_n\right\}_n\right) \to C^*\left(\left\{F_n, \partial_\Gamma F_n\right\}_n\right)$$

by

$$S[a_1 \oplus a_2 \oplus a_3 \oplus \ldots] = [\chi_1 a_2 \chi_1 \oplus \chi_2 a_3 \chi_2 \oplus \chi_3 a_4 \chi_3 \oplus \ldots]$$

for all $[a_1 \oplus a_2 \oplus a_3 \oplus \ldots] \in C^*\left(\left\{F_n, \partial_\Gamma F_n\right\}_n\right)$.

Denote by $C^*\left([0,1], \left\{F_n, \partial_\Gamma F_n\right\}_n\right)$ the C^*-algebra of all continuous functions on $[0,1]$ with values in $C^*\left(\left\{F_n, \partial_\Gamma F_n\right\}_n\right)$.

Definition 5.4. We define the C^*-algebra $C^*_S(\Gamma)$ to be

$$\left\{a \in C^*([0,1], \left\{F_n, \partial_\Gamma F_n\right\}_n) : a(1) = S(a(0))\right\}.$$

Define a $*$-homomorphism $j\colon C^*_S(\Gamma) \to C^*(\left\{F_n, \partial_\Gamma F_n\right\}_n)$ by

$$j(a) = a(0)$$

for every $a \in C^*_S(\Gamma)$.

Proposition 5.5. *We have the following exact sequence*:

$$\cdots \to K_{*+1}(C^*(\left\{F_n, \partial_\Gamma F_n\right\}_n)) \xrightarrow{(Id-S)_*} K_{*+1}(C^*(\left\{F_n, \partial_\Gamma F_n\right\}_n)) \to K_*(C^*_S(\Gamma))$$

$$\xrightarrow{j_*} K_*(C^*(\left\{F_n, \partial_\Gamma F_n\right\}_n)) \xrightarrow{(Id-S)_*} K_*(C^*(\left\{F_n, \partial_\Gamma F_n\right\}_n)) \to \cdots ,$$

where Id is the identity homomorphism.

Proof. Let $A_1 = C^*([0,1], \left\{F_n, \partial_\Gamma F_n\right\}_{n\ odd})$ be the algebra of all continuous functions on the unit interval $[0,1]$ with values in $C^*\left(\left\{F_n, \partial_\Gamma F_n\right\}_{n \text{ odd}}\right)$, let $A_2 = C^*\big([0,1], \left\{F_n, \partial_\Gamma F_n\right\}_{n \text{ even}}\big)$ be the algebra of all continuous functions on the unit interval $[0,1]$ with values in $C^*\left(\left\{F_n, \partial_\Gamma F_n\right\}_{n \text{ even}}\right)$, and $B = C^*\left(\left\{F_n, \partial_\Gamma F_n\right\}_n\right)$.

Define $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$ by

$$f_1[a_1 \oplus a_3 \oplus a_5 \oplus a_7 \oplus \ldots)] = [a_1(1) \oplus \chi_2 a_3(0)\chi_2 \oplus a_3(1) \oplus \chi_4 a_5(0)\chi_4 \oplus a_5(1) \oplus \ldots)],$$

$$f_2[b_2 \oplus b_4 \oplus b_6 \oplus \ldots)] = [\chi_1 b_2(0)\chi_1 \oplus b_2(1) \oplus \chi_3 b_4(0)\chi_3 \oplus b_4(1) \oplus \ldots)].$$

Let $P = \{(a, b) \mid f_1(a) = f_2(b)\} \subseteq A_1 \oplus A_2$. It is not difficult to see that P is isomorphic to $C^*_S(\Gamma)$. Note that f_1 and f_2 are surjective $*$-homomorphisms. By the Mayer–Vietoris sequence [36], we have the following exact sequence:

$$\cdots \longrightarrow K_{*+1}(A_1) \oplus K_{*+1}(A_2) \longrightarrow K_{*+1}(B) \longrightarrow K_*(P)$$
$$\xrightarrow{(g_{1*},\, g_{2*})} K_*(A_1) \oplus K_*(A_2) \xrightarrow{f_{2*} - f_{1*}} K_*(B) \longrightarrow \cdots,$$

where $g_1(a, b) = a$, $g_2(a, b) = b$. This implies Proposition 5.5. $\square$

We define a map

$$\chi : C^*(\Gamma) \to C^*_S(\Gamma)$$

by

$$\chi(a) = \left[\bigoplus_{n=1}^{\infty} \chi_n a \chi_n \right]$$

for all $a \in C^*(\Gamma)$, where $\chi(a)$ is viewed as a constant function on $[0, 1]$.

It is not difficult to prove the following lemma.

Lemma 5.6. *χ is a $*$-homomorphism.*

Proposition 5.5 allows us to localize $\chi_*(K_*(C^*(\Gamma)))$ to finite subspaces of Γ. It is not known if χ_* is an isomorphism.

Next, we shall introduce twisted versions of the algebras of Definitions 5.1–5.3.

Let $f : \Gamma \to X$ be a uniform embedding. For each $n \in \mathbb{N}$ (the set of all natural numbers), let $V_n \subseteq X$ be a finite-dimensional subspace such that $f(F_n) \subseteq V_n$. Let $W_n = V_n \oplus V_n^*$.

Let H be an infinite-dimensional separable Hilbert space, and let $H_{F_n} = l^2(F_n) \otimes H$ be the Hilbert space as in the definition of $C^*(F_n)$.

Recall that, for each $m \geqslant 1$, we define a norm on $\otimes^m W_n$ by

$$\|u\| = \sup\{(\phi_1 \otimes \cdots \phi_m)(u) : \phi_k \in W_n^*, \ \|\phi_k\| \leqslant 1, \ 1 \leqslant k \leqslant m\}$$

for all $u \in \otimes^m W_n$, where W_n^* is the dual (Banach) space of W_n.

We define $T(W_n)$ to be the Banach space

$$\left\{ \bigoplus_{m=0}^{\infty} u_m : u_m \in \otimes^m W_n, \sum_{m=0}^{\infty} \|u_m\| < \infty \right\}$$

endowed with the norm

$$\left\| \bigoplus_{m=0}^{\infty} u_m \right\| = \sum_{m=0}^{\infty} \|u_m\|.$$

Let $T_{\mathbb{C}}(W_n)$ be the complexification of $T(W_n)$.

For each $m \geqslant 1$, we define a norm on $H_{F_n} \otimes (\otimes^m W_n)$ by

$$\begin{aligned} &\|u\| \\ &\quad = \sup\{(\phi_0 \otimes \phi_1 \otimes \cdots \phi_m)(u) : \phi_0 \in (H_{F_n})^*, \\ &\qquad \|\phi_0\| = 1, \ \phi_k \in W_n^*, \ \|\phi_k\| \leqslant 1, \ 1 \leqslant k \leqslant m\} \end{aligned}$$

for all $u \in H_{F_n} \otimes (\otimes^m W_n)$, where $(H_{F_n})^*$ is the dual (Hilbert) space of H_{F_n}, and W_n^* is the dual (Banach) space of W_n.

We define $H_{F_n} \otimes T(W_n)$ to be the Banach space

$$\left\{ \bigoplus_{m=0}^{\infty} u_m : u_m \in H_{F_n} \otimes (\otimes^m W_n), \sum_{m=0}^{\infty} \|u_m\| < \infty \right\}$$

endowed with the norm

$$\left\| \bigoplus_{m=0}^{\infty} u_m \right\| = \sum_{m=0}^{\infty} \|u_m\|.$$

Let $H_{F_n} \otimes T_{\mathbb{C}}(W_n)$ be the complexification of the Banach space $H_{F_n} \otimes T(W_n)$.

Let $H_{F_n} \otimes I_{\mathbb{C}}(W_n)$ be the closed (complex) Banach subspace of $H_{F_n} \otimes T_{\mathbb{C}}(W_n)$ spanned by all elements of the form

$$h \otimes (v_1 \otimes ((w_1 \otimes w_2) + (w_2 \otimes w_1) + 2q(w_1, w_2)) \otimes v_2),$$

where $h \in H_{F_n}$, $w_1 \in W_n$, $w_2 \in W_n$, $v_1 \in \otimes^m W_n$ for some m, $v_2 \in \otimes^l W_n$ for some l, q is the quadratic form on W_n defined in Section 3, and $I_{\mathbb{C}}(W_n)$ is as in Section 3.

Now we define the Banach space $H_{F_n} \otimes Cl(W_n)$ to be quotient Banach space:

$$H_{F_n} \otimes Cl(W_n) = (H_{F_n} \otimes T_{\mathbb{C}}(W_n))/(H_{F_n} \otimes I_{\mathbb{C}}(W_n)),$$

where $Cl(W_n)$ is as in Section 3.

Let $C_0(W_n, H_{F_n} \otimes Cl(W_n))$ be the Banach space of all continuous functions on W_n with values in $H_{F_n} \otimes Cl(W_n)$, vanishing at ∞, where the norm of each element $\xi \in C_0(W_n, H_{F_n} \otimes Cl(W_n))$ is defined by

$$\|\xi\| = \sup_{w \in W_n} \|\xi(w)\|.$$

Let $C^*(F_n) \otimes_{\mathrm{alg}} C_0(W_n, Cl(W_n))$ be the algebraic tensor product of $C^*(F_n)$ with $C_0(W_n, Cl(W_n))$. We shall construct a representation of the algebra $C^*(F_n) \otimes_{\mathrm{alg}} C_0(W_n, Cl(W_n))$ on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$ as follows.

For each $T \in C^*(F_n)$, let $T \otimes 1$ be the bounded operator on $H_{F_n} \otimes Cl(W_n)$ defined by

$$(T \otimes 1)(h \otimes \eta) = (Th) \otimes \eta$$

for all $h \in H_{F_n}$ and $\eta \in Cl(W_n)$.

We define a bounded operator $\psi(T)$ on $C_0(W_n, H_{F_n} \otimes Cl(W_n))$ by

$$((\psi(T))(\xi))(w) = (T \otimes 1)(\xi(w))$$

for all $\xi \in C_0(W_n, H_{F_n} \otimes Cl(W_n))$ and $w \in W_n$.

For every element $v \in Cl(W_n)$, let $1 \otimes v$ be the bounded operator on $H_{F_n} \otimes Cl(W_n)$ defined by

$$(1 \otimes v)(h \otimes \eta) = h \otimes (v\eta)$$

for all $h \in H_{F_n}$ and $\eta \in Cl(W_n)$, where $v\eta$ is the product of v with η in $Cl(W_n)$.

For every $g \in C_0(W_n, Cl(W_n))$, we define a bounded operator M_g on $C_0(W_n, H_{F_n} \otimes Cl(W_n))$ by

$$(M_g(\xi))(w) = (1 \otimes g(w))(\xi(w))$$

for all $\xi \in C_0(W_n, H_{F_n} \otimes Cl(W_n))$ and $w \in W_n$.

We now construct a representation of the algebra $C^*(F_n) \otimes_{\mathrm{alg}} C_0(W_n, Cl(W_n))$ on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$ by

$$(T \otimes g)\xi = \psi(T)(M_g \xi)$$

for all $T \in C^*(F_n)$, $g \in C_0(W_n, Cl(W_n))$, and $\xi \in C_0(W_n, H_{F_n} \otimes Cl(W_n))$.

By the definitions of the operators $\psi(T)$, M_g and the Banach space norm on $C_0(W_n, H_{F_n} \otimes Cl(W_n))$, it is not difficult to verify

$$\|T \otimes g\| \leqslant \|T\| \, \|g\|.$$

We now define the Banach algebra $C^*(F_n, V_n)$ to be the closure of algebraic tensor product of $C^*(F_n) \otimes_{\rm alg} C_0(W_n, Cl(W_n))$ under the operator norm given by the above representation of the algebra $C^*(F_n) \otimes_{\rm alg} C_0(W_n, Cl(W_n))$ on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$.

We remark that $C^*(F_n, V_n)$ is carefully defined for the purpose of constructing the Bott map later on (cf. the definition of the Bott map before Lemma 6.1).

We define the Banach space $C_0(W_n, H \otimes Cl(W_n))$ in a way similar to the above definition of the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$. Let K be the C^*-algebra of all compact operators on the Hilbert space H, let $K \otimes_{\rm alg} C_0(W_n, Cl(W_n))$ be the algebraic tensor product of K with $C_0(W_n, Cl(W_n))$. We can construct a representation of the algebra $K \otimes_{\rm alg} C_0(W_n, Cl(W_n))$ on the Banach space $C_0(W_n, H \otimes Cl(W_n))$ in way similar to the construction of the above representation of the algebra $C^*(F_n) \otimes_{\rm alg} C_0(W_n, Cl(W_n))$ on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$.

We define $K \otimes C_0(W_n, Cl(W_n))$ to be operator norm closure of the algebra $K \otimes_{\rm alg} C_0(W_n, Cl(W_n))$, where the operator norm is given by a representation of the algebra $K \otimes_{\rm alg} C_0(W_n, Cl(W_n))$ on the Banach space $C_0(W_n, H \otimes Cl(W_n))$.

We can identify $C^*(F_n, V_n)$ with the algebra of all functions on $F_n \times F_n$ with values in $K \otimes C_0(W_n, Cl(W_n))$ with the following convolution product:

$$(a \cdot b)(x, y) = \sum_{z \in F_n} a(x, z)\, b(z, y)$$

for all a and b in $C^*(F_n, V_n)$, and $(x, y) \in F_n \times F_n$.

For any $a \in C^*(F_n, V_n)$, we define

$$supp(a) = \{(x, y) \in F_n \times F_n \ : \ a(x, y) \neq 0\},$$

$$propagation(a) = sup\{d(x, y) : (x, y) \in supp(a)\}.$$

For any $a \in C^*(F_n, V_n)$ and $(x, y) \in F_n \times F_n$, we can identify $a(x, y)$ as a function from W_n to $K \otimes Cl(W_n)$ and define

$$support(a(x, y)) = \{\xi \in W_n \ : \ (a(x, y))(\xi) \neq 0\},$$

where $K \otimes Cl(W_n)$ is the operator norm closure of $K \otimes_{\rm alg} Cl(W_n)$ (the operator norm is given by the natural representation of $K \otimes_{\rm alg} Cl(W_n)$ on the Banach space $H \otimes Cl(W_n)$).

Let

$$C^*_{\rm alg}(\{F_n, V_n\}_n) = \Big\{ \bigoplus_{n=1}^{\infty} a_n \ : \ a_n \in C^*(F_n, V_n), \ \sup_n \|a_n\| < +\infty, \ \exists\, r > 0 \text{ such that } propagation(a_n) < r \quad \text{for all } n, \ \text{and } \exists\, R > 0 \text{ such that } support(a_n(x, y)) \subseteq B_{W_n}(f(x) \oplus 0, R) \quad \text{for all } n \Big\}.$$

Endow $C^*_{\mathrm{alg}}(\{F_n, V_n\}_n)$ with the norm

$$\left\| \bigoplus_{n=1}^{\infty} a_n \right\| = \sup_n \|a_n\|.$$

Definition 5.7. We define the Banach algebra $C^*(\{F_n, V_n\}_n)$ to be the norm completion of $C^*_{\mathrm{alg}}(\{F_n, V_n\}_n)$.

Definition 5.8. $C^*(\left\{\partial_\Gamma F_n, V_n\right\}_n)$ is defined to be the closed subalgebra of $C^*(\{F_n, V_n\}_n)$ generated by elements $\oplus_{n=1}^{\infty} a_n$ such that $\exists\, r > 0$ satisfying

$$supp(a_n) \subseteq (F_n \cap B_\Gamma(\Gamma - F_n, r)) \times (F_n \cap B_\Gamma(\Gamma - F_n, r))\,.$$

Note that $C^*(\left\{\partial_\Gamma F_n, V_n\right\}_n)$ is a closed two sided ideal of $C^*(\{F_n, V_n\}_n)$.

Definition 5.9. We define $C^*(\left\{F_n, \partial_\Gamma F_n, V_n\right\}_n)$ to be the quotient algebra

$$C^*(\{F_n, V_n\}_n)/C^*(\left\{\partial_\Gamma F_n, V_n\right\}_n).$$

6. The Bott maps

In this section, we use the family of uniformly almost flat vector bundles introduced in Section 3 to construct certain Bott maps. These Bott maps play a crucial role in the proof of the main result of this paper.

We shall first describe a difference construction in K-theory of Banach algebras. Let B be a unital Banach algebra, let A be a closed two sided ideal in B. Let p and q be idempotents in B such that $p - q \in A$. We shall define a difference element $D(p, q) \in K_0(A)$ associated to the pair p and q. When A and B are C^*-algebras, the difference construction described here is compatible with the difference construction in KK-theory (as explained below).

Let

$$Z(p, q) = \begin{pmatrix} q & 0 & 1-q & 0 \\ 1-q & 0 & 0 & q \\ 0 & 0 & q & 1-q \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We have

$$(Z(p,q))^{-1} = \begin{pmatrix} q & 1-q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1-q & 0 & q & 0 \\ 0 & q & 1-q & 0 \end{pmatrix}.$$

Define

$$D_0(p,q) = (Z(p,q))^{-1} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z(p,q).$$

Let

$$p_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $D_0(p,q)$ is an element in $M_4(A^+)$ and $D_0(p,q) = p_1$ modulo $M_4(A)$. We define

$$D(p,q) = [D_0(p,q)] - [p_1]$$

in $K_0(A)$.

Next we shall explain that, when A and B are C^*-algebras, the difference construction described above is compatible with the difference construction in KK-theory.

Recall that the difference element in $KK(\mathbb{C}, A) \cong K_0(A)$ is represented by the KK module (E, ϕ, F), where $E = A \oplus A$ is the Hilbert module over A with the inner product:

$$< (a_0 \oplus a_1), (b_0 \oplus b_1) >= a_0^* b_0 \oplus a_1^* b_1$$

for all $a_0 \oplus a_1$ and $b_0 \oplus b_1$ in $A \oplus A$, $\phi = \phi_0 \oplus \phi_1$ is the homomorphism from $\mathbb{C}$ to $B(E)$ defined by

$$(\phi_0(c))a = cpa,$$

$$(\phi_1(c))a = cqa$$

for all $c \in \mathbb{C}$ and $a \in A$, and F is the operator acting on E defined by

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Let $E_0' = E_1' = A \oplus A \oplus A \oplus A$ be the Hilbert module over A with the inner product

$$< a_1 \oplus a_2 \oplus a_3 \oplus a_4, b_1 \oplus b_2 \oplus b_3 \oplus b_4 >= a_1^* b_1 \oplus a_2^* b_2 \oplus a_3^* b_3 \oplus a_4^* b_4$$

for all $a_1 \oplus a_2 \oplus a_3 \oplus a_4$ and $b_1 \oplus b_2 \oplus b_3 \oplus b_4$ in $A \oplus A \oplus A \oplus A$.

Let $E' = E_0' \oplus E_1'$ and F' be the operator acting on E' defined by

$$F' = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} ,$$

where I is the identity element in $M_4(\mathbb{C})$.

Let ϕ_0' and ϕ_1' be, respectively, homomorphisms from $\mathbb{C}$ to $B(E_0')$ and $B(E_1')$ defined by

$$(\phi_0'(c))v_0 = c \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_0$$

for all $c \in \mathbb{C}$ and $v_0 \in E_0'$, and

$$(\phi_1'(c))v_1 = c \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_1$$

for all $c \in \mathbb{C}$ and $v_1 \in E_1'$.

Let $\phi' = \phi_0' \oplus \phi_1'$. It is not difficult to see that (E, ϕ, F) is equivalent to (E', ϕ', F') as KK modules in $KK(\mathbb{C}, A)$. This, together with identity $(Z(p,q)^{-1} \oplus Z(p,q)^{-1}) F'(Z(p,q) \oplus Z(p,q)) = F'$, implies that (E, ϕ, F) is equivalent to $(E', (Z(p,q))^{-1} \oplus (Z(p,q))^{-1}) \phi'(Z(p,q) \oplus Z(p,q)), F')$ as Kasparov modules in $KK(\mathbb{C}, A)$, where $Z(p,q)$ is defined as above.

By the formula of $(Z(p,q))^{-1}$ as described above, we can verify

$$p_1 = Z(p,q))^{-1} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z(p,q) ,$$

where p_1 is defined as above.

It follows that (E, ϕ, F) is equivalent to $D(p, q)$ in $KK(\mathbb{C}, A) \cong K_0(A)$.

Let $C^*(F_n) \otimes_{\mathrm{alg}} C_0(W_n, Cl(W_n))^+$ be the algebraic tensor product of $C^*(F_n)$ with $C_0(W_n, Cl(W_n))^+$, where $C_0(W_n, Cl(W_n))^+$ is obtained from $C_0(W_n, Cl(W_n))$ by adjoining the identity.

We can construct a representation of the algebra $C^*(F_n) \otimes_{\mathrm{alg}} C_0(W_n, Cl(W_n))^+$ on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$ exactly in the same way as the representation of the algebra $C^*(F_n) \otimes_{\mathrm{alg}} C_0(W_n, Cl(W_n))$ on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$. We define the Banach algebra $C^*(F_n, V_n^+)$ to be the closure of the algebra $C^*(F_n) \otimes_{\mathrm{alg}} C_0(W_n, Cl(W_n))$ under the operator norm given by the representation of the algebra $C^*(F_n) \otimes_{\mathrm{alg}} C_0(W_n, Cl(W_n))^+$ on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$.

We define

$$C^*_{\mathrm{alg}}(\{F_n, V_n^+\}_n) = \Bigg\{ \bigoplus_{n=1}^{\infty} a_n \, : \, a_n \in C^*(F_n, V_n^+), \ \sup_n \|a_n\| < +\infty, \\ \exists\, r > 0 \text{ such that } propagation(a_n) < r \text{ for all } n, \\ \exists\, R > 0 \text{ such that } a_n(x, y) = c_n I + b_n(x, y) \\ \text{for some } c_n \in \mathbb{C},\ b_n \in C^*(F_n, V_n) \\ \text{satisfying } support(b_n(x, y)) \subseteq B_{W_n}(f(x) \oplus 0, R) \Bigg\},$$

where I is the identity element in $C_0(W_n, Cl(W_n))^+$.

Endow $C^*_{\mathrm{alg}}(\{F_n, V_n^+\}_n)$ with the norm

$$\left\| \bigoplus_{n=1}^{\infty} a_n \right\| = \sup_n \|a_n\|.$$

Let $C^*(\{F_n, V_n^+\}_n)$ be the norm completion of $C^*_{\mathrm{alg}}\left(\{F_n, V_n^+\}_n\right)$. Note that $C^*(\{F_n, V_n\}_n)$ is a closed two sided ideal of $C^*\left(\{F_n, V_n^+\}_n\right)$.

We can similarly define $C^*(\{\partial_\Gamma F_n, V_n^+\}_n)$ and $C^*\left(\{F_n, \partial_\Gamma F_n, V_n^+\}_n\right)$.

Next, we shall define a Bott map

$$\beta_{\{V_n\}} : K_0(C^*(\{F_n\}_n)) \to K_0(C^*(\{F_n, V_n\}_n)).$$

Let $e = \bigoplus_{n=1}^{\infty} e_n \in C^*(\{F_n\}_n)$ be an idempotent representing an element in $K_0(C^*(\{F_n\}_n))$. Given any $\delta > 0$, $\exists\, e' = \bigoplus_{n=1}^{\infty} e'_n \in C^*_{\mathrm{alg}}(\{F_n\}_n)$ such that $\|e' - e\| < \delta$.

Define

$$p_0(e') = \bigoplus_{n=1}^{\infty} a_n \in C^*_{\mathrm{alg}}(\{F_n, V_n^+\}) \otimes M_2(\mathbb{C})$$

by

$$(a_n(x, y))(v \oplus h) = e'_n(x, y) \otimes (index(F_{V_n,\varphi}))((v + f(x)) \oplus h)$$

for all $x, y \in F_n$ and $v \oplus h \in W_n = V_n \oplus V_n^*$, where $index(F_{V_n,\varphi})$ is defined as in Proposition 3.3. Note that $p_0(e')$ depends on φ.

The finiteness of $\sup_n \|a_n\|$ follows from part (1) of Lemma 6.1 below. This, together with Proposition 3.2, implies that $p_0(e')$ is an element in $C^*_{\text{alg}}(\{F_n, V_n^+\}) \otimes M_2(\mathbb{C})$.

Lemma 6.1. (1) *Let* $b = \bigoplus_{n=1}^{\infty} b_n \in C^*_{\text{alg}}(\{F_n\}_n)$, $\phi \in C_0(W_n, Cl(W_n))^+$. *Let* $c_n(b, \phi) = \bigoplus_{n=1}^{\infty} c_n \in C^*_{\text{alg}}(\{F_n, V_n^+\}_n)$ *be defined by*

$$(c_n(x, y))(v \oplus h) = b_n(x, y) \otimes \phi((v + f(x)) \oplus h)$$

for all $x, y \in F_n$ *and* $v \oplus h \in W_n = V_n \oplus V_n^*$, *where* ϕ *is identified with a function on* W_n *with values in* $Cl(W_n)$. *We have*

$$\|c(b, \phi)\| \leqslant \|b\| \ \|\phi\|.$$

(2) *Let* e, e' *and* δ *be as above. If* Γ *has bounded geometry, then* $\forall \varepsilon > 0, \exists\ \delta > 0$ *and* φ *such that*

$$\|(p_0(e'))^2 - p_0(e')\| < \varepsilon,$$

where φ *is as in the definition of* $F_{V_n,\varphi}$, *both* φ *and* δ *are independent of* n.

Proof. For each $m \geqslant 1$, recall that the norm on $\otimes^m W_n$ is defined by

$$\|u\| = \sup\{(\lambda_1 \otimes \cdots \lambda_m)(u) : \ \lambda_k \in W_n^*, \ \|\lambda_k\| \leqslant 1, \ 1 \leqslant k \leqslant m\}$$

for all $u \in \otimes^m W_n$, where W_n^* is the dual (Banach) space of W_n.

We also recall that $T(W_n)$ is the Banach space

$$\left\{ \bigoplus_{m=0}^{\infty} u_m : u_m \in \otimes^m W_n, \sum_{m=0}^{\infty} \|u_m\| < \infty \right\}$$

endowed with the norm

$$\left\| \bigoplus_{m=0}^{\infty} u_m \right\| = \sum_{m=0}^{\infty} \|u_m\|,$$

and $T_{\mathbb{C}}(W_n)$ is the complexification of $T(W_n)$.

Let $H_{F_n} = l^2(F_n) \otimes H$ and $H_{F_n} \otimes T_{\mathbb{C}}(W_n)$ be as in the definitions of $C^*(F_n, V_n) = C^*(F_n) \otimes C_0(W_n, Cl(W_n))$ and $C^*(F_n, V_n^+) = C^*(F_n) \otimes C_0(W_n, Cl(W_n))^+$.

Given a collection of elements $\{w_x\}_{x \in F_n}$ in $T_{\mathbb{C}}(W_n)$, we define an operator A on $H_{F_n} \otimes T_{\mathbb{C}}(W_n)$ by

$$A((\delta_x \otimes \eta) \otimes \xi) = (\delta_x \otimes \eta) \otimes (w_x \otimes \xi),$$

where $x \in F_n$, $\eta \in H$, $\xi \in T_{\mathbb{C}}(W_n)$, and δ_x is the Dirac function at x.

Claim 1. $\|A\| \leqslant \sup_{x \in F_n} \|w_x\|$.

Proof. Let $m \geqslant 1$. Given $v \in H_{F_n} \otimes (\otimes^m (W_n))$, we can write

$$v = \sum_{x \in F_n} (\delta_x \otimes v_x),$$

where $\delta_x \otimes v_x$ is an element in the closed subspace of $H_{F_n} \otimes (\otimes^m (W_n))$ spanned by all vectors $(\delta_x \otimes \eta) \otimes \xi$ for all $\eta \in H$ and $\xi \in \otimes^m (W_n)$.

By the definition of the Banach space norm on $H_{F_n} \otimes T_{\mathbb{C}}(W_n)$, we have

$$\begin{aligned} \|v\| &= \left\| \sum_{x \in F_n} (\delta_x \otimes v_x) \right\| \\ &= \sup_{\mu_x \in H^*,\ \|\mu_x\| \leqslant 1,\ \lambda_k \in W_n^*,\ \|\lambda_k\| \leqslant 1,\ 1 \leqslant k \leqslant m} \left(\sum_{x \in F_n} |(\mu_x \otimes \lambda_1 \otimes \cdots \otimes \lambda_m)(v_x)|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where H^* is the dual (Hilbert) space of H, and W_n^* is the dual (Banach) space of W_n.

Claim 1 follows from the above norm formula. $\square$

Let $H_{F_n} \otimes Cl(W_n)$ be the Banach space as in the definitions of $C^*(F_n, V_n) = C^*(F_n) \otimes C_0(W_n, Cl(W_n))$ and $C^*(F_n, V_n^+) = C^*(F_n) \otimes C_0(W_n, Cl(W_n))^+$.

Given a collection of elements $\{s_x\}_{x \in F_n}$ in $Cl(W_n)$, we define an operator B on $H_{F_n} \otimes Cl(W_n)$ by

$$B((\delta_x \otimes \eta) \otimes \xi) = (\delta_x \otimes \eta) \otimes (s_x\ \xi),$$

where $x \in F_n$, $\eta \in H$, $\xi \in Cl(W_n)$, and δ_x is the Dirac function at x, and $s_x\ \xi$ is the multiplication of s_x with ξ in $Cl(W_n)$.

Claim 1, together with the definition of the norm on $H_{F_n} \otimes Cl(W_n)$, implies the following:

Claim 2. $\|B\| \leqslant \sup_{x \in F_n} \|s_x\|$.

For each $w = v \oplus h \in W_n$, let $M_{\phi,w}$ be the bounded linear operator on $H_{F_n} \otimes Cl(W_n)$ defined by

$$M_{\phi,w}((\delta_x \otimes \eta) \otimes \xi) = (\delta_x \otimes \eta) \otimes (\phi_{x,w}\xi),$$

where $x \in F_n$, $\eta \in H$, $\xi \in Cl(W_n)$, δ_x is the Dirac function at x, $\phi_{x,w}$ is the element in $Cl(W_n)$ defined by $\phi_{x,w} = \phi((v + f(x)) \oplus h)$, and $\phi_{x,w}\xi$ is the multiplication of $\phi_{x,w}$ with ξ in $Cl(W_n)$.

By Claim 2 and the definition of the Banach space norm on $H_{F_n} \otimes Cl(W_n)$, we have

$$\|M_{\phi,w}\| \leqslant \|\phi\|.$$

Let $C_0(W_n, H_{F_n} \otimes Cl(W_n))$ be the Banach space in the definitions of $C^*(F_n, V_n) = C^*(F_n) \otimes C_0(W_n, Cl(W_n))$ and $C^*(F_n, V_n^+) = C^*(F_n) \otimes C_0(W_n, Cl(W_n))^+$.

We now define an operator T_ϕ on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$ by

$$(T_\phi \zeta)(w) = M_{\phi,w}(\zeta(w))$$

for all $\zeta \in C_0(W_n, H_{F_n} \otimes Cl(W_n))$ and $w \in W_n$.

By the above inequality, we have

$$\|T_\phi\| \leqslant \|\phi\|.$$

Let $\psi(b_n)$ be the operator on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$ defined by

$$(\psi(b_n)\zeta)(w) = (b_n \otimes 1)\zeta(w)$$

for all $\zeta \in C_0(W_n, H_{F_n} \otimes Cl(W_n))$ and $w \in W_n$.

By the definition of the norm on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$, we have

$$\|\psi(b_n)\| \leqslant \|b_n\|.$$

Note that $c_n(b_n, \phi)$ is the product of T_ϕ with $\psi(b_n)$ as operators on the Banach space $C_0(W_n, H_{F_n} \otimes Cl(W_n))$.

It follows that

$$\|c_n(b_n, \phi)\| \leqslant \|b_n\| \; \|\phi\|$$

for all $\phi \in C_0(W_n, Cl(W_n))^+$.

This proves part (1) of Lemma 6.1.

Recall that $index(F_{V_n,\varphi})$ is an element in $C_0(W_n, Cl(W_n))^+ \otimes M_2(\mathbb{C})$. By part (2) of Proposition 3.2, there exists $C > 0$ such that $index(F_{V_n,\varphi}) \leqslant C$ for all n. Now part (2) of Lemma 6.1 follows from part (1) of Lemma 6.1, Proposition 3.3, and the bounded geometry property of Γ. $\square$

Define $q_0(e') = \bigoplus_{n=1}^{\infty} b_n \in C^*_{\text{alg}}(\{F_n, V_n^+\}) \otimes M_2(\mathbb{C})$ by

$$(b_n(x, y))(v \oplus h) = e'_n(x, y) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

for all $x, y \in F_n$, and $v \oplus h \in W_n = V_n \oplus V_n^*$.

Let ε be as in Proposition 6.1. Choose ε to be sufficiently small. Let $p(e')$ and $q(e')$ be idempotents in $C^*(\{F_n, V_n^+\}) \otimes M_2(\mathbb{C})$ obtained from $p_0(e')$ and $q_0(e')$ by functional calculus. Note that $p(e') - q(e') \in C^*(\{F_n, V_n\}) \otimes M_2(\mathbb{C})$.

Definition 6.2. We define $\beta_{\{V_n\}}[e] \in K_0(C^*(\{F_n, V_n\}))$ to be the difference element $D(p(e'), q(e'))$.

By suspension, we can similarly define

$$\beta_{\{V_n\}} : K_1(C^*(\{F_n\})) \to K_1(C^*(\{F_n, V_n\})).$$

Note that the homomorphism $\beta_{\{V_n\}} : K_*(C^*(\{F_n\})) \to K_*(C^*(\{F_n, V_n\}))$, is independent of the choice of φ.

We can define the following Bott map by the same method:

$$\beta_{\{V_n\}} : K_* \left(C^* \left(\{F_n, \partial_\Gamma F_n\}\right)\right) \to K_* \left(C^* \left(\{F_n, \partial_\Gamma F_n, V_n\}_n\right)\right).$$

For any $R > 0$, we define $C^*(F_n, V_n)_R$ to be the closed subalgebra of $C^*(F_n, V_n)$ generated by elements $a \in C^*(F_n, V_n)$ such that *propagation* $(a) < R$ and *support* $(a(x, y)) \subseteq B_{W_n}(f(x) \oplus 0, R)$ for all $x, y \in F_n$. Note that there exists $R' > 0$ such that every element in $C^*(F_n, V_n)_R$ has propagation at most R', where R' does not depend on n.

Definition 6.3.

$$C^* \left(\{F_n, V_n\}_n\right)_R = \left\{ \bigoplus_{n=1}^{\infty} a_n \ : \ a_n \in C^* \left(F_n, V_n\right)_R \ , \ \sup_n \|a_n\| < +\infty \right\},$$

where $C^*(\{F_n, V_n\}_n)_R$ is endowed with the norm $\left\| \bigoplus_{n=1}^{\infty} a_n \right\| = \sup_n \|a_n\|$.

We can similarly define the Banach algebras $C^*(\{\partial_\Gamma F_n, V_n\}_n)_R$ and $C^*(\{F_n, \partial_\Gamma F_n, V_n\}_n)_R$.

Lemma 6.4. *We have*

(1) $C^*(\{F_n, V_n\}_n) = \lim_{R\to\infty} C^*(\{F_n, V_n\}_n)_R$;
(2) $C^*(\{\partial_\Gamma F_n, V_n\}_n) = \lim_{R\to\infty} C^*(\{\partial_\Gamma F_n, V_n\}_n)_R$;
(3) $C^*(\{F_n, \partial_\Gamma F_n, V_n\}_n) = \lim_{R\to\infty} C^*(\{F_n, \partial_\Gamma F_n, V_n\}_n)_R$.

Note that there exists a natural homomorphism

$$\lim_{R\to\infty} K_*(C^*(\{F_n, V_n\}_n)_R) \to \lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C^*(F_n, V_n)_R),$$

where $\bigoplus_{n=1}^{\infty} K_*(C^*(F_n, V_n)_R) = \{\bigoplus_{n=1}^{\infty} z_n : z_n \in K_*(C^*(F_n, V_n)_R)\}$.

Similarly there exists a natural homomorphism

$$\lim_{R\to\infty} K_*(C^*(\{F_n, \partial_\Gamma F_n, V_n\}_n)_R) \to \lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C^*(F_n, \partial_\Gamma F_n, V_n)_R),$$

where $\bigoplus_{n=1}^{\infty} K_*(C^*(F_n, \partial_\Gamma F_n, V_n)_R) = \{\bigoplus_{n=1}^{\infty} z_n : z_n \in K_*(C^*(F_n, \partial_\Gamma F_n, V_n)_R)\}$.

Composing $\beta_{\{V_n\}}$ with the above homomorphisms, we obtain homomorphisms (still denoted by $\beta_{\{V_n\}}$)

$$K_*(C^*(\{F_n\}_n)) \to \lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C^*(F_n, V_n)_R),$$

$$K_*(C^*(\{F_n, \partial_\Gamma F_n\}_n)) \to \lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C^*(F_n, \partial_\Gamma F_n, V_n)_R).$$

Proposition 6.5. *Assume that $\{V_n'\}$ is a sequence of finite-dimensional subspaces of X such that $V_n \subseteq V_n'$. For any $[z] \in K_*\left(C^*\left(\{F_n, \partial_\Gamma F_n\}_n\right)\right)$,*

$$\beta_{\{V_n\}}[z] = 0 \text{ in } \lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C^*(F_n, \partial_\Gamma F_n, V_n)_R)$$

if and only if

$$\beta_{\{V_n'\}}[z] = 0 \text{ in } \lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C^*(F_n, \partial_\Gamma F_n, V_n')_R).$$

Roughly speaking, the proof of Proposition 6.5 is based on the idea that the product of Bott elements for two finite-dimensional vector spaces is the Bott element for the product space of the two vector spaces. We need some preparations before we can prove Proposition 6.5.

Next, we shall introduce the concept of a Fredholm pair for Banach algebras and define its index. The concept of Fredholm pair and its index is motivated by KK-theory. In the special case of C^*-algebras, it is compatible with the corresponding construction in KK-theory.

Let B be a graded unital complex Banach algebra with the grading induced by a grading operator ε in B satisfying $\varepsilon^2 = 1$ and $\|\varepsilon\| = 1$. Let A be a graded closed two sided ideal in B.

Let F be an element of degree 1 in B and let e be an idempotent of degree 0 in B.

(F, e) is said to be a Fredholm pair if

(1) $e(F^2 - 1) \in A$;
(2) $eF - Fe \in A$.

We define *Index*$(F, e) \in K_0(A)$ of a Fredholm pair (F, e) as follows.

Let

$$p = \left(\frac{1+\varepsilon}{2}\right) e \left(\frac{1+\varepsilon}{2}\right),$$

$$q = \left(\frac{1-\varepsilon}{2}\right) e \left(\frac{1-\varepsilon}{2}\right),$$

$$\alpha = \left(\frac{1+\varepsilon}{2}\right) F \left(\frac{1-\varepsilon}{2}\right),$$

$$\alpha' = \left(\frac{1-\varepsilon}{2}\right) F \left(\frac{1+\varepsilon}{2}\right).$$

Define

$$a_{11}(F, e) = 1 + (p - p\alpha q\alpha' p) p\alpha p\alpha' p + (p\alpha q\alpha' p - p),$$

$$a_{12}(F, e) = (p - p\alpha q\alpha' p) p\alpha q (q - q\alpha' p\alpha q) + p\alpha q (q - q\alpha p\alpha' q),$$

$$a_{21}(F, e) = (q - q\alpha' p\alpha q) q\alpha' p,$$

$$a_{22}(F, e) = (q - q\alpha' p\alpha q)^2.$$

Define

$$\mathit{index}(F, e) = \begin{pmatrix} a_{11}(F, e) & a_{12}(F, e) \\ a_{21}(F, e) & a_{22}(F, e) \end{pmatrix}.$$

It is not difficult to verify that $index(F, e)$ is an idempotent in $M_2(A)$.
We define

$$Index(F, e) = [index(F, e)] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \in K_0(A),$$

where $K_0(A)$ is the K-group of A considered as a Banach algebra without grading.
We remark that if $e = 1$, then

$$Index(F, e) = Index(F),$$

where $Index(F)$ is defined as in Section 2.
Let A_1 and A_2 be two Banach algebras. Assume that $A_1 \otimes A_2$ is a Banach algebra tensor product of A_1 and A_2. We shall need an explicit construction of the product:

$$K_0(A_1) \times K_0(A_2) \to K_0(A_1 \otimes A_2).$$

Let $\pi_i : A_i^+ \to \mathbb{C}$ be the homomorphism defined by $\pi_i(a + cI) = c$ for any $a \in A_i$, $c \in \mathbb{C}$ and $i = 1, 2$, where A_i^+ is obtained from A_i by adjoining the identity.
Let $\pi : A_1^+ \otimes A_2 + A_1 \otimes A_2^+ \to \mathbb{C} \otimes A_2 + A_1 \otimes \mathbb{C}$ be the homomorphism defined by

$$\pi(a_1 \otimes a_2 + b_1 \otimes b_2) = \pi_1(a_1) \otimes a_2 + b_1 \otimes \pi_2(b_2)$$

for any $a_1 \otimes a_2 \in A_1^+ \otimes A_2$, $b_1 \otimes b_2 \in A_1 \otimes A_2^+$. We have the following split exact sequence:

$$0 \to A_1 \otimes A_2 \to A_1^+ \otimes A_2 + A_1 \otimes A_2^+ \underset{i}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \otimes A_2 + A_1 \otimes \mathbb{C} \to 0,$$

where $i : \mathbb{C} \otimes A_2 + A_1 \otimes \mathbb{C} \to A_1^+ \otimes A_2 + A_1 \otimes A_2^+$ is the inclusion homomorphism.
Next, we shall construct an explicit homomorphism

$$\phi : Ker\,\pi_* \to K_0(A_1 \otimes A_2).$$

Let p be an idempotent in $M_n((A_1^+ \otimes A_2 + A_1 \otimes A_2^+)^+)$ such that $[p] - [i\pi(p)]$ represents an element in $Ker\,\pi_*$.
Let

$$Z = \begin{pmatrix} i\pi(p) & 0 & 1 - i\pi(p) & 0 \\ 1 - i\pi(p) & 0 & 0 & i\pi(p) \\ 0 & 0 & i\pi(p) & 1 - i\pi(p) \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We have

$$Z^{-1} = \begin{pmatrix} i\pi(p) & 1 - i\pi(p) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 - i\pi(p) & 0 & i\pi(p) & 0 \\ 0 & i\pi(p) & 1 - i\pi(p) & 0 \end{pmatrix}.$$

Let

$$\phi_0(p) = Z^{-1} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1 - i\pi(p) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z.$$

We define

$$\phi\ ([p] - [i\pi(p)]) = [\phi_0(p)] - \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \in K_0(A_1 \otimes A_2).$$

Lemma 6.6. *The homomorphism ϕ from $Ker\,\pi_*$* to *$K_0(A_1 \otimes A_2)$ is an isomorphism.*

The product $K_0(A_1) \times K_0(A_2) \to K_0(A_1 \otimes A_2)$ can now be described as follows. Given idempotents p_0, p_1 in $M_n(A_1^+)$ and q_0, q_1 in $M_n(A_2^+)$ representing $[p_0] - [p_1] \in K_0(A_1)$ and $[q_0] - [q_1] \in K_0(A_2)$ such that $\pi_1(p_0) = \pi_1(p_1)$, $\pi_2(q_0) = \pi_2(q_1)$, note that

$$D((p_0 \otimes q_0) \oplus (p_1 \otimes q_1),\ (p_0 \otimes q_1) \oplus (p_1 \otimes q_0)) \in Ker\,\pi_* \subseteq K_0(A_1^+ \otimes A_2 + A_1 \otimes A_2^+).$$

We define the product of $[p_0] - [p_1]$ and $[q_0] - [q_1]$ to be

$$\phi(D((p_0 \otimes q_0) \oplus (p_1 \otimes q_1),\ (p_0 \otimes q_1) \oplus (p_1 \otimes q_0))) \in K_0(A_1 \otimes A_2).$$

Let B_1 and B_2 be two graded unital complex Banach algebras with gradings induced by grading operators ε_1 and ε_2 (respectively in B_1 and B_2) satisfying $\varepsilon_i^2 = 1$ and $\|\varepsilon_i\| = 1$ for $i = 1, 2$. Let A_1 and A_2 be, respectively, graded closed two sided ideals in B_1 and B_2.

Assume that F_i is an element of degree 1 in B_i satisfying

(1) $F_i^2 - 1 \in A_i$,
(2) $(1 - F_i^2)^{1/2}$ is a well defined element in A_i,

where $i = 1,\ 2$.

Assume that $B_1 \otimes B_2$ is a Banach tensor product of B_1 and B_2. Endow $B_1 \otimes B_2$ with the grading induced by the grading operator $\varepsilon_1 \otimes \varepsilon_2$.

Let M and N be elements in $B_1 \otimes B_2$ such that

(1) M and N have degree 0 and respectively commute with $F_1 \otimes 1$ and $1 \otimes F_2$;
(2) $M^2 + N^2 - 1 \in A_1 \otimes A_2$;
(3) $M((1 - F_1^2)^{1/2} \otimes 1)$ and $N(1 \otimes (1 - F_2^2)^{1/2})$ are elements in $A_1 \otimes A_2$.

Define

$$F = M(F_1 \otimes 1) + N(\varepsilon_1 \otimes F_2) \in B_1 \otimes B_2.$$

It is easy to check that F has degree 1 and $F^2 - 1 \in A_1 \otimes A_2$.

Hence we can define

$$Index(F) \in K_0(A_1 \otimes A_2).$$

Proposition 6.7. *Assume that there exist homotopies M_t and N_t ($t \in [0, 1]$) in $B_1 \otimes B_2$ such that*

(1) *M_t and N_t have degree 0 for each $t \in [0, 1]$;*
(2) *M_t and N_t commute, respectively, with $F_1 \otimes 1$ and $1 \otimes F_2$ for each $t \in [0, 1]$;*
(3) *M_t and N_t commute, respectively, with $(1 - F_1^2)^{1/2} \otimes 1$ and $1 \otimes (1 - F_2^2)^{1/2}$ for each $t \in [0, 1]$;*
(4) *$M_t^2 + N_t^2 - 1 \in A_1 \otimes A_2$ for all $t \in [0, 1]$;*
(5) *$M_t((A_1^+ \otimes A_2) \bigoplus (A_1 \otimes A_2^+)) \subseteq (A_1^+ \otimes A_2) \bigoplus (A_1 \otimes A_2^+)$ for all $t \in [0, 1]$,*
$((A_1^+ \otimes A_2) \bigoplus (A_1 \otimes A_2^+))M_t \subseteq (A_1^+ \otimes A_2) \bigoplus (A_1 \otimes A_2^+)$ for all $t \in [0, 1]$,
$N_t((A_1^+ \otimes A_2) \bigoplus (A_1 \otimes A_2^+)) \subseteq (A_1^+ \otimes A_2) \bigoplus (A_1 \otimes A_2^+)$ for all $t \in [0, 1]$,
and $((A_1^+ \otimes A_2) \bigoplus (A_1 \otimes A_2^+))N_t \subseteq (A_1^+ \otimes A_2) \bigoplus (A_1 \otimes A_2^+)$ for all $t \in [0, 1]$;
(6) *$M_0 = M,\ N_0 = N,\ M_1 = N_1 = 1/\sqrt{2}$.*

Then $Index(F)$ is equal to the product of $Index(F_1)$ with $Index(F_2)$ in $K_0(A_1 \otimes A_2)$.

Proof. Let

$$\tilde{F}_i = \begin{pmatrix} F_i & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1,\ 2,$$

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $i = 1, 2$, let $M_2(B_i)$ be endowed with the grading induced by the grading operator:

$$\begin{pmatrix} \varepsilon_i & 0 \\ 0 & -\varepsilon_i \end{pmatrix}.$$

$(\tilde{F}_i, p_0)$ is a Fredholm pair for $i = 1, 2$. We have

$$Index(\tilde{F}_i, p_0) = Index(F_i).$$

Let

$$W_i = \frac{1}{2}\begin{pmatrix} 1+\varepsilon_i & 1-\varepsilon_i \\ 1-\varepsilon_i & 1+\varepsilon_i \end{pmatrix}, \quad i = 1,\ 2,$$

$$\varepsilon_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$W_i^{-1}\begin{pmatrix} \varepsilon_i & 0 \\ 0 & -\varepsilon_i \end{pmatrix} W_i = \varepsilon_0\,, \quad i = 1,\ 2\,.$$

Let

$$E_i = W_i^{-1}\tilde{F}_i W_i, \ \ e_i = W_i^{-1} p_0 W_i$$

for $i = 1,\ 2$.

Note that (E_i, e_i) is a Fredholm pair with respect to the grading on $M_2(B_i)$ induced by the grading operator ε_0 for $i = 1, 2$. We have

$$Index(E_i, e_i) = Index(F_i)$$

for $i = 1,\ 2$.

Let

$$G_i = \begin{pmatrix} E_i & (1-E_i^2)^{1/2} \\ (1-E_i^2)^{1/2} & -E_i \end{pmatrix}, \quad i = 1,\ 2\,.$$

Define

$$F' = M'(G_1 \otimes 1) + N'((\varepsilon_0 \oplus -\varepsilon_0) \otimes G_2)) \in M_2(M_2(B_1)) \otimes M_2(M_2(B_2))\,,$$

where $M' = \sum_{k=1}^{16} M$, $N' = \sum_{k=1}^{16} N \in M_{16}(B_1 \otimes B_2) \cong M_2(M_2(B_1)) \otimes M_2(M_2(B_2))$.

Endow $M_2(M_2(B_1)) \otimes M_2(M_2(B_2))$ with the grading operator

$$\varepsilon' = (\varepsilon_0 \oplus -\varepsilon_0) \otimes (\varepsilon_0 \oplus -\varepsilon_0).$$

Define

$$e' = \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(M_2(B_1)) \otimes M_2(M_2(B_2)) .$$

It is not difficult to verify that (F', e') is a Fredholm pair with respect to the grading given by the grading operator ε'. We have

$$Index(F) = Index(F', e').$$

Define

$$\begin{aligned} F'' &= M''((G_1 \oplus - G_1) \otimes 1) + N''(((\varepsilon_0 \oplus - \varepsilon_0) \oplus (-\varepsilon_0 \oplus \varepsilon_0)) \otimes (G_2 \oplus - G_2)) \\ &\in M_2(M_4(B_1)) \otimes M_2(M_4(B_2)) , \end{aligned}$$

where $M'' = M' \oplus M' \oplus M' \oplus M'$, $N'' = N' \oplus N' \oplus N' \oplus N'$, and $M_2(M_4(B_1)) \otimes M_2(M_4(B_2))$ is endowed with the grading operator

$$\varepsilon'' = ((\varepsilon_0 \oplus - \varepsilon_0) \oplus (-\varepsilon_0 \oplus \varepsilon_0)) \otimes ((\varepsilon_0 \oplus - \varepsilon_0) \oplus (-\varepsilon_0 \oplus \varepsilon_0)) .$$

Define

$$\begin{aligned} e'' &= \left(\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \otimes \left(\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \\ &\in M_2(M_4(B_1)) \otimes M_2(M_4(B_2)) . \end{aligned}$$

Note that (F'', e'') is a Fredholm pair with respect to the grading given by the grading operator ε''. We have

$$Index(F', e') = Index(F'', e'').$$

Let

$$U_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -G_i \\ G_i & 1 \end{pmatrix}$$

for $i = 1, 2$.

We have

$$U_i \begin{pmatrix} G_i & 0 \\ 0 & -G_i \end{pmatrix} U_i^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $i = 1, 2$.

Note that $U_1 \otimes 1$ and $1 \otimes U_2$ have degree 0 in $M_2(M_4(B_1)) \otimes M_2(M_4(B_2))$. Let

$$\begin{aligned} F''' &= (1 \otimes U_2)(U_1 \otimes 1)F''(U_1 \otimes 1)^{-1}(1 \otimes U_2)^{-1} \\ &= M''\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1\right) + N''\left(((\varepsilon_0 \oplus -\varepsilon_0) \oplus (-\varepsilon_0 \oplus \varepsilon_0)) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \end{aligned}$$

$$e''' = \left(U_1\left(\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)U_1^{-1}\right) \otimes \left(U_2\left(\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)U_2^{-1}\right).$$

Note that ε'' commutes with $(1 \otimes U_2)(U_1 \otimes 1)$. It follows that (F''', e''') is a Fredholm pair with respect to the grading induced by the grading operator ε''. We have

$$Index(F''', e''') = Index(F'', e'').$$

Let

$$F_i''' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(M_4(B_i)),$$

$$e_i''' = U_i\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)U_i^{-1} \in M_2(M_4(B_i)), \quad i = 1, 2.$$

Note that (F_i''', e_i''') is a Fredholm pair with respect to the grading given by the grading operator $(\varepsilon_0 \oplus -\varepsilon_0) \oplus (-\varepsilon_0 \oplus \varepsilon_0)$ in $M_2(M_4(B_i))$.

We have

$$Index(F_i) = Index(F_i''', e_i''')$$

for $i = 1, 2$.

Let

$$V = \frac{1}{2}\begin{pmatrix} 1 + (\varepsilon_0 \oplus -\varepsilon_0) & 1 - (\varepsilon_0 \oplus -\varepsilon_0) \\ 1 - (\varepsilon_0 \oplus -\varepsilon_0) & 1 + (\varepsilon_0 \oplus -\varepsilon_0) \end{pmatrix}.$$

We have

$$V^{-1}\begin{pmatrix} \varepsilon_0 \oplus -\varepsilon_0 & 0 \\ 0 & -\varepsilon_0 \oplus \varepsilon_0 \end{pmatrix} V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note also that V commute with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Write

$$V^{-1}e_i'''V = \begin{pmatrix} p_i & 0 \\ 0 & q_i \end{pmatrix} \in M_2(M_4(B_i)), \quad i = 1,\ 2.$$

Let

$$e_i'''' = \begin{pmatrix} p_i & 0 & 0 & 0 \\ 0 & 1-q_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} q_i & 0 & 0 & 0 \\ 0 & 1-q_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_{2\times 4^2}(B_i),$$

$$F'''' = M'''\left(\begin{pmatrix} 0 & I' \\ I' & 0 \end{pmatrix} \otimes I''\right) + N'''\left(\tau \otimes \begin{pmatrix} 0 & I' \\ I' & 0 \end{pmatrix}\right) \in M_{2\times 4^2}(B_1) \otimes M_{2\times 4^2}(B_2),$$

where $I' = I \oplus I \oplus I \oplus I$ (I is the identity element in $M_4(B_1)$ or $M_4(B_2)$), $I'' = I' \oplus I'$, $M''' = M'' \oplus M'' \oplus M'' \oplus M''$, $N''' = N'' \oplus N'' \oplus N'' \oplus N''$, and

$$\tau = \begin{pmatrix} I' & 0 \\ 0 & -I' \end{pmatrix}.$$

Let

$$e'''' = e_1'''' \otimes e_2'''' \in M_{2\times 4^2}(B_1) \otimes M_{2\times 4^2}(B_2).$$

Observe that (F'''', e'''') is a Fredholm pair with respect to the grading induced by the grading operator $\tau \otimes \tau$. We have

$$Index(F''', e''') = Index(F'''', e'''').$$

Let

$$Z(p_i, q_i) = \begin{pmatrix} q_i & 0 & 1-q_i & 0 \\ 1-q_i & 0 & 0 & q_i \\ 0 & 0 & q_i & 1-q_i \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Write

$$(Z(p_i, q_i) \oplus Z(p_i, q_i))^{-1} e_i''''(Z(p_i, q_i) \oplus Z(p_i, q_i)) = \begin{pmatrix} p_i' & 0 \\ 0 & q_i' \end{pmatrix} \in M_2(M_{4^2}(A_i)).$$

We have

$$p_i' - q_i' \in M_{4^2}(A_i) \qquad \cdots\cdots\cdots \qquad (J).$$

Note that $(Z(p_1,q_1)\otimes 1)\bigoplus(Z(p_1,q_1)\otimes 1)$ and $(1\otimes Z(p_2,q_2))\bigoplus(1\otimes Z(p_2,q_2))$ commute with F''''.

Define

$$H(t) = M_t'\left(\begin{pmatrix} 0 & I' \\ I' & 0\end{pmatrix}\otimes I''\right) + N_t'\left(\tau\otimes\begin{pmatrix} 0 & I' \\ I' & 0\end{pmatrix}\right) \in M_{2\times 4^2}(B_1)\otimes M_{2\times 4^2}(B_2),$$

where $M_t' = \bigoplus_{k=1}^{2\times 4^4} M_t$ and $N_t' = \bigoplus_{k=1}^{2\times 4^4} N_t$.

Let

$$g = \begin{pmatrix} p_1' & 0 \\ 0 & q_1'\end{pmatrix}\otimes\begin{pmatrix} p_2' & 0 \\ 0 & q_2'\end{pmatrix} \in M_{2\times 4^2}(B_1)\otimes M_{2\times 4^2}(B_2).$$

We have

$$H(t)g - gH(t) \in (M_{2\times 4^2}(A_1^+)\otimes M_{2\times 4^2}(A_2)) + (M_{2\times 4^2}(A_1)\otimes M_{2\times 4^2}(A_2^+)),$$

$$H(t)^2 - 1 \in M_{2\times 4^2}(A_1)\otimes M_{2\times 4^2}(A_2).$$

Hence, by the formula of $index(H(t),g)$, condition (5) of the proposition and the above property (J) of p_i' and q_i', we know that $index(H(t),g)$ is a homotopy of idempotents in $M_2(((M_{2\times 4^2}(A_1^+)\otimes M_{2\times 4^2}(A_2)) + (M_{2\times 4^2}(A_1)\otimes M_{2\times 4^2}(A_2^+)))^+)$ (note that although $(H(t),g)$ may not be a Fredholm pair, $index(H(t),g)$ is still well defined as an idempotent for each t).

It follows that

$$Index(H(1),g) = Index(H(0),g)$$

in $K_0((M_{2\times 4^2}(A_1^+)\otimes M_{2\times 4^2}(A_2)) + (M_{2\times 4^2}(A_1)\otimes M_{2\times 4^2}(A_2^+)))$.

This implies that

$$Index(H(1),g) \in Ker\,\pi_*,$$

where π is the natural homomorphism from $(M_{2\times 4^2}(A_1^+)\otimes M_{2\times 4^2}(A_2)) + (M_{2\times 4^2}(A_1)\otimes M_{2\times 4^2}(A_2^+))$ to $(M_{2\times 4^2}(\mathbb{C})\otimes M_{2\times 4^2}(A_2)) + (M_{2\times 4^2}(A_1)\otimes M_{2\times 4^2}(\mathbb{C}))$.

Note that

$$H(1) = \sqrt{\frac{1}{2}}\left(\left(\begin{pmatrix} 0 & I' \\ I' & 0 \end{pmatrix} \otimes I''\right) + \left(\tau \otimes \begin{pmatrix} 0 & I' \\ I' & 0 \end{pmatrix}\right)\right).$$

It is not difficult to see that there exists a homotopy $H_1(t)$ in $M_{2\times 4^2}(\mathbb{C})\otimes M_{2\times 4^2}(\mathbb{C}) \subseteq M_{2\times 4^2}(B_1) \otimes M_{2\times 4^2}(B_2)$ such that

(1) $H_1(t)$ has degree one for all $t \in [0, 1]$;
(2) $H_1(0) = H(1)$;
(3) $H_1(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(M_2((M_{2\times 4}(B_1) \otimes M_{2\times 4}(B_2))))$, where $M_2(M_2((M_{2\times 4}(B_1) \otimes M_{2\times 4}(B_2))))$ is a graded algebra with the grading operator $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and is identified with $M_{2\times 4^2}(B_1) \otimes M_{2\times 4^2}(B_2)$ as graded algebras;
(4) $(H_1(t))^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;
(5) $H_1(t)g - gH_1(t) \in M_2(M_2((M_{2\times 4}(A_1^+)\otimes M_{2\times 4}(A_2)) + (M_{2\times 4}(A_1)\otimes M_{2\times 4}(A_2^+)))) \subseteq M_2(M_2((M_{2\times 4}(B_1) \otimes M_{2\times 4}(B_2))))$ for all $t \in [0, 1]$.

By the formula of *index*$(H_1(t), g)$ and the above properties of $H_1(t)$, we know that *index*$(H_1(t), g)$ is a homotopy of idempotents in $M_2(((M_{2\times 4^2}(A_1^+) \otimes M_{2\times 4^2}(A_2)) + (M_{2\times 4^2}(A_1) \otimes M_{2\times 4^2}(A_2^+)))^+)$ (note that although $(H_1(t), g)$ may not be a Fredholm pair, *index*$(H_1(t), g)$ is still well defined as an idempotent for each t).

It follows that

$$\mathit{Index}(H_1(1), g) = \mathit{Index}(H_1(0), g) = j_*(\mathit{Index}(F))$$

in $K_0((M_{2\times 4^2}(A_1^+) \otimes M_{2\times 4^2}(A_2)) + (M_{2\times 4^2}(A_1) \otimes M_{2\times 4^2}(A_2^+)))$, where j is the natural inclusion homomorphism from $A_1 \otimes A_2$ to $(M_{2\times 4^2}(A_1^+) \otimes M_{2\times 4^2}(A_2)) + (M_{2\times 4^2}(A_1) \otimes M_{2\times 4^2}(A_2^+))$.

Now Proposition 6.7 follows from the fact that $\phi(\mathit{Index}(H_1(1), g))$ is the product of *Index*(F_1''', e_1''') with *Index*(F_2''', e_2'''), where ϕ is as in Lemma 6.6. $\square$

Let V_n and V_n' be as in Proposition 6.5. Let V_n'' be a finite-dimensional subspace of X such that $V_n' = V_n \oplus V_n''$.

Define

$$\|x\|_1 = \sqrt{\|x_1\|^2 + \|x_2\|^2}$$

$$x^{*,1} = x_1^* \oplus x_2^*$$

for any $x = x_1 \oplus x_2 \in V_n \oplus V_n''$, and

$$\|h\|_1 = \sqrt{\|h_1\|^2 + \|h_2\|^2}$$

$$h^{*,1} = h_1^* \oplus h_2^*$$

for any $h = h_1 \oplus h_2 \in (V_n')^* = V_n^* \oplus (V_n'')^*$.

We define $F_{V_n',\varphi,1} \in C_{\mathrm{b}}(W_n', Cl(W_n'))$ by

$$F_{V_n',\varphi,1}(x \oplus h) = \frac{\varphi\left(\sqrt{\|x\|_1^2 + \|h\|_1^2}\right)}{\sqrt{\|x\|_1^2 + \|h\|_1^2 + i(h(x) - x^{*,1}(h^{*,1}))}} \left(\frac{(h^{*,1} \oplus x^{*,1}) - x \oplus h}{2} + i\frac{x \oplus h + (h^{*,1} \oplus x^{*,1})}{2}\right)$$

for all $x \oplus h \in W_n' = V_n' \oplus (V_n')^*$.

Let $W_n' = V_n' \oplus (V_n')^*$, $W_n = V_n \otimes (V_n)^*$, and $W_n'' = V_n'' \oplus (V_n'')^*$.

For each $k \geqslant 1$ and $l \geqslant 1$, we define a norm on $(\otimes^k W_n) \otimes (\otimes^l W_n'')$ by

$$\|u\| = \sup_{\lambda_i \in W_n^*, \mu_j \in W_n'', \|\lambda_i\| \leqslant 1, \|\mu_j\| \leqslant 1, 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l} ((\lambda_1 \otimes \cdots \otimes \lambda_k) \otimes (\mu_1 \otimes \cdots \otimes \mu_l))(u)$$

for all $u \in (\otimes^k W_n) \otimes (\otimes^l W_n'')$.

This norm can be extended to construct Banach algebra tensor products $T(W_n) \otimes T(W_n')$ and $T_{\mathbb{C}}(W_n) \otimes T_{\mathbb{C}}(W_n')$. The Banach algebra norm on $T_{\mathbb{C}}(W_n) \otimes T_{\mathbb{C}}(W_n')$ induces a Banach algebra norm on $Cl(W_n) \otimes Cl(W_n'')$ by a quotient construction. It is not difficult to see that $Cl(W_n) \otimes Cl(W_n')$ is naturally isomorphic to $Cl(W_n')$.

We define a norm on the algebraic tensor product $C_0(W_n, Cl(W_n)) \otimes_{\mathrm{alg}} C_0(W_n'', Cl(W_n''))$ by

$$\left\|\sum_k f_k \otimes g_k\right\| = \sup_{w \in W_n, w'' \in W_n''} \left\|\sum_k (f_k(w) \otimes g_k(w''))\right\|$$

for all $\sum_k f_k \otimes g_k \in C_0(W_n, Cl(W_n)) \otimes C_0(W_n'', Cl(W_n''))$.

We define $C_0(W_n, Cl(W_n)) \otimes C_0(W_n'', Cl(W_n''))$ to be the norm closure $C_0(W_n, Cl(W_n)) \otimes_{\mathrm{alg}} C_0(W_n'', Cl(W_n''))$. Observe that $C_0(W_n, Cl(W_n)) \otimes C_0(W_n'', Cl(W_n''))$ is isomorphic to $C_0(W_n', Cl(W_n'))$ as Banach algebras.

For each natural number k, the above Banach algebra isomorphism can be naturally extended to a Banach algebra isomorphism from $M_k(C_0(W_n, Cl(W_n)) \otimes C_0(W_n'', Cl(W_n'')))$ to $M_k(C_0(W_n', Cl(W_n')))$.

We can similary define the graded Banach tensor product $C_0(W_n, Cl(W_n))\widehat{\otimes} C_0(W_n'', Cl(W_n''))$, where the gradings on $C_0(W_n, Cl(W_n))$ and $C_0(W_n'', Cl(W_n''))$ are induced by the natural gradings on the Clifford algebras.

Proposition 6.8. *Let $F_{V_n',\varphi,1}$ be as above. Given $r > 0$ and $\varepsilon > 0$, there exist φ and a natural number m such that $index(F_{V_n',\varphi,1}) \oplus p_m$ is homotopy equivalent to $\phi_0(D_0((index(F_{V_n,\varphi}) \otimes index(F_{V_n'',\varphi})) \oplus (p_1 \otimes p_1), (index(F_{V_n,\varphi}) \otimes p_1) \oplus (p_1 \otimes index(F_{V_n'',\varphi})))) \oplus p_m$ in $M_k((C_0(W_n', Cl(W_n')))^+)$ through a homotopy of idempotents which are (r, ε)-flat relative to W_n, where ϕ_0 is as in the definition of ϕ in Lemma 6.6, D_0 is as in the definition of the difference construction D in this section, and p_m is the direct sum of m copies of the identity 1 and m copies of 0.*

Proof. Let

$$M = \frac{\sqrt{\|x_1\|^2 + \|h_1\|^2 + i(h_1(x_1) - x_1^*(h_1^*))}\ \varphi\left(\sqrt{\|x\|_1^2 + \|h\|_1^2}\right)}{\sqrt{\|x\|_1^2 + \|h\|_1^2 + i(h(x) - x^{*,1}(h^{*,1}))}},$$

$$N = \frac{\sqrt{\|x_2\|^2 + \|h_2\|^2 + i(h_2(x_2) - x_2^*(h_2^*))}\ \varphi\left(\sqrt{\|x\|_1^2 + \|h\|_1^2}\right)}{\sqrt{\|x\|_1^2 + \|h\|_1^2 + i(h(x) - x^{*,1}(h^{*,1}))}},$$

for all $x = x_1 \oplus x_2 \in V_n' = V_n \oplus V_n''$ and $h = h_1 \oplus h_2 \in (V_n')^* = (V_n)^* \oplus (V_n'')^*$.

Let ε be the grading operator in $C_b(W_n, Cl(W_n))$ induced by the natural grading of $Cl(W_n)$, where $W_n = V_n \oplus V_n^*$.

Define

$$F_{V_n',\varphi,0} = M((F_{V_n,\varphi} \otimes 1) + N(\varepsilon \otimes F_{V_n'',\varphi})),$$

where ε is the grading operator.

Let

$$F(t) = tF_{V_n',\varphi,1} + (1-t)F_{V_n',\varphi,0}$$

for $t \in [0, 1]$.

Observe that $index(F(t))$ is a homotopy between $index(F_{V_n',\varphi,1})$ and $index(F_{V_n',\varphi,1})$. Furthermore, $index(F(t))$ is (r, ε)-flat relative to W_n for a suitable choice of φ.

Let

$$M_t = \sqrt{(1-t)M^2 + \frac{t}{2}},$$

$$N_t = \sqrt{(1-t)N^2 + \frac{t}{2}}$$

for all $t \in [0, 1]$.

It is not difficult to see that M_t and N_t satisfy the conditions of Proposition 6.7. Now Proposition 6.8 follows from Proposition 6.7 and its proof. $\square$

Proof of Proposition 6.5. We shall prove the K_0 case. The K_1 case can be proved in a similar way by a suspension argument.

Let $V_n' = V_n \oplus V_n''$. For any $g \in V_n^*$, we extend g to an element in $(V_n')^*$ by defining $g(x) = 0$ if $x \in V_n''$. Thus we can identify V_n^* with a subspace of $(V_n')^*$. Similarly we identify $(V_n'')^*$ with a subspace of $(V_n')^*$. We have

$$(V_n')^* = V_n^* \oplus (V_n'')^*.$$

Define

$$\|x\|_1 = \sqrt{\|x_1\|^2 + \|x_2\|^2}$$

for any $x = x_1 \oplus x_2 \in V_n \oplus V_n''$, and

$$\|g\|_1 = \sqrt{\|g_1\|^2 + \|g_2\|^2}$$

for any $g = g_1 \oplus g_2 \in (V_n')^* = V_n^* \oplus (V_n'')^*$.

Define

$$\|x\|_t^2 = t(\|x\|_1)^2 + (1-t)\|x\|^2,$$

$$\|g\|_t^2 = t(\|g\|_1)^2 + (1-t)\|g\|^2,$$

for all $t \in [0, 1]$, $x = x_1 \oplus x_2 \in V_n \oplus V_n''$ and $g = g_1 \oplus g_2 \in (V_n')^* = V_n^* \oplus (V_n'')^*$.

For each $t \in [0, 1]$, let $W_n' = V_n' \oplus (V_n')^*$ be the given norm,

$$\|x \oplus g\|_t = \sqrt{\|x\|_t^2 + \|g\|_t^2}$$

for all $x \oplus g \in W_n' = V_n' \oplus (V_n')^*$.

Let

$$x^{*,t} = t(x_1^* \oplus x_2^*) + (1-t)x^*,$$

$$g^{*,t} = t(g_1^* \oplus g_2^*) + (1-t)g^*,$$

for any $t \in [0, 1]$, $x = x_1 \oplus x_2 \in V_n \oplus V_n''$, $g = g_1 \oplus g_2 \in (V_n')^* = V_n^* \oplus (V_n'')^*$.

Let φ be a continuous function on $\mathbb{R}$ such that $0 \leqslant \varphi(t) \leqslant 1$, $\exists\, 0 < c_1 < c_2$ satisfying $\varphi(t) = 0$ if $t \leqslant c_1$ and $\varphi(t) = 1$ if $t \geqslant c_2$.

For each $t \in [0, 1]$, let $F_{V_n',\varphi,t}, \in C_{\mathrm{b}}(W_n', Cl(W_n'))$ be defined by

$$F_{V_n',\varphi,t}(0 \oplus 0) = 0,$$

$$\begin{aligned} &F_{V_n',\varphi,t}(x \oplus h) \\ &\quad = \frac{\varphi\left(\sqrt{\|x\|_t^2 + \|h\|_t^2}\right)}{\sqrt{\|x\|_t^2 + \|h\|_t^2 + i(h(x) - x^{*,t}(h^{*,t}))}} \\ &\quad\quad \left(\frac{(h^{*,t} \oplus x^{*,t}) - x \oplus h}{2} + i\,\frac{x \oplus h + (h^{*,t} \oplus x^{*,t})}{2}\right), \end{aligned}$$

for all nonzero $x \oplus h \in W_n' = V_n' \oplus (V_n')^*$.

For each $t \in [0, 1]$, let $\|\cdot\|_t$ be the Banach algebra norm on $C_{\mathrm{b}}(W_n', Cl(W_n'))$ induced by the Banach space norm $\|\cdot\|_t$ on W_n'.

By the definition of $index(F_{V_n',\varphi,t})$, it is not difficult to verify that, given $r > 0$, $\varepsilon > 0$, there exists φ such that φ is independent of n, and $index(F_{V_n',\varphi,t})$ is (r, ε)-flat relative to $V_n \oplus 0 \subseteq W_n'$ with respect to $\|\cdot\|_t$, i.e.

$$\|(index(F_{V_n',\varphi,t}))(u_1) - (index(F_{V_n',\varphi,t}))(u_2)\|_t < \varepsilon$$

if $u_1, u_2 \in V_n \oplus 0 \subseteq W_n'$ and $\|u_1 - u_2\|_t \leqslant r$.

Let

$$index(F_{V_n',\varphi,t}) = a_{V_n',\varphi,t} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

for some $a_{V_n',\varphi,t} \in C_0(W_n', Cl(W_n'))$.

Note that $\|x\|_t \geqslant \|x\|$ for all $x \in V_n \oplus V_n''$, and $\|g\|_t \geqslant \|g\|$ for all $g \in V_n^* \oplus (V_n'')^*$. This, together with the definition of $index(F_{V_n',\varphi,t})$, implies that there exists $R > 0$ such that R is independent of n, and

$$supp(a_{V_n',\varphi,t}) \subseteq B_{W_n'}(0, R) = \{\xi \in W_n' : \|\xi\| < R\}$$

for all n, where $supp(a_{V_n',\varphi,t}) = \{\xi \in W_n' : a_{V_n',\varphi,t}(\xi) \neq 0\}$.

For each $t \in [0,1]$, in the definition of $\beta_{\{V_n'\}}$, we replace $F_{V_n',\varphi}$ by $F_{V_n',\varphi,t}$ to define $\beta_{\{V_n'\},t}$. Note that $\|x\oplus 0\|_t = \|x\oplus 0\|$ for all $x\oplus 0 \in V_n\oplus 0 \subseteq W_n' = V_n'\oplus (V')_n^*$, and, for each n, the topology on $C^*(F_n, \partial_\Gamma F_n, V_n)_R$ induced by $\|\cdot\|_t$ is equivalent to the topology induced by $\|\cdot\|$.

The above facts, together with the (r,ε)-flatness of $index(F_{V_n',\varphi,t})$) and the above property of the support of $a_{V_n',\varphi,t}$, imply that $\beta_{\{V_n'\},t}$ is a well-defined homomorphism from $K_0(C^*(\{F_n, \partial_\Gamma F_n\}_n))$ to $\lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_0(C^*(F_n, \partial_\Gamma F_n, V_n)_R)$.

It is easy to see that $\beta_{\{V_n'\},0} = \beta_{\{V_n'\}}$, and $\beta_{\{V_n'\},0} = \beta_{\{V_n'\},1}$ as homomorphisms from $K_0(C^*(\{F_n, \partial_\Gamma F_n\}_n))$ to $\lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_0(C^*(F_n, \partial_\Gamma F_n, V_n)_R)$.

Let $W_n'' = V_n''\oplus (V_n'')^*$. Let $C_0(W_n, Cl(W_n))\widehat{\otimes}\, C_0(W_n'', Cl(W_n''))$ be the graded Banach algebra tensor product $C_0(W_n, Cl(W_n))$ and $C_0(W_n'', Cl(W_n''))$ as defined in the paragraphs before Proposition 6.8, where the gradings of $C_0(W_n, Cl(W_n))$ and $C_0(W_n'', Cl(W_n''))$ are induced by the natural gradings of the Clifford algebras. Note that the isomorphism from $C_0(W_n, Cl(W_n))\widehat{\otimes}\, C_0(W_n'', Cl(W_n''))$ to $C_0(W_n', Cl(W_n'))$ is a graded Banach algebra isomorphism.

Recall that the map: $a\widehat{\otimes} b \to a\tau^{\partial b}\otimes b$, is an isomorphism from

$$C_0(W_n, Cl(W_n))\widehat{\otimes}\, C_0(W_n'', Cl(W_n''))$$

to

$$C_0(W_n, Cl(W_n))\otimes C_0(W_n'', Cl(W_n'')),$$

where a and b are, respectively, homogeneous elements in $C_0(W_n, Cl(W_n))$ and $C_0(W_n'', Cl(W_n''))$, τ is the grading operator in $C_b(W_n, Cl(W_n))$ induced by the natural grading of the Clifford algebra $Cl(W_n)$, and ∂b is the degree of b. This fact, together with Proposition 6.8, implies that $\beta_{\{V_n'\},1}[z]$ is the product of $\beta_{\{V_n\}}[z]$ with the direct sum of the Bott elements associated to $\{W_n''\}_n$. Hence Proposition 6.5 follows. $\square$

7. The proof of the main result

The purpose of this section is to prove the main result of this paper.

We need some preparations before we prove the main result.

Given $d \geqslant 0$, let $C^*_{L,\mathrm{alg}}(\{P_d(F_n)\}_n)$ be the algebra of all elements $\bigoplus_{n=1}^{\infty} a_n$ such that

(1) $a_n \in C^*_L(P_d(F_n))$;
(2) $\sup_n \|a_n\| < +\infty$;
(3) The map, $t \to \bigoplus_{n=1}^{\infty} a_n(t)$, is uniformly continuous on $[0,\infty)$;
(4) $\sup_{n,t} propagation(a_n(t)) < +\infty$ and $\sup_n propagation(a_n(t)) \to 0$ as $t\to\infty$.

Endow $C^*_{L,\mathrm{alg}}(\{P_d(F_n)\}_n)$ with the norm

$$\left\| \bigoplus_{n=1}^{\infty} a_n \right\| = \sup_n \|a_n\| .$$

Define $C^*_L(\{P_d(F_n)\}_n)$ to be the norm completion of $C^*_{L,\mathrm{alg}}(\{P_d(F_n)\}_n)$. Let $C^*_{L,\mathrm{alg}}(\{P_d(\partial_\Gamma F_n)\}_n)$ be the subalgebra of $C^*_{L,\mathrm{alg}}(\{P_d(F_n)\}_n)$ consisting of elements $\oplus_{n=1}^{\infty} a_n$ such that $\exists\, R \geqslant 0$,

$$supp(a_n(t)) \subseteq P_d((F_n \cap B_\Gamma(\Gamma - F_n, R)) \times P_d((F_n \cap B_\Gamma(\Gamma - F_n, R)) ,$$

for any natural number n and $t \in [0, \infty)$, where $B_\Gamma(\Gamma - F_n, R) = \{x \in \Gamma : d(x, \Gamma - F_n) < R\}$ if $\Gamma - F_n \neq \emptyset$, and $B_\Gamma(\Gamma - F_n, R) = \emptyset$ if $\Gamma - F_n = \emptyset$.

Define $C^*_L(\{P_d(\partial_\Gamma F_n)\}_n)$ to be the norm closure of $C^*_{L,\mathrm{alg}}(\{P_d(\partial_\Gamma F_n)\}_n)$. Note that $C^*_L(\{P_d(\partial_\Gamma F_n)\}_n)$ is a two-sided ideal of $C^*_L(\{P_d(F_n)\}_n)$.

Throughout the rest of this paper, we fix $x_0 \in \Gamma$ and choose

$$F_n = B(x_0, n) = \{x \in \Gamma : d(x, x_0) \leqslant n\}$$

for each nonnegative integer n.

For any $d \geqslant 0$, let $\partial_{\Gamma,d} F_n$ be

$$\{x \in \Gamma : n - 10d \leqslant d(x, x_0) \leqslant n\}.$$

By the choice of F_n, $P_d(F_n) \backslash P_d(\partial_{\Gamma,d} F_n)$ is an open subset of $P_d(F_{n+1}) \backslash P_d(\partial_{\Gamma,d} F_{n+1})$. Let $i_{n,d}$ be the inclusion homomorphism from $C_0(P_d(F_n) \backslash P_d(\partial_{\Gamma,d} F_n))$ into $C_0(P_d(F_{n+1}) \backslash P_d(\partial_{\Gamma,d} F_{n+1}))$. By the definition of relative K-homology group in Section 4, $i_{n,d}$ induces a homomorphism

$$(i_{n,d})^* : \ K_*(P_d(F_{n+1}), P_d(\partial_{\Gamma,d} F_{n+1})) \to K_*(P_d(F_n), P_d(\partial_{\Gamma,d} F_n)).$$

We have

$$K_*\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n) \right) = \oplus_n K_*(P_d(F_n), P_d(\partial_{\Gamma,d} F_n)),$$

where $\oplus_n K_*(P_d(F_n), P_d(\partial_{\Gamma,d} F_n))$ is defined to be

$$\{\oplus_n z_n : \ z_n \in K_*(P_d(F_n), P_d(\partial_{\Gamma,d} F_n))\}.$$

For each $d \geqslant 0$, we define a homomorphism

$$s_d : K_*\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n)\right) \to K_*\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n)\right)$$

by

$$s_d(\oplus_n z_n) = \oplus_n (i_{n,d})^*(z_{n+1})$$

for all $\oplus_n z_n \in K_*(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n))$.

Note that if $d_1 > d_2 \geqslant 0$, then

$$s_{d_1}(\oplus_n z_n) = s_{d_2}(\oplus_n z_n) \in \lim_{d\to\infty} K_*\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n)\right)$$

for all $\oplus_n z_n \in K_*(\prod_n P_{d_2}(F_n), \prod_n P_{d_2}(\partial_{\Gamma,d_2} F_n))$.

It follows that $\{s_d\}_d$ induces a well-defined homomorphism

$$s\colon \lim_{d\to\infty} K_*\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n)\right) \to \lim_{d\to\infty} K_*\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n)\right).$$

Observe that $P_d(F_n)\backslash P_d(\partial_{\Gamma,d} F_n)$ is an open subset of $P_d(\Gamma)$. Let $j_{n,d}$ be the inclusion homomorphism from $C_0(P_d(F_n)\backslash P_d(\partial_{\Gamma,d} F_n))$ to $C_0(P_d(\Gamma))$. $j_{n,d}$ induces a homomorphism

$$(j_{n,d})^* : K_*(P_d(\Gamma)) \to K_*(P_d(F_n), P_d(\partial_{\Gamma,d} F_n)).$$

We define a homomorphism

$$r : \lim_{d\to\infty} K_*(P_d(\Gamma)) \to \lim_{d\to\infty} K_*\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n)\right)$$

by

$$r(z) = \oplus_n (j_{n,d})^* z$$

for all $z \in K_*(P_d(\Gamma))$.

Proposition 7.1. *We have the following exact sequence*:

$$\to \lim_{d\to\infty} K_{*+1}\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d}F_n)\right)$$

$$\overset{Id-s}{\to} \lim_{d\to\infty} K_{*+1}\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d}F_n)\right)$$

$$\to \lim_{d\to\infty} K_*(P_d(\Gamma)) \overset{r}{\to} \lim_{d\to\infty} K_*\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d}F_n)\right)$$

$$\overset{Id-s}{\to} \lim_{d\to\infty} K_*\left(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d}F_n)\right) \to,$$

where Id is the identity homomorphism.

The above proposition follows from the standard lim^1-sequence for K-homology [36].

Definition 7.2. We define the C^*-algebra $C^*_{L,d}\left(\{F_n, \partial_\Gamma F_n\}_n\right)$ to be the quotient algebra of $C^*_L(\{P_d(F_n)\}_n)$ over $C^*_L\left(\{P_d(\partial_\Gamma F_n)\}_n\right)$.

We define a map

$$\chi_L : C^*_L(P_d(\Gamma)) \to C^*_{L,d}\left(\{F_n, \partial_\Gamma F_n\}\right)$$

by

$$\chi_L(a) = \left[\bigoplus_{n=1}^{\infty} \chi_n a \chi_n\right]$$

for all $a \in C^*(\Gamma)$, where χ_n is the characteristic function of F_n.

Proposition 7.3. *We have the following exact sequence*:

$$\to \lim_{d\to\infty} K_{*+1}\left(C^*_{L,d}\left(\{F_n, \partial_\Gamma F_n\}_n\right)\right) \overset{(Id-S)_*}{\to} \lim_{d\to\infty} K_{*+1}\left(C^*_{L,d}\left(\{F_n, \partial_\Gamma F_n\}_n\right)\right)$$

$$\to \lim_{d\to\infty} K_*\left(C^*_L(P_d(\Gamma))\right) \overset{(\chi_L)_*}{\to} \lim_{d\to\infty} K_*\left(C^*_{L,d}\left(\{F_n, \partial_\Gamma F_n\}_n\right)\right)$$

$$\overset{(Id-S)_*}{\to} \lim_{d\to\infty} K_*\left(C^*\left(\{F_n, \partial_\Gamma F_n\}_n\right)\right) \cdots,$$

where Id is the identity homomorphism.

Proof. We have the following commutative diagram:

$$\begin{array}{ccc}
\lim_{d\to\infty} K_{*+1}(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n)) & \stackrel{Id-s}{\to} & \lim_{d\to\infty} K_{*+1}(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n)) \\
\downarrow & & \downarrow \\
\lim_{d\to\infty} K_{*+1}(C^*_{L,d}(\{F_n, \partial_\Gamma F_n\})) & \stackrel{(Id-S)_*}{\to} & \lim_{d\to\infty} K_{*+1}(C^*_{L,d}(\{F_n, \partial_\Gamma F_n\}))
\end{array}$$

$$\begin{array}{ccccc}
\to \lim_{d\to\infty} K_*(P_d(\Gamma)) & \stackrel{r}{\to} & \lim_{d\to\infty} K_*(\prod_n P_d(F_n), \prod_n P_d(\partial_{\Gamma,d} F_n)) & \stackrel{Id-s}{\to} \\
\downarrow & & \downarrow & \\
\to \lim_{d\to\infty} K_*(C^*_L(P_d(\Gamma))) & \stackrel{(\chi_L)_*}{\to} & \lim_{d\to\infty} K_*(C^*_{L,d}(\{F_n, \partial_\Gamma F_n\})) & \stackrel{(Id-S)_*}{\to}
\end{array}$$

By Theorem 4.5 and Proposition 4.6, the vertical maps in the above diagram are isomorphisms. Note that the first horizontal sequence is exact. This, together with the commutativity of the diagram, implies that the second horizontal sequence is exact. □

Note that $\lim_{d\to\infty} C^*(\{P_d(F_n)\}_n)$ is naturally $*$-isomorphic to $C^*(\{F_n\}_n)$ (cf. [25]).

Let e be the evaluation homomorphism from $C^*_L(\{P_d(F_n)\}_n)$ to $C^*(\{F_n\}_n)$ defined by

$$e(\bigoplus_{n=1}^{\infty} a_n) = \bigoplus_{n=1}^{\infty} a_n(0)$$

for all $\bigoplus_{n=1}^{\infty} a_n \in C^*_L(\{P_d(F_n)\}_n)$, where $\bigoplus_{n=1}^{\infty} a_n(0)$ is identified with an element of $C^*(\{F_n\}_n)$ by the above isomorphism.

e induces a $*$-homomorphism (still denoted by e) from $C^*_{L,d}\left(\{F_n, \partial_\Gamma F_n\}_n\right)$ to $C^*(\{F_n, \partial_\Gamma F_n\}_n)$.

Lemma 7.4. *$\beta_{\{V_n\}} \circ e_*$ is an isomorphism of* $\lim_{d\to\infty} K_*\left(C^*_L(\{P_d(F_n)\}_n)\right)$ *onto* $\lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C^*(F_n, V_n)_R)$.

We need some preparations to prove Lemma 7.4.

For any $d \geqslant 0$, let $C^*(P_d(F_n) \otimes_{\text{alg}} C_0(W_n, Cl(W_n))$ be the algebraic tensor product of $C^*(P_d(F_n)$ with $C_0(W_n, Cl(W_n))$. We can construct a representation of the algebra $C^*(P_d(F_n)) \otimes_{\text{alg}} C_0(W_n, Cl(W_n))$ on the Banach space $C_0(W_n, (l^2(\Gamma_{d,n}) \otimes H) \otimes Cl(W_n))$, where $C^*(P_d(F_n))$ is defined as in Section 4 using a countable dense subset $\Gamma_{d,n}$ of $P_d(F_n)$, and $C_0(W_n, (l^2(\Gamma_{d,n}) \otimes H) \otimes Cl(W_n))$ is the Banach space defined as in the definition of $C^*(F_n, V_n)$. We define $C^*(P_d(F_n)) \otimes C_0(W_n, Cl(W_n))$ to be the operator norm closure of $C^*(P_d(F_n)) \otimes_{\text{alg}} C_0(W_n, Cl(W_n))$, where the operator norm is induced by the representation of the algebra $C^*(P_d(F_n)) \otimes_{\text{alg}} C_0(W_n, Cl(W_n))$ on the Banach space $C_0(W_n, (l^2(\Gamma_{d,n}) \otimes H) \otimes Cl(W_n))$.

For any η in $l^\infty(\Gamma_{d,n})$, we can define an operator N_η on the Banach space $C_0(W_n, (l^2(\Gamma_{d,n}) \otimes H) \otimes Cl(W_n))$ by

$$(N_\eta \zeta)(w) = ((\eta \otimes I) \otimes 1)(\zeta(w))$$

for all $\zeta \in C_0(W_n, (l^2(\Gamma_{d,n}) \otimes H) \otimes Cl(W_n))$ and $w \in W_n$, where η acts on $l^2(\Gamma_{d,n})$ by multiplication.

We can verify that N_η is a bounded operator (using an argument similar to the proof of Claim 2 in the proof of Lemma 6.1).

For every pair (x, y) in $\Gamma_{d,n} \times \Gamma_{d,n}$, let δ_x and δ_y be, respectively, the Dirac functions in $l^\infty(\Gamma_{d,n})$ at x and y. For any $a \in C^*(P_d(F_n), V_n)$, we write

$$a(x, y) = N_{\delta_x} a N_{\delta_y}.$$

We define

$$supp(a) = \{(x, y) \in \Gamma_d \times \Gamma_d \mid a(x, y) \neq 0\},$$

$$propagation(a) = sup\{d(x, y) \mid (x, y) \in supp(a)\}.$$

For any g in $C_0(W_n)$ and a in $C^*(P_d(F_n), V_n)$, we define a bounded operator ga on the Banach space $C_0(W_n, (l^2(\Gamma_{d,n}) \otimes H) \otimes Cl(W_n))$ by

$$((ga)\zeta)(w) = g(w)\ (a\zeta)(w)$$

for all $\zeta \in C_0(W_n, (l^2(\Gamma_{d,n}) \otimes H) \otimes Cl(W_n))$ and $w \in W_n$.

We define the support of $a(x, y)$, $support(a(x, y))$, to be the complement of the set of all points $\xi \in W_n$ such that there exists $g \in C_0(W_n)$ satisfying $g(\xi) \neq 0$ and $(ga)(x, y) = 0$.

We define $C^*_{\mathrm{alg},L}(P_d(F_n), V_n)$ to be the algebra of all uniformly continuous functions

$$a : [0, \infty) \to C^*(P_d(F_n), V_n)$$

satisfying $\sup_t propagation(a(t)) < +\infty$ and $propagation(a(t)) \to 0$ as t goes to ∞.

We endow $C^*_{\mathrm{alg},L}(P_d(F_n), V_n)$ with the norm

$$\|a\| = sup_{t \in [0,\infty)} \|a(t)\|.$$

We define $C^*_L(P_d(F_n), V_n)$ to be the norm closure of $C^*_{\mathrm{alg},L}(P_d(F_n), V_n)$.

For each n and $R > 0$, we define $C_L^*(P_d(F_n), V_n)_R$ to be the closed subalgebra of $C_L^*(P_d(F_n), V_n)$ generated by all elements $a \in C_L^*(P_d(F_n), V_n)$ such that

(1) *propagation*$(a(t)) < R$ for all $t \in [0, \infty)$,
(2) *support*$((a(t))(x, y)) \subseteq B_{W_n}(f(x)\oplus 0, R)$ for all $t \in [0, \infty)$, all x and y in $\Gamma_{d,n}$, where $f : P_d(\Gamma) \to X$, is the convex linear extension of the uniform embedding $f : \Gamma \to X$.

We can define the Bott map

$$\beta_{\{V_n\},L} : K_*(C_L^*(\{P_d(F_n)\}_n)) \to \lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C_L^*(P_d(F_n), V_n)_R),$$

in a way similar to the definition of $\beta_{\{V_n\}}$.

Lemma 7.5. *$\beta_{\{V_n\},L}$ is an isomorphism between $K_*(C_L^*(\{P_d(F_n)\}_n))$ and $\lim_{R\to\infty} \bigoplus_{n=1}^{\infty}$ $K_*(C_L^*(P_d(F_n), V_n)_R)$.*

Proof. By a Mayer–Vietoris sequence argument and induction on the dimension of the skeletons, the general case can be reduced to the zero-dimensional case, i.e., if $\Gamma_n \subseteq P_d(F_n)$ is δ-separated (meaning $d(x, y) \geqslant \delta$ if $x \neq y \in \Gamma_n$) for some $\delta > 0$, then $\beta_{\{V_n\},L}$ is an isomorphism from $K_*(C_L^*(\{\Gamma_n\}_n))$ to $K_*(C_L^*(\{\Gamma_n, V_n\}))$. This zero-dimensional case follows from the facts that

$$K_*(C_L^*(\{\Gamma_n\}_n)) \cong \bigoplus_{n=1}^{\infty} \bigoplus_{z\in\Gamma_n} K_*(C_L^*(\{z\})),$$

$$\lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C_L^*(\Gamma_n, V_n)_R) \cong \bigoplus_{n=1}^{\infty} \bigoplus_{z\in\Gamma_n} K_*(C_L^*(\{z\}, V_n)),$$

and $\beta_{\{V_n\},L}$ is an isomorphism from $K_*(C_L^*(\{z\}))$ to $K_*(C_L^*(\{z\}, V_n))$ by Lemma 3.1 and the Bott periodicity. □

Proof of Lemma 7.4. Note that $\lim_{d\to\infty} C^*(P_d(F_n), V_n)_R$ is naturally isomorphic to $C^*(F_n, V_n)_R$.

Let e_{V_n} be the homomorphism

$$C_L^*(P_d(F_n), V_n)_R \to C^*(F_n, V_n)_R$$

defined by

$$e_{V_n}(a) = a(0)$$

for each $a \in C_L^*(P_d(F_n), V_n)_R$, where $a(0)$ is identified with an element in $C^*(F_n, V_n)_R$ by the above isomorphism.

The family of homomorphisms $\{e_{V_n}\}$ induces a homomorphism $\left(e_{\{V_n\}}\right)_*$:

$$\lim_{d\to\infty}\lim_{R\to\infty}\bigoplus_{n=1}^{\infty} K_*(C_L^*(P_d(F_n), V_n)_R) \to \lim_{R\to\infty}\bigoplus_{n=1}^{\infty} K_*(C^*(F_n, V_n)_R)\,.$$

Clearly we have

$$\left(e_{\{V_n\}}\right)_* \circ \beta_{\{V_n\},L} = \beta_{\{V_n\}} \circ e_*\ .$$

By Lemma 7.5, it is enough to prove that $\left(e_{\{V_n\}}\right)_*$ is an isomorphism. Assume that for each n, O_n is an open subset of $W_n = V_n \oplus V_n^*$. We define $C^*(F_n, O_n)$ to be a closed subalgebra of $C^*(F_n, V_n)$ consisting of elements $a \in C^*(F_n, V_n)$ such that *support*$(a(x, y)) \subseteq O_n$ for all $x, y \in F_n$.

Similarly we can define $C^*(F_n, O_n)_R$ for each $R > 0$. For each $r > 0$, let

$$O_{n,r} = \bigcup_{x\in F_n} B_{W_n}(f(x)\oplus 0\,,\ r)\,.$$

By the bounded geometry property of Γ and the fact that f is a uniform embedding, there exists a natural number m (independent of n) such that

(1) $O_{n,r} = \bigcup_{k=1}^{m} O_{n,r}^{(k)}$;
(2) for each k, $O_{n,r}^{(k)}$ is a disjoint union of open balls of the form $B_{W_n}(f(x)\oplus 0, r)$.

By a Mayer–Vietoris sequence argument, it suffices to prove that $(e_{\{V_n\}})_*$ is an isomorphism,

$$\lim_{d\to\infty}\lim_{R\to\infty}\bigoplus_{n=1}^{\infty} K_*(C_L^*(P_d(F_n), O_{n,r}^{(k)})_R) \to \lim_{R\to\infty}\bigoplus_{n=1}^{\infty} K_*(C^*(F_n, O_{n,r}^{(k)})_R)$$

for each k and r.

Assume that $O_{n,r}^{(k)}$ is the disjoint union of $B_{W_n}(f(z)\oplus 0, r)$ ($z \in F_{n,r}^{(k)}$), where $F_{n,r}^{(k)}$ is a subset of F_n.

For each $D > 0$, we define $A_{D,n,r}^{(k)}$ to be the Banach algebra consisting of elements $\bigoplus_{z\in F_{n,r}^{(k)}} b_z$ such that

(1) $b_z \in C^*(B_{F_n}(z, D), V_n)$, where $B_{F_n}(z, D) = \{z \in F_n\ :\ d(x, z) < D\}$;
(2) *support*$(b_z(x, y)) \subseteq B_{W_n}(f(z)\oplus 0, r)$ for all $x,\ y \in B_{F_n}(z, D)$.

$A_{D,n,r}^{(k)}$ is endowed with the norm

$$\left\| \bigoplus_{z\in F_{n,r}^{(k)}} b_z \right\| = \sup_{z\in F_{n,r}^{(k)}} \|b_z\|\,.$$

It is easy to see that

$$\lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C^*(F_n, O_{n,r}^{(k)})_R)$$

is naturally isomorphic to

$$\lim_{D\to\infty} \bigoplus_{n=1}^{\infty} K_*\left(A_{D,n,r}^{(k)}\right).$$

For each n, we define $A_{L,D,n,r}^{(k)}$ to be the Banach algebra generated by elements $\bigoplus_{z\in F_{n,r}^{(k)}} a_z$ such that

(1) $a_z \in C_L^*(P_{D'}(B_{F_n}(z, D)), V_n)$ where $D' = diameter(B_{F_n}(z, D))$;
(2) $\sup_{z\in F_{n,r}^{(k)}} \|a_z\| < +\infty$;
(3) $support(a_z(x, y)) \subseteq B_{W_n}(f(z)\oplus 0, r)$ for all $z \in F_{n,r}^{(k)}$ and $x,\ y \in B_{F_n}(z, D)$;
(4) $t \to \bigoplus_{z\in F_{n,r}^{(k)}} a_z(t)$ is uniformly continuous;
(5) $\sup_z propagation(a_z(t)) < +\infty$, and $\sup_z propagation(a_z(t)) \to 0$ as $t \to \infty$, where $A_{L,D,n,r}^{(k)}$ is endowed with the norm

$$\left\| \bigoplus_{z\in F_{n,r}^{(k)}} a_z \right\| = \sup_{z\in F_{n,r}^{(k)}} \|a_z\|.$$

It is easy to see that

$$\lim_{d\to\infty} \lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*(C_L^*(P_d(F_n), O_{n,r}^{(k)})_R)$$

is naturally isomorphic to

$$\lim_{D\to\infty} \bigoplus_{n=1}^{\infty} K_*\left(A_{L,D,n,r}^{(k)}\right).$$

Using the fact that $P_{D'}(B_{F_n}(z, D))$ is (Lipschitz) contractible, it is easy to see that the evaluation homomorphism (at 0) induces an isomorphism from

$$\lim_{D\to\infty} \bigoplus_{n=1}^{\infty} K_*\left(A_{L,D,n,r}^{(k)}\right)$$

to

$$\lim_{D\to\infty} \bigoplus_{n=1}^{\infty} K_*\left(A_{D,n,r}^{(k)}\right).$$

This implies Lemma 7.4. □

The following result can be proved in a way similar to Lemma 7.4.

Lemma 7.6. $\beta_{\{V_n\}} \circ e_*$ *is an isomorphism between*

$$\lim_{d\to\infty} K_*\left(C_L^*\left(\left\{P_d(\partial_\Gamma F_n)\right\}_n\right)\right)$$

and

$$\lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*\left(C^*\left(\{\partial_\Gamma F_n, V_n\}\right)_R\right).$$

Proposition 7.7. $\beta_{\{V_n\}} \circ e_*$ *is an isomorphism between*

$$\lim_{d\to\infty} K_*\left(C_{L,d}^*(\{F_n, \partial_\Gamma F_n\}_n)\right)$$

and

$$\lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_*\left(C^*\left(F_n, \partial_\Gamma F_n, V_n\right)_R\right).$$

Proof. The proposition follows from Lemmas 7.4 and 7.6, and a five lemma argument. □

Proof of Theorem 1.1. Let χ_L be as in Proposition 7.3. Let e be the evaluation homomorphism from $C_L^*(P_d(\Gamma))$ to $C^*(P_d(\Gamma))$ defined by: $e(a) = a(0)$. e induces a homomorphism (still denoted by e_*) from $K_*(C_L^*(P_d(\Gamma)))$ to $K_*(C^*(\Gamma))$.

Let $e'_* = \chi_* \circ e_*$ be the homomorphism from $K_*(C_L^*(P_d(\Gamma)))$ to $K_*(C_S^*(\Gamma))$, where χ is as in Lemma 5.6. It is enough to prove that e'_* is injective from $\lim_{d\to\infty} K_*(P_d(\Gamma))$ to $K_*(C_S^*(\Gamma))$.

We have the following commutative diagram:

$$\begin{array}{ccccc}
\lim_{d\to\infty} K_{*+1}(C_{L,d}^*(\left\{F_n, \partial_\Gamma F_n\right\})) & \stackrel{(Id-S)_*}{\longrightarrow} & \lim_{d\to\infty} K_{*+1}(C_{L,d}^*(\left\{F_n, \partial_\Gamma F_n\right\})) & \stackrel{\gamma}{\longrightarrow} \\
\downarrow e_* & & \downarrow e_* & \\
K_{*+1}(C^*(\left\{F_n, \partial_\Gamma F_n\right\})) & \stackrel{(Id-S)_*}{\longrightarrow} & K_{*+1}(C^*(\left\{F_n, \partial_\Gamma F_n\right\})) & \stackrel{\gamma}{\longrightarrow} \\
\downarrow \beta_{\{V_n\}} & & \downarrow \beta_{\{V_n\}} & \\
\lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_{*+1}(C^*(F_n, \partial_\Gamma F_n, V_n)_R) & & \lim_{R\to\infty} \bigoplus_{n=1}^{\infty} K_{*+1}(C^*(F_n, \partial_\Gamma F_n, V_n)_R) &
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\gamma} & \lim\limits_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) & \xrightarrow{(\chi_L)_*} & \lim\limits_{d\to\infty} K_*(C_{L,d}^*(\{F_n, \partial_\Gamma F_n\})) \\
& \downarrow e'_* & & \downarrow e_* \\
\xrightarrow{\gamma} & K_*(C_S^*(\Gamma)) & \xrightarrow{j_*} & K_*(C^*(\{F_n, \partial_\Gamma F_n\})) \\
& & & \downarrow \beta_{\{V_n\}} \\
& & & \lim\limits_{R\to\infty} \bigoplus\limits_{n=1}^{\infty} K_*(C^*(F_n, \partial_\Gamma F_n, V_n)_R)
\end{array}$$

By Propositions 7.3 and 5.5, the first and second horizontal sequences in the above diagram are exact.

Let $[x] \in \lim_{d\to\infty} K_*(C_L^*(P_d(\Gamma)))$ such that $e'_*[x] = 0$. We need to prove that $[x] = 0$.

We first claim that $(\chi_L)_*[x] = 0$. This follows from the identity

$$\beta_{\{V_n\}}(e_*((\chi_L)_*[x])) = \beta_{\{V_n\}}(j_*(e'_*[x])) = 0$$

(by the commutativity of the diagram), and the fact that $\beta_{\{V_n\}} \circ e_*$ is an isomorphism (by Proposition 7.7).

By exactness, $\exists\,[x'] \in \lim_{d\to\infty} K_{*+1}(C_{L,d}^*(\{F_n, \partial_\Gamma F_n\}))$ such that $\gamma[x'] = [x]$. By the commutativity of the diagram, we have

$$\gamma(e_*[x']) = e'_*(\gamma[x']) = e'_*[x] = 0.$$

Hence by exactness, $\exists\,[y] \in K_{*+1}(C^*(\{F_n, \partial_\Gamma F_n\}))$ such that

$$(Id - S)_*[y] = e_*[x'].$$

The fact that $\beta_{\{V_n\}} \circ e_*$ is an isomorphism implies that

$$\exists\,[x''] \in \lim_{d\to\infty} K_{*+1}(C_{L,d}^*(\{F_n, \partial_\Gamma F_n\}))$$

such that

$$\beta_{\{V_n\}} \circ e_*[x''] = \beta_{\{V_n\}}[y].$$

Hence we have

$$\beta_{\{V_n\}}(e_*[x''] - [y]) = 0.$$

This, together with Proposition 6.5, implies that

$$\beta_{\{V_n\}}(Id - S)_*(e_*[x''] - [y]) = 0.$$

By the commutativity of the diagram and the property of y, it follows that

$$\beta_{\{V_n\}}(e_*(Id - S)_*[x''] - e_*[x']) = 0.$$

Hence

$$\beta_{\{V_n\}} \circ e_*((Id - S)_*[x''] - [x']) = 0.$$

By Proposition 7.7, we have

$$(Id - S)_*[x''] - [x'] = 0.$$

This, together with exactness and the identity $[x] = \gamma[x']$, implies that $[x] = 0$. $\square$

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