Periodic solution and almost periodic solution for a nonautonomous Lotka–Volterra dispersal system with infinite delay

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Abstract

This paper studies a nonautonomous Lotka–Volterra dispersal systems with infinite time delay which models the diffusion of a single species into n patches by discrete dispersal. Our results show that the system is uniformly persistent under an appropriate condition. The sufficient condition for the global asymptotical stability of the system is also given. By using Mawhin continuation theorem of coincidence degree, we prove that the periodic system has at least one positive periodic solution, further, obtain the uniqueness and globally asymptotical stability for periodic system. By using functional hull theory and directly analyzing the right functional of almost periodic system, we show that the almost periodic system has a unique globally asymptotical stable positive almost periodic solution. We also show that the delays have very important effects on the dynamic behaviors of the system.

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1. Introduction

The classical Lotka–Volterra type systems form a significant component of the models of population dynamics. A hallmark of observed population densities in the field is their permanent, stable and periodic or almost periodic behaviors. A main purpose of modeling population interactions is to understand what causes such behaviors. Systems with time delay involving delay differential equations have attracted the interest of many researchers in the past twenty years [1–13,27,28]. The results of [27,28] implied that the time delays had no effect on the permanence of the system, that is, these delays are “harmless” for the permanence. What is the effect of time delays on the permanence of systems with delays? Then it is significant for us to study the permanence of systems. Recently, diffusions [8,14–20] have been introduced into Lotka–Volterra type systems. The effect of environment change in the growth and diffusion of a species

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in a heterogeneous habitat is a subject of considerable interest in the ecological literature [8,14–16,19,20,24]. Because of the ecological effects of human activities and industry, more and more habitats are broken into patches and some of them are polluted. Negative feedback crowding or the effect of the past life history of the species on its present birth rate are common examples illustrating the biological meaning of time delays and justifying their use in these systems.

Since biological and environmental parameters are naturally subject to fluctuation in time, the effects of a periodically or almost periodically varying environment are considered as important selective forces on systems in a fluctuating environment. Therefore, on the one hand, models should take into account both the seasonality of the periodically changing environment and the effects of time delays [1–13]. However, on the other hand, in fact, it is more realistic to consider almost periodic system than periodic system. The most basic and important questions to ask for these systems in the theory of mathematical ecology are the persistence, extinctions, global asymptotic behaviors, and existences of coexistence states (for example, the positive equilibrium, strictly positive solution, positive periodic solution, periodic solution and almost periodic solution, etc.) of population [1–24]. In this paper, we will investigate the persistence, global asymptotic behaviors, strictly positive solution, strictly positive periodic solution and strictly positive almost periodic solution of population in model.

There are many works on the study of the Lotka–Volterra type periodic systems that have been developed in [2,3,5,8–10,14–17]. Recently, by using the definition of almost periodic solution or the contraction mapping and fixed point theory, some authors have done many good works in theory on almost periodic systems [11,12,21]. In this paper, by using a new method which is almost periodic functional hull theorem, we directly analyze the right functional of almost periodic system, and obtain the new weak sufficient conditions with delay for the existence and uniqueness of almost periodic solution.

In this paper, we shall consider the case of combined effects: dispersion, time delays, periodicity and almost periodicity of the environment. Namely, we investigate the following general nonautonomous Lotka–Volterra type dispersal system with discrete and continuous infinite time delays which models the diffusion of a single species $x_i$ into $n$ patches connected by discrete dispersal:

$$
\dot{x}_i(t) = x_i(t) \left[ r_i(t) - a_i(t)x_i(t) - b_i(t)x_i(t - \tau_i(t)) - \int_{-\infty}^{0} k_i(t,s)x_i(t+s) ds \right] + \sum_{j=1}^{n} D_{ij}(t)(x_j(t) - x_i(t)),
$$

where $x_i(t)$ represents the density of the species in $i$th patch, $D_{ij}(t)$ is the dispersion rate of the species from patch $j$ to patch $i$, $r_i(t)$ is the intrinsic growth rate of the species in patch $i$, $a_i(t)$ is the death rate (or density-dependent) of the species in patch $i$, the terms $b_i(t)x_i(t - \tau_i(t))$ and $\int_{-\infty}^{0} k_i(t,s)x_i(t+s) ds$ represent the negative feedback crowding and the effect of all the past life history of the species on its present birth rate, respectively.

Suppose $h(t)$ is a bounded function defined on $R$. Define $h^u = \lim_{t \to -\infty} \sup h(t)$, $h^l = \lim_{t \to -\infty} \inf h(t)$.

In system (1), we always assume that for all $i, j = 1, 2, \ldots, n$:

(H1) The bounded functions $r_i(t)$, $a_i(t)$, $b_i(t)$, and $D_{ij}(t)(D_{ii}(t) = 0)$ are nonnegative and continuous for all $t \in R$, and $a_i^l \geq 0$, $b_i^l \geq 0$, $a_i^u + b_i^l > 0$.

(H2) The bounded functions $k_i(t, s)$ are defined on $R \times (-\infty, 0]$ and nonnegative and continuous with respect to $t \in R$ and integrable with respect to $s$ on $(-\infty, 0]$ such that $\int_{-\infty}^{0} k_i(t,s) ds$ is continuous and bounded with respect to $t \in R$.

(H3) $\tau_i(t)$ is continuous and differentiable bounded functions on $R$, and $\dot{\tau}_i(t)$ is uniformly continuous with respect to $t$ on $R$ and $\inf_{t \in R} \{1 - \dot{\tau}_i(t)\} > 0$.

Let $\tau^* = \sup \{\tau_i(t): t \in [0, +\infty), i = 1, 2, \ldots, n\}$, then we have $0 \leq \tau^* < +\infty$. Let $\sigma_i(t) = t - \tau_i(t)$, then the function $\sigma_i^{-1}(t)$ is an inverse function of the function $\sigma_i(t)$.

Following notations and definitions will be used in the rest of this paper.

We use $C_h = \{\phi(t) \in C(R_+; R^n): \int_{-\infty}^{0} \sup_{1 \leq i \leq n} |\phi(t)| ds < \infty\}$. Let $C_h^+ = \{\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in C_h: \phi_i(s) (i = 1, 2, \ldots, n) \text{ is nonnegative, bounded, and continuous function for } s \in (-\infty, 0], \phi_i(0) > 0\}$.

Motivated by the application of system (1) to population dynamics, we assume that solutions of system (1) satisfy the initial conditions.
\[ x_i(\theta) = \phi_i(\theta), \quad \theta \in (-\infty, 0], \quad i = 1, 2, \ldots, n, \]  
(2)

where \( \phi = (\phi_1, \phi_2, \ldots, \phi_n) \in C^+_h \). Therefore, if one chooses the initial function space of system (1) to be \( C^+_h \), we easily see that for any \( \phi = (\phi_1, \phi_2, \ldots, \phi_n) \in C^+_h \) and \( \phi(0) > 0 \), according to the infinite delay functional basic theorem [11,12], there exists a unique solution \( x(t, \phi) \) satisfying the initial conditions (2) which remains positive, so, such solutions of system (1) are called positive solutions. Hence, in the rest of this paper we always assume \( \phi \) satisfies
\[
\phi \in C^+_h, \quad \phi(0) > 0.
\]  
(3)

The organization of this paper is as follows. In the next section, we present the sufficient conditions for uniform persistence of system (1). Further, sufficient condition for global asymptotic stability of the solution of system (1) is obtained. In Section 4, using Mawhin continuation theorem of coincidence degree, we investigate the periodic system of system (1) and show that it has a unique globally asymptotically stable periodic solution. In Section 5, by using almost periodic functional hull theory, we study the almost periodic system of system (1) and show that it has a unique globally asymptotically stable almost periodic solution. Finally, we give conclusions and two special cases.

2. Uniform persistence

**Definition 2.1.** System (1) is said to be uniformly persistent if there exists a compact region \( D \subset \text{Int} \mathbb{R}^n_+ \), such that every solution \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) for system (1) with initial conditions (3) eventually enters and remains in the region \( D \).

**Definition 2.2.** Suppose \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) is any solution for system (1), \( x(t) \) is said to be a strictly positive solution in \( D \), if for \( t \in D \) and \( i = 1, 2, \ldots, n \) such that
\[
0 < \inf_{t \in D} x_i(t) \leq \sup_{t \in D} x_i(t) < \infty.
\]

For system (1), we will consider two cases, \( a^j_i > 0, \quad b^j_i \geq 0 \) and \( a^j_i \geq 0, \quad b^j_i > 0 \) respectively, then we obtain Lemmas 2.1–2.3.

**Lemma 2.1.** If \( a^j_i > 0 \), then there exist two positive constants \( m^* \) and \( M^* \) such that
\[
m^* \leq \lim_{t \to \infty} \inf_{i=1,2,\ldots,n} x_i(t) \leq \lim_{t \to \infty} \sup_{i=1,2,\ldots,n} x_i(t) \leq M^*, \quad i = 1, 2, \ldots, n,
\]
provided
\[
p_i = r^u_i - \left( b^u_i + \int_{-\infty}^{0} k^u_i(s) ds \right) M^* - \sum_{j=1}^{n} D^u_{ij} > 0,
\]
where
\[
M^* = \max_{i=1,2,\ldots,n} \left\{ \frac{p_i}{a^u_i} \right\}, \quad m^* = \min_{i=1,2,\ldots,n} \left\{ \frac{p_i}{a^u_i} \right\}.
\]

**Proof.** Suppose \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) is any positive solution for system (1) with initial conditions (3). Define \( V(t) = \max\{x_j(t)\} \) for \( t \geq 0, \quad j = 1, 2, \ldots, n \). Suppose \( V(t) = x_i(t) \), calculating the upper right derivative of \( V(t) \) follows from the positive solution for system (1), and we have
\[
D^+ V(t) = \dot{x}_i(t)
\]
\[
\leq x_i(t)(r^u_i - a^u_i x_i(t))
\]
\[
\leq x_i(t)(r^u_i - a^u_i x_i(t)).
\]
By using comparability theorem, we have
\[
\lim_{t \to \infty} \sup_{i=1,2,\ldots,n} x_i(t) \leq M_i, \quad i = 1, 2, \ldots, n,
\]
where \( M_i = \frac{r_i^u}{a_i^u} \). Taken \( M^* = \max_{i=1,2,...,n} \{ M_i \} \), we have

\[
\lim_{t \to \infty} \sup_{i=1,2,...,n} x_i(t) \leq M^*, \quad i = 1, 2, \ldots, n.
\]

Hence, there exists a positive constant \( T \), for \( t \geq T \) such that

\[
\dot{x}_i(t) \geq x_i(t) \left[ r_i^u - \left( b_i^u + \int_{-\infty}^{0} k_i(s) \, ds \right) M^* - \sum_{j=1}^{n} D_{ij}^u - a_i^u x_i(t) \right].
\]

By using comparability theorem, we have

\[
\lim_{t \to \infty} \inf_{i=1,2,...,n} x_i(t) \geq m_i, \quad i = 1, 2, \ldots, n,
\]

where \( m_i = \frac{r_i^u - (b_i^u + \int_{-\infty}^{0} k_i(s) \, ds) M^* - \sum_{j=1}^{n} D_{ij}^u}{a_i^u} > 0 \). Taken \( m^* = \min_{i=1,2,...,n} \{ m_i \} \), we have

\[
\lim_{t \to \infty} \inf_{i=1,2,...,n} x_i(t) \geq m^*, \quad i = 1, 2, \ldots, n.
\]

From (4) and (5), we have

\[
m^* \leq \lim_{t \to \infty} \inf_{i=1,2,...,n} x_i(t) \leq \lim_{t \to \infty} \sup_{i=1,2,...,n} x_i(t) \leq M^*, \quad i = 1, 2, \ldots, n.
\]

The proof is complete. \( \Box \)

**Lemma 2.2.** If \( b_i^j > 0 \), then there exists a positive constant \( M^{**} \) such that

\[
\lim_{t \to \infty} \sup_{i=1,2,...,n} x_i(t) \leq M^{**}, \quad i = 1, 2, \ldots, n,
\]

where

\[
M^{**} = \max_{i=1,2,...,n} \left\{ \frac{r_i^u}{b_i^u} e^{\tau_i^u \tau^*} \right\}.
\]

**Proof.** If \( a_i^j \geq 0, b_i^j > 0 \), then let \( V(t) = \max_{j=1,2,...,n} \{ x_j(t) \} \) for \( t \geq 0 \), \( j = 1, 2, \ldots, n \). Suppose \( V(t) = x_i(t) \), calculating the upper right derivative of \( V(t) \) follows from the positive solution for system (1), and we have

\[
D^+ V(t) = \dot{x}_i(t) \leq x_i(t) \left[ r_i^u (r_i^u - b_i x_i(t) - \tau_i(t)) \right] \leq x_i(t) \left( r_i^u - b_i^u x_i(t) - \tau_i(t) \right).
\]

Take \( M_i^* = \frac{r_i^u}{b_i^u} (1 + l_i) \), where \( 0 < l_i < \exp[r_i^u \tau^*] - 1 \). Suppose \( x_i(t) \) is not oscillatory about \( M_i^* \), that is, there exists a \( T_1 > 0 \), for \( t > T_1 \) such that

\[
x_i(t) < M_i^* \tag{7},
\]

or

\[
x_i(t) > M_i^*. \tag{8}
\]

If (7) holds, then system (1) follows. Suppose (8) holds, then from (6) it follows

\[
\dot{x}_i(t) \leq -l_i r_i^u x_i(t) \quad \text{for all } t \geq T_1 + \tau^*.
\]

And so, there exists a \( T'_1 > T_1 \) such that \( x_i(t) < M_i^* \) for \( t \geq T'_1 \), which is a contradiction. Hence (8) could not hold. Now assume \( x_i(t) \) is oscillatory about \( M_i^* \) for \( t \geq T_1 \), that is, there exists a time sequence \( \{ n_k \} \) such that \( \tau < t_1 < t_2 < \cdots < t_n < \cdots \) is a sequence of zeros of \( x_i(t_n) - M_i^* \) with \( \lim_{n \to \infty} t_n = \infty \) and \( x_i(t_n) = M_i^* \). Our strategy is to establish the upper bound in each interval \( (t_n, t_{n+1}) \). For this, let \( \tilde{t}_n \) be a point where \( x_i(t) \) attends its maximum in \( (t_n, t_{n+1}) \). Now, since \( x_i(t_n) \) is the maximum, we have \( x_i(t) \geq x_i(t_n) = M_i^* \). Since \( x_i'(\tilde{t}_n) = 0 \), it follows from (6) that we have

\[
0 = \frac{dx_i(t)}{dt} \bigg|_{t=\tilde{t}_n} \leq x_i(\tilde{t}_n)(r_i^u - b_i^u x_i(\tilde{t}_n - \tau_i(\tilde{t}_n))). \tag{9}
\]
This leads to
\[ x_i (\tilde{t}_n - \tau_i (\tilde{t}_n)) \leq \frac{r_i^u}{b_i^l}. \] (10)

Integrating both sides of (6) on the interval \([\tilde{t}_n - \tau_i (\tilde{t}_n), \tilde{t}_n]\), we have
\[ \ln \left[ \frac{x_i (\tilde{t}_n)}{x_i (\tilde{t}_n) - b_i^l x_i (t - \tau_i (t))} \right] \leq \int_{\tilde{t}_n - \tau_i (\tilde{t}_n)}^{\tilde{t}_n} (r_i^u - b_i^l x_i (t - \tau_i (t))) \, dt \leq \int_{\tilde{t}_n - \tau_i (\tilde{t}_n)}^{\tilde{t}_n} (r_i^u) \, dt = r_i^u \tau_i (\tilde{t}_n). \]

Thus
\[ x_i (\tilde{t}_n) \leq \frac{r_i^u}{b_i^l} e^{r_i^u \tau^*} =: M_i. \] (11)

Since the right-hand side of (11) is independent of \( t \) and \( x_i (\tilde{t}_n) \) is an arbitrary local maximum of \( x_i(t) \), we can conclude that there is a \( T_2 > 0 \) such that
\[ x_i(t) \leq M_i (i = 1, 2, \ldots, n) \text{ for all } t \geq T_2. \]

Thus we have
\[ \lim_{t \to \infty} \sup x_i(t) \leq M^{**}. \] (12)

This completes the proof. \( \square \)

**Lemma 2.3.** If \( b_i^l > 0 \), then there exists a positive constant \( m^{**} \) such that
\[ \lim_{t \to \infty} \inf x_i(t) \geq m^{**}, \quad i = 1, 2, \ldots, n, \]
provided
\[ p_i = r_i^l - \left( a_i^u + \int_{-\infty}^{0} k_i^u (s) \, ds \right) M^{**} - \sum_{j=1}^{n} D_{ij}^{nu} > 0, \]
where
\[ M^{**} = \max_{i=1,2,\ldots,n} \left\{ \frac{r_i^u}{b_i^l} e^{r_i^u \tau^*} \right\}, \quad m^{**} = \min_{i=1,2,\ldots,n} \left\{ \frac{r_i^l}{2b_i^l}, m_i \right\}, \quad m_i \leq \left\{ \frac{p_i}{2b_i^l}, i = 1, 2, \ldots, n \right\}. \]

**Proof.** By the proof of Lemma 2.2 and from the \( i \)th equation of system (1), we have
\[ \dot{x}_i(t) \geq x_i(t) \left[ r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^{0} k_i^u (s) \, ds \right) M_i - \sum_{j=1}^{n} D_{ij}^{nu} \right] \text{ for } t \geq T_2 + \tau^*, \] (13)
where \( M_i = \frac{r_i^u}{b_i^l} e^{r_i^u \tau^*} \).

Assume that
\[ r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^{0} k_i^u (s) \, ds \right) M_i - \sum_{j=1}^{n} D_{ij}^{nu} > 0, \]
then from the positivity of the solution and (13), it follows that
\[ \dot{x}_i(t) > 0 \quad \text{for } t \geq T_2 + \tau^*. \]

Hence
\[ x_i(t) \geq x_i(T_2 + \tau^*) \exp \left\{ \left( r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^{0} k_i^u (s) \, ds \right) M_i - \sum_{j=1}^{n} D_{ij}^{nu} \right) (t - (T_2 + \tau^*)) \right\}, \]
then we have \( x_i(t) \to +\infty \) as \( t \to +\infty \), which is contract with (12).
Therefore,
\[ r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^{0} k_i^u(s) \, ds \right) M_i - \sum_{j=1}^{n} D_{ij}^u < 0. \]

If
\[ r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^{0} k_i^u(s) \, ds \right) M_i - \sum_{j=1}^{n} D_{ij}^u = 0, \]
then from the positivity of the solution and (13), it follows that
\[ \dot{x}_i(t) = 0 \quad \text{for} \quad t \geq T_2 + \tau^*. \]

Then (12) together with (14) implies that \( \lim_{t \to +\infty} x_i(t) \) exists. If \( \lim_{t \to +\infty} x_i(t) := q_i < \frac{r_i^l}{b_i^u} \), then there exists \( T_3 > T_2 \), such that for \( t \geq T_3 \)
\[ x_i(t) < q_i + \frac{r_i^l/b_i^u - q_i}{2} < \frac{r_i^l}{b_i^u}. \]

The \( i \)th equation of system (1) together with (15) leads to
\[ \dot{x}_i(t) \geq x_i(t) \left( r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^{0} k_i^u(s) \, ds \right) \frac{r_i^l/b_i^u + q_i}{2} - \sum_{j=1}^{n} D_{ij}^u \right) \quad \text{for} \quad t \geq T_3 + \tau^*. \]

Noting that \( \frac{r_i^l/b_i^u + q_i}{2} < M_i \), and similar to case (i), we have \( x_i(t) \to +\infty \) as \( t \to +\infty \), which is contradict with (12). Hence we have
\[ \lim_{t \to +\infty} x_i(t) := q_i \geq \frac{r_i^l}{b_i^u}, \]
which implies that there is a \( T_3^* \) such that \( x_i(t) \geq \frac{r_i^l}{2b_i^u} \) for \( t \geq T_3^* \).

If
\[ r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^{0} k_i^u(s) \, ds \right) M_i - \sum_{j=1}^{n} D_{ij}^u < 0, \]
then from the positivity of the solution and the \( i \)th equation of system (1), for \( t \geq T_2 + \tau^* \), it follows that
\[ \dot{x}_i(t) \geq x_i(t) \left[ r_i^l - \left( a_i^u + \int_{-\infty}^{0} k_i^u(s) \, ds \right) M_i - \sum_{j=1}^{n} D_{ij}^u - b_i^u x_i(t - \tau_i(t)) \right]. \]

Let
\[ m_i^* = \frac{r_i^l - (a_i^u + \int_{-\infty}^{0} k_i^u(s) \, ds) M_i - \sum_{j=1}^{n} D_{ij}^u (1 - \delta_i)}{b_i^u} > 0, \]
where
\[ 0 < \delta_i < 1 - \exp \left\{ \left( r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^{0} k_i^u(s) \, ds \right) M_i - \sum_{j=1}^{n} D_{ij}^u \right) \tau^* \right\}. \]
Suppose \( x_i(t) \) is not oscillatory about \( m_i^* \), that is, there exists \( T_4 > T_2 \), for \( t > T_4 \) such that
\[
x_i(t) > m_i^*,
\] (17)
or
\[
x_i(t) < m_i^*.
\] (18)

If (17) holds, then our aim is obtained. Suppose (18) holds, then from (16) it follows
\[
\dot{x}_i(t) \geq \delta_i \left[ r_i^l - \left( a_i^u + \int_{-\infty}^0 k_i^u(s) \, ds \right) M_i - \sum_{j=1}^n D_{ij}^u \right] x_i(t) \quad \text{for} \quad t \geq T_4 + \tau^*.
\] (19)

Then there exists \( T_{4}^* > T_4 + \tau^* \), for \( t > T_{4}^* \) such that \( x_i(t) > m_i^* \), which is a contradiction. Therefore, (18) could not hold.

Suppose \( x_i(t) \) is oscillatory about \( m_i^* \) for \( t \geq T_2 + \tau^* \), that is, there exists a time sequence \( \{t_n\} \) such that \( \tau < t_1 < t_2 < \cdots < t_n < \cdots \) is a sequence of zeros of \( x_i(t_n) - m_i^* \) with \( \lim_{n \to \infty} t_n = \infty \) and \( x_i(t_n) = m_i^* \). Our strategy is to establish the lower bound in each interval \((t_n, t_{n+1})\). For this, let \( \tilde{t}_n \) be a point where \( x_i(t) \) attains its minimum in \((t_n, t_{n+1})\). Now, since \( x_i(\tilde{t}_n) \) is the minimum, we have \( x_i(\tilde{t}_n) \leq x_i(t_n) = m_i^* \). Since \( x_i'(\tilde{t}_n) = 0 \), it follows from (16) that we have
\[
0 = \frac{dx_i(t)}{dt} \bigg|_{t = \tilde{t}_n} \geq x_i(\tilde{t}_n) \left( r_i^l - \left( a_i^u + \int_{-\infty}^0 k_i^u(s) \, ds \right) M_i - \sum_{j=1}^n D_{ij}^u \right) - b_i^u x_i(t - \tau_i(t)).
\] (20)

which leads to
\[
x_i(\tilde{t}_n - \tau_i(\tilde{t}_n)) \geq \frac{r_i^l - \left( a_i^u + \int_{-\infty}^0 k_i^u(s) \, ds \right) M_i - \sum_{j=1}^n D_{ij}^u}{b_i^u}. \tag{21}
\]

Integrating both sides of (16) on the interval \([\tilde{t}_n - \tau_i(\tilde{t}_n), \tilde{t}_n]\), we have
\[
\ln \left[ \frac{x_i(\tilde{t}_n)}{x_i(\tilde{t}_n - \tau_i(\tilde{t}_n))} \right] \geq \int_{\tilde{t}_n - \tau_i(\tilde{t}_n)}^{\tilde{t}_n} \left( r_i^l - \left( a_i^u + \int_{-\infty}^0 k_i^u(s) \, ds \right) M_i - \sum_{j=1}^n D_{ij}^u \right) dt \geq \int_{\tilde{t}_n - \tau_i(\tilde{t}_n)}^{\tilde{t}_n} \left( r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^0 k_i^u(s) \, ds \right) M_i - \sum_{j=1}^n D_{ij}^u \right) dt = \left( r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^0 k_i^u(s) \, ds \right) M_i - \sum_{j=1}^n D_{ij}^u \right) \tau_i(\tilde{t}_n).
\]

Thus
\[
x_i(\tilde{t}_n) \geq \frac{r_i^l - \left( a_i^u + \int_{-\infty}^0 k_i^u(s) \, ds \right) M_i - \sum_{j=1}^n D_{ij}^u}{b_i^u} \exp \left\{ \left( r_i^l - \left( a_i^u + b_i^u + \int_{-\infty}^0 k_i^u(s) \, ds \right) M_i \sum_{j=1}^n D_{ij}^u \right) \tau_i(\tilde{t}_n) \right\} \tag{22}
\]

Since the right-hand side of (22) is independent of \( t \) and \( x_i(\tilde{t}_n) \) is an arbitrary local minimum of \( x_i(t) \), we can get that there exists a positive constant \( m_{i'} (m_i \leq m_{i'}^{**} \) such that \( x_i(t) \geq m_{i'} \) for all \( t \geq T_4^* \).

Taken \( m_{i'}^{**} = \min_{1, 2, \ldots, n} \left\{ \frac{r_i^l}{2a_i^u}, m_i \right\} \), we have
\[
\lim_{t \to \infty} \inf x_i(t) \geq m_{i'}^{**}. \tag{23}
\]

This completes the proof. \( \Box \)
Denote (H4):

\[ a_i^j > 0, \quad p_i = r_i^j - \left(b_i^j + \int_{-\infty}^0 k_i^j(s) \, ds \right) \left(M^* - \sum_{j=1}^n D_{ij}^* \right) > 0, \quad \text{where} \quad M^* = \max_{i=1,2,\ldots,n} \left\{ \frac{r_i^j}{a_i^j} \right\}; \]

or

\[ b_i^j > 0, \quad p_i = r_i^j - \left(a_i^j + \int_{-\infty}^0 k_i^j(s) \, ds \right) \left(M^{**} - \sum_{j=1}^n D_{ij}^* \right) > 0, \quad \text{where} \quad M^{**} = \max_{i=1,2,\ldots,n} \left\{ \frac{r_i^j}{b_i^j} e^{r_i^j t} \right\}. \]

**Theorem 2.1.** Assume that system (1) satisfies (H1)–(H3) and (H4), then system (1) is uniformly persistent.

**Proof.** Suppose \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) is any positive solution for system (1) with initial conditions (3). Let \( M = \max\{M^*, M^{**}\} \) and \( m = \min\{m^*, m^{**}\} \). By Lemmas 2.1–2.3, if system (1) satisfies (H1)–(H3) and (H4), then we have

\[ m \leq \lim_{t \to \infty} \inf x_i(t) \leq \lim_{t \to \infty} \sup x_i(t) \leq M, \quad i = 1, 2, \ldots, n. \]

Set

\[ D = \left\{ (x_1(t), x_2(t), \ldots, x_n(t)) : m \leq x_i(t) \leq M, \quad i = 1, 2, \ldots, n \right\}. \]

Then \( D \) is a bounded compact region which has positive distance from coordinate hyperplanes. By the proof of Lemmas 2.1–2.3, one obtains that every positive solution of system (1) with the initial condition (3) eventually enters and remains in the region \( D \). The proof is completed.

Theorem 2.1 and (H4) imply that if the intrinsic growth rate of the species in each patch is larger than the sum of the death rate and dispersion rate of the species under the negative feedback crowding and the effect of all the past life history of the species on its present birth rate, then the species population in each patch is uniformly persistent.

### 3. Global asymptotic stability

Now we further discuss the global asymptotic stability of system (1). Denote (H5):

\[ \lim_{t \to \infty} \inf A_i(t) = \lim_{t \to \infty} \inf \left\{ c_i a_i(t) - c_i \left( \frac{b_i(\sigma_i^{-1}(t))}{1 - \tau_i(\sigma_i^{-1}(t))} \right) + \int_{-\infty}^0 k_i(t-s, s) \, ds \right\} - \sum_{j=1}^n \frac{c_j D_{ji}(t)}{m} \geq \delta, \]

where constants \( \delta > 0 \) and \( c_i > 0, \quad i = 1, 2, \ldots, n \).

**Theorem 3.1.** System (1) is globally asymptotically stable provided (H1)–(H4) and (H5) hold.

**Proof.** For two arbitrary positive nontrivial solutions of system (1) \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) and \( y(t) = (y_1(t), y_2(t), \ldots, y_n(t)) \), according to Theorem 2.1, there exist positive constants \( T, m \) and \( M \) such that for all \( t \geq T \)

\[ m \leq x_i(t) \leq M, \quad m \leq y_i(t) \leq M, \quad i = 1, 2, \ldots, n. \]

Construct the Lyapunov function

\[ V(t) = \sum_{i=1}^n c_i \left[ \left| \ln x_i(t) - \ln y_i(t) \right| + \int_{\tau_i(t)}^t \frac{b_i(\sigma_i^{-1}(s))}{1 - \tau_i(\sigma_i^{-1}(s))} \left| x_i(s) - y_i(s) \right| \, ds \right. \]

\[ + \left. \int_{-\infty}^0 \int_{t+s}^t k_i(\theta - s, s) \left| x_i(\theta) - y_i(\theta) \right| \, d\theta \, ds \right]. \]
Calculating the upper right derivative of $V(t)$ along the solution of (1), it follows that

$$D^+ V(t) = \sum_{i=1}^{n} c_i \left[ \left( \frac{\dot{x}_i(t)}{x_i(t)} - \frac{\dot{y}_i(t)}{y_i(t)} \right) \text{sgn}(x_i(t) - y_i(t)) + \frac{b_i(\sigma_i^{-1}(t))}{1 - \tilde{t}_i(\sigma_i^{-1}(t))} |x_i(t) - y_i(t)| ight. $$

$$\left. - b_i(t)|x_i(t - \tau_i(t)) - y_i(t - \tau_i(t))| + \int_{-\infty}^{0} k_i(t - s, s)|x_i(t) - y_i(t)| \, ds \right]$$

$$- \int_{-\infty}^{0} k_i(t, s)|x_i(t + s) - y_i(t + s)| \, ds \right]$$

$$\leq \sum_{i=1}^{n} c_i \left[ -a_i(t)|x_i(t) - y_i(t)| + \frac{b_i(\sigma_i^{-1}(t))}{1 - \tilde{t}_i(\sigma_i^{-1}(t))} |x_i(t) - y_i(t)| + \int_{-\infty}^{0} k_i(t - s, s)|x_i(t) - y_i(t)| \, ds \right.$$

$$\left. + \text{sgn}(x_i(t) - y_i(t)) \sum_{j=1}^{n} D_{ij}(t) \frac{x_j(t) y_i(t) - x_i(t) y_j(t)}{x_i(t) y_i(t)} \right]. \quad (26)$$

In fact, we have

$$\text{sgn}(x_i(t) - y_i(t)) \sum_{j=1}^{n} D_{ij}(t) \frac{x_j(t) y_i(t) - x_i(t) y_j(t)}{x_i(t) y_i(t)} \leq \sum_{j=1}^{n} D_{ij}(t) \frac{|x_j(t) - y_j(t)|}{m}.$$ \quad (27)

In the following, we will show (27) in four cases.

Case (i): If $y_i < x_i, y_j < x_j$, then we have

$$\text{sgn}(x_i(t) - y_i(t)) \sum_{j=1}^{n} D_{ij}(t) \frac{x_j(t) y_i(t) - x_i(t) y_j(t)}{x_i(t) y_i(t)} \leq \sum_{j=1}^{n} D_{ij}(t) \frac{|x_j(t) - y_j(t)|}{m}.$$ \quad (27)

Case (ii): If $y_i < x_i, y_j > x_j$, then we have

$$\text{sgn}(x_i(t) - y_i(t)) \sum_{j=1}^{n} D_{ij}(t) \frac{x_j(t) y_i(t) - x_i(t) y_j(t)}{x_i(t) y_i(t)} \leq 0.$$ \quad (27)

hence

$$\text{sgn}(x_i(t) - y_i(t)) \sum_{j=1}^{n} D_{ij}(t) \frac{x_j(t) y_i(t) - x_i(t) y_j(t)}{x_i(t) y_i(t)} \leq \sum_{j=1}^{n} D_{ij}(t) \frac{|x_j(t) - y_j(t)|}{m}.$$ \quad (27)

Case (iii): if $y_i > x_i, y_j < x_j$ and Case (iv): if $y_i > x_i, y_j > x_j$ have the same result as Cases (i) and (ii).

From (26), (27) and (H5), we have that

$$D^+ V(t) \leq \sum_{i=1}^{n} c_i \left[ -a_i(t)|x_i(t) - y_i(t)| + \frac{b_i(\sigma_i^{-1}(t))}{1 - \tilde{t}_i(\sigma_i^{-1}(t))} |x_i(t) - y_i(t)| + \int_{-\infty}^{0} k_i(t - s, s) \, ds |x_i(t) - y_i(t)| \right.$$
\[ \leq - \sum_{i=1}^{n} A_i(t) |x_i(t) - y_i(t)| \leq - \sum_{i=1}^{n} \delta |x_i(t) - y_i(t)| \text{ for } t \geq T^* > T. \] (28)

Integrating both sides of (28) on interval \([T^*, t] \), for \( t \geq T^* \)

\[ V(t) + \int_{T^*}^{t} \sum_{i=1}^{n} \delta |x_i(s) - y_i(s)| ds \leq V(T^*). \]

Therefore, \( V(t) \) is bounded on \([T^*, \infty)\) and

\[ \int_{T^*}^{\infty} |x_i(s) - y_i(s)| ds < \infty \text{ for } i = 1, 2, \ldots, n. \] (29)

According to Theorem 2.1, we know that \( x(t) \) and \( y(t) \) are bounded on \([T^*, \infty)\). Hence \( |x_i(t) - y_i(t)| \) \((i = 1, 2, \ldots, n)\) are bounded and uniformly continuous on \([T^*, \infty)\). By using Lemma 1.22 in [26], we can conclude that for \( i = 1, 2, \ldots, n \)

\[ \lim_{t \to \infty} |x_i(t) - y_i(t)| = 0. \]

This completes the proof. \( \Box \)

In the following section, we consider the periodic solution of system (1).

4. Periodic solution of periodic system

If system (1) is a periodic system, then throughout this section we will assume that for all \( i, j = 1, 2, \ldots, n \):

- **(H\textsubscript{6})** The coefficients \( r_i(t), b_i(t) \) and \( D_{ij}(t) (D_{ii}(t) = 0) \) are nonnegative, continuous and periodic functions with period \( \omega \), and \( a_i(t) \) is a positive periodic function with period \( \omega \).
- **(H\textsubscript{7})** The functions \( k_i(t, s) \) are nonnegative, continuous and \( \omega \)-periodic with respect to \( t \) and integrable with respect to \( s \) on \((-\infty, 0] \) such that \( \int_{-\infty}^{0} k_i(t, s) ds \) is continuous with respect to \( t \) on \([0, \omega] \).
- **(H\textsubscript{8})** The functions \( \tau_i(t) \) is nonnegative, continuously differentiable and \( \omega \)-periodic with respect to \( t \) such that \( \inf_{[0, \omega]} \{1 - \dot{\tau}_i(t)\} > 0 \).

Define

\[ h^u = \max_{t \in [0, \omega]} h(t), \quad h^l = \min_{t \in [0, \omega]} h(t), \quad \bar{h} = \frac{1}{\omega} \int_{0}^{\omega} h(t) dt, \]

and we set

\[ \sigma_i(t) = t - \tau_i(t) \text{ for } t \in R, \quad \bar{k}_i(s) = \frac{1}{\omega} \int_{0}^{\omega} k_i(t, s) dt \text{ for } s \in (-\infty, 0], \quad \bar{k}_i = \int_{-\infty}^{0} \bar{k}_i(s) ds. \]

In this section, by using the Mawhin’s continuation theorem (see [25]), we shall show the existence of at least one positive periodic solution of system (1). To do so, we need to make some preparations.

Let \( X, Z \) be real Banach spaces, \( L : \text{dom } L \subset X \to Z \) be a linear mapping with domain \( \text{dom } L \subset X \) and \( N : X \to Z \) a continuous mapping. \( L \) is called a Fredholm mapping of index zero if \( \dim \text{Ker } L = \text{codim Im } L < +\infty \) and \( \text{Im } L \) is closed in \( Z \). If \( L \) is a Fredholm mapping of index zero and there are continuous projectors \( P : X \to X, \ Q : Z \to Z \) such that \( \text{Im } P = \text{ker } L, \ \text{Ker } Q = \text{Im } L = \text{Im} (I - Q) \), then the restriction \( L_P \) of \( L \) to \( \text{dom } L \cap \text{Ker } P \) is invertible.
Denote by $J$ the isomorphism $J : \text{Im } Q \to \text{Ker } L$, $K_P : \text{Im } L \to \text{Ker } P \cap \text{dom } L$ the inverse to $L_P$. If $\Omega$ is an open bounded subset of $X$, $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact, then the mapping $N$ will be called $L$-compact on $\overline{\Omega}$.

For convenience, we first introduce Mawhin’s continuation theorem [25].

**Lemma 4.1 (Continuation Theorem).** Let $\Omega \subset X$ be an open bounded set, let $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on $\overline{\Omega}$. Assume

(a) For each $\lambda \in (0, 1)$ any solution $x$ of the equation $Lx = \lambda Nx$ is such that $x \notin \partial \Omega$.
(b) For each $x \in \text{Ker } L \cap \partial \Omega$, $QN_x \neq 0$.
(c) $\deg \{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$.

**Lemma 4.2.** Let $z = (z_1, z_2, \ldots, z_n)^T$ be a solution of the equation

$$\tilde{r}_i - \tilde{a}_i e^{z_i} - \lambda \tilde{b}_i e^{z_i} - \lambda e^{z_i} \int_{-\infty}^{0} \tilde{k}_i(s) ds + \lambda \sum_{j=1}^{n} \tilde{D}_{ij} e^{z_j - z_i} - \sum_{j=1}^{n} \tilde{D}_{ij} = 0, \quad i = 1, 2, \ldots, n, \quad (30)$$

where $\lambda \in [0, 1]$. Assume that

$$(H_9) \quad \tilde{r}_i - \sum_{j=1}^{n} \tilde{D}_{ij} > 0, \quad i = 1, 2, \ldots, n, \quad (31)$$

holds, then $\eta_1 \leq z_i \leq \eta_2$, $i = 1, 2, \ldots, n$, where

$$\eta_1 = \ln \left( \frac{\min_{i=1,2,\ldots,n} (\tilde{r}_i - \sum_{j=1}^{n} \tilde{D}_{ij})}{\max_{i=1,2,\ldots,n} (\tilde{a}_i + \tilde{b}_i + \tilde{k}_i)} \right), \quad \eta_2 = \ln \left( \frac{\max_{i=1,2,\ldots,n} (\tilde{r}_i)}{\min_{i=1,2,\ldots,n} (\tilde{a}_i)} \right).$$

**Proof.** From (30) we have

$$\tilde{a}_i e^{z_i} \geq \tilde{r}_i - \lambda \tilde{b}_i e^{z_i} - \lambda e^{z_i} \int_{-\infty}^{0} \tilde{k}_i(s) ds - \sum_{j=1}^{n} \tilde{D}_{ij},$$

hence

$$\left( \tilde{a}_i + \lambda \tilde{b}_i + \lambda \int_{-\infty}^{0} \tilde{k}_i(s) ds \right) e^{z_i} \geq \tilde{r}_i - \sum_{j=1}^{n} \tilde{D}_{ij}. \quad (32)$$

Since $\lambda \leq 1$, from (32) we get that $z_i \geq \eta_1$, $i = 1, 2, \ldots, n$.

From (30) we have

$$\tilde{a}_i e^{z_i} \leq \tilde{r}_i + e^{-z_i} \sum_{j=1}^{n} \tilde{D}_{ij} e^{z_j} - \sum_{j=1}^{n} \tilde{D}_{ij}.$$

Setting $e^{z_k} = \max_{i=1,2,\ldots,n} e^{z_i}$, from (30) we have

$$\tilde{a}_k e^{z_k} \leq \tilde{r}_k + e^{-z_k} \sum_{j=1}^{n} \tilde{D}_{ij} e^{z_j} - \sum_{j=1}^{n} \tilde{D}_{ij} \leq \tilde{r}_k + \sum_{j=1}^{n} \tilde{D}_{ij} - \sum_{j=1}^{n} \tilde{D}_{ij} = \tilde{r}_k,$$

then we get that $z_i \leq \eta_2$, $i = 1, 2, \ldots, n$. This completes the proof. $\square$

**Theorem 4.1.** If $(H_9)$ holds true, then periodic system (1) with initial condition (3) has at least one strictly positive $\omega$-periodic solution.
Proof. Since solution of (1) remain positive for all \( t \geq 0 \), we let
\[
z_i(t) = \ln[x_i(t)], \quad i = 1, 2, \ldots, n.
\]
Substituting (33) into system (1), we derive
\[
\dot{z}_i(t) = r_i(t) - a_i(t)e^{z_i(t)} - b_i(t)e^{z_i(t-\tau_i(t))} - \int_{-\infty}^{0} k_i(t,s)e^{z_i(t+s)} \, ds + \sum_{j=1}^{n} D_{ij}(t)e^{z_j(t)-z_i(t)} - \sum_{j=1}^{n} D_{ij}(t),
\]
i = 1, 2, \ldots, n.
\]
It is easily to see that if (34) has one \( \omega \)-periodic solution \( z^*(t) = (z_1^*(t), z_2^*(t), \ldots, z_n^*(t))^T \), then \( x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T = (\exp[z_1^*(t)], \exp[z_2^*(t)], \ldots, \exp[z_n^*(t)])^T \) is a positive \( \omega \)-periodic solution of periodic system (1). Therefore, to complete the proof, it suffices to show that system (34) has one \( \omega \)-periodic solution. Let
\[
L : \text{dom} L \rightarrow X, \quad \dot{z}_1(t), \dot{z}_2(t), \ldots, \dot{z}_n(t))^T \rightarrow (\dot{z}_1, \dot{z}_2, \ldots, \dot{z}_n)^T,
\]
where \( \text{dom} L = \{(z_1(t), z_2(t), \ldots, z_n(t))^T \in X \cap C^1(\mathbb{R}, \mathbb{R}^n)\} \) and the mapping \( N : X \rightarrow X \) by
\[
N z = (N_1 z, N_2 z, \ldots, N_n z),
\]
and \( z = (z_1, z_2, \ldots, z_n)^T \). It is clear that \( N \) is continuous and it maps \( X \) into itself.

Define two projectors \( P \) and \( Q \) as
\[
P, Q : X \rightarrow X, \quad Pz = Qz = (P_1 z, P_2 z, \ldots, P_n z),
\]
where
\[
P_i z = \int_{0}^{\omega} z_i(t) \, dt, \quad i = 1, 2, \ldots, n.
\]
It is clear that
\[
\text{Ker} L = \{ z : z \in X, \quad z = h, \quad h \in \mathbb{R}^n \},
\]
\[
\text{Im} L = \{ z : z \in X, \int_{0}^{\omega} z(t) \, dt = 0 \}
\]
is closed in \( X \),
\[
\dim \text{Ker} L = \text{codim} \text{Im} L = n.
\]
Therefore, \( L \) is a Fredholm mapping of index zero. It is easy to show that \( P \) and \( Q \) are continuous projectors such that
\[
\text{Im} P = \text{Ker} L, \quad \text{Ker} Q = \text{Im} L = \text{Im}(I - Q).
\]
Furthermore, the inverse \( K_P \) of \( L_P \) exists and is given by \( K_P : \text{Im} L \rightarrow L \cap \text{Ker} P \),
\[
K_P z = \int_{0}^{l} z(s) \, ds - \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{l} z(s) \, ds \, dt.
\]
Then $QN : X \to X$ and $KP(I - Q)N : X \to X$ read

$$QNz = \frac{1}{\omega} \int_0^\omega \left\{ r_i(t) - a_i(t)e^{z_i(t)} - b_i(t)e^{z_i(t-\tau_i(t))} - \int_{-\infty}^0 k_{i,j}(t,s)e^{z_i(t+s)}ds + \sum_{j=1}^n D_{ij}(t)e^{z_j(t)} - z_i(t) \right\} dt,$$

$$KP(I - Q)Nz = \int_0^t Nz(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t Nz(s)dsdt - \frac{(t - \omega)}{2} \int_0^\omega Nz(s)ds.$$

Clearly, $QN$ and $KP(I - Q)N$ are continuous. By Ascoli–Arzela Theorem, if $\Omega$ is an open bounded set, then $QN\bar{\Omega}$ and $KP(I - Q)N\bar{\Omega}$ are precompact sets in $X$. Hence we deduce that $N$ is $L$-compact on $\bar{\Omega}$.

In the following, we will prove Theorem 4.1 by means of the continuation theorem. Suppose that $z = (z_1, z_2, \ldots, z_n)^T$ is a solution of the equation $Lz = \lambda Nz$ for some $\lambda \in (0, 1)$, and we have

$$\dot{z}_i(t) = \lambda \left\{ r_i(t) - a_i(t)e^{z_i(t)} - b_i(t)e^{z_i(t-\tau_i(t))} - \int_{-\infty}^0 k_{i,j}(t,s)e^{z_i(t+s)}ds + \sum_{j=1}^n D_{ij}(t)e^{z_j(t)} - z_i(t) \right\},$$

$i = 1, 2, \ldots, n$.  \hfill (36)

Integrating system (36) over $[0, \omega]$, for $i = 1, 2, \ldots, n$ we have

$$\int_0^\omega a_i(t)e^{z_i(t)}dt + \int_0^\omega b_i(t)e^{z_i(t-\tau_i(t))}dt + \int_0^\omega k_{i,j}(t,s)e^{z_i(t+s)}dsdt = \int_0^\omega r_i(t)dt + \sum_{j=1}^n \int_0^\omega D_{ij}(t)e^{z_j(t)}dt - \sum_{j=1}^n D_{ij}(t)dt.$$ \hfill (37)

Denote by $z_i(\bar{t}_i) = \max_{[0,\omega]} z_i(t)$ and $z_i(\bar{t}_i) = \min_{[0,\omega]} z_i(t)$. Then from (37) we have

$$\omega \bar{a}_i e^{z_i(\bar{t}_i)} + \omega \bar{b}_i e^{z_i(\bar{t}_i)} + \omega e^{z_i(\bar{t}_i)} \int_{-\infty}^0 \bar{k}_i(s)ds \geq \omega \bar{r}_i - \omega \sum_{j=1}^n \bar{D}_{ij}, \quad i = 1, 2, \ldots, n,$$

and so,

$$e^{z_i(\bar{t}_i)} \geq m_1,$$ \hfill (38)

where

$$m_1 = \frac{\min_{i=1,2,\ldots,n} (\bar{r}_i - \sum_{j=1}^n \bar{D}_{ij})}{\max_{i=1,2,\ldots,n} (\bar{a}_i + \bar{b}_i + \bar{k}_i)} > 0.$$  

Multiplying (37) by $e^{z_i(t)}$ and integrating over $(0, \omega)$, for $i = 1, 2, \ldots, n$ we have

$$\int_0^\omega a_i(t)e^{z_i(t)}dt = \int_0^\omega r_i(t)e^{z_i(t)}dt - \int_0^\omega b_i(t)e^{z_i(t-\tau_i(t))}dt - \int_{-\infty}^0 k_{i,j}(t,s)e^{z_i(t+s)}e^{z_j(t)}dsdt + \sum_{j=1}^n \int_0^\omega D_{ij}(t)e^{z_j(t)}dt - \sum_{j=1}^n \int_0^\omega D_{ij}(t)e^{z_i(t)}dt.$$ \hfill (39)
From (39) and by Hölder inequality, we have
\[
\frac{d^2}{\omega} \left( \int_0^\omega e^{z_i(t)} \, dt \right)^2 \leq \int_0^\omega d_i(t) e^{2z_i(t)} \, dt \leq r^\omega \int_0^\omega e^{z_i(t)} \, dt + \sum_{j=1}^n D_{ij}^\omega \int_0^\omega e^{z_j(t)} \, dt.
\]

Let \( \int_0^\omega e^{z_i(t)} \, dt = \max_{i=1, \ldots, n} \int_0^\omega e^{z_i(t)} \, dt \), then we have
\[
\frac{d^2}{\omega} \left( \int_0^\omega e^{z_i(t)} \, dt \right)^2 \leq r^\omega \int_0^\omega e^{z_i(t)} \, dt + \sum_{j=1}^n D_{ij}^\omega \int_0^\omega e^{z_j(t)} \, dt.
\]

From (40) we have
\[
\int_0^\omega e^{z_i(t)} \, dt \leq M_1,
\]
where
\[
M_1 = \omega \frac{\max_{i=1, \ldots, n} \{r^\omega_i + \sum_{j=1}^n D_{ij}^\omega \}}{\min_{i=1, \ldots, n} \{d_i^\omega \}} > 0,
\]
then we have
\[
\int_0^\omega e^{z_i(t)} \, dt \leq M_1, \quad i = 1, 2, \ldots, n.
\]

Since \( z_i(t) \) is \( \omega \)-periodic, we have
\[
\int_0^\omega \int_{-\infty}^0 k_i(t, s) e^{z_i(t+s)} \, ds \, dt = \int_0^\omega \int_{-\infty}^0 k_i(t, s) e^{z_i(t)} \, ds \, dt = \int_0^\omega \int_0^\omega k_i(t, s) \, ds \, dt,
\]
and
\[
\int_0^\omega \int_{-\infty}^0 e^{z_i(t-\tau(t))} \, ds \, dt = \int_0^\omega \int_{\tau_i(0)}^\omega e^{z_i(s)} \frac{1}{1 - \dot{\tau}_i(\sigma_i^{-1}(s))} \, ds \leq \frac{1}{\xi_i^\theta} \int_0^\omega \int_{-\tau_i(0)}^\omega e^{z_i(s)} \, ds \, dt,
\]
where \( \xi_i^\theta = \inf_{[0, \omega]} (1 - \dot{\tau}_i(t)) > 0 \).

From (37), (41), (42) and (43), it follows that
\[
\sum_{j=1}^n \int_0^\omega D_{ij}(t) e^{z_j(t-\tau(t))} \, dt \leq \left( a_i^\mu + \frac{b_i^\mu}{\xi_i^\theta} + \int_{-\infty}^0 k_i^\mu(s) \, ds \right) M_1 + \omega \sum_{j=1}^n \bar{D}_{ij}.
\]
Hence there exists a positive constant \( M_2 \) which is independent of \( \lambda \) such that
\[
\sum_{j=1}^n \int_0^\omega D_{ij}(t) e^{z_j(t)-\tau(t)} \, dt \leq M_2, \quad i = 1, 2, \ldots, n.
\]

It follows from (36), (41)–(44) that
\[
\int_0^\omega |z_i(t)| \, dt \leq \omega \tilde{r}_i + \left( a_i^\mu + \frac{b_i^\mu}{\xi_i^\theta} + \int_{-\infty}^0 k_i^\mu(s) \, ds \right) M_1 + M_2 + \omega \sum_{j=1}^n \bar{D}_{ij}.
\]
Hence there exists a positive constant $M_3$ which is independent of $\lambda$ such that
\[
\int_0^\omega |\tilde{z}_i(t)| \, dt \leq M_3, \quad i = 1, 2, \ldots, n.
\] (45)

By means of the mean value theorem, (38) and (41), we have
\[
z_i(\tilde{t}_i) \geq \ln\left(\frac{m_1}{\omega}\right), \quad z_i(\bar{t}_i) \leq \ln\left(\frac{M_1}{\omega}\right).
\] (46)

Since
\[
z_i(t) \leq z_i(\bar{t}_i) + \int_0^\omega |\tilde{z}_i(t)| \, dt, \quad z_i(t) \geq z_i(\tilde{t}_i) - \int_0^\omega |\tilde{z}_i(t)| \, dt
\]
for $i = 1, 2, \ldots, n$ and $t \in [0, \omega]$, then it follows from (45) and (46) that
\[
\ln\left(\frac{m_1}{\omega}\right) - M_3 \leq z_i(t) \leq \ln\left(\frac{M_1}{\omega}\right) + M_3.
\]

Hence there exists a positive constant $M_4$ which is independent of $\lambda$ such that
\[
\|z\|_X = \sum_{j=1}^n \max_{[0,\omega]} \left|z_i(t)\right| \leq M_4,
\] (47)

where $z = (z_1(t), z_2(t), \ldots, z_n(t))^T$.

Now we take $\Omega = \{z \in X : \|z\| < B\}$, where $B = \max\{M_4, n|\eta_1|, n|\eta_2|\} + 1 > 0$. It is clear that $\Omega$ is an open bounded set of $X$ and $z \notin \partial \Omega$ for $\lambda \in (0, 1)$. Hence, the condition (a) of Lemma 4.1 is satisfied.

Let $z \in \partial \Omega \cap \text{Ker} L$, then $z = (z_1, z_2, \ldots, z_n)^T$ is a constant vector in $\mathbb{R}^n$ such that $\|z\| = \sum_{j=1}^n |z_j| = B$. Suppose that $QNz = 0$, we have
\[
\bar{r}_i - \bar{a}_i e^{zi} - \bar{b}_i e^{zi} - e^{zi} \int_{-\infty}^0 \bar{k}_i(s) \, ds + \sum_{j=1}^n \bar{D}_{ij} e^{z_j} - z_i - \sum_{j=1}^n \bar{D}_{ij} = 0, \quad i = 1, 2, \ldots, n.
\] (48)

By Lemma 4.2 with $\lambda = 1$, we conclude
\[
\|z\| = \sum_{j=1}^n |z_j| \leq \max\{n|\eta_1|, n|\eta_2|\} < B,
\]

which is a contradiction. Therefore, $QNz \neq 0$ and the condition (b) of Lemma 4.1 is satisfied. Finally, we will prove that the condition (c) of Lemma 4.1 is satisfied. To this end, we define $\phi : \text{dom} L \times [0, 1] \to X$, $\phi(z, \lambda) = (\phi_1(z, \lambda), \phi_2(z, \lambda), \ldots, \phi_n(z, \lambda))$ with $z = (z_1, z_2, \ldots, z_n)^T$ and
\[
\phi_i(z, \lambda) = \bar{r}_i - \bar{a}_i e^{zi} - \bar{b}_i e^{zi} - \lambda e^{zi} \int_{-\infty}^0 \bar{k}_i(s) \, ds + \lambda \sum_{j=1}^n \bar{D}_{ij} e^{z_j} - z_i - \sum_{j=1}^n \bar{D}_{ij}, \quad i = 1, 2, \ldots, n,
\]

where $\lambda$ is a parameter and $\phi$ is continuous with respect to $z$ and $\lambda$. In the following, we shall show that $\phi_i(z, \lambda) \neq 0$ for $z \in \partial \Omega \cap \text{Ker} L$ and any $\lambda \in [0, 1]$. Suppose that there exists $z \in \partial \Omega \cap \text{Ker} L$ such that
\[
\bar{r}_i - \bar{a}_i e^{zi} - \bar{b}_i e^{zi} - \lambda e^{zi} \int_{-\infty}^0 \bar{k}_i(s) \, ds + \lambda \sum_{j=1}^n \bar{D}_{ij} e^{z_j} - z_i - \sum_{j=1}^n \bar{D}_{ij} = 0, \quad i = 1, 2, \ldots, n.
\]
By means of Lemma 4.2, we conclude
\[ \|z\| = \sum_{j=1}^{n} |z_i| \leq \max\{n|\eta_1|, n|\eta_2|\} < B, \]
which is a contradiction. Hence \( \phi_i(z, \lambda) \neq 0 \) for \( z \in \partial \Omega \cap \text{Ker } L \) and \( \lambda \in [0, 1] \).

By using the invariance property of the topological degree by homotopy (see [25]), we get
\[
\deg(JQN(z_1, z_2, \ldots, z_n)^T, \Omega \cap \text{Ker } L, (0, 0, \ldots, 0)^T)
= \deg(\phi(z_1, z_2, \ldots, z_n, 1), \Omega \cap \text{Ker } L, (0, 0, \ldots, 0)^T)
= \deg(\phi(z_1, z_2, \ldots, z_n, 0), \Omega \cap \text{Ker } L, (0, 0, \ldots, 0)^T)
= \deg\left( \bar{r}_1 - \bar{a}_1 e^{\bar{z}_1} - \sum_{j=1}^{n} \bar{D}_1 j, \ldots, \bar{r}_n - \bar{a}_n e^{\bar{z}_n} - \sum_{j=1}^{n} \bar{D}_n j \right)^T, \Omega \cap \text{Ker } L, (0, 0, \ldots, 0)^T. \]

Under assumption (H9), one can easily show that the system of algebraic equations
\[
\bar{r}_i - \sum_{j=1}^{n} \bar{D}_{ij} - \bar{a}_i e^{\bar{z}_i} = 0, \quad i = 1, 2, \ldots, n
\]
has a unique solution \( z^* = (z_1^*, z_2^*, \ldots, z_n^*)^T \) which satisfies
\[
z_i^* = \ln\left( \frac{\bar{r}_i - \sum_{j=1}^{n} \bar{D}_{ij}}{\bar{a}_i} \right).
\]
Since \( \eta_1 \leq z_i^* \leq \eta_2 \), then \( z^* \in \Omega \cap \text{Ker } L \). A direct calculation shows that
\[
\deg(JQN(z_1, z_2, \ldots, z_n)^T, \Omega \cap \text{Ker } L, (0, 0, \ldots, 0)^T) = \text{sgn}\left( \prod_{i=1}^{n} (-\bar{a}_i e^{\bar{z}_i}) \right) = (-1)^n \neq 0.
\]
Therefore, the condition (c) of Lemma 4.1 is satisfied. The proof is completed. \( \square \)

We note that if (H4) holds true, then (H9) must hold true. By Theorems 3.1 and 4.1, we can deduce the following corollary.

**Corollary 4.1.** If the periodic system (1) satisfies the assumptions (H4) and (H5), then the periodic system (1) has a unique strictly positive periodic solution which is globally attractive.

In the following section, by means of the almost periodic functional hull theory, we directly analyze the right functions of system (1) to discuss the uniqueness and global asymptotic stability of the almost periodic solution of almost periodic system (1).

5. Almost periodic solution of almost periodic system

Suppose that \( h(t) \) is an almost periodic function defined on \( R \). Define \( H(h(t)) \) denotes the hull of \( h(t) \).

If system (1) is a almost periodic system, then throughout this section we will assume that for all \( i, j = 1, 2, \ldots, n \):

\begin{itemize}
  \item [(H_{10})] The coefficients \( r_i(t), a_i(t), b_i(t) \) and \( D_{ij}(t)(D_{ii}(t) = 0) \) are nonnegative, continuous and almost periodic functions for all \( t \) on \( R \) and \( H(r_i(t)) > 0 \).
  \item [(H_{11})] The almost periodic functions \( k_i(t, s) \) are defined on \( R \times (-\infty, 0] \) and nonnegative and continuous with respect to \( t \) on \( R \) and integrable with respect to \( s \) on \( (-\infty, 0] \). There are nonnegative and continuous functions \( h_i(s) \) defined on \( (-\infty, 0] \) satisfying \( 0 < \int_{-\infty}^{0} (-s)h_i(s) \, ds < \infty \) such that \( k_i(t, s) \leq h_i(s) \) for all \( (t, s) \in R \times (-\infty, 0] \).
  \item [(H_{12})] \( \tau_i(t) \) is continuous and differentiable almost periodic functions on \( R \), and \( \bar{\tau}_i(t) \) is uniformly continuous with respect to \( t \) on \( R \) and \( \inf_{t \in R} [1 - \bar{\tau}_i(t)] > 0 \).
\end{itemize}
If system (1) satisfies the assumptions (H10)–(H12), then system (1) also satisfies (H1)–(H3).

Suppose that \( r^*_i(t) \in H(r_i(t)) \), \( a^*_i(t) \in H(a_i(t)) \), \( b^*_i(t) \in H(b_i(t)) \), \( \tau^*_i(t) \in H(\tau_i(t)) \), \( k^*_i(t, s) \in H(k_i(t, s)) \), \( D^*_i(t) \in H(D_{ij}(t)) \) are selected such that there is a time sequence \( \{ t_n \} \) and \( r_i(t + t_n) \to r^*_i(t) \), \( a_i(t + t_n) \to a^*_i(t) \), \( b_i(t + t_n) \to b^*_i(t) \), \( \tau_i(t + t_n) \to \tau^*_i(t) \), \( k_i(t + t_n, s) \to k^*_i(t, s) \), \( D_{ij}(t + t_n) \to D^*_i(t) \) as \( n \to \infty \), i.e., \( t_n \to \infty \) for all \( t \in R \) and \( s \in (-\infty, 0] \). Then we have a hull equation of system (1) as follows:

\[
\dot{x}_i(t) = x_i(t) \left[ r^*_i(t) - a^*_i(t)x_i(t) - b^*_i(t)x_i(t - \tau^*_i(t)) - \int_{-\infty}^{0} k^*_i(t, s)x_i(t + s) \, ds \right] + \sum_{j=1}^{n} D^*_i(t)(x_j(t) - x_i(t)), \quad i = 1, 2, \ldots, n. \tag{49}
\]

According to the almost periodic theory, we can conclude that if system (1) satisfies (H4), (H5), (H10)–(H12), then the following almost periodic differential equation

\[
\frac{dx}{dt} = f(t, x),
\tag{50}
\]

where \( f(t, x) \in C(R \times S, \Omega) \) is an almost periodic function, and \( \Omega \) is compact subset of \( R^n \).

By Theorem 3.2 in [23], we can easily obtain the lemma as follows.

**Lemma 5.1.** If each of hull equation of system (50) has a unique strictly positive solution, then system (50) has a unique strictly positive almost periodic solution.

**Proof.** Suppose \( \varphi(t) \) is a strictly positive solution of system (50) for \( t \) on \( R \). There exist sequences of real values \( \bar{\alpha} \) and \( \bar{\beta} \) which have common subsequence \( \alpha \subset \bar{\alpha} \) and \( \beta \subset \bar{\beta} \) such that \( T_{a + \beta}f(t, x) = T_{a}T_{\beta}f(t, x) \) for \( t \) on \( R \) and \( x \in R^n \). \( T_{a + \beta}\varphi(t) \) and \( T_{a}T_{\beta}\varphi(t) \) exist uniformly on compact set \( R \). Then \( T_{a + \beta}\varphi(t) \) and \( T_{a}T_{\beta}\varphi(t) \) are solutions of the following common hull equation of system (50)

\[
\frac{dx}{dt} = T_{a + \beta}f(t, x).
\]

Therefore, we have \( T_{a + \beta}\varphi(t) = T_{a}T_{\beta}\varphi(t) \), then \( \varphi(t) \) is an almost periodic solution of system (50). Since \( \alpha \subset \bar{\alpha} = \{ \bar{\alpha}_n \} \) and \( \bar{\alpha}_n \to +\infty \) as \( n \to +\infty \), \( T_{\bar{\alpha}}f(t, x) = f(t, x) \) is uniformly tenable with respect to \( t \) on \( R \) and \( x \in R^n \). For the sequences \( \bar{\alpha} \) and \( \alpha \subset \bar{\alpha} \), we conclude that \( T_{\alpha}\varphi(t) = \psi(t) \) is uniformly tenable with respect to \( t \) on \( R \) and \( \psi(t) \in R^n \). According to the uniqueness of solution and \( T_{\alpha}\psi(t) = \psi(t) \), one obtains that \( \varphi(t) = \psi(t) \). The proof is completed. \( \square \)

**Theorem 5.1.** If almost periodic system (1) satisfies (H4) and (H5), then almost periodic system (1) has a unique strictly positive almost periodic solution which is global asymptotically stable.

**Proof.** By Lemma 5.1, we only need to prove that each of hull equation of almost periodic system (1) has a unique strictly positive solution, hence we need firstly prove that each of hull equation of almost periodic system (1) has at least a strictly positive solution (existence), then we further prove that each of hull equation of system (1) has a unique strictly positive solution (uniqueness).

Now we prove the existence of strictly positive solution of any hull equation (49). According to the almost periodic hull theory (see [23]), there exists a time sequence \( \{ t_n \} \) such that \( r_i(t + t_n) \to r^*_i(t) \), \( a_i(t + t_n) \to a^*_i(t) \), \( b_i(t + t_n) \to b^*_i(t) \), \( \tau_i(t + t_n) \to \tau^*_i(t) \), \( k_i(t + t_n, s) \to k^*_i(t, s) \), \( D_{ij}(t + t_n) \to D^*_i(t) \) as \( n \to \infty \), i.e., \( t_n \to \infty \) for all \( t \in R \) and \( s \in (-\infty, 0] \). Suppose \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) is any positive solution of hull equation (49). By the proof of Theorem 2.1, we have

\[
0 < \inf_{t \in [0, \infty)} x_i(t) \leq \sup_{t \in [0, \infty)} x_i(t) < \infty \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{51}
\]
Let \( x_n(t) = x(t + t_n) \) for all \( t \geq -t_n, \ n = 1, 2, \ldots, \) such that

\[
\frac{dx_i(t)}{dt} = x_i(t) \left[ r_i^*(t + t_n) - a_i^*(t + t_n)x_i(t) - b_i^*(t + t_n)x_i(t - \tau_i^*(t + t_n)) - \int_{-\infty}^{0} k_i^*(t + t_n, s)x_i(t + s) \, ds \right] + \sum_{j=1}^{n} D_{ij}^*(t + t_n)(x_j(t) - x_i(t)), \quad i = 1, 2, \ldots, n.
\]

From the inequality (51) and the assumptions \((H_4)\) and \((H_{10})-(H_{12})\), there exists a positive constant \( K \) which is independent of \( n \) such that \( dx_n(t)/dt \leq K \) for all \( t \geq -t_n, \ n = 1, 2, \ldots. \) Therefore, for any positive integer \( r \) sequence \( \{x_n(t): n \geq r\} \) is uniformly bounded and equicontinuous on \([-t_n, \infty)\). According to Ascoli–Arzelà Theorem, one can conclude that there exists a subsequence \( \{t_k\} \) of \( \{t_n\} \) such that sequence \( \{x_k(t)\} \) not only converges on \( R \) as \( k \to \infty \). Suppose \( \lim_{k \to \infty} x_k(t) = x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t)) \), then \( x^*(t) \) is continuous on \( R \), and we have

\[
0 < \inf_{t \in R} x_i^*(t) \leq \sup_{t \in R} x_i^*(t) < \infty \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

From differential equation (52) and the assumptions \((H_4)\) and \((H_{10})-(H_{12})\), we can easily see that \( x^*(t) \) is the solution of hull equation (49), hence each of hull equation of almost periodic system (1) has at least a strictly positive solution.

In the following section, we will prove the uniqueness of strictly positive solution of any hull equation (49). Suppose that the hull equation (49) has two arbitrary strictly positive solutions \( x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t)) \) and \( y^*(t) = (y_1^*(t), y_2^*(t), \ldots, y_n^*(t)) \). Now we define a Lyapunov function \( V^*(t) \) on \( t \in R \) as

\[
V^*(t) = \sum_{i=1}^{n} c_i \left[ \int x_i^*(t) - y_i^*(t) \right] + \int_{t-\tau_i(t)}^{t} \left( \frac{b_i^*(\sigma_i^*(s))}{1 - \tau_i^*(\sigma_i^*(s))} \right) \left| x_i^*(s) - y_i^*(s) \right| ds \left| x_i^*(s) - y_i^*(s) \right| ds + \int_{-\infty}^{t} \int_{t+s}^{t} k_i^*(\theta - s, s) \left| x_i^*(\theta) - y_i^*(\theta) \right| d\theta ds.
\]

It is similar to the discussion of (28). Calculating the upper right derivative of \( V^*(t) \) along the solution of differential equation (49), one has

\[
D^+ V^*(t) \leq -\sum_{i=1}^{n} \delta \left| x_i^*(t) - y_i^*(t) \right| \quad \text{for} \quad t \in R.
\]

From (54), we can see that \( V^*(t) \) is a nonincreasing function on \( R \). Integrating both side of (54) on interval \([t, 0]\), we have that

\[
\int_{t}^{0} \sum_{i=1}^{n} \delta \left| x_i^*(s) - y_i^*(s) \right| ds \leq V^*(t) - V^*(0) \quad \text{for} \quad t < 0.
\]

Hence we have

\[
\int_{-\infty}^{0} \left| x_i^*(t) - y_i^*(t) \right| dt \leq \infty, \quad i = 1, 2, \ldots, n.
\]

Then \( x_i^*(t) \to y_i^*(t) \) as \( t \to -\infty \). Let \( m^* = \inf_{t \in R} \{x_i^*(t), y_i^*(t): i = 1, 2, \ldots, n\} \). It follows from (53) and the assumptions \((H_4), (H_{10})-(H_{12})\) that
By assumptions (H10)–(H12), we obtain that 
\[ V^*(t) \leq \sum_{i=1}^{n} c_i \left[ \frac{1}{m^s} |x_i^*(t) - y_i^*(t)| + \tau_i(t) \max_{s \leq t} \int_{-\infty}^{0} \left( -sh_i(s) \right) ds \right] + \max_{s \leq t} |x_i^*(s) - y_i^*(s)| \int_{-\infty}^{0} \left( -sh_i(s) \right) ds \]

\[ \leq \sum_{i=1}^{n} c_i \left[ \frac{1}{m^s} + \tau_i(t) \max_{t \in R} \left\{ \frac{b_i(t)}{1 - \tilde{\tau}_i(t)} \right\} + \int_{-\infty}^{0} \left( -sh_i(s) \right) ds \right] \left[ x_i^*(t) - y_i^*(t) \right]. \tag{55} \]

Let
\[ \rho(t) = \sum_{i=1}^{n} c_i \left[ \frac{1}{m^s} + \tau_i(t) \max_{t \in R} \left\{ \frac{b_i(t)}{1 - \tilde{\tau}_i(t)} \right\} + \int_{-\infty}^{0} \left( -sh_i(s) \right) ds \right]. \]

By assumptions (H10)–(H12), we obtain that \( \rho(t) \) is bounded for \( t < 0 \). Hence \( V^*(t) \to 0 \) as \( t \to -\infty \). On the other hand, by the proof of Section 3, one can also obtain that \( V^*(t) \to 0 \) as \( t \to +\infty \). Hence \( V^*(t) \to 0 \) as \( t \to \infty \), thus \( V^*(t) \) is a nonincreasing and nonnegative function on \( R \), then \( V^*(t) = 0 \). That is, \( x_i^*(t) = y_i^*(t) \) for all \( t \) on \( R \) and \( i = 1, 2, \ldots, n \). It is proved that any hull equation of system (1) has a unique strictly positive solution.

Summarizing the inference above, we know that each of hull equation of system (1) has a unique strictly positive almost periodic solution which is global asymptotically stable. The proof is completed. \( \square \)

6. Conclusions and examples

In this section, we present two particular cases of system (1).

Example I. If \( k_i(t, s) = 0 \) and \( \tau_i(t) = 0 \) for \( i = 1, 2, \ldots, n \), then system (1) can be rewritten as the following dispersal system [14,24]
\[ \dot{x}_i(t) = x_i(t) \left[ r_i(t) - a_i(t)x_i(t) \right] + \sum_{j=1}^{n} D_{ij}(t) \left( x_j(t) - x_i(t) \right), \quad i = 1, 2, \ldots, n. \]

Clearly, Example I is the case, \( a_i(t) > 0 \) and \( b_i(t) = 0 \) in this paper. The similar results on Lemma 2.1 are given in [14,24].

Example II. If \( k_i(t, s) = 0 \) for \( s \geq -\sigma \), that is the \( \int_{-\sigma}^{0} k_i(t, s)x_i(t+s)ds \) term which represents the effect of a period of past life history of the species on its present birth rate, and \( \tau_i(t) = \tau(t) \) for \( i = 1, 2, \ldots, n \), then system (1) can be written as the following dispersal system with delays [18,27]
\[ \dot{x}_i(t) = x_i(t) \left[ r_i(t) - a_i(t)x_i(t) - b_i(t)x_i(t - \tau(t)) - \int_{-\sigma}^{0} k_i(t, s)x_i(t+s)ds \right] + \sum_{j=1}^{n} D_{ij}(t) \left( x_j(t) - x_i(t) \right), \]

\[ i = 1, 2, \ldots, n. \]

The conclusions or results of the theorems in this paper depend on time delays, which is different from that of the theorems in [18] and [27]. The similar results on Lemma 2.1 with the particular case \( a_i(t) > 0 \) are given in [14,24].

In this paper, we consider the system as composed of patches connected by discrete diffusions, each patch is assumed to be occupied by a single species. Moreover, we have introduced the negative feedback crowding and the effect of the period of past life history of the species on its present birth rate into the model. By fundamental theory of delay differential equations, we get sufficient conditions for the permanence of the population in every patches. Further, by constructing suitable Lyapunove functional, we show that the system is globally asymptotically stable.
under some appropriate conditions. Using Mawhin continuation theorem of coincidence degree, we prove that the periodic system has at least one positive periodic solution, further, obtain the uniqueness and globally asymptotical stability for periodic system. By using almost periodic functional hull theory, we obtain sufficient conditions for the existence, uniqueness and globally asymptotical stability of almost periodic solution. This implies that the population in every patch present stable periodic or almost periodic fluctuating. We note that the conditions for the permanence, global stability of system and the existence, uniqueness of positive almost periodic solution are dependent of delays and diffusions. This phenomena is different from that in [13,14,24,27]. Our results show that the population in each patch will not go to extinction under some appropriate conditions. We also note that the permanence of system is dependent of delays, which show that the effect of the negative feedback crowding and all the past life history of the species on its present birth rate restricts the permanence of the population. Moreover, we also conclude that overmany population diffuse out a patch can cause the population in the patch to go to extinction.

To conclude, we make the following remarks:

(i) Usually one of the conditions for the permanence of the system is that the coefficient of density-dependent term (with $a_i(t)$) must be positive [14,18,27], however, in this paper we allow the coefficient to be zero therefore the study of the permanence of the population becomes technically more difficult.

(ii) Under assumption (H_9), system (1) has at least one positive periodic solution, thus the solution of system is not necessarily permanent. However, the condition is necessary in [14,18,24,27], where the Fixed Point Theorem was employed to prove existence of periodic solutions.

(iii) The right-hand side of system (1) is not exponential, the dispersal term $D_{ij}(x_j(t) - x_i(t))$ generates some technical difficulties in using Mawhin continuation theorem.

(iv) If system (1) is a periodic system and the assumptions (H_4)–(H_9) are satisfied, then system (1) has not only a unique strictly positive globally asymptotically stable periodic solution, but also a unique strictly positive almost periodic solution which is globally asymptotically stable. Hence the results of almost periodic system (1) contains that of the periodic system.

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References