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Tannaka duality for proper Lie groupoids☆

Giorgio Trentinaglia *

Department of Mathematics, Utrecht University, P.O. Box 80010, 3508 TA Utrecht, The Netherlands

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ABSTRACT

By replacing the category of smooth vector bundles of finite rank over a manifold with the category of what we call smooth Euclidean fields, which is a proper enlargement of the former, and by considering smooth actions of Lie groupoids on smooth Euclidean fields, we are able to prove a Tannaka duality theorem for proper Lie groupoids. The notion of smooth Euclidean field we introduce here is the smooth, finite dimensional analogue of the usual notion of continuous Hilbert field.

0. Introduction

The classical duality theory of Tannaka leads to the result that a compact group can be reconstructed from a purely discrete, algebraic object, namely the ring of its continuous, finite dimensional representations or, more precisely, the algebra of its representative functions. Compare [1,2]. The same theory can be efficiently recast in categorical terms. This alternative point of view on Tannaka duality stems from Grothendieck's theory of motives in algebraic geometry [3–5]. In this approach, one starts by considering, for an arbitrary locally compact group G, the category formed by the continuous representations of G on finite dimensional vector spaces, endowed with the symmetric monoidal structure arising from the usual tensor product of representations, and then one tries to recover G as the group of all tensor preserving natural endomorphisms of the standard forgetful functor which assigns each G-module the underlying vector space. See, for instance, [6]. When G is a compact Lie group, in particular, it follows that G can be reconstructed in this way up to isomorphism, as the C^∞-manifold structure of a Lie group is determined by the underlying topology.

It is natural to ask for a generalization of the aforesaid duality result to the realm of Lie groupoids, in which proper groupoids are expected to play the same role as compact groups. When trying to extend Tannaka duality from Lie groups to Lie groupoids, however, one is first of all confronted with the problem of choosing a suitable notion of representation for the latter. Now, the notion of smooth or, equivalently, continuous, finite dimensional representation has an obvious naive generalization to the Lie groupoid setting: the representations of a Lie groupoid $\mathcal{G}$ could be defined to be Lie groupoid homomorphisms $\mathcal{G} \longrightarrow GL(E)$ from $\mathcal{G}$ into the linear groupoids associated with smooth vector bundles E of finite rank over the base manifold of $\mathcal{G}$. Unfortunately, this naive generalization turns out to be inadequate for the purpose of carrying forward Tannaka duality to Lie groupoids; cf. Section 2 of [7] for a thorough discussion. This state of affairs forces us to introduce a different notion of representation for Lie groupoids. It seems reasonable to require that the new notion should be as close as possible to the naive notion recalled above, and that moreover in the case of groups one should recover the usual notion of smooth representation on a finite dimensional vector space. In the present paper, it is our purpose to work out all these problems.

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* Corresponding address: Mathematisches Institut, Georg-August-Universität Göttingen, Bunsenstraße 3–5, 37073 Göttingen, Germany. Tel.: +49 0551 397817; fax: +49 0551 392985.

E-mail address: gtrentin@uni-math.gwdg.de.

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To begin with, we construct, for each smooth manifold X, a category whose objects we call *smooth Euclidean fields* over X (Section 3). Our notion of smooth Euclidean field is the analogue, in the smooth and finite dimensional setting we confine ourselves to, of the notion of continuous Hilbert field introduced by Dixmier and Douady [8]. (The most recent work known to the author where representations of groupoids on continuous Hilbert fields are considered is [9]; the reader will find a number of additional references therein.) The category of smooth Euclidean fields over X is, for any paracompact manifold X, a proper enlargement of the category of smooth vector bundles over X.

One has a natural notion of representation of a Lie groupoid on a smooth Euclidean field (Section 4). Such representations form, for each Lie groupoid $\mathcal{G}$, a symmetric monoidal category, which, by construction, is connected to the category of smooth Euclidean fields over the base manifold of $\mathcal{G}$ by a canonical forgetful functor. These are the categorical data out of which we intend to recover $\mathcal{G}$ (under the assumption that $\mathcal{G}$ is proper).

By generalizing the procedure outlined at the beginning, one obtains, for each Lie groupoid $\mathcal{G}$, a "reconstructed groupoid" $\mathcal{T}(\mathcal{G})$. We call $\mathcal{T}(\mathcal{G})$ the *Tannakian bidual of* $\mathcal{G}$ (Definition 6.1). This groupoid comes equipped with a natural candidate for a differentiable structure on its space of arrows, namely, a sheaf of algebras of continuous real valued functions stable under composition with arbitrary smooth functions of several variables. (Any topological space endowed with such a structure constitutes what we call a C^∞*-space*; see 1.1.) There is a canonical homomorphism $\boldsymbol{\pi}_{\mathcal{G}}$ from $\mathcal{G}$ into its Tannakian bidual $\mathcal{T}(\mathcal{G})$, which proves to be also a morphism of C^∞-spaces (Definition 6.2). Now, our duality result can be stated as follows:

Theorem 6.9. For an arbitrary proper Lie groupoid $\mathcal{G}$, the canonical homomorphism $\boldsymbol{\pi}_{\mathcal{G}}$ from $\mathcal{G}$ into the respective Tannakian bidual $\mathcal{T}(\mathcal{G})$ is an isomorphism of both groupoids and C^∞-spaces. It follows that $\mathcal{T}(\mathcal{G})$ is a Lie groupoid, isomorphic to $\mathcal{G}$.

The argument we give here is complementary to the proof of the classical Tannaka duality theorem. Our efforts are mainly directed into showing how the classical theorem of Tannaka implies the surjectivity of the above-mentioned canonical homomorphism, and then into establishing the claim about the C^∞-space structure. By contrast, the fact that the canonical homomorphism is injective is a more or less direct consequence of a theorem of Zung [10], which, in fact, should be regarded as a counterpart for proper Lie groupoids of the classical Peter–Weyl theorem. Compare [7], Section 5.

1. Proper Lie groupoids

The present section is essentially introductory. Its purpose is to recall the necessary background, and to fix some notations that we shall be using throughout the paper. The standard references on the general theory of Lie groupoids are [11,12]. Among the other sources we shall be following in this section, we mention here [13,14,10].

The term *groupoid* refers to a small category where every arrow is invertible. A *Lie* groupoid can be approximately described as an internal groupoid in the category of smooth manifolds. To construct a Lie groupoid $\mathcal{G}$, one has to give a pair of manifolds of class C^∞, $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$, respectively called *manifold of objects* and *manifold of arrows,* and a list of smooth maps, called *structure maps;* the basic items in this list are the *source map* $\boldsymbol{s} : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$ and the *target map* $\boldsymbol{t} : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$, which have to meet the requirement that the fibred product $\mathcal{G}^{(2)} := \mathcal{G}^{(1)}{}_{\boldsymbol{s}}\times_{\boldsymbol{t}} \mathcal{G}^{(1)}$ exists in the category of C^∞-manifolds; then, one has to give a *composition law* $\boldsymbol{c} : \mathcal{G}^{(2)} \to \mathcal{G}^{(1)}$, a *unit section* $\boldsymbol{u} : \mathcal{G}^{(0)} \to \mathcal{G}^{(1)}$, and an *inverse map* $\boldsymbol{i} : \mathcal{G}^{(1)} \to \mathcal{G}^{(1)}$, for which the familiar algebraic laws must be satisfied.

The points $x = \boldsymbol{s}(g)$ and $x' = \boldsymbol{t}(g)$ are resp. called the *source* and the *target* of the *arrow* g. We let $\mathcal{G}(x, x')$ denote the set of all the arrows whose source is x and whose target is x', and we use the abbreviation $\mathcal{G}|_x$ for the *isotropy* (or *vertex*) group $\mathcal{G}(x, x)$. Notationally, we will often identify a point $x \in \mathcal{G}^{(0)}$ and the corresponding unit arrow $\boldsymbol{u}(x) \in \mathcal{G}^{(1)}$. It is customary to write $g' \cdot g$ or $g'g$ for the composition $\boldsymbol{c}(g', g)$, and g^{-1} for the inverse $\boldsymbol{i}(g)$.

Our description of the notion of Lie groupoid is still incomplete. It turns out that a couple of additional requirements are needed in order to get a reasonable definition.

Recall that a manifold M is said to be paracompact, if it is Hausdorff and there exists an ascending sequence of open subsets with compact closure $\cdots \subset U_i \subset \overline{U}_i \subset U_{i+1} \subset \cdots$ such that $M = \cup_{i=0}^{\infty} U_i$. A Hausdorff manifold is paracompact if, and only if, it possesses a countable basis of open subsets. Any open cover of a paracompact manifold admits a locally finite refinement. Any paracompact manifold admits partitions of unity of class C^∞ (subordinated to any given open cover). Compare [15].

In order to make the fibred product $\mathcal{G}^{(1)}{}_{\boldsymbol{s}}\times_{\boldsymbol{t}} \mathcal{G}^{(1)}$ meaningful as a manifold, and for other purposes related to our study, we shall include the following additional conditions in the definition of Lie groupoid:

1. The source map $\boldsymbol{s} : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$ is a submersion with Hausdorff fibres;
2. The manifold $\mathcal{G}^{(0)}$ is paracompact.

Note that we do not require the manifold of arrows $\mathcal{G}^{(1)}$ to be Hausdorff or paracompact. Actually, this manifold is neither Hausdorff nor second countable in many examples of interest. The first condition implies at once that the domain of the composition map is a submanifold of the Cartesian product $\mathcal{G}^{(1)} \times \mathcal{G}^{(1)}$ and that the target map is a submersion with Hausdorff fibres. Thus, the source fibres $\mathcal{G}(x, -) = \boldsymbol{s}^{-1}(x)$ and the target fibres $\mathcal{G}(-, x') = \boldsymbol{t}^{-1}(x')$ are closed Hausdorff submanifolds of $\mathcal{G}^{(1)}$. A Lie groupoid $\mathcal{G}$ is said to be *Hausdorff* if the manifold $\mathcal{G}^{(1)}$ is Hausdorff.

Some more Terminology: The manifold $\mathcal{G}^{(0)}$ is usually called the *base* of the groupoid $\mathcal{G}$. One also says that $\mathcal{G}$ is a groupoid over the manifold $\mathcal{G}^{(0)}$. We shall often use the notation $\mathcal{G}^x = \mathcal{G}(x, -) = \boldsymbol{s}^{-1}(x)$ for the fibre of the source map over a point $x \in \mathcal{G}^{(0)}$. More generally, we shall write

$$\mathcal{G}(S, S') = \{g \in \mathcal{G}^{(1)} : \boldsymbol{s}(g) \in S \,\&\, \boldsymbol{t}(g) \in S'\}, \qquad \mathcal{G}|_S = \mathcal{G}(S, S) \tag{1}$$

and $\mathcal{G}^S = \mathcal{G}(S, -) = \mathcal{G}(S, \mathcal{G}^{(0)}) = \boldsymbol{s}^{-1}(S)$ for all subsets $S, S' \subset \mathcal{G}^{(0)}$.

Example: Let G be a Lie group acting smoothly (from the left) on a manifold M. Then the *action* (or *translation*) *groupoid* $G \ltimes M$ is defined to be the Lie groupoid over M whose manifold of arrows is the Cartesian product $G \times M$, whose source and target maps are respectively the projection onto the second factor $(g, x) \mapsto x$ and the action $(g, x) \mapsto gx$, and whose composition law is the operation

$$(g', x')(g, x) = (g'g, x). \tag{2}$$

There is a similar construction $M \rtimes G$ associated with right actions.

A homomorphism of Lie groupoids is a smooth functor. More precisely, a homomorphism $\phi : \mathcal{G} \to \mathcal{H}$ consists of two smooth maps $\phi^{(0)} : \mathcal{G}^{(0)} \to \mathcal{H}^{(0)}$ and $\phi^{(1)} : \mathcal{G}^{(1)} \to \mathcal{H}^{(1)}$ compatible with the groupoid structure in the sense that $\boldsymbol{s} \circ \phi^{(1)} = \phi^{(0)} \circ \boldsymbol{s}$, $\boldsymbol{t} \circ \phi^{(1)} = \phi^{(0)} \circ \boldsymbol{t}$ and $\phi^{(1)}(g' \cdot g) = \phi^{(1)}(g') \cdot \phi^{(1)}(g)$. Lie groupoids and their homomorphisms form a category. A homomorphism $\phi : \mathcal{G} \to \mathcal{H}$ is said to be a *Morita equivalence* when

$$\begin{tikzcd} \mathcal{G}^{(1)} \arrow[r, "\phi^{(1)}"] \arrow[d, "(\boldsymbol{s},\boldsymbol{t})"'] & \mathcal{H}^{(1)} \arrow[d, "(\boldsymbol{s},\boldsymbol{t})"] \\ \mathcal{G}^{(0)} \times \mathcal{G}^{(0)} \arrow[r, "\phi^{(0)} \times \phi^{(0)}"] & \mathcal{H}^{(0)} \times \mathcal{H}^{(0)} \end{tikzcd} \tag{3}$$

is a pullback diagram in the category of C^∞-manifolds and the map

$$\boldsymbol{t}_{\mathcal{H}} \circ pr_2 : \mathcal{G}^{(0)}{}_{\phi^{(0)}} \times_{\boldsymbol{s}_{\mathcal{H}}} \mathcal{H}^{(1)} \to \mathcal{H}^{(0)} \tag{4}$$

is a surjective submersion.

There is also a notion of topological groupoid. This is just an internal groupoid in the category of topological spaces and continuous mappings. In the continuous case the definition is much simpler, since one need not worry about the domain of definition of the composition map. With the obvious notion of homomorphism, topological groupoids constitute a category.

Let $\mathcal{G}$ be a Lie groupoid and let x be a point of its base manifold $\mathcal{G}^{(0)}$. The *orbit of* $\mathcal{G}$ (or $\mathcal{G}$*-orbit*) *through* x is the subset

$$\mathcal{G}x \overset{\text{def}}{=} \mathcal{G} \cdot x \overset{\text{def}}{=} \boldsymbol{t}(\mathcal{G}^x) = \{x' \in \mathcal{G}^{(0)} | \exists g : x \to x'\}. \tag{5}$$

Note that the isotropy group $\mathcal{G}|_x$ acts from the right on the manifold $\mathcal{G}^x$. This action is clearly free and transitive along the fibres of the restriction of the target map to $\mathcal{G}^x$. The following facts hold, cf. [11] p. 115: (a) $\mathcal{G}(x, x')$ is a closed submanifold of $\mathcal{G}^{(1)}$ (b) $G_x = \mathcal{G}|_x$ is a Lie group (c) the $\mathcal{G}$-orbit through x is an immersed submanifold of $\mathcal{G}^{(0)}$ and the target map $\boldsymbol{t} : \mathcal{G}^x \to \mathcal{G}x$ determines a principal G_x-bundle over it (the set $\mathcal{G}x$ can obviously be identified with the homogeneous space $\mathcal{G}^x/G_x$, and it can be proved that there exists a possibly non-Hausdorff manifold structure on this quotient space such that the quotient map turns out to be a principal bundle).

1.1 (C^∞*-Spaces*). Recall that a *functionally structured space* is a topological space X endowed with a sheaf $\mathscr{F}$ of real algebras of continuous real valued functions on X (functional structure). There is an obvious notion of morphism for such spaces. (Compare [13], p. 297.)

Let $\mathscr{F}$ be an arbitrary functional structure on a topological space X. Let $\mathscr{F}^\infty$ denote the sheaf of continuous real valued functions on X generated by the following presheaf (of such functions):

$$U \mapsto \{f(a_1|_U, \ldots, a_d|_U) : f : \mathbb{R}^d \to \mathbb{R} \text{ of class } C^\infty, a_1, \ldots, a_d \in \mathscr{F}(U)\}; \tag{6}$$

here, the expression $f(a_1|_U, \ldots, a_d|_U)$ stands, of course, for the function $u \mapsto f(a_1(u), \ldots, a_d(u))$ on U. The pair $(X, \mathscr{F}^\infty)$ constitutes a functionally structured space, to which we shall refer as a C^∞*-functionally structured space* or, in short, C^∞*-space.*[1] More correctly, a C^∞-space should be defined as a functionally structured space $(X, \mathscr{F})$ such that $\mathscr{F} = \mathscr{F}^\infty$. Observe that smooth manifolds can be defined as topological spaces endowed with a C^∞-functional structure locally isomorphic to that of smooth functions on $\mathbb{R}^n$. C^∞-Spaces have, in general, better categorical properties than smooth manifolds. Note that the latter form, within C^∞-spaces, a full subcategory.

[1] A more abstract notion of C^∞-ring was introduced by Moerdijk and Reyes [16] in the context of smooth infinitesimal analysis. We do not feel competent enough to embark into a discussion of the literature here. In any case, we are not making any claim to originality in connection with the notion of C^∞-space.

1.2 (*C^∞-Groupoids*). For any continuous map $f : S \to T$ into a functionally structured space $(T, \mathscr{T})$, one has the functional structure $f^*\mathscr{T}$ on S *induced by* $\mathscr{T}$ *along* f. This is defined to be the sheaf of all the continuous functions on S which can be written locally in the form $\alpha \circ f$ with α a local section of $\mathscr{T}$. If $(X, \mathscr{F})$ is a C^∞-space, so is $(S, \mathscr{F}|_S)$ for every subspace S of X, where $\mathscr{F}|_S := i_S{}^*\mathscr{F}$ denotes the functional structure on S induced by $\mathscr{F}$ along the inclusion $i_S : S \hookrightarrow X$.

Caution! It is perhaps good to stress that $f^*\mathscr{T}$ is *not* the usual inverse image sheaf $f^{-1}\mathscr{T}$ associated with the presheaf $U \mapsto \varinjlim_{V \supset f(U)} \mathscr{T}(V)$ of all germs of local sections of $\mathscr{T}$ along the image $f(S)$. The above notion of induced functional structure, and the corresponding notation, are borrowed from Bredon [13, Section VI.1].

Next, we note that if $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ are two functionally structured spaces then so is their Cartesian product endowed with the sheaf $\mathscr{F} \otimes \mathscr{G}$ locally generated by the functions $(\varphi \otimes \psi)(x, y) = \varphi(x)\psi(y)$. It follows that $(\mathscr{F}^\infty \otimes \mathscr{G}^\infty)^\infty$ is a C^∞-functional structure on $X \times Y$, turning this into the product of $(X, \mathscr{F}^\infty)$ and $(Y, \mathscr{G}^\infty)$ in the category of C^∞-spaces.

By combining the preceding constructions, one can show that the category of C^∞-spaces is closed under fibred products (pullbacks). Notice that when X and Y are smooth manifolds or $S \subset X$ is a submanifold, one recovers the correct manifold structures, so that all these constructions on C^∞-spaces agree with the usual ones on manifolds whenever the latter make sense.

We shall use the term C^∞-*groupoid* to indicate a groupoid whose set of objects and of arrows are each endowed with the structure of a C^∞-space so that all the maps belonging to the groupoid structure (source, target, composition, unit section, inverse) are morphisms of C^∞-spaces. The base spaces of our C^∞-groupoids will always be smooth manifolds in practice (with C^∞-functional structure given by the sheaf of smooth functions). Every Lie groupoid is an example of a C^∞-groupoid.

A Lie (topological, C^∞-) groupoid $\mathcal{G}$ is said to be *proper* if it is Hausdorff and the combined source–target map $(\boldsymbol{s}, \boldsymbol{t}) : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$ is proper (in the familiar sense: the inverse image of a compact subset is compact). When $\mathcal{G}$ is proper, every $\mathcal{G}$-orbit is a closed submanifold.

Normalized Haar systems on proper Lie groupoids are the analogue of Haar probability measures on compact Lie groups. We will now recall the definition and the construction of Haar systems on proper Lie groupoids. Our exposition is based on [14]. Let $\mathcal{G}$ be a Lie groupoid over a manifold M.

Definition 1.3. A *positive Haar system* on $\mathcal{G}$ is a family of positive Radon measures $\{\mu^x\}$ $(x \in M)$, each one supported by the respective source fibre $\mathcal{G}^x = \mathcal{G}(x, -) = \boldsymbol{s}^{-1}(x)$, satisfying the following conditions:

(i) $\int \varphi \, \mathrm{d}\mu^x > 0$ for all nonnegative real $\varphi \in C_c{}^\infty(\mathcal{G}^x)$ with $\varphi \neq 0$;

(ii) for each $\varphi \in C_c{}^\infty(\mathcal{G}^{(1)})$ the function Φ on M defined by setting

$$\Phi(x) = \int_{\mathcal{G}^x} \varphi|_{\mathcal{G}^x} \mathrm{d}\mu^x \tag{7}$$

is of class C^∞;

(iii) (right invariance) for all $g \in \mathcal{G}(x, y)$ and $\varphi \in C_c{}^\infty(\mathcal{G}^x)$ one has

$$\int_{\mathcal{G}^y} \varphi \circ \tau^g \mathrm{d}\mu^y = \int_{\mathcal{G}^x} \varphi \, \mathrm{d}\mu^x \tag{8}$$

where $\tau^g : \mathcal{G}(y, -) \to \mathcal{G}(x, -)$ denotes right translation $h \mapsto hg$.

The existence of positive Haar systems on a Lie groupoid $\mathcal{G}$ can be established when $\mathcal{G}$ is proper. One way to do this is the following. One starts by fixing a Riemann metric on the vector bundle $\mathfrak{g} \to M$, where $\mathfrak{g}$ is the Lie algebroid of $\mathcal{G}$ (cf. [14], or Chap. 6 of [11]; note the use of paracompactness). Right translations determine isomorphisms $T\mathcal{G}(x, -) \approx \boldsymbol{t}^*\mathfrak{g}|_{\mathcal{G}(x,-)}$ for all $x \in M$. These isomorphisms can be used to induce, on the source fibres $\mathcal{G}(x, -)$, Riemann metrics whose associated volume densities provide the desired system of measures.

Definition 1.4. A *normalized Haar system* on $\mathcal{G}$ is a family of positive Radon measures $\{\mu^x\}$ $(x \in M)$, each one with support concentrated in the respective source fibre $\mathcal{G}^x$, enjoying the following properties: (a) All smooth functions on $\mathcal{G}^x$ are integrable with respect to μ^x, that is to say

$$C^\infty(\mathcal{G}^x) \subset L^1(\mu^x). \tag{9}$$

(b) Condition (ii), respectively (iii) of the preceding definition holds for an arbitrary smooth function φ on $\mathcal{G}^{(1)}$, respectively $\mathcal{G}^x$ (c) The following normalization condition is satisfied:

(i*) $\int \mathrm{d}\mu^x = 1$ for every $x \in M$.

Every proper Lie groupoid admits normalized Haar systems. One can prove this by using a cut-off function, namely a positive, smooth function c on the base M of the groupoid such that the source map restricts to a proper map on $\mathrm{supp}(c \circ \boldsymbol{t})$ and $\int (c \circ \boldsymbol{t}) \, \mathrm{d}\nu^x = 1$ for all $x \in M$, where $\{\nu^x\}$ is an arbitrary positive Haar system chosen in advance. The system of positive measures $\mu^x \equiv (c \circ \boldsymbol{t}) \, \nu^x$ will have the desired properties.

1.5 (*Slices and Local Linearizability*). Let $\mathcal{G}$ be a Lie groupoid and let M be its base manifold. We say that a submanifold N of M is a *slice at a point* $z \in N$ if the orbit immersion $\mathcal{G}z \hookrightarrow M$ is transversal to N at z. A submanifold S of M will be called a *slice* if it is a slice at all of its points. Note that if N is a submanifold of M and $g \in \mathcal{G}^N = \boldsymbol{s}^{-1}(N)$ then N is a slice at $z = \boldsymbol{s}(g)$ if and only if the intersection $\mathcal{G}^N \cap \boldsymbol{t}^{-1}(z')$, $z' = \boldsymbol{t}(g)$ is transversal at g. From this remark it follows that for each submanifold N the subset of all points at which N is a slice forms an open subset of N. If a submanifold S of M is a slice then the intersection $\boldsymbol{s}^{-1}(S) \cap \boldsymbol{t}^{-1}(S)$ is transversal, so that the restriction $\mathcal{G}|_S$ is a Lie groupoid over S; moreover, $\mathcal{G} \cdot S$ is an invariant open subset of M. For the proof of the following result, we refer the reader to [10].

Local Linearizability Theorem (*Zung*). *Let $\mathcal{G}$ be a proper Lie groupoid. Let x be a base point which is not moved by the tautological action of $\mathcal{G}$ on its own base. Then there exists a continuous linear representation $G \to GL(V)$ of the isotropy group $G = \mathcal{G}|_x$ on a finite dimensional vector space V such that for some open neighbourhood U of x one can find an isomorphism of Lie groupoids $\mathcal{G}|_U \approx G \ltimes V$ which makes x correspond to zero.*

1.6 (*Remark*). Consider two slices S, S' in M with, let us say, $\dim S \leqq \dim S'$. Suppose $g \in \mathcal{G}(S, S')$. Put $x = \boldsymbol{s}(g) \in S$ and $x' = \boldsymbol{t}(g) \in S'$. It is not difficult to see that there is a smooth target section $\tau : B' \to \mathcal{G}^{(1)}$ defined over some open neighbourhood B' of x' in S' such that $\tau(x') = g$ and the composite map $\boldsymbol{s} \circ \tau$ induces a submersion of B' onto an open neighbourhood of x in S. Thus, when $\mathcal{G}$ is proper, it follows from the preceding theorem that for each point $x \in M$ there are a finite dimensional linear representation $G \to GL(V)$ of a compact Lie group G and a $\mathcal{G}$-invariant open neighbourhood U of x in M for which there exists a Morita equivalence $\iota : G \ltimes V \hookrightarrow \mathcal{G}|_U$ such that $\iota^{(0)} : V \hookrightarrow U$ is an embedding of manifolds mapping the origin of V to x.

2. The language of tensor categories

This section consists of two parts. The first one contains a concise description of some standard categorical notions: tensor category, tensor functor, and tensor preserving natural transformation. On these topics, our exposition will follow the standard sources [17,4,5]. In the second part, and precisely from 2.2 onwards, we establish a couple of new results. These will play a central role in the proof of our reconstruction theorem, Theorem 6.9.

The term "tensor category" is introduced here just as a convenient abbreviation for "additive, complex linear, symmetric monoidal category, also endowed with a conjugation endofunctor". The description of the latter notion will involve several steps. We shall start by defining tensor categories simply as symmetric monoidal categories, and we shall add the rest of the structure along the way.

A tensor structure on a category $\mathcal{C}$ consists of the following data:

$$\text{a bifunctor } \otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \quad \text{a distinguished object } 1 \in \mathrm{Ob}(\mathcal{C}) \tag{10}$$

and a list of natural isomorphisms, called *constraints,*

$$\begin{aligned} &\alpha_{R,S,T} : R \otimes (S \otimes T) \xrightarrow{\sim} (R \otimes S) \otimes T, \\ &\gamma_{R,S} : R \otimes S \xrightarrow{\sim} S \otimes R, \\ &\lambda_R : R \xrightarrow{\sim} 1 \otimes R \quad \text{and} \quad \rho_R : R \xrightarrow{\sim} R \otimes 1 \end{aligned} \tag{11}$$

satisfying MacLane's coherence conditions [17]. A *tensor category* is a category endowed with a tensor structure. In the terminology of [17], the present notion corresponds to that of symmetric monoidal category. The natural isomorphism α (resp. γ) is called the *associativity* (resp. *commutativity*) constraint. The isomorphisms λ and ρ are called *unit* constraints.

Recall that a k-linear category, where k is any number field, is a category $\mathcal{C}$ in which all hom sets are endowed with a structure of k-vector space so that the composition of arrows is bilinear. One also says that $\mathcal{C}$ is a category endowed with a k-linear structure. A k-linear tensor category is a tensor category endowed with a k-linear structure such that the bifunctor $\otimes$ is bilinear. A *real* (*complex*) *tensor category* is a $\mathbb{R}$-linear ($\mathbb{C}$-linear) tensor category. From now on, in this paper, "tensor category" will mean "*additive* linear tensor category". Thus, in particular, there will be a zero object, and for all objects R, S there will be a direct sum $R \oplus S$.

Let $\mathcal{C}$, $\mathcal{C}'$ be tensor categories. A *tensor functor* of $\mathcal{C}$ into $\mathcal{C}'$ is obtained by attaching, to an ordinary functor $F : \mathcal{C} \to \mathcal{C}'$, two isomorphisms

$$\begin{aligned} &\tau_{R,S} : F(R) \otimes F(S) \xrightarrow{\sim} F(R \otimes S) \quad (\text{natural in } R, S) \quad \text{and} \\ &\upsilon : 1' \xrightarrow{\sim} F(1), \end{aligned} \tag{12}$$

also called *constraints,* which are to satisfy certain conditions expressing their compatibility with the constraints of the tensor categories $\mathcal{C}$ and $\mathcal{C}'$. The reader is referred to [loc. cit.] for a discussion of these conditions. Recall that a functor of k-linear categories is said to be k-linear if the induced maps on hom sets are k-linear. A k-linear functor between additive k-linear categories will preserve zero objects and direct sums. We agree that an assumption of linearity on the functor $F : \mathcal{C} \to \mathcal{C}'$ is part of our definition of the notion of tensor functor.

Let F, F' be tensor functors from $\mathcal{C}$ into $\mathcal{C}'$. A natural transformation $\lambda : F \to F'$ is said to be *tensor preserving* if the following diagrams commute:

$$\begin{tikzcd} F(R)\otimes F(S) \arrow[r, "\lambda(R)\otimes\lambda(S)"] \arrow[d, "\tau_{R,S}"] & F'(R)\otimes F'(S) \arrow[d, "\tau'_{R,S}"] \\ F(R\otimes S) \arrow[r, "\lambda(R\otimes S)"] & F'(R\otimes S) \end{tikzcd} \qquad \begin{tikzcd} 1' \arrow[r, equal] \arrow[d, "\upsilon"] & 1' \arrow[d, "\upsilon'"] \\ F(1) \arrow[r, "\lambda(1)"] & F'(1). \end{tikzcd} \tag{13}$$

The collection of all tensor preserving natural transformations $F \to F'$ shall be denoted by $\mathrm{Hom}^{\otimes}(F, F')$. Note that a natural transformation of F into F' is necessarily additive, i.e., satisfies $\lambda(R \oplus S) = \lambda(R) \oplus \lambda(S)$.

2.1 (*Complex Conjugation*). By an *anti-involution* on a complex tensor category $\mathcal{C}$, we mean an anti-linear tensor functor

$$\mathcal{C} \longrightarrow \mathcal{C}, \qquad R \mapsto \overline{R} \tag{14}$$

with the property that there exists a tensor preserving natural isomorphism

$$\iota_R : \overline{\overline{R}} \simeq R \quad \text{with } \iota_{\overline{R}} = \overline{\iota_R}. \tag{15}$$

By fixing one such isomorphism once and for all, one obtains a mathematical structure to which for the moment we shall refer as a *tensor category with conjugation.* A morphism of tensor categories with conjugation is obtained by attaching, to an ordinary (complex linear) tensor functor F, a tensor preserving natural isomorphism

$$\xi_R : \overline{F(R)} \xrightarrow{\sim} F\big(\overline{R}\big) \tag{16}$$

such that the following diagram commutes:

$$\begin{tikzcd} \overline{\overline{F(R)}} \arrow[r, "\overline{\xi_R}"] \arrow[dr, "\iota_{F(R)}"'] & \overline{F\big(\overline{R}\big)} \arrow[r, "\xi_{\overline{R}}"] & F\Big(\overline{\overline{R}}\Big) \arrow[dl, "F(\iota_R)"] \\ & F(R). & \end{tikzcd} \tag{17}$$

A *self-conjugate* tensor preserving natural transformation is defined to be a tensor preserving natural transformation λ making

$$\begin{tikzcd} \overline{F(R)} \arrow[r, "\overline{\lambda(R)}"] \arrow[d, "\xi_R"] & \overline{F'(R)} \arrow[d, "\xi'_R"] \\ F\big(\overline{R}\big) \arrow[r, "\lambda(\overline{R})"] & F'\big(\overline{R}\big) \end{tikzcd} \tag{18}$$

commute. We shall write $\lambda \in \mathrm{Hom}^{\overline{\otimes}}(F, F')$.

Example: The category of vector spaces. For any complex vector space V, we let $\overline{V}$ denote the complex vector space obtained by retaining the additive structure of V but changing the scalar multiplication into $zv^* = (\overline{z}v)^*$; the star here indicates that a vector of V is to be regarded as one of $\overline{V}$. Since any linear map $f : V \to W$ also maps $\overline{V}$ linearly into $\overline{W}$, we can as well regard f as a linear map $\overline{f} : \overline{V} \to \overline{W}$. Moreover, the unique linear map of $\overline{V} \otimes \overline{W}$ into $\overline{(V \otimes W)}$ that sends $v^* \otimes w^*$ to $(v \otimes w)^*$ is an isomorphism, and complex conjugation sets up a linear bijection between $\mathbb{C}$ and $\overline{\mathbb{C}}$. This turns vector spaces into a tensor category with conjugation, with $\overline{\overline{V}} = V$. We denote it by $\underline{\mathsf{Vec}}_{\mathbb{C}}$.

Example: Vector bundles. By generalizing the previous example, one constructs the category $\underline{\mathsf{Vec}}_{\mathbb{C}}(M)$ of smooth complex vector bundles of locally finite rank over a smooth manifold M. We shall identify $\underline{\mathsf{Vec}}_{\mathbb{C}}(\star)$ (where $\star$ denotes the one-point manifold) with the category $\underline{\mathsf{Vec}}_{\mathbb{C}}$ introduced above. Notice that the pullback of vector bundles along a smooth mapping $f : N \to M$ determines a morphism $f^* : \underline{\mathsf{Vec}}_{\mathbb{C}}(M) \longrightarrow \underline{\mathsf{Vec}}_{\mathbb{C}}(N)$ of tensor categories with conjugation.

Let $\mathcal{C}$ be a tensor category with conjugation. By a *real structure* on an object $R \in \mathrm{Ob}(\mathcal{C})$, we mean an isomorphism $\mu : R \simeq \overline{R}$ in $\mathcal{C}$ such that the composite $\overline{\mu} \circ \mu$ equals the identity on R modulo the canonical identification $\overline{\overline{R}} \simeq R$ provided by (15). Let $\mathbb{R}(\mathcal{C})$ denote the category whose objects are the pairs (R, μ) consisting of an object $R \in \mathrm{Ob}(\mathcal{C})$ together with a real structure μ on R and whose morphisms $a : (R, \mu) \to (S, \nu)$ are the morphisms $a : R \to S$ in $\mathcal{C}$ such that $\nu \circ a = \overline{a} \circ \mu$. Note that $\mathbb{R}(\mathcal{C})$ is naturally an $\mathbb{R}$-linear category. Further, there is an obvious induced tensor structure on it, which turns it into an $\mathbb{R}$-linear tensor category.

As an example, observe that one has an equivalence of (real) tensor categories between $\underline{\mathsf{Vec}}_{\mathbb{R}}$ and $\mathbb{R}(\underline{\mathsf{Vec}}_{\mathbb{C}})$. In one direction, to any real vector space V one can assign the pair $\big(\mathbb{C} \otimes V, z \otimes v \mapsto \overline{z} \otimes v\big)$. Conversely, any real structure $\mu : U \simeq \overline{U}$ on a complex vector space U determines the real eigenspace $U^{\mu} \subset U$ of all μ-invariant vectors. There is an analogous equivalence between $\underline{\mathsf{Vec}}_{\mathbb{R}}(M)$ and $\mathbb{R}\big(\underline{\mathsf{Vec}}_{\mathbb{C}}(M)\big)$ for each smooth manifold M.

Notice that a morphism $F : \mathcal{C} \to \mathcal{D}$ of tensor categories with conjugation induces an $\mathbb{R}$-linear tensor functor $\mathbb{R}(F) : \mathbb{R}(\mathcal{C}) \to \mathbb{R}(\mathcal{D})$. For any pair $F, G : \mathcal{C} \to \mathcal{D}$ of such morphisms, the map $\lambda \mapsto \widetilde{\lambda}$, where $\widetilde{\lambda}(R, \mu) = \lambda(R)$, is a bijection

$$\mathrm{Hom}^{\overline{\otimes}}(F, G) \xrightarrow{\simeq} \mathrm{Hom}^{\otimes}\big(\mathbb{R}(F), \mathbb{R}(G)\big) \tag{19}$$

between the self-conjugate tensor preserving transformations $F \to G$ and the tensor preserving transformations $\mathbb{R}(F) \to \mathbb{R}(G)$. Indeed, by exploiting the additivity of the category $\mathcal{C}$, one can construct a functor $\mathcal{C} \to \mathbb{R}(\mathcal{C})$ which plays the same role as the functor that assigns a complex vector space the underlying real vector space. One chooses, for each pair R, S of objects of $\mathcal{C}$, a direct sum $R \oplus S$; then, the obvious isomorphism $R \oplus \overline{R} \simeq \overline{(R \oplus \overline{R})}$ is a real structure on $R \oplus \overline{R}$. Observe that the functor $\mathbb{R}(\mathcal{C}) \to \mathcal{C}$, $(R, \mu) \mapsto R$ has an analogous interpretation. One therefore sees that the formalism of tensor categories with conjugation is essentially equivalent to that of real tensor categories.

Convention. Throughout the rest of the paper, we shall deal exclusively with complex tensor categories with conjugation. Therefore, from now on, "tensor category" shall mean "additive $\mathbb{C}$-linear tensor category with conjugation". Accordingly, "tensor functor" shall mean "morphism of $\mathbb{C}$-linear tensor categories with conjugation".

The next results are original. They will be put to use in the proof of our main theorem, in the final section.

2.2 (*Terminology*). Let $\mathcal{C}$ be a tensor category and let $F : \mathcal{C} \to \underline{\mathrm{Vec}}_{\mathbb{C}}$ be a tensor functor with values into (finite dimensional) complex vector spaces. Let H be a topological group, and suppose a homomorphism of monoids is given

$$\pi : H \longrightarrow \mathrm{End}^{\overline{\otimes}}(F). \tag{20}$$

We say that π is *continuous* if for every object $R \in \mathrm{Ob}(\mathcal{C})$ the induced representation

$$\pi_R : H \longrightarrow \mathrm{End}(F(R)) \tag{21}$$

defined by setting $\pi_R(h) = \pi(h)(R)$ is continuous.

Proposition 2.3. *Let $\mathcal{C}$, F, H and π be as in 2.2, with π continuous, and suppose, in addition, that H is a compact Lie group. Assume that the following condition is satisfied:*

(*) *for each pair of objects $R, S \in \mathrm{Ob}(\mathcal{C})$, and for each H-equivariant homomorphism $A : F(R) \to F(S)$, there exists some arrow $R \xrightarrow{a} S$ in $\mathcal{C}$ with $A = F(a)$.*

Then the homomorphism π is surjective. In particular, the monoid $\mathrm{End}^{\overline{\otimes}}(F)$ is a group.

Proof. Put $K = \mathrm{Ker}\,\pi \subset H$. This is a closed normal subgroup, because it coincides with the intersection $\bigcap \mathrm{Ker}\,\pi_R$ over all objects R of $\mathcal{C}$. On the quotient $G = H/K$ there is a unique (compact) Lie group structure such that the quotient homomorphism $H \to G$ becomes a Lie group homomorphism. Every π_R can be indifferently thought of as a continuous representation of H or a continuous representation of G, and every linear map $A : F(R) \to F(S)$ is a morphism of G-modules if and only if it is a morphism of H-modules. Being continuous, every π_R is also smooth.

We claim there exists an object R_0 of $\mathcal{C}$ such that the corresponding π_{R_0} is faithful as a representation of G. Indeed, by the compactness of the Lie group G, we can find $R_1, \ldots, R_\ell \in \mathrm{Ob}(\mathcal{C})$ with the property that

$$\mathrm{Ker}\,\pi_{R_1} \cap \cdots \cap \mathrm{Ker}\,\pi_{R_\ell} = \{e\}, \tag{22}$$

where e denotes the unit of G; compare [2], p. 136. Then, if we set $R_0 = R_1 \oplus \cdots \oplus R_\ell$, the representation π_{R_0} will be faithful because of the existence of an isomorphism of G-modules

$$F(R_1 \oplus \cdots \oplus R_\ell) \approx F(R_1) \oplus \cdots \oplus F(R_\ell). \tag{23}$$

Now, it follows that the G-module $F(R_0)$ is a "generator" for the tensor category $\underline{\mathrm{Rep}}_{\mathbb{C}}(G)$ of all continuous, finite dimensional, complex G-modules. Indeed, every irreducible such G-module V embeds as a submodule of some tensor power $F(R_0)^{\otimes k} \otimes (F(R_0)^*)^{\otimes \ell}$, see for instance [2], p. 137. Since each $\pi(h)$ is, by assumption, self-conjugate and tensor preserving, this tensor power will be naturally isomorphic to $F\left({R_0}^{\otimes k} \otimes ({R_0}^*)^{\otimes \ell}\right)$ as a G-module and hence for each object V of $\underline{\mathrm{Rep}}_{\mathbb{C}}(G)$ there will be some object R of $\mathcal{C}$ such that V embeds into $F(R)$ as a submodule.

Next, consider an arbitrary natural transformation $\lambda \in \mathrm{End}(F)$. Let R be an object of the category $\mathcal{C}$, and let $V \subset F(R)$ be a submodule. The choice of a complement to V in $F(R)$ determines an endomorphism of modules $P_V : F(R) \to V \hookrightarrow F(R)$, which, by the assumption (*), will come from some endomorphism of R in $\mathcal{C}$. This implies that the linear operators $\lambda(R)$ and P_V on the space $F(R)$ commute with one another and, consequently, that $\lambda(R)$ maps the subspace V into itself. We will usually omit any reference to R and write simply λ_V for the linear map that $\lambda(R)$ induces on V by restriction. Note finally that, given another submodule $W \subset F(S)$ and any equivariant map $B : V \to W$, one has

$$B \cdot \lambda_V = \lambda_W \cdot B. \tag{24}$$

To prove this identity, one first extends B to an equivariant map $F(R) \to F(S)$ and then invokes (*), as before.

Let F_G denote the tensor functor $\underline{\mathsf{Rep}}_{\mathbb{C}}(G) \longrightarrow \underline{\mathsf{Vec}}_{\mathbb{C}}$ that assigns each G-module the underlying vector space. We will now define an isomorphism of complex algebras

$$\theta : \mathrm{End}(F) \xrightarrow{\sim} \mathrm{End}(\mathsf{F}_G) \tag{25}$$

so that the following diagram commutes

$$\begin{tikzcd} H \arrow[r, "\pi"] \arrow[d, "\text{proj.}"'] & \mathrm{End}(F) \arrow[d, "\theta", "\simeq"'] \\ G \arrow[r, "\boldsymbol{\pi}_G"] & \mathrm{End}(\mathsf{F}_G), \end{tikzcd} \tag{26}$$

where $\boldsymbol{\pi}_G(g)$, for any $g \in G$, is the natural transformation of F_G into itself that assigns left multiplication by g on V to each G-module V. For each G-module V, there exists an object R of $\mathcal{C}$ together with an embedding $V \hookrightarrow F(R)$, so we could define $\theta(\lambda)(V)$ as the restriction λ_V of $\lambda(R)$ to V. Of course, it is necessary to check that this does not depend on the choices involved. Let two objects $R, S \in \mathrm{Ob}(\mathcal{C})$ be given, along with two equivariant embeddings of V into $F(R)$ and $F(S)$ respectively. Since it is always possible to embed everything equivariantly into $F(R \oplus S)$ without affecting the induced λ_V, it is no loss of generality to assume $R = S$. Now, it follows from remark (24) above that the two embeddings actually determine the same linear endomorphism of V. This shows that θ is well defined. (24) also implies that $\theta(\lambda) \in \mathrm{End}(\mathsf{F}_G)$. On the other hand put, for $\mu \in \mathrm{End}(\mathsf{F}_G)$ and $R \in \mathrm{Ob}(\mathcal{C})$, $\mu^F(R) = \mu(F(R))$. Then $\mu^F \in \mathrm{End}(F)$ and $\theta(\mu^F) = \mu$, because of the existence of embeddings of the form $V \hookrightarrow F(R)$ and because of the naturality of μ. This shows that θ is surjective, and also injective since $\lambda(R) = \theta(\lambda)(F(R))$. Finally, it is straightforward to check that (26) commutes.

In order to conclude the proof it will be enough to check that θ induces a bijection between $\mathrm{End}^{\overline{\otimes}}(F)$ and $\mathrm{End}^{\overline{\otimes}}(\mathsf{F}_G)$, for then our claim that π is surjective will follow immediately from the commutativity of (26) and the classical Tannaka duality theorem for compact groups (which says that $\boldsymbol{\pi}_G$ establishes a bijection between G and $\mathrm{End}^{\overline{\otimes}}(\mathsf{F}_G)$; see for example [6] or [2] for a proof). This can safely be left to the reader. $\square$

The argument we used in the foregoing proof to construct a "generator" tells us something interesting even in the noncompact case.

Proposition 2.4. *Let $\mathcal{C}$ and F be as in 2.2. Let G be a Lie group, but not necessarily compact, and let $\pi : G \to \mathrm{End}(F)$ be a faithful continuous homomorphism. Then there exists an object $R_0 \in \mathrm{Ob}(\mathcal{C})$ such that $\mathrm{Ker}\,\pi_{R_0}$ is a discrete subgroup of G or, equivalently, such that the representation*

$$\pi_{R_0} : G \to GL(F(R_0)) \tag{27}$$

is faithful in some open neighbourhood of the unit element in G.

Proof. By the faithfulness of π, one can always find, for a given object $R \in \mathrm{Ob}(\mathcal{C})$, another object S such $\mathrm{Ker}\,\pi_{R\oplus S}$ is a submanifold of $\mathrm{Ker}\,\pi_R$ of strictly smaller dimension, unless $\dim \mathrm{Ker}\,\pi_R = 0$. An inductive argument using the additivity of the category $\mathcal{C}$ will yield the required object R_0. $\square$

3. Smooth euclidean fields

In this section, we introduce our notion of smooth Euclidean field. These objects shall replace smooth vector bundles in the definition of the notion of Lie groupoid representation in Section 4. As we already pointed out in the course of the introduction, representations on vector bundles are inadequate for the purpose of generalizing the duality theorem of Tannaka to proper Lie groupoids. In fact, one can easily construct examples of inequivalent proper Lie groupoids whose respective categories of vector bundle representations are indistinguishable, i.e., isomorphic; on these matters, the reader may consult [7], Section 2. The present approach is, therefore, entirely justified.

Conventions. The capital letters $X, Y, Z, \ldots$ shall denote manifolds of class C^∞, the corresponding lower-case letters $x, x', \ldots, y$ etc. shall denote points on these manifolds. For practical purposes, we need to consider manifolds which are possibly neither Hausdorff nor paracompact. The symbol $\mathscr{C}^\infty_X$ shall stand for the sheaf of smooth complex valued functions on X. We shall occasionally refer to sheaves of $\mathscr{C}^\infty_X$-modules as *sheaves of modules over X*.

Definition 3.1. By a *smooth Hilbert field*, we mean an object $\mathscr{H}$ consisting of (1) a family $\{H_x\}$ of complex Hilbert spaces indexed by the set of points of a manifold X and (2) a sheaf $\boldsymbol{\Gamma}\mathscr{H}$ of $\mathscr{C}^\infty_X$-modules of local sections of $\{H_x\}$ subject to the following conditions:

(i) $\{\zeta(x) : \zeta \in (\boldsymbol{\Gamma}\mathscr{H})_x\}$, where $(\boldsymbol{\Gamma}\mathscr{H})_x$ indicates the stalk at x, is a dense linear subspace of H_x;
(ii) for each open subset U, and for all sections $\zeta, \zeta' \in \boldsymbol{\Gamma}\mathscr{H}(U)$, the function $\langle \zeta, \zeta' \rangle$ on U defined by $u \mapsto \langle \zeta(u), \zeta'(u) \rangle_{H_u}$ is smooth.

We refer to the manifold X as the *base* of $\mathscr{H}$; we will also say that $\mathscr{H}$ is a smooth Hilbert field over X. The sheaf $\boldsymbol{\Gamma}\mathscr{H}$ shall be called the *sheaf of smooth sections of* $\mathscr{H}$ and the elements of $\boldsymbol{\Gamma}\mathscr{H}(U)$, accordingly, shall be referred to as *smooth sections of* $\mathscr{H}$ *over* U.

Note on terminology: By a *section* (without further qualifications) of a family of spaces $\{H_x\}_{x\in X}$ over an open subset $U \subset X$ we simply mean a mapping $\zeta \in \prod_{x\in U} H_x$ which to each $x \in U$ assigns a vector $\zeta(x) \in H_x$. The sections of $\{H_x\}$ form, of course, a sheaf of $\mathscr{C}^\infty_X$-modules.

Let $\mathscr{H}$ and $\mathscr{H}'$ be smooth Hilbert fields over a manifold X. A *morphism* of $\mathscr{H}$ into $\mathscr{H}'$ is given by a family of bounded linear maps $\{a_x : H_x \to H'_x\}$ indexed by the set of points of X such that for each open subset U and for each $\zeta \in \boldsymbol{\Gamma}\mathscr{H}(U)$ the section over U given by $u \mapsto a_u \cdot \zeta(u)$ belongs to $\boldsymbol{\Gamma}\mathscr{H}'(U)$. Thus, in particular, a morphism $a : \mathscr{H} \to \mathscr{H}'$ of smooth Hilbert fields over X determines a morphism of sheaves of $\mathscr{C}^\infty_X$-modules, which we shall denote by $\boldsymbol{\Gamma}a : \boldsymbol{\Gamma}\mathscr{H} \to \boldsymbol{\Gamma}\mathscr{H}'$, between the corresponding sheaves of smooth sections. Smooth Hilbert fields over X and their morphisms form a category, which we shall denote by $\underline{\mathfrak{Hilb}}(X)$.

Suppose $\mathscr{H}$ and $\mathscr{G}$ are smooth Hilbert fields over a manifold X. Consider the family of tensor products $\{H_x \otimes G_x\}$. For any pair of smooth sections $\zeta \in \boldsymbol{\Gamma}\mathscr{H}(U)$ and $\eta \in \boldsymbol{\Gamma}\mathscr{G}(U)$, we let $\zeta \otimes \eta$ denote the section over U of the family $\{H_x \otimes G_x\}$ defined by $u \mapsto \zeta(u) \otimes \eta(u)$. The correspondence

$$U \mapsto \mathscr{C}^\infty(U)\Big\{\zeta \otimes \eta : \zeta \in \boldsymbol{\Gamma}\mathscr{H}(U),\ \eta \in \boldsymbol{\Gamma}\mathscr{G}(U)\Big\} \tag{28}$$

defines a sub-presheaf of the sheaf of sections of $\{H_x \otimes G_x\}$; here $\mathscr{C}^\infty(U)\{\cdots\}$ stands for the $\mathscr{C}^\infty(U)$-module spanned by $\{\cdots\}$. Let $\mathscr{H} \otimes \mathscr{G}$ denote the smooth Hilbert field over X given by the family $\{H_x \otimes G_x\}$ together with the sheaf of sections generated by the presheaf (28). We call $\mathscr{H} \otimes \mathscr{G}$ the *tensor product* of $\mathscr{H}$ and $\mathscr{G}$. Observe that for all morphisms $\mathscr{H} \xrightarrow{\alpha} \mathscr{H}'$ and $\mathscr{G} \xrightarrow{\beta} \mathscr{G}'$ of smooth Hilbert fields over X, the family of bounded linear maps $\{a_x \otimes b_x\}$ yields a morphism $\alpha \otimes \beta$ of $\mathscr{H} \otimes \mathscr{G}$ into $\mathscr{H}' \otimes \mathscr{G}'$.

We define the conjugate $\overline{\mathscr{H}}$ of a smooth Hilbert field $\mathscr{H}$ by taking the family $\{\overline{H_x}\}$ of conjugate spaces along with the local smooth sections of $\mathscr{H}$ regarded as local smooth sections of $\{\overline{H_x}\}$.

With the obvious tensor unit $\underline{\mathbb{C}}$ and the obvious tensor category constraints, these operations turn $\underline{\mathfrak{Hilb}}(X)$ into a tensor category.

Next, let $f : X \to Y$ be a smooth mapping. Let $\mathscr{G}$ be a smooth Hilbert field over Y. The *pullback of* $\mathscr{G}$ *along* f, to be denoted by $f^*\mathscr{G}$, is the smooth Hilbert field over X whose associated family of Hilbert spaces is $\{G_{f(x)}\}$ and whose associated sheaf of smooth sections is generated by the following presheaf of sections of the family $\{G_{f(x)}\}$:

$$U \mapsto \mathscr{C}^\infty_X(U)\Big\{\eta \circ f : \eta \in \boldsymbol{\Gamma}\mathscr{G}(V), V \supset f(U)\Big\}. \tag{29}$$

For each morphism $\beta : \mathscr{G} \to \mathscr{G}'$ of smooth Hilbert fields over Y, the family of bounded linear maps $\{b_{f(x)}\}$ defines a morphism $f^*\beta : f^*\mathscr{G} \to f^*\mathscr{G}'$ of smooth Hilbert fields over X. The operation $\mathscr{G} \mapsto f^*\mathscr{G}$ defines a *strict* tensor functor of $\underline{\mathfrak{Hilb}}(Y)$ into $\underline{\mathfrak{Hilb}}(X)$, in other words one has the identities

$$f^*(\mathscr{G} \otimes \mathscr{G}') = f^*\mathscr{G} \otimes f^*\mathscr{G}', \qquad f^*(1_Y) = 1_X \quad \text{and} \quad f^*\big(\overline{\mathscr{G}}\big) = \overline{f^*\mathscr{G}}.$$

Finally, the following identities of tensor functors hold:

$$(g \circ f)^* = f^* \circ g^* \quad \text{and} \quad {id_X}^* = Id.$$

Let $i_U : U \hookrightarrow X$ denote the inclusion of an open subset. We shall put, for every smooth Hilbert field $\mathscr{H}$ and morphism a in the category $\underline{\mathfrak{Hilb}}(X)$, $\mathscr{H}|_U := {i_U}^*\mathscr{H}$ and $a|_U := {i_U}^*a$. More generally, we shall make use of the same abbreviations for the inclusion $i_S : S \hookrightarrow X$ of an arbitrary submanifold.

Definition 3.2. We shall say that a smooth Hilbert field $\mathscr{E}$ over X is *locally finite* if $\boldsymbol{\Gamma}\mathscr{E}$ is a locally finitely generated sheaf of $\mathscr{C}^\infty_X$-modules, in other words, if X can be covered by open subsets U such that there are epimorphisms of sheaves of modules

$$\underbrace{\mathscr{C}^\infty_U \oplus \cdots \oplus \mathscr{C}^\infty_U}_{\text{finite direct sum}} \xrightarrow{\text{epi}} (\boldsymbol{\Gamma}\mathscr{E})|_U. \tag{30}$$

Let $\underline{\mathfrak{Euc}}(X)$ denote the full subcategory of $\underline{\mathfrak{Hilb}}(X)$ consisting of all locally finite smooth Hilbert fields. We refer to the objects of this subcategory as *smooth Euclidean fields over* X.

From the preceding definition and the condition (i) of Definition 3.1, it follows that for every smooth Euclidean field $\mathscr{E} = \big(\{E_x\}, \boldsymbol{\Gamma}\mathscr{E}\big)$ the Hilbert spaces E_x are finite dimensional. It will be convenient to have a name for these spaces; we shall refer to E_x as the *fibre of* $\mathscr{E}$ *at* x, occasionally written $\mathscr{E}_x$.

Note that $\underline{\mathfrak{Euc}}(X)$ is a *tensor subcategory* of the tensor category $\underline{\mathfrak{Hilb}}(X)$, i.e., it is closed under taking tensor products and conjugates; indeed, by the definition of the tensor product of smooth Hilbert fields, the locally finitely generated $\mathscr{C}^\infty_X$-module $\boldsymbol{\Gamma}\mathscr{E} \otimes_{\mathscr{C}^\infty_X} \boldsymbol{\Gamma}\mathscr{E}'$ surjects onto the $\mathscr{C}^\infty_X$-module $\boldsymbol{\Gamma}(\mathscr{E} \otimes \mathscr{E}')$. For similar reasons, for any smooth map $f : X \to Y$, the pullback functor $f^* : \underline{\mathfrak{Hilb}}(Y) \to \underline{\mathfrak{Hilb}}(X)$ must carry $\underline{\mathfrak{Euc}}(Y)$ into $\underline{\mathfrak{Euc}}(X)$, and therefore induces a strict tensor functor $f^* : \underline{\mathfrak{Euc}}(Y) \to \underline{\mathfrak{Euc}}(X)$.

Definition 3.3. We say that a morphism $\phi : \mathscr{E} \otimes \overline{\mathscr{E}} \to \underline{\mathbb{C}}$ in the category $\underline{\mathfrak{Euc}}(X)$ is a *metric on* $\mathscr{E}$, if for every point x the form ϕ_x induced on E_x

$$E_x \otimes_{\mathbb{C}} \overline{E_x} = \left(\mathscr{E} \otimes \overline{\mathscr{E}}\right)_x \xrightarrow{\phi_x} (\underline{\mathbb{C}})_x = \mathbb{C} \tag{31}$$

is positive definite Hermitian.

It follows at once, from our definitions, that for every smooth Euclidean field $\mathscr{E}$ one has a *canonical* metric on $\mathscr{E}$. However, the reader should keep in mind that these canonical metrics do not play any explicit role in our theory; what really matters is just the existence of at least one metric on each smooth Euclidean field. For example, no reference to metrics was really made in our definition of morphisms of smooth Euclidean fields, which, in fact, was given only in terms of the sheaves of smooth sections; nor will it be made in our definition of representations of Lie groupoids in Section 4 (our representations will be "smooth", but not unitary).

Let ϕ be a metric on a smooth Euclidean field $\mathscr{E}$. By a *ϕ-orthonormal frame for $\mathscr{E}$ about a point x in X*, we mean a list of smooth sections $\zeta_1, \dots, \zeta_d \in \boldsymbol{\Gamma}\mathscr{E}(U)$ defined over a neighbourhood U of x such that for all $u \in U$ the vectors $\zeta_1(u), \dots, \zeta_d(u)$ are orthonormal in E_u and

$$\mathrm{Span}\{\zeta_1(x), \dots, \zeta_d(x)\} = E_x. \tag{32}$$

We observe that orthonormal frames for $\mathscr{E}$ exist about each point x. Indeed, over some neighbourhood V of x, one can find local smooth sections $\zeta_1, \dots, \zeta_d$ with the property that the vectors $\zeta_1(x), \dots, \zeta_d(x)$ form a basis for E_x. Since for all $v \in V$ the vectors $\zeta_1(v), \dots, \zeta_d(v)$ are linearly dependent if and only if there is a d-tuple of complex numbers $(z_1, \dots, z_d)$ with $|z_1|^2 + \cdots + |z_d|^2 = 1$ and $\sum_{i=1}^d z_i\, \zeta_i(v) = 0$, the continuous function

$$V \times S^{2d-1} \to \mathbb{R}, \qquad (v; s_1, t_1, \dots, s_d, t_d) \mapsto \left| \sum_{k=1}^{d} (s_k + it_k)\zeta_k(v) \right|$$

must have a positive minimum at $v = x$, hence a positive lower bound on a suitable neighbourhood U of x, so that the vectors $\zeta_1(u), \dots, \zeta_d(u)$ must be linearly independent for all $u \in U$. Now, in order to obtain an orthonormal frame over U, it will be enough to apply the Gram–Schmidt process.

Definition 3.4. We shall say that a smooth Euclidean field $\mathscr{E}$ over X is *locally trivial*, if for each point x there is an open neighbourhood U such that one can find an isomorphism of smooth Euclidean fields over U

$$\mathscr{E}|_U \simeq \underline{\mathbb{C}}_U \oplus \cdots \oplus \underline{\mathbb{C}}_U.$$

Such an isomorphism will be called a *local trivialization* for $\mathscr{E}$ over U.

If, for a given smooth Euclidean field $\mathscr{E}$, the dimension of the spaces E_x is locally constant over X, then $\mathscr{E}$ is locally trivial. This is an immediate consequence of the preceding remark about the existence of local orthonormal frames, and of the following remark.

Consider an *embedding* $e : \mathscr{E}' \hookrightarrow \mathscr{E}$ of smooth Euclidean fields; that is to say, a morphism such that the linear map $e_x : E'_x \hookrightarrow E_x$ is injective for all x. (An embedding is a monomorphism. The converse need not be true, because the functor $\mathscr{E} \mapsto E_x$, for fixed x, does not enjoy any exactness properties. For example, let a be a smooth function on $\mathbb{R}$ such that $a(t) = 0$ if and only if $t = 0$. Then a, regarded as a morphism in $\mathrm{End}(\underline{\mathbb{C}})$, is both mono and epi in $\underline{\mathfrak{Euc}}(\mathbb{R})$, while $a_0 = 0 : \mathbb{C} \to \mathbb{C}$ is neither injective nor surjective.) Suppose $\mathscr{E}'$ is locally trivial. Then e admits a *cosection*, i.e., there exists a morphism $p : \mathscr{E} \to \mathscr{E}'$ with $p \circ e = id$. For example, one can define p_x, for each x, to be the orthogonal projection from E_x onto E'_x, where ϕ is a fixed global metric on $\mathscr{E}$. One can check this is indeed a morphism of smooth fields, locally at any point, by fixing a local orthonormal frame for $\mathscr{E}'$ relative to the metric ϕ' induced by ϕ on $\mathscr{E}'$ and then by writing down the explicit formula for the orthogonal projection in terms of this orthonormal frame.

Lemma 3.5. *Let X be a paracompact manifold, and let $i_S : S \hookrightarrow X$ be a closed submanifold. Let $\mathscr{E}, \mathscr{F}$ be smooth Euclidean fields over X, and suppose that the restriction $\mathscr{E}' := \mathscr{E}|_S$ is locally trivial; also, put $\mathscr{F}' := \mathscr{F}|_S$. Then every morphism $a' : \mathscr{E}' \to \mathscr{F}'$ in $\underline{\mathfrak{Euc}}(S)$ can be extended to a morphism $a : \mathscr{E} \to \mathscr{F}$ of smooth Euclidean fields over X.*

Proof. Fix a point $s_0 \in S$. By the local triviality assumption on $\mathscr{E}'$, there will be an open neighbourhood S_0 of s in S such that one has a trivialization $\mathscr{E}'|_{S_0} \simeq \underline{\mathbb{C}} \oplus \cdots \oplus \underline{\mathbb{C}}$ over S_0. Let $\zeta'_1, \dots, \zeta'_d \in \boldsymbol{\Gamma}\mathscr{E}'(S_0)$ be the corresponding frame of local smooth sections of $\mathscr{E}'$.

Let us fix an open subset U_0 of X such that $U_0 \cap S = S_0$. After taking U_0 and S_0 smaller about s_0 if necessary, it will be no loss of generality to assume that there are local smooth sections $\zeta_1, \dots, \zeta_d \in \boldsymbol{\Gamma}\mathscr{E}(U_0)$ with $\zeta'_k = \zeta_k \circ i_S$, $k = 1, \dots, d$. (This is clear from the construction of the pullback of smooth fields.)

Put $x_0 = i_S(s_0)$. The vectors $\zeta_k(x_0)$, $k = 1, \dots, d$ are linearly independent in the space E_{x_0}, because, by definition, they are precisely the vectors $\zeta'_k(s_0)$, $k = 1, \dots, d$ in the space E'_{s_0}. Hence, if U_0 is small enough, the morphism

$$\zeta_1 \oplus \cdots \oplus \zeta_d : \underline{\mathbb{C}}_{U_0} \oplus \cdots \oplus \underline{\mathbb{C}}_{U_0} \longrightarrow \mathscr{E}|_{U_0}$$

is an embedding in $\underline{\mathfrak{Euc}}(U_0)$ and therefore admits a cosection

$$p : \mathscr{E}|_{U_0} \longrightarrow \underline{\mathbb{C}}_{U_0} \oplus \cdots \oplus \underline{\mathbb{C}}_{U_0}.$$

Put $\eta'_k := \boldsymbol{\Gamma} a'(S_0)(\zeta'_k) \in \boldsymbol{\Gamma}\mathscr{F}'(S_0)$. As before, it is no loss of generality to assume that there are smooth sections $\eta_1, \ldots, \eta_d \in \boldsymbol{\Gamma}\mathscr{F}(U_0)$ with $\eta'_k = \eta_k \circ i_S$, $k = 1, \ldots, d$. These can be combined into a morphism

$$\eta_1 \oplus \cdots \oplus \eta_d : \underline{\mathbb{C}}_{U_0} \oplus \cdots \oplus \underline{\mathbb{C}}_{U_0} \longrightarrow \mathscr{F}|_{U_0}$$

(d-fold direct sum). Then one can take the composition

$$\mathscr{E}|_{U_0} \xrightarrow{p} \underline{\mathbb{C}}_{U_0} \oplus \cdots \oplus \underline{\mathbb{C}}_{U_0} \xrightarrow{\eta_1 \oplus \cdots \oplus \eta_d} \mathscr{F}|_{U_0}$$

as a local extension of a' about the given point s_0.

One concludes the proof by resorting to a smooth partition of unity (whose existence is ensured by the paracompactness of the manifold X) subordinated to the open cover given by the complement $\complement_X S$ and the open neighbourhoods U_0 where a local extension of a' can be constructed as above. □

4. Foundations of representation theory

We proceed to develop the theory of Lie groupoid representations on smooth Euclidean fields. The topics to be discussed here are essentially standard, the only unconventional aspect being our use of smooth Euclidean fields in place of ordinary smooth vector bundles. We choose the "cocycle description" of representations (which is well known, perhaps only less familiar; cf. for instance [5]), as this lends itself to an immediate extension to the new framework. After briefly reviewing a bunch of basic definitions, we discuss in some detail the invariance of our theory under Morita equivalence.

Definition 4.1. Let $\mathcal{G}$ be a Lie groupoid, and let M be its base manifold. By a *representation of $\mathcal{G}$ on a smooth Euclidean field,* we mean a pair $(\mathscr{E}, \varrho)$ consisting of a smooth Euclidean field $\mathscr{E}$ over M and a morphism

$$\varrho : \boldsymbol{s}^*\mathscr{E} \longrightarrow \boldsymbol{t}^*\mathscr{E} \quad \text{in the category } \underline{\mathfrak{Euc}}\big(\mathcal{G}^{(1)}\big)$$

such that the following two conditions are satisfied:

(i) $\boldsymbol{u}^*\varrho = id_{\mathscr{E}}$;
(ii) $\boldsymbol{c}^*\varrho = {pr_0}^*\varrho \circ {pr_1}^*\varrho$.

Recall that $\boldsymbol{s}, \boldsymbol{t} : \mathcal{G}^{(1)} \to M$ are the source and target map, $\boldsymbol{u} : M \to \mathcal{G}^{(1)}$ is the unit section, and $\boldsymbol{c} : \mathcal{G}^{(2)} \to \mathcal{G}^{(1)}$ is the composition map; $\mathcal{G}^{(2)} = \mathcal{G}^{(1)}{}_{\boldsymbol{s}} \times_{\boldsymbol{t}} \mathcal{G}^{(1)}$ denotes the manifold of composable arrows; $pr_0, pr_1 : \mathcal{G}^{(2)} \to \mathcal{G}^{(1)}$ denote the left and right projection, respectively.

It follows from the conditions (i) and (ii) in the last definition that the morphism $\varrho : \boldsymbol{s}^*\mathscr{E} \to \boldsymbol{t}^*\mathscr{E}$ is necessarily an *isomorphism.*

In this paper, the notion of representation introduced above is the only one we shall be considering. So, from now on, we shall omit the specification "on a smooth Euclidean field".

Definition 4.2. Let $(\mathscr{E}, \varrho)$ and $(\mathscr{E}', \varrho')$ be representations of a Lie groupoid $\mathcal{G}$. By a *morphism of representations* $(\mathscr{E}, \varrho) \to (\mathscr{E}', \varrho')$, we mean a morphism $a : \mathscr{E} \to \mathscr{E}'$ of smooth Euclidean fields over the base M of $\mathcal{G}$ such that the following condition is satisfied:

$$\boldsymbol{t}^*a \circ \varrho = \varrho' \circ \boldsymbol{s}^*a. \tag{33}$$

With this notion of morphism, the representations of a Lie groupoid $\mathcal{G}$ constitute a category, hereafter denoted by $\underline{\mathfrak{Rep}}(\mathcal{G})$.

The category $\underline{\mathfrak{Rep}}(\mathcal{G})$, endowed with the $\mathbb{C}$-linear structure inherited from $\underline{\mathfrak{Euc}}(M)$, is clearly additive.

Let $R = (\mathscr{E}, \varrho)$ and $S = (\mathscr{F}, \varsigma)$ be representations of the Lie groupoid $\mathcal{G}$. We put $R \otimes S := (\mathscr{E} \otimes \mathscr{F}, \varrho \otimes \varsigma)$. This also is a representation of $\mathcal{G}$, which we shall call the *tensor product* of R and S. For all morphisms of representations $a : (\mathscr{E}, \varrho) \to (\mathscr{E}', \varrho')$ and $b : (\mathscr{F}, \varsigma) \to (\mathscr{F}', \varsigma')$, the tensor product $a \otimes b$ is a morphism of representations $R \otimes S \to R' \otimes S'$. We define the unit object for this tensor product to be the pair $1_{\mathcal{G}} := (\underline{\mathbb{C}}, \text{identity})$. The tensor category constraints associated with the category of smooth Euclidean fields over M will provide the constraints for the tensor product bifunctor on $\underline{\mathfrak{Rep}}(\mathcal{G})$. One also has an inherited operation of conjugation on $\underline{\mathfrak{Rep}}(\mathcal{G})$. This completes the construction of $\underline{\mathfrak{Rep}}(\mathcal{G})$ as a tensor category.

Let $\phi : \mathcal{G} \to \mathcal{H}$ be a homomorphism of Lie groupoids, and let $f : M \to N$ be the smooth map induced by ϕ on the base manifolds.

Suppose $S = (\mathscr{F}, \varsigma) \in \mathrm{Ob}\underline{\mathfrak{Rep}}(\mathcal{H})$. Consider the following morphism of smooth Euclidean fields over the manifold $\mathcal{G}^{(1)}$

$$\boldsymbol{s}_{\mathcal{G}}{}^*(f^*\mathscr{F}) = \phi^*\boldsymbol{s}_{\mathcal{H}}{}^*\mathscr{F} \xrightarrow{\phi^*\varsigma} \phi^*\boldsymbol{t}_{\mathcal{H}}{}^*\mathscr{F} = \boldsymbol{t}_{\mathcal{G}}{}^*(f^*\mathscr{F}). \tag{34}$$

The identities $f \circ \boldsymbol{s}_{\mathcal{G}} = \boldsymbol{s}_{\mathcal{H}} \circ \phi$ etc. account, of course, for those in (34). It is routine to check that the pair $\phi^*(S) := (f^*\mathscr{F}, \phi^*\varsigma)$ constitutes an object of the category $\underline{\mathfrak{Rep}}(\mathcal{G})$ and that if $b : (\mathscr{F}, \varsigma) \to (\mathscr{F}', \varsigma')$ is a morphism of representations of $\mathcal{H}$ then f^*b is a morphism of $(f^*\mathscr{F}, \phi^*\varsigma)$ into $(f^*\mathscr{F}', \phi^*\varsigma')$ in $\underline{\mathfrak{Rep}}(\mathcal{G})$. Hence, we get a functor

$$\phi^* : \underline{\mathfrak{Rep}}(\mathcal{H}) \longrightarrow \underline{\mathfrak{Rep}}(\mathcal{G}), \tag{35}$$

which we shall call the *inverse image* (or *pullback*) *functor along the homomorphism* ϕ. The identities

$$\phi^*(1_{\mathcal{H}}) = 1_{\mathcal{G}}, \qquad \phi^*(S \otimes S') = \phi^*(S) \otimes \phi^*(S') \quad \text{and} \quad \phi^*(\overline{S}) = \overline{\phi^*(S)}$$

are an immediate consequence of the fact that the pullback of smooth Euclidean fields along any smooth map is a strict tensor functor. Hence (35) is in fact a strict tensor functor. Moreover, one has the following identities of strict tensor functors, for all pairs of composable Lie groupoid homomorphisms:

$$(\psi \circ \phi)^* = \phi^* \circ \psi^* \quad \text{and} \quad id_{\mathcal{G}}{}^* = Id.$$

Recall [11] that a *transformation* $\theta : \phi \simeq \psi$ of homomorphisms of Lie groupoids $\phi, \psi : \mathcal{G} \to \mathcal{H}$ is given by a smooth map $\theta : M \to \mathcal{H}^{(1)}$ such that $\theta(x) : \phi^{(0)}(x) \to \psi^{(0)}(x)$ for all $x \in M$ and

$$\psi(g) \cdot \theta(x) = \theta(x') \cdot \phi(g) \tag{36}$$

for all arrows $g \in \mathcal{G}(x, x')$.

Suppose a representation $S = (\mathscr{F}, \varsigma) \in \mathrm{Ob}\underline{\mathfrak{Rep}}(\mathcal{H})$ is given. Then we can apply θ^* to the *isomorphism* $\varsigma : \boldsymbol{s}^*\mathscr{F} \to \boldsymbol{t}^*\mathscr{F}$ to obtain an isomorphism

$$[\phi^{(0)}]^*\mathscr{F} = \theta^*\boldsymbol{s}^*\mathscr{F} \xrightarrow{\theta^*\varsigma} \theta^*\boldsymbol{t}^*\mathscr{F} = [\psi^{(0)}]^*\mathscr{F} \tag{37}$$

of smooth Euclidean fields over the base of $\mathcal{G}$. By rewriting (36) as an identity between suitable smooth maps, one sees that (37) is a morphism of representations $\theta^*(S) : \phi^*(S) \xrightarrow{\sim} \psi^*(S)$. Thus, we obtain an isomorphism of tensor functors $\theta^* : \phi^* \simeq \psi^*$.

4.3 (*Invariance Under Morita Equivalence*). We observe next that the inverse image functor $\phi^* : \underline{\mathfrak{Rep}}(\mathcal{H}) \to \underline{\mathfrak{Rep}}(\mathcal{G})$ associated with a Morita equivalence $\phi : \mathcal{G} \to \mathcal{H}$ is an equivalence of tensor categories. (Recall that a tensor functor $\Phi : \mathcal{C} \to \mathcal{D}$ is said to be a *tensor equivalence* in case there exists a tensor functor $\Psi : \mathcal{D} \to \mathcal{C}$ for which there are tensor preserving natural isomorphisms $\Psi \circ \Phi \simeq Id_{\mathcal{C}}$ and $\Phi \circ \Psi \simeq Id_{\mathcal{D}}$.) Clearly, this is tantamount to saying that ϕ^* is an ordinary categorical equivalence. Although the procedure to obtain a quasi-inverse for ϕ^* follows a well known pattern, we review it for the reader's convenience. In fact, we know of no adequate standard reference for this precise argument.

The condition that the map (4) should be a surjective submersion will of course be satisfied when $\phi^{(0)}$ itself is a surjective submersion. As a first step, we show how the task of constructing a quasi-inverse may be reduced to the special case where $\phi^{(0)}$ is precisely a surjective submersion. To this end, consider the weak pullback (see [11], pp. 123–132)

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\psi"] \arrow[d, "\chi"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & \mathcal{G} \arrow[d, "\phi"] \\ \mathcal{H} \arrow[r, equal] & \mathcal{H}. \end{tikzcd} \tag{38}$$

Let P be the base manifold of the Lie groupoid $\mathcal{P}$. It is well known ([11], p. 130) that the Lie groupoid homomorphisms ψ and χ are Morita equivalences with the property that the respective base maps $\psi^{(0)} : P \to M$ and $\chi^{(0)} : P \to N$ are surjective submersions. Now, if we prove that ψ^* and χ^* are categorical equivalences then, since by the remarks about transformations we have

$$\chi^* \simeq (\phi \circ \psi)^* = \psi^* \circ \phi^*, \tag{39}$$

the same will be true of ϕ^*.

From now on, we work under the hypothesis that the Morita equivalence ϕ determines a surjective submersion $f : M \to N$ on the base manifolds. This being the case, there exists an open cover of the manifold $N = \cup_{i \in I} V_i$ by open subsets V_i such that for each V_i one can find a smooth section $s_i : V_i \hookrightarrow M$ to f. We fix such a cover and such sections once and for all.

Let $R = (\mathscr{E}, \varrho) \in \mathrm{Ob}\underline{\mathfrak{Rep}}(\mathcal{G})$ be given. For each $i \in I$, one can take the pullback $\mathscr{E}_i := s_i{}^*\mathscr{E} \in \mathrm{Ob}\underline{\mathfrak{Euc}}(V_i)$. Fix a couple of indices $i, j \in I$. Then, since (3) is a pullback diagram, for each $y \in V_i \cap V_j$ there is exactly one arrow $g(y) : s_i(y) \to s_j(y)$ such that $\phi(g(y)) = y$. More precisely, let $y \mapsto g(y) = g_{ij}(y)$ be the smooth mapping defined as the unique solution to the following universal problem (in the category of manifolds of class C^∞)

$$\begin{tikzcd} V_{ij} \arrow[dr, dashed, "g_{ij}"] \arrow[drr, bend left, "\boldsymbol{u}|V_{ij}"] \arrow[ddr, bend right, "(s_i,s_j)"'] & & \\ & \mathcal{G}^{(1)} \arrow[r, "\phi^{(1)}"] \arrow[d, "(\boldsymbol{s},\boldsymbol{t})"] & \mathcal{H}^{(1)} \arrow[d, "(\boldsymbol{s},\boldsymbol{t})"] \\ & M \times M \arrow[r, "f\times f"] & N \times N, \end{tikzcd} \tag{40}$$

where $\boldsymbol{u} : N \to \mathcal{H}$ denotes the unit section and $V_{ij} = V_i \cap V_j$. Then, putting $\mathscr{E}_i|_j = \mathscr{E}_i|_{V_i \cap V_j}$ and $\mathscr{E}_j|_i = \mathscr{E}_j|_{V_i \cap V_j}$, one may pull the action ϱ back along the map g_{ij} so as to get an isomorphism of smooth Euclidean fields over V_{ij}

$$\theta_{ij} := {g_{ij}}^* \varrho : \mathscr{E}_i|_j \xrightarrow{\sim} \mathscr{E}_j|_i. \tag{41}$$

From the remark that for an arbitrary third index $k \in I$ one has $g_{ik}|_j = \boldsymbol{c} \circ \left(g_{jk}|_i, g_{ij}|_k\right)$ where $g_{ik}|_j$ denotes the restriction of g_{ik} to V_{ijk}, and from the multiplicative axiom (ii) for ϱ, it follows that the system of isomorphisms $\{\theta_{ij}\}$ constitutes a "cocycle" or "glueing datum" for the family $\{\mathscr{E}_i : i \in I\}$ relative to the open cover $\coprod_{i\in I} V_i \to N$. Since N is a paracompact manifold, there exists a smooth Euclidean field $\mathscr{F}$ over N and a system of isomorphisms $\theta_i : \mathscr{F}|_{V_i} \xrightarrow{\sim} \mathscr{E}_i$ in $\underline{\mathfrak{Euc}}(V_i)$ compatible with the $\{\theta_{ij}\}$ in the sense that

$$\theta_j|_{V_{ij}} = \theta_{ij} \circ \theta_i|_{V_{ij}}. \tag{42}$$

Note that the paracompactness of the base manifold N is needed here in order to construct a metric for the smooth field $\mathscr{F}$ (by means of a fixed partition of unity subordinated to the open cover $\{V_i\}$).

Next, we construct a morphism $\varsigma : \boldsymbol{s}_{\mathcal{H}}{}^* \mathscr{F} \to \boldsymbol{t}_{\mathcal{H}}{}^* \mathscr{F}$. For each pair $V_i, V_{i'}$, let us introduce the shorthand $\mathcal{H}_{i,i'} := \mathcal{H}(V_i, V_{i'})$. We shall also write $\mathcal{H}_{ij,i'j'} := \mathcal{H}(V_{ij}, V_{i'j'})$. Then, the subsets $\{\mathcal{H}_{i,i'} : i, i' \in I\}$ constitute an open cover for the manifold $\mathcal{H}^{(1)}$. Now, let $g_{i,i'} : \mathcal{H}_{i,i'} \to \mathcal{G}$ be the smooth map obtained by solving the following universal problem:

$$\begin{tikzcd} \mathcal{H}_{i,i'} \arrow[d, "(\boldsymbol{s},\boldsymbol{t})"'] \arrow[dr, dashed, "g_{i,i'}"] \arrow[drr, bend left, "\text{inclusion}"] & & \\ V_i \times V_{i'} \arrow[dr, bend right, "s_i \times s_{i'}"'] & \mathcal{G}^{(1)} \arrow[r, "\phi^{(1)}"] \arrow[d, "(\boldsymbol{s},\boldsymbol{t})"] & \mathcal{H}^{(1)} \arrow[d, "(\boldsymbol{s},\boldsymbol{t})"] \\ & M \times M \arrow[r, "f\times f"] & N \times N. \end{tikzcd} \tag{43}$$

We can use this map to define a morphism $\varsigma_{i,i'} : (\boldsymbol{s}_{\mathcal{H}}{}^* \mathscr{F})|_{i,i'} \to (\boldsymbol{t}_{\mathcal{H}}{}^* \mathscr{F})|_{i,i'}$ of smooth Euclidean fields over the manifold $\mathcal{H}_{i,i'}$; namely, we put

$$\varsigma_{i,i'} := \left[(\boldsymbol{t}_{\mathcal{H}}|_{i,i'})^* {\theta_{i'}}^{-1}\right] \circ [{g_{i,i'}}^* \varrho] \circ \left[(\boldsymbol{s}_{\mathcal{H}}|_{i,i'})^* \theta_i\right]. \tag{44}$$

By taking into account the equality of mappings

$$g_{i,i'}|_{j,j'} = \left(g_{j'i'} \circ \boldsymbol{t}_{\mathcal{H}}|_{ij,i'j'}\right) \cdot g_{j,j'}|_{i,i'} \cdot \left(g_{ji} \circ \boldsymbol{s}_{\mathcal{H}}|_{ij,i'j'}\right) \tag{45}$$

and the identities (41), (42) and (44), one sees that $\varsigma_{i,i'}|_{j,j'} = \varsigma_{j,j'}|_{i,i'}$ in the category $\underline{\mathfrak{Euc}}(\mathcal{H}_{ij,i'j'})$. Hence the $\varsigma_{i,i'}$ glue together into a unique ς.

This essentially completes the description of a quasi-inverse functor. We will leave the rest of the construction as an exercise.

5. Construction of equivariant maps

It is our purpose here to prove various interrelated results about the existence of morphisms (of representations on smooth Euclidean fields) with certain specific properties, like invariant metrics, equivariant extensions of invariant partial sections, and so on.

Lemma 5.1. *Let $\mathcal{G}$ be a (locally) transitive Lie groupoid. Let X be its base. Let any representation $(\mathscr{E}, \varrho) \in \mathrm{Ob}\underline{\mathfrak{Rep}}(\mathcal{G})$ be given. Then $\mathscr{E}$ is a locally trivial smooth Euclidean field over X (cf. Definition 3.4).*

Proof. Local transitivity means that the mapping $(\boldsymbol{s}, \boldsymbol{t}) : \mathcal{G} \to X \times X$ is a submersion. Fix a point $x \in X$. Since (x, x) lies in the image of the map $(\boldsymbol{s}, \boldsymbol{t})$, the latter admits a local smooth section $U \times U \to \mathcal{G}$ over some open neighbourhood of (x, x). Let us consider the restriction $g : U \to \mathcal{G}$ of this section to $U = U \times \{x\}$.

Let $x : \star \to X$ denote the map $\star \mapsto x$. Fix a trivialization – i.e., a vector space basis – for $x^*\mathscr{E}$ in $\underline{\mathfrak{Euc}}(\star)$. We pull ϱ back to U along the smooth map g, and observe that there is a unique factorization of $\boldsymbol{t} \circ g$ through $\star$ (namely, collapse $c : U \to \star$ followed by $x : \star \to X$). Since ϱ is an isomorphism,

$$\begin{aligned}\mathscr{E}|_U &= i_U{}^*\mathscr{E} = (\boldsymbol{s} \circ g)^*\mathscr{E} = g^*\boldsymbol{s}^*\mathscr{E} \xrightarrow{g^*\varrho} g^*\boldsymbol{t}^*\mathscr{E} = (\boldsymbol{t} \circ g)^*\mathscr{E}\\ &= (x \circ c)^*\mathscr{E} = c^*(x^*\mathscr{E}) \approx c^*(\mathbb{C} \oplus \cdots \oplus \mathbb{C}) = \underline{\mathbb{C}}_U \oplus \cdots \oplus \underline{\mathbb{C}}_U\end{aligned}$$

provides a trivialization for $\mathscr{E}|_U$ in $\underline{\mathfrak{Euc}}(U)$. $\square$

Let $i_S : S \hookrightarrow X$ be an invariant immersed submanifold. Observe that $\mathcal{G}|_S = \mathcal{G}^S = \boldsymbol{s}_{\mathcal{G}}{}^{-1}(S)$; then the pullback $\mathcal{G}|_S$ of $\mathcal{G}$ along i_S is defined, and is a Lie subgroupoid of $\mathcal{G}$. (A *Lie subgroupoid* is a Lie groupoid homomorphism (ϕ, f) such that both ϕ and f are injective immersions. Compare for instance [18].) In the special case of an orbit immersion, $\mathcal{G}|_S$ will be transitive over S. Then our lemma says that for any $(\mathscr{E}, \varrho) \in \mathrm{Ob}\underline{\mathfrak{Rep}}(\mathcal{G})$ the restriction $\mathscr{E}|_S$ is a locally trivial smooth Euclidean field over S.

So far, we have been working with representations in a completely intrinsic way. We were able to prove all results by means of purely formal arguments involving only manipulations of commutative diagrams. For the purposes of the present section, however, we need to change our point of view.

Let $\mathcal{G}$ be a Lie groupoid. Consider a representation $\varrho : \boldsymbol{s}^*\mathscr{E} \to \boldsymbol{t}^*\mathscr{E}$ of $\mathcal{G}$ on a smooth Euclidean field $\mathscr{E}$. By definition, the morphism of smooth Euclidean fields ϱ is given by a collection of linear maps

$$\left\{\varrho(g) : E_{\boldsymbol{s}(g)} \to E_{\boldsymbol{t}(g)} \mid g \in \mathcal{G}^{(1)}\right\}. \tag{46}$$

It is routine to check that the conditions (i) and (ii) in Definition 4.1 say that the correspondence $g \mapsto \varrho(g)$ is *multiplicative*, i.e., that $\varrho(g'g) = \varrho(g') \circ \varrho(g)$ and $\varrho(x) = id$ for all base points x.

Fix an arbitrary arrow g_0. Let $\zeta \in \boldsymbol{\Gamma}\mathscr{E}(U)$ be any local smooth section, defined over an open neighbourhood U of $\boldsymbol{s}(g_0)$. Then, ζ will determine a local smooth section $\zeta \circ \boldsymbol{s}$ of the pullback Euclidean field $\boldsymbol{s}^*\mathscr{E}$, defined over $\mathcal{G}^U = \boldsymbol{s}^{-1}(U)$. The morphism of smooth Euclidean fields ϱ can be evaluated on the latter, so as to get a smooth section of $\boldsymbol{t}^*\mathscr{E}$ over $\mathcal{G}^U$. Now, by the definition of the pullback of smooth Euclidean fields, there will be an open neighbourhood Γ of g_0 in $\mathcal{G}^U$ over which this section can be expressed as a finite linear combination, with coefficients in $C^\infty(\Gamma)$, of sections of the form $\zeta_i' \circ \boldsymbol{t}$, for some smooth sections $\{\zeta_i' : i = 1, \ldots, d\}$ of $\mathscr{E}$ over $\boldsymbol{t}(\Gamma)$. In symbols,

$$\varrho(g) \cdot \zeta(\boldsymbol{s}\, g) = \sum_{i=1}^{d} r_i(g)\, \zeta_i'(\boldsymbol{t}\, g) \tag{47}$$

for all $g \in \Gamma$, where $r_1, \ldots, r_d \in C^\infty(\Gamma)$ and $\zeta_1', \ldots, \zeta_d' \in \boldsymbol{\Gamma}\mathscr{E}\big(\boldsymbol{t}(\Gamma)\big)$.

Conversely, any multiplicative operation $g \mapsto \varrho(g)$ which is locally expressible in the form (47) determines a representation ϱ of $\mathcal{G}$ on $\mathscr{E}$.

Overall assumption. For the remainder of the section, we let $\mathcal{G}$ be a *proper* Lie groupoid. As before, X shall denote the base of $\mathcal{G}$.

Fix a base point $x_0 \in X$, and let G_0 denote the isotropy group for $\mathcal{G}$ at x_0. It is evident from (47) that

$$\varrho_0 : G_0 \to GL(E_0), \qquad g \mapsto \varrho(g) \tag{48}$$

is a continuous representation of the compact Lie group G_0 on the vector space E_0 (the fibre of $\mathscr{E}$ at x_0).

Let another representation $(\mathscr{F}, \varsigma)$ be given, along with a G_0-equivariant linear map $A_0 : E_0 \to F_0$. Let $S_0 \hookrightarrow X$ be the orbit through x_0. By our remarks on the invariance under Morita equivalence in Section 4.3, there is a unique morphism $a' : (\mathscr{E}', \varrho') \to (\mathscr{F}', \varsigma')$ in $\underline{\mathfrak{Rep}}(\mathcal{G}')$ (the primes here signify that we are taking the corresponding restrictions to S_0) such that $a'_0 = A_0$. In fact, for every point $z \in S_0$ and every arrow $g \in \mathcal{G}(x_0, z)$, one has

$$a'_z = \varsigma(g) \circ A_0 \circ \varrho(g)^{-1} : E_z \to F_z. \tag{49}$$

By Lemma 5.1, $\mathscr{E}'$ is a locally trivial object of $\underline{\mathfrak{Euc}}(S_0)$. Then Lemma 3.5 yields a global morphism $a : \mathscr{E} \to \mathscr{F}$ extending a' and hence, a fortiori, A_0. We proceed to "average out" this a to make it equivariant, as follows.

Choose an arbitrary normalized Haar system $\mu = \{\mu^x\}$ on $\mathcal{G}$ (Definition 1.4). We proceed to construct a linear operator, depending on μ,

$$\mathrm{Av} : \mathrm{Hom}_{\underline{\mathfrak{Euc}}(X)}(\mathscr{E}, \mathscr{F}) \longrightarrow \mathrm{Hom}_{\underline{\mathfrak{Rep}}(\mathcal{G})}(R, S) \tag{50}$$

(averaging operator), with the property that $\mathrm{Av}(a) = a$ whenever a already belongs to $\mathrm{Hom}_{\underline{\mathfrak{Rep}}(\mathcal{G})}(R, S)$ (more generally, $\mathrm{Av}(a)|_S = a|_S$ whenever S is an invariant submanifold over which a restricts to an equivariant morphism); here of course we put $R = (\mathscr{E}, \varrho)$ and $S = (\mathscr{F}, \varsigma)$.

We start from a very simple remark. Suppose we are given local smooth sections $\zeta \in \boldsymbol{\Gamma}\mathscr{E}(U)$ and $\eta_1, \ldots, \eta_n \in \boldsymbol{\Gamma}\mathscr{F}(U)$ such that $\eta_1, \ldots, \eta_n$ are generators for $\boldsymbol{\Gamma}\mathscr{F}$ over U. Then, for each arrow g_0 in $\mathcal{G}^U$, there exist smooth functions $\varphi_1, \ldots, \varphi_n$

on some open neighbourhood $\Gamma \subset \mathcal{G}^U$ of g_0 with

$$\left[\varsigma(g)^{-1} \circ a_{\boldsymbol{t}(g)} \circ \varrho(g)\right] \cdot \zeta(\boldsymbol{s}\,g) = \sum_{j=1}^{n} \varphi_j(g)\,\eta_j(\boldsymbol{s}\,g) \tag{51}$$

for all $g \in \Gamma$. To see this, note that, by Eq. (47), there are an open neighbourhood Γ of g_0 contained in $\mathcal{G}^U$ and smooth sections $\zeta'_1, \dots, \zeta'_m$ of $\mathscr{E}$ over $U' := \boldsymbol{t}(\Gamma)$ such that $\varrho(g) \cdot \zeta(\boldsymbol{s}\,g) = \sum_{i=1}^{m} r_i(g)\,\zeta'_i(\boldsymbol{t}\,g)$ for some functions $r_1, \dots, r_m \in C^\infty(\Gamma)$. For each $i = 1, \dots, m$, put $\eta'_i := \boldsymbol{\Gamma} a\,(U')(\zeta'_i) \in \boldsymbol{\Gamma}\mathscr{F}(U')$. Since Γ^{-1} is a neighbourhood of ${g_0}^{-1}$, we can also assume, in view of the hypothesis that the η_j are generators, Γ to be so small that for each $i = 1, \dots, m$ there exist $s_{1,i}, \dots, s_{n,i} \in C^\infty(\Gamma^{-1})$ with $\varsigma(g^{-1}) \cdot \eta'_i(\boldsymbol{t}\,g) = \sum_{j=1}^{n} s_{j,i}(g^{-1})\,\eta_j(\boldsymbol{s}\,g)$ for all $g \in \Gamma$. This proves (51).

Put $\alpha := \boldsymbol{\Gamma} a$. We can use the last remark to obtain a new morphism of sheaves of modules $\widetilde{\alpha} : \boldsymbol{\Gamma}\mathscr{E} \to \boldsymbol{\Gamma}\mathscr{F}$, as follows. Let ζ be a local smooth section of $\mathscr{E}$ over an open subset U so small that there exists a system $\eta_1, \dots, \eta_n$ of local generators for $\boldsymbol{\Gamma}\mathscr{F}$ over U. For each $g_0 \in \mathcal{G}^U$, select an open neighbourhood $\Gamma(g_0)$ contained in $\mathcal{G}^U$ on which one can find smooth functions $\varphi_1^{g_0}, \dots, \varphi_n^{g_0}$ satisfying (51). Then, choose a smooth partition of unity $\{\theta_i : i \in I\}$ over $\mathcal{G}^U$ subordinated to the open cover given by the subsets $\Gamma(g_0)$, and put

$$\widetilde{\alpha}(U)(\zeta) := \sum_{j=1}^{n} \Phi_j \eta_j \quad \text{where } \Phi_j(u) := \int_{\mathcal{G}^u} \sum_{i\in I} \theta_i(g)\,{\varphi^i}_j(g)\mathrm{d}\mu^u(g). \tag{52}$$

Some arbitrary choices are involved here, so one has to make sure that this is a good definition. If we look at (51) for $x := \boldsymbol{s}(g)$ fixed, we recognize that the operation $g \mapsto \varsigma(g)^{-1} \circ a_{\boldsymbol{t}(g)} \circ \varrho(g) \cdot \zeta(x)$ defines a smooth mapping from the manifold $\mathcal{G}^x$ into the vector space F_x. For each $v \in E_x$, there is some local smooth section ζ about x such that $\zeta(x) = v$, so one is allowed to take the integral

$$\varkappa_x(v) := \int_{\mathcal{G}^x} \left[\sigma(g)^{-1} \circ a_{\boldsymbol{t}(g)} \circ \varrho(g)\right] \cdot v\ \mathrm{d}\mu^x(g). \tag{53}$$

This defines, for each base point x, a linear map $\varkappa_x : E_x \to F_x$. Now,

$$\begin{aligned}
\left[\widetilde{\alpha}(U)(\zeta)\right](u) &= \sum_{j=1}^{n} \Phi_j(u)\,\eta_j(u) = \sum_{j=1}^{n} \int_{\mathcal{G}^u} \sum_{i\in I} \theta_i(g)\,{\varphi^i}_j(g)\,\mathrm{d}\mu^u(g)\eta_j(u) \\
&= \int_{\mathcal{G}^u} \sum_{i\in I} \theta_i(g) \sum_{j=1}^{n} {\varphi^i}_j(g)\,\eta_j(\boldsymbol{s}\,g)\mathrm{d}\mu^u(g) \\
&= \int_{\mathcal{G}^u} \sum_{i\in I} \theta_i(g)\Big[\sigma(g)^{-1} \circ a_{\boldsymbol{t}(g)} \circ \varrho(g) \cdot \zeta(\boldsymbol{s}\,g)\Big]\,\mathrm{d}\mu^u(g) \\
&= \varkappa_u \cdot \zeta(u).
\end{aligned}$$

It follows that the section $\widetilde{\alpha}(U)(\zeta)$ in (52) does not depend on the auxiliary choices we made in order to define it (as the $\varkappa_u$ do not).

We define $\mathrm{Av}(a) : \mathscr{E} \to \mathscr{F}$ as the morphism of smooth Euclidean fields over X given by the bundle of linear maps $\{\varkappa_x\}$. It remains to check that Av is a projection operator onto $\mathrm{Hom}_{\underline{\mathfrak{Rep}}(\mathcal{G})}(R, S)$. We leave the verification to the reader. Summing up, we have shown

Proposition 5.2. *Suppose $\mathcal{G}$ is proper. Let x_0 be a base point of $\mathcal{G}$, and let G_0 denote the isotropy group at x_0. Let $R = (\mathscr{E}, \varrho)$ and $S = (\mathscr{F}, \varsigma)$ be two representations. Then any G_0-equivariant linear map $E_0 \to F_0$ can be extended to a morphism $R \to S$ in $\underline{\mathfrak{Rep}}(\mathcal{G})$.*

By applying the averaging operator to a randomly chosen Hilbert metric, we get the existence of invariant metrics:

Proposition 5.3. *Let $R = (\mathscr{E}, \varrho)$ be a representation of a proper Lie groupoid $\mathcal{G}$. Then there exists a $\mathcal{G}$-invariant metric on $\mathscr{E}$, that is, a metric on $\mathscr{E}$ which is at the same time a morphism $R \otimes \overline{R} \to 1_{\mathcal{G}}$ in $\underline{\mathfrak{Rep}}(\mathcal{G})$.*

Let $(\mathscr{E}, \varrho)$ be a representation of the groupoid $\mathcal{G}$. By a *ϱ-invariant partial section of $\mathscr{E}$* over an invariant submanifold S, we mean a global section of $\mathscr{E}|_S$ which is at the same time a morphism in $\underline{\mathfrak{Rep}}(\mathcal{G}|_S)$.

Proposition 5.4. *Let S be a closed invariant submanifold of the base of a proper Lie groupoid $\mathcal{G}$. Let $R = (\mathscr{E}, \varrho)$ be a representation of $\mathcal{G}$. Then each ϱ-invariant partial section of $\mathscr{E}$ over S can be extended to a global ϱ-invariant section of $\mathscr{E}$.*

We call a function φ on an arbitrary subset S of the manifold X *smooth*, when for each $x \in X$ one can find an open neighbourhood U of x in X and a smooth function on U which agrees with φ on the intersection $U \cap S$.

Proposition 5.5. *Let S be an invariant subset of the base manifold X of a proper Lie groupoid $\mathcal{G}$. Suppose φ is a smooth invariant (viz., constant along the $\mathcal{G}$-orbits) function on S. Then there exists a smooth invariant function extending φ on all of X.*

Proof. Average out any smooth extension of φ obtained by means of a smooth partition of unity over X. $\square$

6. Proof of the reconstruction theorem

Let $\mathcal{G}$ be a proper Lie groupoid, and let M denote its base manifold. The forgetful functor

$$\mathsf{F}_{\mathcal{G}} : \underline{\mathfrak{Rep}}(\mathcal{G}) \longrightarrow \underline{\mathfrak{Euc}}(M), \qquad (\mathscr{E}, \varrho) \mapsto \mathscr{E} \tag{54}$$

is a strict tensor functor, by the definition of the tensor category structure on $\mathfrak{Rep}(\mathcal{G})$.

Let $x_0 \in M$ be a base point. We shall identify x_0 with the map $\{\star\} \to M, \star \mapsto x_0$ from the one-point manifold. Consider the (strict) tensor functor

$$x_0{}^* : \underline{\mathfrak{Euc}}(M) \longrightarrow \underline{\mathsf{Vec}}_{\mathbb{C}}, \qquad \mathscr{E} = \big(\{E_x\}, \boldsymbol{\Gamma}\mathscr{E}\big) \mapsto E_{x_0}. \tag{55}$$

Let $\mathsf{F}_{\mathcal{G},x_0}$ denote the composite tensor functor

$$\underline{\mathfrak{Rep}}(\mathcal{G}) \xrightarrow{\mathsf{F}_{\mathcal{G}}} \underline{\mathfrak{Euc}}(M) \xrightarrow{x_0{}^*} \underline{\mathsf{Vec}}_{\mathbb{C}}, \qquad (\mathscr{E}, \varrho) \mapsto \mathscr{E}_{x_0}. \tag{56}$$

Definition 6.1. The *Tannakian bidual* of $\mathcal{G}$ is the groupoid $\mathcal{T}(\mathcal{G})$ over M defined as follows. For each pair of base points $x, x' \in M$, put

$$\mathcal{T}(\mathcal{G})(x, x') := \mathrm{Iso}^{\overline{\otimes}}\big(\mathsf{F}_{\mathcal{G},x}, \mathsf{F}_{\mathcal{G},x'}\big). \tag{57}$$

(The right-hand side denotes the set of all self-conjugate, tensor preserving natural *isomorphisms.*) As to the groupoid structure, let the composition law be $(\lambda' \cdot \lambda)(R) := \lambda'(R) \circ \lambda(R)$.

Let $R = (\mathscr{E}, \varrho) \in \mathrm{Ob}\underline{\mathfrak{Rep}}(\mathcal{G})$. Let ϕ be a metric on $\mathscr{E}$. (Definition 3.3.) Let ζ and ζ' be smooth sections of $\mathscr{E}$ over M. Then, let us introduce the following function on the set of arrows of the bidual groupoid $\mathcal{T}(\mathcal{G})$:

$$r_{R,\phi,\zeta,\zeta'} : \mathcal{T}(\mathcal{G})^{(1)} \to \mathbb{C}, \qquad \lambda \mapsto \Big\langle \lambda(R) \cdot \zeta(\boldsymbol{s}\,\lambda), \zeta'(\boldsymbol{t}\,\lambda) \Big\rangle := \phi_{\boldsymbol{t}(\lambda)}\Big(\lambda(R) \cdot \zeta(\boldsymbol{s}\,\lambda), \zeta'(\boldsymbol{t}\,\lambda)\Big). \tag{58}$$

We put

$$\mathscr{R} := \Big\{ r_{R,\phi,\zeta,\zeta'} \mid R = (\mathscr{E}, \varrho) \in \mathrm{Ob}\underline{\mathfrak{Rep}}(\mathcal{G}),\ \phi \text{ metric on } \mathscr{E}, \zeta, \zeta' \in \boldsymbol{\Gamma}\mathscr{E}(M) \Big\}. \tag{59}$$

We shall call the elements of $\mathscr{R}$ *representative functions.* Observe that $\mathscr{R}$ is a complex algebra of functions on the set of arrows of the groupoid $\mathcal{T}(\mathcal{G})$, closed under the operation of complex conjugation. In particular, the real and imaginary parts of any function in $\mathscr{R}$ must belong to $\mathscr{R}$ as well.

We endow the set of arrows of the groupoid $\mathcal{T}(\mathcal{G})$ with the smallest topology making all the representative functions continuous. As a consequence of the existence of metrics on smooth Euclidean fields, the resulting topological space will necessarily be Hausdorff. The real valued representative functions generate, on this topological space, a functional structure, which we shall complete to a C^∞-space structure $\mathscr{R}^\infty$ as explained in Section 1.1.

It is immediate to check that the source and the target as well as the unit section of the groupoid $\mathcal{T}(\mathcal{G})$ are smooth mappings relative to the C^∞-structure $\mathscr{R}^\infty$. In fact, we will see (Proposition 6.7) that $\mathcal{T}(\mathcal{G})$ is a C^∞-groupoid relative to this C^∞-structure, at least for $\mathcal{G}$ proper. (This is not as easy to check.)

Definition 6.2. Let the *canonical homomorphism* $\boldsymbol{\pi}_{\mathcal{G}} : \mathcal{G} \to \mathcal{T}(\mathcal{G})$ be

$$\boldsymbol{\pi}_{\mathcal{G}}(g)(\mathscr{E}, \varrho) := \varrho(g). \tag{60}$$

It is not difficult to see that $\boldsymbol{\pi}_{\mathcal{G}}{}^{(1)}$ is a smooth mapping from the manifold $\mathcal{G}^{(1)}$ into the C^∞-space $\big(\mathcal{T}(\mathcal{G})^{(1)}, \mathscr{R}^\infty\big)$.

Theorem 6.3. *Suppose $\mathcal{G}$ is proper. Then the canonical homomorphism $\boldsymbol{\pi}_{\mathcal{G}} : \mathcal{G} \to \mathcal{T}(\mathcal{G})$ is a surjection.*

Proof. To begin with, we prove that whenever $\mathcal{G}(x, x')$ is empty, so must be $\mathcal{T}(\mathcal{G})(x, x')$. Let $\varphi : \mathcal{G}x \cup \mathcal{G}x' \to \mathbb{C}$ be the function which takes the value one on the orbit $\mathcal{G}x$ and the value zero on the orbit $\mathcal{G}x'$. This function is well defined, because $\mathcal{G}(x, x')$ is empty. By Proposition 5.5, there is a global invariant smooth function Φ extending φ. Being invariant, Φ determines an endomorphism a of the trivial representation $1_{\mathcal{G}} \in \mathrm{Ob}\underline{\mathfrak{Rep}}(\mathcal{G})$ such that $a_z = \Phi(z)\, id_{\mathbb{C}}$ for all z (thus, in particular, $a_x = id$ and $a_{x'} = 0$). Now, suppose $\lambda \in \mathcal{T}(\mathcal{G})(x, x')$. Because of the naturality of λ, the existence of the morphism a contradicts the invertibility of the linear map $\lambda(1)$.

We are therefore reduced to proving that the induced isotropy group homomorphisms $\boldsymbol{\pi}_{\mathcal{G}}|_x : \mathcal{G}|_x \to \mathcal{T}(\mathcal{G})|_x$ are surjective. This is a direct consequence of the two Propositions 2.3 and 5.2. □

Definition 6.4. We say that a Lie groupoid $\mathcal{G}$ is *reflexive,* when the canonical homomorphism $\boldsymbol{\pi}_{\mathcal{G}}$ is an isomorphism of topological groupoids.

Proposition 6.5. *Let $\mathcal{G}$ be a proper Lie groupoid. Then, in order that $\mathcal{G}$ may be reflexive, it is enough that its canonical homomorphism be injective.*

Proof. The continuity of $\boldsymbol{\pi}_{\mathcal{G}}$ being established, what we really have to show is that for each open subset Γ of $\mathcal{G}$ and for each arrow $g_0 \in \Gamma$ the image $\boldsymbol{\pi}_{\mathcal{G}}(\Gamma)$ is a neighbourhood of $\boldsymbol{\pi}_{\mathcal{G}}(g_0)$ in $\mathcal{T}(\mathcal{G})$.

Let $g_0 \in \mathcal{G}(x_0, x_0')$. We start by observing that, since we are assuming that $\boldsymbol{\pi}_{\mathcal{G}}$ is injective, it must be possible to find a representation $R = (\mathscr{E}, \varrho)$ whose associated x_0th isotropy homomorphism $\varrho_0 : \mathcal{G}|_0 \to GL(E_0)$ is injective. (Compare the proof of Proposition 2.3.) Fix an arbitrary metric ϕ on $\mathscr{E}$, and two local ϕ-orthonormal frames for $\mathscr{E}$

$$\zeta_1, \ldots, \zeta_d \quad \text{about } x_0 \quad \text{and} \quad \zeta_1', \ldots, \zeta_d' \quad \text{about } x_0'.$$

Choose two compactly supported smooth functions $0 \leqq a \leqq 1$ and $0 \leqq a' \leqq 1$, with support lying close enough to x_0 and x_0' respectively, such that $a(z) = 1 \Leftrightarrow z = x_0$ and $a'(z) = 1 \Leftrightarrow z = x_0'$. Then, put

$$\varrho_{i,i'} := r_{i,i'} \circ \boldsymbol{\pi}_{\mathcal{G}} := r_{R,\phi,\zeta_i,\zeta_{i'}'} \circ \boldsymbol{\pi}_{\mathcal{G}} \quad \text{and} \quad \varrho_{\iota,\iota'} := r_{\iota,\iota'} \circ \boldsymbol{\pi}_{\mathcal{G}} := (a \circ \boldsymbol{s}_{\mathcal{G}})(a' \circ \boldsymbol{t}_{\mathcal{G}})$$

for $\iota = 0$ or $\iota' = 0$. Finally, let $\omega_{\iota,\iota'} := \varrho_{\iota,\iota'}(g_0)$ for $0 \leqq \iota, \iota' \leqq d$.

We claim that there exist open disks $D_{\iota,\iota'}$ in the complex plane, with $D_{\iota,\iota'}$ centred at $\omega_{\iota,\iota'}$, such that

$$\bigcap_{0 \leqq \iota,\iota' \leqq d} {\varrho_{\iota,\iota'}}^{-1}(D_{\iota,\iota'}) \subset \Gamma. \tag{61}$$

Once this claim is proven, the statement that $\boldsymbol{\pi}_{\mathcal{G}}(\Gamma)$ is a neighbourhood of $\boldsymbol{\pi}_{\mathcal{G}}(g_0)$ will be proven as well; indeed, by Theorem 6.3, we will then have

$$\begin{aligned} \bigcap {r_{\iota,\iota'}}^{-1}(D_{\iota,\iota'}) &= \boldsymbol{\pi}_{\mathcal{G}}\, {\boldsymbol{\pi}_{\mathcal{G}}}^{-1}\left(\bigcap {r_{\iota,\iota'}}^{-1}(D_{\iota,\iota'})\right) \\ &= \boldsymbol{\pi}_{\mathcal{G}}\left(\bigcap {\varrho_{\iota,\iota}}^{-1}(D_{\iota,\iota'})\right) \subset \boldsymbol{\pi}_{\mathcal{G}}(\Gamma), \end{aligned}$$

where the ${r_{\iota,\iota'}}^{-1}(D_{\iota,\iota'})$ are open neighbourhoods of $\boldsymbol{\pi}_{\mathcal{G}}(g_0)$.

In order to establish (61), we fix, for each $0 \leqq \iota, \iota' \leqq d$, a decreasing sequence of open disks centred at $\omega_{\iota,\iota'}$

$$\cdots \subset {D_{\iota,\iota'}}^{p+1} \subset {D_{\iota,\iota'}}^{p} \subset \cdots \subset {D_{\iota,\iota'}}^{1} \subset \mathbb{C} \tag{62}$$

with radius $\to 0$. If we agree that ${D_{\iota,\iota'}}^1$ has radius $\frac{1}{2}$, then (for $p \geqq 1$)

$$\Sigma^p := \bigcap {r_{\iota,\iota'}}^{-1}\left(\overline{{D_{\iota,\iota'}}^p}\right) - \Gamma \tag{63}$$

is a closed subset of the compact space $\mathcal{G}(K, K')$, where $K = \operatorname{supp} a$ and $K' = \operatorname{supp} a'$. The intersection $\bigcap_{p=1}^{\infty} \Sigma^p$ is empty, because of the injectivity of the map $\mathcal{G}(x_0, x_0') \to \operatorname{Iso}(E_{x_0}, E_{x_0'})$ that sends $g \mapsto \varrho(g)$, and because of the choice of a and a'. Thus, there exists p such that $\Sigma^p = \varnothing$. This proves the claim. $\square$

Hereafter, we shall freely make use of the notations introduced in the course of the preceding proof. Let us define the smooth mappings

$$\begin{aligned} &\varrho^{\zeta}_{\zeta'} : \mathcal{G} \longrightarrow M \times M \times \operatorname{End}(\mathbb{C}^d), \\ &g \mapsto \left(\boldsymbol{s}(g); \boldsymbol{t}(g); \varrho_{1,1}(g), \ldots, \varrho_{i,i'}(g), \ldots, \varrho_{d,d}(g)\right), \end{aligned} \tag{64}$$

where we have put $\zeta := \zeta_1, \ldots, \zeta_d$, $\zeta' := \zeta_1', \ldots, \zeta_d'$. If the homomorphism $\boldsymbol{\pi}_{\mathcal{G}}$ is faithful, Proposition 2.4 implies that for each arrow g_0 there is a representation $R = (\mathscr{E}, \varrho)$ such that the map $\mathcal{G}(x_0, x_0') \longrightarrow \operatorname{Iso}(E_{x_0}, E_{x_0'})$, $g \mapsto \varrho(g)$ becomes injective when restricted to a sufficiently small open neighbourhood of g_0.

Lemma 6.6. *Suppose the map $\mathcal{G}(x_0, x_0') \to \operatorname{Iso}(E_{x_0}, E_{x_0'})$, $g \mapsto \varrho(g)$ is injective near g_0. Then (64) is an immersion at g_0.*

Proof. Fix open balls U and U' centred at x_0 and x_0' respectively, so small that the sections $\zeta_1, \ldots, \zeta_d$ (resp. $\zeta_1', \ldots, \zeta_d'$) form a local orthonormal frame for $\mathscr{E}$ about x_0 (resp. x_0') with domain U (resp. U'). Up to local diffeomorphism, the map (64) has the following form near g_0, provided U is chosen small enough:

$$U \times \mathbb{R}^k \to U \times U' \times \operatorname{End}(\mathbb{C}^d), \qquad (u, v) \mapsto \left(u; u'(u, v); \boldsymbol{\varrho}(u, v)\right), \tag{65}$$

where $\boldsymbol{\varrho}(g)$ denotes the matrix $\{\varrho_{i,i'}(g)\}_{1 \leq i,i' \leq d}$. Evidently, (65) is immersive at $g_0 = (x_0, 0)$ if and only if the partial map $v \mapsto \left(u'(x_0, v); \boldsymbol{\varrho}(x_0, v)\right)$ is immersive at zero. We are therefore reduced to showing that the restriction of (64) to $\mathcal{G}(x_0, -)$ is immersive at g_0.

Let G be the isotropy group of $\mathcal{G}$ at x_0. By choosing a local equivariant trivialization $\mathcal{G}(x_0, S) \approx S \times G$, where S is a submanifold of U' passing through x_0', the restriction of (64) to $\mathcal{G}(x_0, -)$ takes the form

$$S \times G \to U' \times \mathrm{End}(\mathbb{C}^d), \qquad (s, g) \mapsto \big(s; \boldsymbol{\varrho}(s, g)\big). \tag{66}$$

This map is immersive at $g_0 = (x_0', e)$ if and only if so is at e the partial map $g \mapsto \boldsymbol{\varrho}(x_0{}', g)$, where e is the unit of the group G. Thus, it suffices to show that the isotropy representation $G \to GL(E_{x_0})$ induced by ϱ is immersive at e. By hypothesis, this representation is injective in an open neighbourhood of e and hence our claim follows at once. $\square$

Let an arrow $\lambda_0 \in \mathcal{T}(\mathcal{G})$ be given. We contend that, under the assumption that $\mathcal{G}$ is reflexive, there exists an open neighbourhood Ω of λ_0 such that $(\Omega, \mathscr{R}^\infty|_\Omega)$ is isomorphic, as a C^∞-space, to a smooth manifold $(X, \mathscr{C}^\infty_X)$.

Since $\mathcal{G}$ is reflexive, there is a unique $g_0 \in \mathcal{G}$ such that $\lambda_0 = \boldsymbol{\pi}_\mathcal{G}(g_0)$. By Lemma 6.6 and the remarks preceding it, we can find some R for which there exists an open neighbourhood Γ of g_0 in $\mathcal{G}$ such that $\varrho^\zeta_{\zeta'}$ induces a diffeomorphism of Γ onto a submanifold X of $M \times M \times \mathrm{End}(\mathbb{C}^d)$. Define

$$r^\zeta_{\zeta'} = r^{\zeta_1,\ldots,\zeta_d}_{\zeta'_1,\ldots,\zeta'_d} : \mathcal{T}(\mathcal{G}) \longrightarrow M \times M \times \mathrm{End}(\mathbb{C}^d),$$
$$\lambda \mapsto \Big(\boldsymbol{s}(\lambda); \boldsymbol{t}(\lambda); r_{1,1}(\lambda), \ldots, r_{i,i'}(\lambda), \ldots, r_{d,d}(\lambda)\Big). \tag{67}$$

This map is evidently a morphism of C^∞-spaces. By the reflexivity of the groupoid $\mathcal{G}$, the homomorphism $\boldsymbol{\pi}_\mathcal{G}$ induces a homeomorphism between Γ and the open subset $\Omega := \boldsymbol{\pi}_\mathcal{G}(\Gamma) \subset \mathcal{T}(\mathcal{G})$. Clearly, $\varrho^\zeta_{\zeta'}|_\Gamma = r^\zeta_{\zeta'}|_\Omega \circ \boldsymbol{\pi}_\mathcal{G}|_\Gamma$ and so $r^\zeta_{\zeta'}|_\Omega$ yields a homeomorphism between Ω and X.

We claim that the map $r^\zeta_{\zeta'}|_\Omega$ is the desired isomorphism of C^∞-spaces. In one direction, suppose $f \in C^\infty(X)$. Because of the local character of the claim, it is no loss of generality to assume that f admits a smooth extension

$$\tilde{f} \in C^\infty\big(M \times M \times \mathrm{End}(\mathbb{C}^d)\big),$$

thus $f \circ r^\zeta_{\zeta'}|_\Omega = \tilde{f} \circ r^\zeta_{\zeta'}|_\Omega$ is evidently an element of $\mathscr{R}^\infty(\Omega)$. Conversely, let $f : X \to \mathbb{C}$ be a function such that $f \circ r^\zeta_{\zeta'}|_\Omega$ belongs to $\mathscr{R}^\infty(\Omega)$. Since $\boldsymbol{\pi}_\mathcal{G}$ is a morphism of C^∞-spaces, the composite $f \circ r^\zeta_{\zeta'}|_\Omega \circ \boldsymbol{\pi}_\mathcal{G}|_\Gamma = f \circ \varrho^\zeta_{\zeta'}|_\Gamma$ belongs to $C^\infty(\Gamma)$. As $\varrho^\zeta_{\zeta'}|_\Gamma$ is a diffeomorphism, it follows that $f \in C^\infty(X)$. The claim is proven.

We summarize our conclusions as follows:

Proposition 6.7. *Let $\mathcal{G}$ be a reflexive Lie groupoid (Definition 6.4). Then, the canonical homomorphism $\boldsymbol{\pi}_\mathcal{G}$ is an isomorphism of C^∞-spaces. In particular, the Tannakian bidual $\mathcal{T}(\mathcal{G})$ is a Lie groupoid, isomorphic to $\mathcal{G}$.*

We shall now turn our attention to the injectivity of the canonical homomorphism. Clearly, $\boldsymbol{\pi}_\mathcal{G}$ is injective if and only if the groupoid $\mathcal{G}$ admits *enough representations;* this means that for each $x \in M$ and $g \neq x$ in the xth isotropy group of $\mathcal{G}$ there is a representation $(\mathscr{E}, \varrho)$ such that $\varrho(g) \neq id \in \mathrm{Aut}(E_x)$. We contend that each proper Lie groupoid admits enough representations on smooth Euclidean fields.

6.8 (*Cut-off Functions*). We begin with some preliminary remarks of a purely topological nature. Let $\mathcal{G}$ be a proper Lie groupoid over a manifold M. Recall that a subset $S \subset M$ is said to be *invariant* when $s \in S \Rightarrow g \cdot s \in S$ for all arrows g. If S is any subset of M, we let $\mathcal{G} \cdot S$ denote the *saturation* of S, that is to say the smallest invariant subset of M containing S. The saturation of an open subset is also open. It is an easy exercise to show that $\mathcal{G} \cdot \overline{V} = \overline{\mathcal{G} \cdot V}$ for all open subsets V with compact closure. It follows that if U is an invariant open subset of M then U coincides with the union over all invariant open subsets V whose closure is contained in U. The last remark applies to the construction of $\mathcal{G}$-invariant partitions of unity over M; for our purposes, it will be enough to illustrate a special case of this construction. Consider an arbitrary point $x_0 \in M$ and let U be an open invariant neighbourhood of x_0. Choose another open neighbourhood V of x_0, invariant and with closure contained in U. The orbit $\mathcal{G} \cdot x_0$ and the set-theoretic complement $\complement V$ are invariant disjoint closed subsets of M, so by Proposition 5.5 there exists an invariant smooth function on M which takes the value one at x_0 and vanishes outside V.

Let $\mathcal{G}$ be a proper Lie groupoid, with base M. Suppose we are given a representation $(\mathscr{E}_U, \varrho_U)$ of $\mathcal{G}|_U$ on a smooth Euclidean field $\mathscr{E}_U$ over U, where U is an invariant open neighbourhood of a point $x_0 \in M$. We contend that there exists a representation $(\mathscr{E}, \varrho)$ of $\mathcal{G}$ on a smooth Euclidean field $\mathscr{E}$ over M such that $(\mathscr{E}_U)_{x_0}$ and $\mathscr{E}_{x_0}$ are isomorphic G_0-modules; here, as usual, we let G_0 denote the isotropy group of $\mathcal{G}$ at x_0; compare (48).

To begin with, we fix any invariant smooth function $a \in C^\infty(M)$ with $a(x_0) = 1$ and $\operatorname{supp} a \subset U$ (cut-off function). Let V denote the set of all x such that $a(x) \neq 0$. Define $\mathscr{E}_x$ to be the fibre $(\mathscr{E}_U)_x$ if $x \in V$, and $\{0\}$ otherwise. Let $\boldsymbol{\Gamma}\mathscr{E}$ be the following sheaf of sections of the bundle $\{\mathscr{E}_x\}$:

$$W \mapsto \big\{\text{“prolongation of } a\zeta \text{ by zero”} : \zeta \in \boldsymbol{\Gamma}(\mathscr{E}_U)(U \cap W)\big\}. \tag{68}$$

These data define a smooth Euclidean field $\mathscr{E}$ over M. Define $\varrho(g)$ to be $\varrho_U(g)$ if $g \in \mathcal{G}|_V$, and the zero map otherwise. The bundle of linear maps

$$\big\{\varrho(g) : (\boldsymbol{s}^*\mathscr{E})_g \xrightarrow{\sim} (\boldsymbol{t}^*\mathscr{E})_g\big\}$$

will provide an action of $\mathcal{G}$ on $\mathcal{E}$ as long as it is a morphism of smooth Euclidean fields over $\mathcal{G}$ from $\boldsymbol{s}^*\mathcal{E}$ into $\boldsymbol{t}^*\mathcal{E}$. Now, by the invariance of a and the local expression (47) for ϱ_U, one has

$$\varrho(g)\big[a\zeta(\boldsymbol{s}\,g)\big] = a(\boldsymbol{s}\,g)\varrho(g)\zeta(\boldsymbol{s}\,g) = a(\boldsymbol{t}\,g)\sum_{i=1}^{d} r_i(g)\,\zeta_i'(\boldsymbol{t}\,g) = \sum_{i=1}^{d} r_i(g)\big[a\zeta_i'(\boldsymbol{t}\,g)\big],$$

so this is clear. By construction, $\mathcal{E}_{x_0} = (\mathcal{E}_U)_{x_0}$ as G_0-modules.

From Proposition 6.5, 6.7, Zung's theorem 1.5, remark 1.6 and what has just been said, we conclude:

Theorem 6.9 (*Reconstruction Theorem*). *Every proper Lie groupoid is reflexive. In other words, every proper Lie groupoid $\mathcal{G}$ is isomorphic to its own Tannakian bidual $\mathcal{T}(\mathcal{G})$ via the canonical homomorphism $\boldsymbol{\pi}_{\mathcal{G}} : \mathcal{G} \to \mathcal{T}(\mathcal{G})$.*

Proof. It will be enough to observe that, in general, a faithful representation of a compact Lie group G on a finite dimensional vector space $\boldsymbol{E}$ induces, for any smooth action of G on a smooth manifold V, a faithful representation of the action groupoid $G \ltimes V$ on the vector bundle $V \times \boldsymbol{E}$. $\square$

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