Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Physics Letters B 628 (2005) 275–280
PHYSICS LETTERS B
www.elsevier.com/locate/physletb

Special symmetries of the charged Kerr–AdS black hole of $D = 5$ minimal gauged supergravity

Paul Davis, Hari K. Kunduri *, James Lucietti

DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Rd., Cambridge CB3 0WA, UK

Received 16 September 2005; accepted 26 September 2005

Available online 30 September 2005

Editor: M. Cvetič

Abstract

In this Letter we prove that the Hamilton–Jacobi equation in the background of the recently discovered charged Kerr–AdS black hole of $D = 5$ minimal gauged supergravity is separable, for arbitrary values of the two rotation parameters. This allows us to write down an irreducible Killing tensor for the spacetime. As a result, we also show that the Klein–Gordon equation in this background is separable. We also consider the Dirac equation in this background in the special case of equal rotation parameters and show it has separable solutions. Finally, we discuss the near-horizon geometry of the supersymmetric limit of the black hole.

It is a curious fact that the Kerr–Newman black hole possesses a hidden symmetry which renders geodesic motion integrable [1]. This is related to the existence of a second rank Killing tensor $K_{\mu\nu}$; by definition such a tensor satisfies $\nabla_{(\mu} K_{\nu\rho)} = 0$. Given a Killing tensor one may construct the quantity $K = K_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ which is conserved along geodesics $x^\mu(\tau)$. Carter was the first to systematically analyse the consequence of separability of solutions to Einstein's equations, and indeed this is how the Kerr–(A)dS black hole and its charged counterpart were first discovered [2]. Higher-dimensional Kerr–(A)dS metrics have only been recently constructed [3]. The existence of a Killing tensor has been verified in five dimensions for arbitrary rotation parameters [4] and in all dimensions for the special cases of equal sets of rotation parameters [5,6]. This renders both the Hamilton–Jacobi (HJ) and Klein–Gordon (KG) equations separable. The charged counterparts of the Kerr–AdS black holes in $D = 5$ minimal gauged supergravity are far more difficult to construct. Progress was first made by tackling the special case where the rotation parameters are equal [7], and a reducible Killing tensor for this black hole was found in [8]. Further, the

* Corresponding author.
E-mail addresses: p.davis@damtp.cam.ac.uk (P. Davis), h.k.kunduri@damtp.cam.ac.uk (H.K. Kunduri), j.lucietti@damtp.cam.ac.uk (J. Lucietti).

0370-2693
doi:10.1016/j.physletb.2005.09.062

same Killing tensor was found for black holes with equal rotation parameters in the more general $U(1)^3$ theory [9]. Only very recently has a charged Kerr–AdS black hole been found with arbitrary rotation parameters [10]. The purpose of this Letter is to show that this black hole also has a Killing tensor rendering geodesic motion integrable and the KG equation separable. We also show that the Dirac equation admits separable solutions in the special case of equal rotation parameters. Finally, we discuss the near-horizon geometry of the supersymmetric limit of this black hole. In the case of equal rotation parameters we show that it has a symmetry algebra $\mathfrak{sl}(2,\mathbb{R})\times\mathfrak{su}(2)\times\mathfrak{u}(1)$ as is the case for the BMPV black hole [11].

In [10] it was shown that $D=5$ minimal gauged supergravity with the Lagrangian density

$$\mathcal{L}=\left(R+12g^2\right)*1-\frac{1}{2}F\wedge *F+\frac{1}{3\sqrt{3}}F\wedge F\wedge A, \tag{1}$$

where $F=dA$, admits a black hole solution parameterised by its mass, charge and two rotation parameters. Explicitly, the metric is given by:

$$\begin{aligned} ds^2 = {} & -\frac{\Delta_\theta[(1+g^2r^2)\rho^2\,dt+2q\nu]\,dt}{\Xi_a\Xi_b\rho^2}+\frac{2q\nu\omega}{\rho^2}+\frac{f}{\rho^4}\left(\frac{\Delta_\theta\,dt}{\Xi_a\Xi_b}-\omega\right)^2+\frac{\rho^2\,dr^2}{\Delta_r}+\frac{\rho^2\,d\theta^2}{\Delta_\theta} \\ & +\frac{r^2+a^2}{\Xi_a}\sin^2\theta\,d\phi^2+\frac{r^2+b^2}{\Xi_b}\cos^2\theta\,d\psi^2, \end{aligned} \tag{2}$$

$$A=\frac{\sqrt{3}q}{\rho^2}\left(\frac{\Delta_\theta\,dt}{\Xi_a\Xi_b}-\omega\right), \tag{3}$$

where

$$\begin{aligned} &\nu=b\sin^2\theta\,d\phi+a\cos^2\theta\,d\psi,\qquad \omega=a\sin^2\theta\,\frac{d\phi}{\Xi_a}+b\cos^2\theta\,\frac{d\psi}{\Xi_b},\qquad \rho^2=r^2+a^2\cos^2\theta+b^2\sin^2\theta, \\ &\Delta_\theta=1-a^2g^2\cos^2\theta-b^2g^2\sin^2\theta,\qquad \Delta_r=\frac{(r^2+a^2)(r^2+b^2)(1+g^2r^2)+q^2+2abq}{r^2}-2m, \\ &f=2m\rho^2-q^2+2abqg^2\rho^2,\qquad \Xi_a=1-a^2g^2,\qquad \Xi_b=1-b^2g^2. \end{aligned} \tag{4}$$

The metric is written in Boyer–Lindquist type coordinates, although we should emphasise that it is in a non-rotating frame at asymptotic infinity. For general rotation parameters a and b this metric has three commuting Killing vectors, namely ∂_t, ∂_ϕ and ∂_ψ. Remarkably, one can check that the determinant of the metric is independent of the charge parameter q and is thus given by the same expression as in the uncharged case,

$$\sqrt{-\det g}=\frac{r\rho^2\sin\theta\cos\theta}{\Xi_a\Xi_b}. \tag{5}$$

A tedious calculation allows one to write the inverse metric as:

$$\begin{aligned} \rho^2g^{tt} = {} & -\frac{(a^2+b^2)(2mr^2-q^2)}{r^2\Delta_r}-\frac{(r^2+a^2)(r^2+b^2)[r^2(1-g^2(a^2+b^2))-a^2b^2g^2]}{r^2\Delta_r} \\ & -\frac{2ma^2b^2}{r^2\Delta_r}-\frac{2abqr^2}{r^2\Delta_r}-\frac{a^2\cos^2\theta\,\Xi_a+b^2\sin^2\theta\,\Xi_b}{\Delta_\theta}, \\ \rho^2g^{t\phi} = {} & \frac{aq^2-[2ma+bq(1+a^2g^2)](r^2+b^2)}{r^2\Delta_r}, \\ \rho^2g^{t\psi} = {} & \frac{bq^2-[2mb+aq(1+b^2g^2)](r^2+a^2)}{r^2\Delta_r}, \end{aligned}$$

$$
\begin{aligned}
\rho^2 g^{\phi\phi} &= \frac{a^2 g^2 q^2}{r^2 \Delta_r} + \frac{\Xi_a}{\sin^2\theta} + \frac{\Xi_a}{r^2\Delta_r}\big(1+g^2r^2\big)\big(r^2+b^2\big)\big(b^2-a^2\big) \\
&\quad - \frac{2m}{r^2\Delta_r}\big(a^2g^2r^2+b^2\big) - \frac{2abq}{\Xi_b r^2\Delta_r}\big(\Xi_b g^2\big(r^2-a^2\big) - b^4g^4+1\big), \\
\rho^2 g^{\psi\psi} &= \frac{b^2 g^2 q^2}{r^2 \Delta_r} + \frac{\Xi_b}{\cos^2\theta} + \frac{\Xi_b}{r^2\Delta_r}\big(1+g^2r^2\big)\big(r^2+a^2\big)\big(a^2-b^2\big) \\
&\quad - \frac{2m}{r^2\Delta_r}\big(b^2g^2r^2+a^2\big) - \frac{2abq}{\Xi_a r^2\Delta_r}\big(\Xi_a g^2\big(r^2-b^2\big) - a^4g^4+1\big), \\
\rho^2 g^{\phi\psi} &= \frac{abg^2q^2 - (1+g^2r^2)(2mab+(a^2+b^2)q)}{r^2\Delta_r}, \\
\rho^2 g^{\theta\theta} &= \Delta_\theta, \qquad \rho^2 g^{rr} = \Delta_r.
\end{aligned} \tag{6}
$$

An important fact, that we will use shortly, is that the component functions $\rho^2 g^{\mu\nu}$ are additively separable as functions of r and θ.

The Hamiltonian describing the motion of free uncharged particles in the background metric $g_{\mu\nu}$ is simply $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$. The corresponding Hamilton–Jacobi equation is then

$$
\frac{\partial S}{\partial \tau} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0, \tag{7}
$$

where S is Hamilton's principal function and τ is the parameter along the worldline of the particle. Due to the presence of the isometries one may immediately separate out the dependence on t, ϕ, ψ leaving

$$
S = \frac{1}{2} M^2 \tau - Et + L_1\phi + L_2\psi + F(r,\theta), \tag{8}
$$

where M^2, E and L_i are constants. Remarkably, it turns out that S is completely separable so $F(r,\theta) = S_r(r) + S_\theta(\theta)$. The proof of this simply relies on the non-trivial fact that $\rho^2 g^{\mu\nu}$ is additively separable as a function of r and θ. This implies that the HJ equation is separable after multiplying it through by ρ^2. The θ-dependent part of the HJ equation is

$$
\Delta_\theta \left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{L_1^2 \Xi_a}{\sin^2\theta} + \frac{L_2^2 \Xi_b}{\cos^2\theta} - \frac{E^2}{\Delta_\theta}\big(a^2\Xi_a\cos^2\theta + b^2\Xi_b\sin^2\theta\big) + M^2\big(a^2\cos^2\theta + b^2\sin^2\theta\big) = K, \tag{9}
$$

whilst the r-dependent part is

$$
\Delta_r \left(\frac{dS_r}{dr}\right)^2 + V(r; E, L_i, M) = -K, \tag{10}
$$

where K is the separation constant, and we have defined an "effective" potential V which is a complicated function of r; as we shall not use it directly, we shall not display it for the sake of brevity. From the θ equation one may easily read off a Killing tensor for the spacetime using $K = K^{\mu\nu} p_\mu p_\nu$ and $g^{\mu\nu} p_\mu p_\nu = -M^2$. This gives

$$
\begin{aligned}
K^{\mu\nu} &= -g^{\mu\nu}\big(a^2\cos^2\theta + b^2\sin^2\theta\big) - \frac{1}{\Delta_\theta}\big(a^2\Xi_a\cos^2\theta + b^2\Xi_b\sin^2\theta\big)\delta_t^\mu\delta_t^\nu + \frac{\Xi_a}{\sin^2\theta}\delta_\phi^\mu\delta_\phi^\nu + \frac{\Xi_b}{\cos^2\theta}\delta_\psi^\mu\delta_\psi^\nu \\
&\quad + \Delta_\theta \delta_\theta^\mu \delta_\theta^\nu.
\end{aligned} \tag{11}
$$

This is an irreducible Killing tensor. Note that this has a smooth limit as $g \to 0$ and when $q = 0$ coincides with the Killing tensor found in [4], up to terms which are outer products of the Killing vectors. In contrast to [4], here it was unnecessary to add outer products of Killing vectors to the Killing tensor in order to obtain a smooth limit. This is presumably related to the fact that we are in a non-rotating frame at infinity, whereas the metric in [4] was in a rotating frame. It is a curious result that the Killing tensor does not depend explicitly on the charge, although this does also occur for the four-dimensional Kerr–Newman solution. Moreover, as we will discuss shortly, there exist supersymmetric solutions with $a \neq b$. Such black holes thus possess an *irreducible* Killing tensor, as do the supersymmetric Kerr–Newman–AdS black holes in four dimensions [12]. We should note that from the Hamiltonian point of view the functions H, K, p_t, p_ϕ, p_ψ are in involution thus establishing Liouville integrability. The general solution to geodesic motion can easily be deduced from the generating function S by differentiating with respect to K, M^2, E, L_i, respectively.

As in the uncharged case, the additive separability of $\rho^2 g^{\mu\nu}$ allows for separable solutions to the Klein–Gordon equation which governs quantum field theory of massive, spinless particles on this background. Writing the KG equation as

$$\frac{1}{\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}\, g^{\mu\nu}\partial_\nu \Phi\right) = M^2\Phi, \tag{12}$$

and taking the following standard ansatz $\Phi = e^{-i\omega t}e^{i\alpha\phi}e^{i\beta\psi}R(r)\Theta(\theta)$, renders the KG equation separable. The details of this are rather similar to the uncharged Kerr–(A)dS [4]. By making the change of variable $z = \sin^2\theta$, the θ equation can be rewritten as

$$\begin{aligned}
&\frac{d^2\Theta}{dz^2} + \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-d}\right)\frac{d\Theta}{dz} \\
&\quad + \left[\frac{\omega^2(a^2\Xi_a + z(b^2\Xi_b - a^2\Xi_a))}{4z(1-z)\Delta_z^2} - \frac{1}{4z(1-z)\Delta_z}\left(\frac{\alpha^2\Xi_a}{z} + \frac{\beta^2\Xi_b}{1-z}\right) + \frac{M^2}{4g^2 z(1-z)} - \frac{k'}{4z(1-z)\Delta_z}\right]\Theta \\
&\quad = 0,
\end{aligned} \tag{13}$$

where $d = \Xi_a/(g^2(b^2 - a^2))$, $\Delta_z = \Xi_a + g^2 z(a^2 - b^2)$ and $k' = k + M^2/g^2$ with k being the separation constant. This equation has four regular singular points and can easily be put in the form of Heun's equation. The special case $a = b$ simplifies this equation and the solutions are Jacobi polynomials.

Having discussed the separability of the Klein–Gordon equation, the next thing to consider is the Dirac equation on this background. We find that the Dirac equation separates in the special case of equal rotation parameters, $a = b$, and can be written as

$$(D_r + D_{\theta'})\Psi = 0, \tag{14}$$

where D_r and $D_{\theta'}$ are linear differential operators depending only on r and θ' respectively, once the following ansatz has been made:

$$\Psi = e^{-i\omega t}e^{im_1\phi'}e^{im_2\psi'}\chi(r,\theta'). \tag{15}$$

The angular coordinates (θ', ϕ', ψ') are Euler angles following the notation of [8]. This then admits solutions which are separable in the sense that

$$\chi(r,\theta') = \begin{pmatrix} R_1(r)S_+(\theta') \\ R_2(r)S_-(\theta') \\ R_3(r)S_+(\theta') \\ R_4(r)S_-(\theta') \end{pmatrix}, \tag{16}$$

where the radial functions form a complicated, coupled system and the functions $S_\pm$ are eigenfunctions of the differential operators

$$\partial_{\theta'}^2 + \cot\theta'\,\partial_{\theta'} - \frac{1}{2\sin^2\theta'} \mp \frac{i(m_1\cos\theta' - m_2)}{\sin^2\theta'} + \frac{\cot^2\theta'}{4} + \frac{(m_1 - m_2\cos\theta')^2}{\sin^2\theta'}. \tag{17}$$

In four dimensions the separability of the Dirac equation leads to the construction of an operator that commutes with the Dirac operator, and is intimately related to the existence of a Yano tensor for the spacetime [13]. Remarkably, the four-dimensional Kerr–Newman Killing tensor admits a decomposition in terms of a Yano tensor, arising from the fact that there is a non-trivial supersymmetry on the worldline of a spinning particle [14]. It is not hard to show that the five-dimensional Schwarzschild's Killing tensor K does not admit a Yano tensor, and hence this suggests that the full black hole we have been considering does not either. However, one actually should try to construct an operator that commutes with the Dirac operator. One expects this to exist due to the presence of an extra (separation) constant of the system. In four dimensions this is readily achieved, but seems to rely crucially on the existence of Weyl spinors, and we have been unable to find such an operator in the five-dimensional case.

Finally, we will briefly discuss the near-horizon geometry of the supersymmetric limit of the black hole. As discussed in [10], the metric given in Eq. (2) can be rewritten as:

$$ds^2 = -\frac{\Delta_r \Delta_\theta r^2 \sin^2 2\theta}{4(\Xi_a \Xi_b)^2 B_\phi B_\psi} dt^2 + \rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) + B_\psi (d\psi + \nu_1 \, d\phi + \nu_2 \, dt)^2 + B_\phi (d\phi + \nu_3 \, dt)^2, \tag{18}$$

where

$$B_\psi = g_{\psi\psi}, \qquad B_\phi = g_{\phi\phi} - \frac{g_{\phi\psi}^2}{g_{\psi\psi}}, \qquad \nu_1 = \frac{g_{\phi\psi}}{g_{\psi\psi}}, \qquad \nu_2 = \frac{g_{t\psi}}{g_{\psi\psi}}, \qquad \nu_3 = \frac{g_{t\phi} g_{\psi\psi} - g_{\phi\psi} g_{t\psi}}{g_{\phi\phi} g_{\psi\psi} - g_{\phi\psi}^2}. \tag{19}$$

In the supersymmetric limit, some simplification of the metric occurs due to the constraints imposed upon the parameters q and m, namely,

$$q = \frac{m}{1 + ag + bg}, \qquad m = \frac{(a+b)(1+ag)(1+bg)(1+ag+bg)}{g}. \tag{20}$$

With these restrictions in place, we find that at the horizon $r_0^2 = g^{-1}(a + b + abg)$, $\nu_3 + g = 0$, $g + g\nu_1 + \nu_2 = 0$ and all the other functions in the metric are complicated functions of θ and the rotation parameters. To investigate the near-horizon geometry of this metric we first need to go to a frame which is corotating with the horizon. This is effected by the redefinitions $\tilde{t} = t$, $\tilde{\phi} = \phi - gt$, and $\tilde{\psi} = \psi - gt$. Then we set $r - r_0 = \epsilon R$ and $\tilde{t} = T/\epsilon$ and take the limit $\epsilon \to 0$. The near-horizon geometry is then

$$ds_{\mathrm{NH}}^2 = \rho^2(\theta) \left(-c_1 R^2 \, dT^2 + c_2 \frac{dR^2}{R^2} + \frac{d\theta^2}{\Delta_\theta} \right) + B_\psi(\theta) \big(d\tilde{\psi} + \nu_1(\theta) \, d\tilde{\phi} + f(\theta) R \, dT \big)^2 + B_\phi(\theta) (d\tilde{\phi} + c_3 R \, dT)^2, \tag{21}$$

where, in general, we denote $F(\theta) \equiv F(r_0, \theta)$. The function $f(\theta)$ as well as the constants c_1, c_2, c_3 are complicated and rather unenlightening. The resulting geometry is similar to the product of AdS_2 with a squashed sphere, which appears to be a generic property of extremal, rotating (possibly charged) black holes [15]. A trivial time rescaling $T = \sqrt{c_2/c_1}\tilde{T}$ puts the $\tilde{T}R$ part of the metric into a form conformal to AdS_2 in Poincaré coordinates. Thus, in addition to the obvious isometries generated by $\partial/\partial\tilde{T}$, $\partial/\partial\tilde{\phi}$, and $\partial/\partial\tilde{\psi}$, (21) is also invariant under dilations $\tilde{T} \to \alpha\tilde{T}$, $R \to R/\alpha$. An obvious question is whether the near-horizon limit has all the symmetries of AdS_2. Following [15], one might try to introduce global coordinates on the AdS_2, in order to show the near-horizon limit has an (analogue) of the global time translation. This needs to be accompanied by a corresponding coordinate transformation for $(\tilde{\psi}, \tilde{\phi})$. We find that this method does not work in this case, due to the θ-dependence of the metric.

Nevertheless, we can show that the near-horizon limit has all the symmetries of AdS_2 in the special case $a = b$ as follows. Let us write the near-horizon limit in terms of left-invariant forms on $SU(2)$ as in [16]. It is of the form

$$ds^2 = -(R \, d\tau + j\sigma_3)^2 + \frac{dR^2}{R^2} + L^2 \big(\sigma_1^2 + \sigma_2^2 \big) + \lambda^2 \sigma_3^2, \tag{22}$$

where j, λ, L are constants related to the horizon radius and the cosmological constant. One should note that this metric is a deformation of the near-horizon limit of BMPV as found in [11]. One may easily check that in addition to the time translation $k = \frac{\partial}{\partial\tau}$ and the dilation operator $l = -\tau\frac{\partial}{\partial\tau} + R\frac{\partial}{\partial R}$, there is a third isometry analogous to the one for pure AdS_2:

$$m = \frac{2}{R^2}\left(1 - \frac{j^2}{\lambda^2}\right)\partial_\tau + 2\tau^2\partial_\tau - 4\tau R\partial_R + \frac{4j}{R\lambda^2}\partial_{\psi'}. \tag{23}$$

One may check that these Killing vectors satisfy

$$[l, k] = k, \qquad [l, m] = -m, \qquad [k, m] = -4l. \tag{24}$$

Furthermore, the gauge field A is regular in the near-horizon limit and one can easily check that $\pounds_k F = \pounds_l F = \pounds_m F = 0$. Therefore, the algebra of the isometry group of the near-horizon limit which preserves the field strength, in the $a = b$ case, is $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$. It would be most interesting to see whether the general case retains all the symmetries of AdS_2. Further, an interesting problem is to determine the full superalgebra of the near-horizon limit, as was done for the BMPV case in [11].

The coordinates we are using are not really suitable on the horizon. One should really be using Gaussian null coordinates adapted to the Killing horizon, which would also allow direct comparison with the near-horizon geometries derived in [16]. One expects the "parameter" Δ used therein to be non-constant for the metric at hand and thus would fall outside their analysis.

While we have studied certain special symmetries of the general charged Kerr–AdS black holes, we doubt that the short list presented here is exhaustive. The existence of supersymmetric black hole solutions with spherical topology having non-equal angular momentum in two orthogonal planes seems unique to gauged supergravity. Given the natural link between supersymmetry and special geometric structures, it seems likely there are further non-trivial symmetries of these black holes.

Acknowledgements

P.D. would like to thank PPARC for financial support. H.K.K. would like to thank St. John's College for financial support. We would like to thank Gary Gibbons and Harvey Reall for useful comments.

References

[1] B. Carter, Phys. Rev. 174 (1968) 1559.
[2] B. Carter, Commun. Math. Phys. 10 (1968) 280.
[3] G.W. Gibbons, H. Lu, D.N. Page, C.N. Pope, J. Geom. Phys. 53 (2005) 49, hep-th/0404008.
[4] H.K. Kunduri, J. Lucietti, Phys. Rev. D 71 (2005) 104021, hep-th/0502124.
[5] M. Vasudevan, K.A. Stevens, D.N. Page, Class. Quantum Grav. 22 (2005) 339, gr-qc/0405125.
[6] M. Vasudevan, K.A. Stevens, gr-qc/0507096.
[7] M. Cvetič, H. Lu, C.N. Pope, Phys. Lett. B 598 (2004) 273, hep-th/0406196.
[8] H.K. Kunduri, J. Lucietti, Nucl. Phys. B 724 (2005) 343, hep-th/0504158.
[9] M. Vasudevan, gr-qc/0507092.
[10] Z.W. Chong, M. Cvetič, H. Lu, C.N. Pope, hep-th/0506029.
[11] J.P. Gauntlett, R.C. Myers, P.K. Townsend, Class. Quantum Grav. 16 (1999) 1, hep-th/9810204.
[12] V.A. Kostelecky, M.J. Perry, Phys. Lett. B 371 (1996) 191, hep-th/9512222.
[13] B. Carter, R.G. Mclenaghan, Phys. Rev. D 19 (1979) 1093.
[14] G.W. Gibbons, R.H. Rietdijk, J.W. van Holten, Nucl. Phys. B 404 (1993) 42, hep-th/9303112.
[15] J.M. Bardeen, G.T. Horowitz, Phys. Rev. D 60 (1999) 104030, hep-th/9905099.
[16] J.B. Gutowski, H.S. Reall, JHEP 0402 (2004) 006, hep-th/0401042.