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Strict partitions and discrete dynamical systems

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Abstract

We prove that the set of partitions with distinct parts of a given positive integer under dominance ordering can be considered as a configuration space of a discrete dynamical model with two transition rules and with the initial configuration being the singleton partition. This allows us to characterize its lattice structure, fixed point, and longest chains as well as their length, using Chip Firing Game theory. Finally, we study the recursive structure of infinite extension of the lattice of strict partitions.

Keywords: Strict partition; Discrete dynamical system; Chip Firing Game; Poset; Lattice

1. Introduction

A partition of a positive integer n is a sequence of non-increasing positive integers $a = (a_1, \ldots, a_m)$ such that $a_1 + \cdots + a_m = n$; the integers a_i are called *parts* of a. The set of all such partitions of n is denoted by $\mathcal{P}(n)$. $\mathcal{P}(n)$ is equipped with a partial order called the *dominance order* as follows: $a \geq b$ if $\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i$, for all $j \geq 1$ (by convention, $a_k = 0$ for $k > m$). This order has been shown to have many applications to problems in combinatorics as well as group representation theory, among other fields. The structure of this poset was studied by Brylawski [3] who showed in particular that it is a lattice. Since then, other properties such as longest chains and fixed points have also been characterized in [3,7,4]. In [9], Latapy and Phan constructed its infinite extension and obtained a construction algorithm.

In this paper, we study the structure of an interesting class $\mathcal{SP}(n)$ of partitions of n called *strict partitions*, *i.e.* partitions with distinct parts, from the point of view of discrete dynamical systems. For any strict partition a of n, one can apply on a the following transition rules so that the resulting partition is also strict (see Fig. 1):

– Vertical transition (V-transition):

$(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) \to (a_1, \ldots, a_i - 1, a_{i+1} + 1, \ldots, a_n)$, if $a_i - a_{i+1} \geq 3$.

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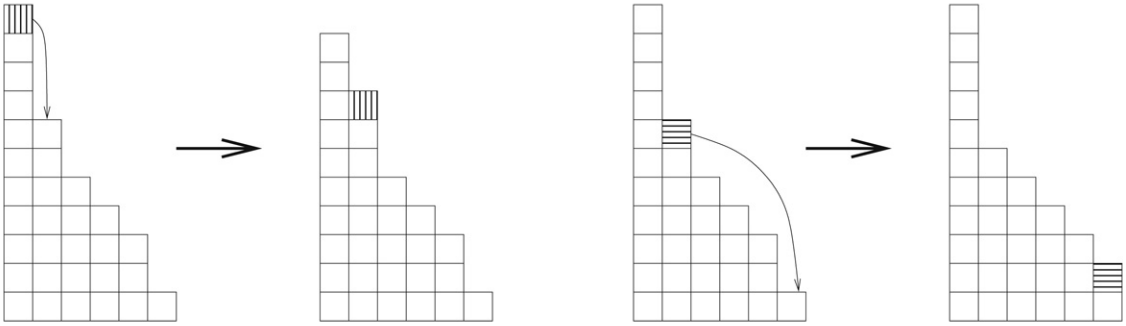

Fig. 1. Vertical transition and horizontal transition.

– Horizontal transition (H-transition) with length ℓ (for $\ell \geq 2$):
$(a_1, \ldots, p+\ell+1, p+\ell-1, p+\ell-2, \ldots, p+2, p+1, p-1, \ldots, a_n) \rightarrow$
$(a_1, \ldots, p+\ell, p+\ell-1, p+\ell-2, \ldots, p+2, p+1, p, \ldots, a_n)$
and Horizontal transition with length 1:
$(a_1, \ldots, p+2, p-1, \ldots, a_n) \rightarrow (a_1, \ldots, p+1, p, \ldots, a_n)$. Note that an H-transition of length 1 is also a V-transition. We define the cover relation as follows. A strict partition a *covers* another strict partition b (write $a \succ_S b$) if b can be obtained from a by applying a transition rule. It is evident that the reflexive and transitive closure of this relation is an order relation. We denote it by $\geq_S$. Moreover, we call a sequence of transitions in this system a *chain*, and a *longest chain* from a to b is a chain of greatest length. By convention, a chain of one element (with no transitions) is of length 0. We show that all strict partitions can be obtained from the initial configuration (n) by applying transition rules. In particular, we show that $\mathcal{SP}(n)$ is a subposet of $\mathcal{P}(n)$ and it is also a lattice. Furthermore, unlike $\mathcal{P}(n)$, $\mathcal{SP}(n)$ is not self-dual. Using the fact that our dynamical model can be viewed as a "composition" of two Chip Firing Games in the sense of [1] (see also [9,6]), we are able to characterize the fixed point explicitly, the longest chains as well as their length in $\mathcal{SP}(n)$. Moreover, we obtain an infinite extension of $\mathcal{SP}(n)$ and an algorithm for constructing $\mathcal{SP}(n)$ in linear time.

2. Lattice structure of $\mathcal{SP}(n)$

Theorem 2.1. *The set $\mathcal{SP}(n)$ is exactly the set of all strict partitions reachable from (n) by applying two transition rules V and H.*

Proof. Let $a = (a_1, \ldots, a_m)$ be a strict partition. It suffices to show that if a is different from (n) itself, then there exists another strict partition a' such that $a' \succ_S a$. First of all, observe that if there is a subsequence $(a_i, a_{i+1}, \ldots, a_j)$ of consecutive numbers in a, where $i = 1$, or else $a_{i-1} - a_i \geq 2$, and similarly $j = m$ or else $a_j - a_{j+1} \geq 2$, then we can choose

$$a' = (a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots, a_m),$$

so that a' is again strict. Furthermore, one recovers a from a' by applying an H-transition. On the other hand, if no such subsequence exists, then $a_1 - a_2 \geq 2$ and either $m = 2$ or $a_2 - a_3 \geq 2$. In this case, we can simply choose $a' = (a_1 + 1, a_2 - 1, a_3, \ldots, a_m)$. It is easy to check that a' is a strict partition and that a V-transition applied on a' at the first position gives back a. The theorem is then proved. $\square$

Proposition 2.2. *$\mathcal{SP}(n)$ is a subposet of $\mathcal{P}(n)$.*

Proof. It is sufficient to show that if $a, b \in \mathcal{SP}(n)$ and $a > b$ then $a >_S b$, *i.e.* there exists a chain from a to b. For this purpose, we prove that there exists a strict partition a' such that $a \succ_S a'$ and $a' \geq b$.

Since $a > b$, we have $\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i$ for all $1 \leq j \leq n$. Let j be the smallest index where $a_j > b_j$. Then let ℓ be the smallest index such that $\ell > j$ and $\sum_{i=1}^{\ell} a_i = \sum_{i=1}^{\ell} b_i$. Such a number ℓ exists because $\ell = n$ satisfies both conditions above. It is clear that $a_\ell < b_\ell$ because of the choice of ℓ.

We claim that we can apply a transition on a at some position between j and ℓ, so that the newly constructed partitions a' are identical with a outside this range. If this is possible, then we are done, because it is easy to verify, using the definition of j and ℓ, that $a' \geq b$ in $\mathcal{P}(n)$.

To construct a', observe that if there is an index $j \leq i \leq \ell$ such that $a_i - a_{i+1} \geq 3$, then a V-transition can be applied at position i. Suppose now that $a_i - a_{i+1} \leq 2$ for all $j \leq i < \ell$. Since b is a strict partition and $b_i \geq 1$ for all i, we have $b_j - b_\ell \geq \ell - j$. But $a_\ell < b_\ell$ and $a_j > b_j$; hence $a_j - a_\ell \geq \ell - j + 2$. It follows that there exist at least two indices $j \leq r < s < \ell$ such that $a_r - a_{r+1} = a_s - a_{s+1} = 2$. Furthermore, by choosing a different pair of indices if necessary, we can even assume that $a_i - a_{i+1} = 1$ for all $r < i < s$. But in this case, the subsequence $(a_r, \ldots, a_s)$ is of exactly the form where one can apply an H-transition. The proof is finished. □

Because of the above result, we can now write $b \leq a$ instead of $b \leq_S a$ for any two strict partitions a and b.

Theorem 2.3. *$\mathcal{SP}(n)$ is a lattice. Moreover, the meet operation in $\mathcal{SP}(n)$ is the same as that in $\mathcal{P}(n)$.*

Proof. Since $\mathcal{SP}(n)$ contains a maximal element, it is enough to prove that any pair of elements (a, b) in $\mathcal{SP}(n)$ has a meet. Of course, their meet $c = a \wedge b$ in $\mathcal{P}(n)$ does exist, but is it true that c is again a strict partition? By definition, c is defined by the formula $\sum_{i=1}^{j} c_i = \min(\sum_{i=1}^{j} a_i, \sum_{i=1}^{j} b_i)$ for all $j \geq 1$. Suppose that $c_j > 0$. Without loss of generality, assume that $\sum_{i=1}^{j} c_i = \sum_{i=1}^{j} a_i$. Then $c_{j+1} \leq a_{j+1}$ while $a_j \leq c_j$. Thus $c_{j+1} < c_j$ because $a_{j+1} < a_j$. Hence c is also a strict partition. The proof above clearly also implies that the meet operation in $\mathcal{SP}(n)$ is the same as that in $\mathcal{P}(n)$. □

Remark 2.4. $\mathcal{SP}(n)$ is **not** a sublattice of $\mathcal{P}(n)$. In fact, the joint operations in $\mathcal{SP}(n)$ and $\mathcal{P}(n)$ are different. For example, $(8, 4, 3, 1) \vee (7, 5, 4) = (8, 4, 4)$ which is not a strict partition. Nevertheless, we still have $a \vee_S b \geq a \vee b$ for any a and b.

While studying discrete dynamical systems, one important question is whether it has (a unique) fixed point (configurations on which no transition is possible). In the case of $\mathcal{SP}(n)$, because it is a lattice, it has a unique minimal element and this is its unique fixed point. We finish this section by giving an explicit formula for this fixed point. Let p be the unique integer such that $\frac{1}{2}p(p+1) \leq n < \frac{1}{2}(p+1)(p+2)$. Then let $q = n - \frac{1}{2}p(p+1)$. One verifies easily that $q < p$. Now let Π be the following partition:

$$\Pi = (p+1, p, \ldots, p-q+2, p-q, p-q-1, \ldots, 2, 1). \tag{2.1}$$

It is evident that Π is a strict partition on which no transition can be applied.

Proposition 2.5. *Π is the fixed point of $\mathcal{SP}(n)$.*

3. Longest chains

In this section, we calculate the greatest length of chains in $\mathcal{SP}(n)$. The longest chains in $\mathcal{P}(n)$ were studied by Greene and Kleitman [7] where they introduced the notion of a VH-chain (*i.e.* a chain of V-transitions followed by a chain of H-transitions) and proved that all VH-chains are longest chains. It turns out that the same is true for strict partitions. Our proof, however, is different. The proof in [7] makes use of a series of delicate lemmas which basically consider the differences of consecutive parts of partitions. We believe that our proof, which is based on the theory of the Chip Firing Game on a directed graph [1], is simpler and probably can be adapted in other contexts.

3.1. V- (H-)chain

Let us first introduce some definitions. A V- (resp. H-)chain is a chain of V- (resp. H-)transitions, and a VH-chain is a concatenation of a V-chain and an H-chain. If there is a V-chain from a strict partition a to another b, then we say that b is V-reachable from a. But a partition d H-reaching c means that there is an H-chain from d to c.

We will also need the two functions *V-weight* $w_V(a)$ and *H-weight* $w_H(a)$ on a strict partition a. From the Ferrers diagram for a, let

$$w_V(a) = \sum (i-1)a_i \quad \text{and} \tag{3.1}$$

$$w_H(a) = \sum (k-1)\tilde{a}_k, \tag{3.2}$$

where $\tilde{a}_k$ is the number of cells (i, j) on the diagonal k: $i + j - 1 = k$, $i \geq 1$, $j \geq 1$. It is easy to see that a V-transition increases the V-weight by 1, but decreases the H-weight by at least 1. On the other hand, an H-transition decreases the H-weight by 1, and increases the V-weight by at least 1. This simple observation shows that:

Lemma 3.1. *A V-chain or an H-chain between two strict partitions is a longest chain.*

The structure of this section is the following. First, by using a CFG model, we prove that there exists a unique smallest strict partition $\lfloor a \rfloor_b$ which is V-reachable from a in any interval $b \leq a$ (Proposition 3.4); moreover these exists a VH-chain $\mathcal{C} : a \xrightarrow{V} \lfloor a \rfloor_b \xrightarrow{H} b$. On the other hand, we give Lemma 3.5 showing that there is a VH-chain from a to b which is a longest chain. And at the end, in Theorem 3.8, we prove that all VH-chains from a to b have the same length (as $\mathcal{C}$), and then they are all longest chains.

3.2. Chip Firing Game

We now give a brief overview of the theory of the Chip Firing Game (CFG for short). In particular, we show that the dynamical models consisting only of V-transitions (resp. H-transitions) are examples of CFG. For a more detailed account of the theory of CFG, we refer the reader to [2,1,9,5]. A CFG is a discrete dynamical system defined on a (directed) graph $G = (V, E)$, where each configuration consists of a partition of n chips on the vertices V, and obeys the following rule, called the *firing rule*: a vertex containing at least at many chips as its outgoing degree (*i.e.* the number of outgoing edges) transfers one chip along each of its outgoing edges. This rule defines a natural partial order on the space of configurations by declaring that a configuration b is smaller than a if b can be obtained from a by iterating the firing rule. A *fixed point* of a CFG is a configuration where no firing is possible. The following is one important result in the theory of CFG:

Theorem 3.2 (*[1,9]*). *The set of all configurations reachable from the initial one of a CFG with no closed components is a lattice.*

A closed component of a graph is a strongly connected component (of at least two vertices) without outgoing edges. The key of the proof is that a closed component can create a loop in the CFG. Note that if the graph contains components of one vertex without an outgoing edge, this theorem remain valid because these components do not create any loop.

One can also characterize the natural order defined above using the notion of a shot vector. If $b \leq a$, then the *shot vector* $k(a, b)$ is the vector in $\mathbb{N}^{|V|}$ whose entry $k_v(a, b)$ is the number of firings at vertex v on a sequence of firings from a to b. This vector depends only on a and b but not on a chosen sequence of firings. We then have:

Lemma 3.3. *[9] Let c and d be two configurations reachable from the same initial configuration a in a CFG. Then $c \geq d$ if and only if $k_v(a, c) \leq k_v(a, d)$ for all vertices $v \in V$.*

Using the above results for the CFG model, we can now study our model.

Proposition 3.4. *The dynamical model consisting of only V-transitions is a CFG. And the dynamical model consisting of only inverse H-transitions is also a CFG.*

Proof. Let us prove this assertion for the case of inverse H-transitions; the other case is similar. Consider the graph $G = (V, E)$ with $n + 1$ vertices defined pictorially as follows: $\overset{v_0}{\circ} \leftarrow \overset{v_1}{\circ} \leftrightarrows \overset{v_2}{\circ} \cdots \leftrightarrows \overset{v_n}{\circ}$. Thus each vertex of G, beside v_0 and v_n, has outgoing degree 2. Let b be a configuration, *i.e.* a strict partition of n; we put $\tilde{d}_i = \tilde{b}_i - \tilde{b}_{i+1}$ chips at vertex v_i for all $i \geq 1$ and no chip at v_0, where the $\tilde{b}_i$ are as in (3.2). Note that $\tilde{d}_i \geq -1$. An inverse H-transition on b can be described as follows: if $\tilde{b}_i - \tilde{b}_{i+1} \geq 2$, then the rightmost grain in the diagonal i slides up to the diagonal $i + 1$. For example, let $b = (10, 7, 6, 5, 4, 1)$; then $\tilde{b} = (1, 2, 3, 4, 5, 6, 5, 5, 1, 1)$, and one can apply an inverse H-transition on the diagonal 8 of b to obtain the strict partition $c = (10, 8, 6, 5, 3, 1)$ with $\tilde{c} = (1, 2, 3, 4, 5, 6, 5, 4, 2, 1)$. The necessary condition for applying an inverse H-transition at diagonal i on b is $\tilde{b}_i - \tilde{b}_{i+1} \geq 2$, or equivalently $\tilde{d}_i \geq 2$ which is the same as the condition for applying the CFG firing rule on v_i. It is easy to see that the space of configurations reachable from b for this CFG is exactly the set of strict partitions that are H-reaching b. In particular, the unique fixed point of this CFG corresponds to the greatest strict partition which is H-reaching b. This implies that in any interval $b \leq a$ of $\mathcal{SP}(n)$ there is a unique greatest strict partition $\lceil b \rceil^a$ which is H-reaching b. Similarly there is a unique smallest strict partition $\lfloor a \rfloor_b$ which is V-reachable from a. $\square$

3.3. VH-chains are longest chains

First of all, it is not hard to show, as in [7], that there exists a longest chain which is a VH-chain. The point is that any chain $\mathcal{C}$ of two transitions (H,V) from a to b can be replaced by a VH-chain of length 2 or 3 from a to b. In fact, if the H-transition in $\mathcal{C}$ is also a V-transition, then $\mathcal{C}$ is a chain of two transitions (V,V), or if the V-transition in $\mathcal{C}$ is also an H-transition, then $\mathcal{C}$ is a chain of two transitions (H,H); hence we have a VH-chain. Let us consider the case where in $\mathcal{C}$ the H-transition is of the form $(p+\ell+1, p+\ell-1, \ldots, p+1, p-1) \to (p+\ell, p+\ell-1, \ldots, p+1, p)$ with $\ell \geq 2$ and the V-transition is of the form $(r, r-k) \to (r-1, r-k+1)$ with $k \geq 4$. If $r < p$ or $r-k > p+l$ then the two transitions can be commuted and we obtain a VH-chain. For the remaining cases ($r = p$ or $r-k = p+l$) one can simply replace $\mathcal{C}$ by a new chain of the form (V,V,H).

Thus for any chain of transitions between two partitions, there is a VH-chain of at least the same length, and we have the following lemma:

Lemma 3.5. *If $b \leq a$ in $\mathcal{SP}(n)$ then there exists a VH-chain from a to b which is a longest chain.*

It remains to show that any VH-chain is a longest chain.

Lemma 3.6. *Let c and d be two partitions which are V-reachable from a. If $d \leq c$, then d is V-reachable from c.*

Proof. We compute the shot vector $k(a, c)$ and $k(a, d)$ in the corresponding CFG. It is easy to see that $k_i(a, c) = k_{i-1}(a, c) + a_i - c_i$ for all $i \geq 1$, which implies that $k_i(a, c) = \sum_{j=1}^{i} a_j - \sum_{j=1}^{i} c_j$. Similarly, $k_i(a, d) = \sum_{j=1}^{i} a_j - \sum_{j=1}^{i} d_j$. On the other hand, $\sum_{j=1}^{i} c_j \geq \sum_{j=1}^{i} d_j$ because $c \geq d$. It follows that $k_i(a, c) \leq k_i(a, d)$ and so d is V-reachable from c by Lemma 3.3. $\square$

Lemma 3.7. *If $a \geq b$, then $\lfloor a \rfloor_b$ is H-reaching b and $\lceil b \rceil^a$ is V-reachable from a.*

Proof. There is a VH-chain $\mathcal{C}$ from $\lfloor a \rfloor_b$ to b: $\lfloor a \rfloor_b \xrightarrow{V} c \xrightarrow{H} b$. Since $\lfloor a \rfloor_b$ is the smallest strict partition which is V-reachable from a in interval $b \leq a$ so $c = \lfloor a \rfloor_b$ and $\mathcal{C}$ is an H-chain, and then $\lfloor a \rfloor_b$ is H-reaching b. A similar argument applies for $\lceil b \rceil^a$. $\square$

As an immediate corollary, we see that there is a VH-chain $a \to \lceil b \rceil^a \to b$ from a to b of length $w_V(a) - w_V(\lceil b \rceil^a) + w_H(\lceil b \rceil^a) - w_H(b)$. We can now state the main result of this section:

Theorem 3.8. *All VH-chains from a to b in $\mathcal{SP}(n)$ have the same length and this length is maximal.*

Proof. Suppose that $a \xrightarrow{V} c \xrightarrow{H} b$ is a VH-chain from a to b with length $w_V(c) - w_V(a) + w_H(b) - w_H(c)$. We will show that it has the same length as the VH-chain $a \xrightarrow{V} \lceil b \rceil^a \xrightarrow{H} b$. In particular, its length depends only on a and b and is maximal.

It is clear from the definition of $\lfloor a \rfloor_b$ and $\lceil b \rceil^a$ that $\lfloor a \rfloor_b \leq c \leq \lceil b \rceil^a$. Since both $\lceil b \rceil^a$ and c are V-reachable from a and $\lceil b \rceil^a \geq c$, then there is a V-chain from $\lceil b \rceil^a$ to c by Lemma 3.3. On the other hand, there is also an H-chain from $\lceil b \rceil^a$ to c because $\lceil b \rceil^a$ is the minimum element of the lattice of all H-reaching strict partitions from b which contains c. By Lemma 3.1, the two chains are both of maximal length; hence $w_V(c) - w_V(\lceil b \rceil^a) = w_H(\lceil b \rceil^a) - w_H(c)$. The required result immediately follows from the equalities

$$
\begin{aligned}
w_V(c) - w_V(a) &= w_V(\lceil b \rceil^a) - w_V(a) + w_V(c) - w_V(\lceil b \rceil^a) \\
w_H(c) - w_H(b) &= w_H(\lceil b \rceil^a) - w_H(b) - (w_H(\lceil b \rceil^a) - w_H(c)). \quad \square
\end{aligned}
$$

3.4. The length of a longest chain

Once we know that all VH-chains are longest chains, it is sufficient to calculate the length of a well-chosen VH-chains from (n) to Π. The VH-chain that we will use is $(n) \xrightarrow{V} \lfloor (n) \rfloor_\Pi \xrightarrow{H} \Pi$. For the point $P = \lfloor (n) \rfloor_\Pi$, which is the fixed point of the dynamical model consisting of only V transitions with initial configuration (n), together with the length of the V-chain from $(n) \to \lfloor (n) \rfloor_\Pi$ which was already computed in [6], our model corresponds to the model

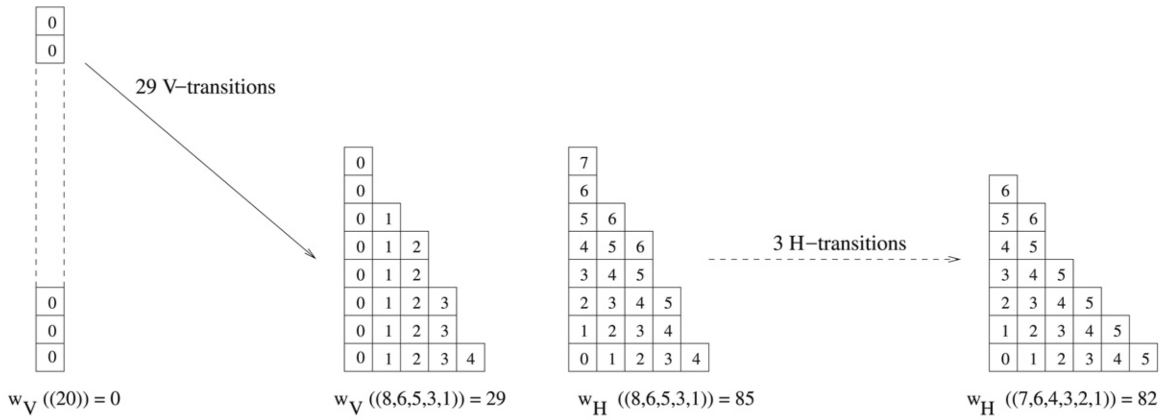

Fig. 2. A longest chain in $\mathcal{SP}(23)$ containing a V-chain from (23) to $P = (8, 6, 5, 3, 1)$ and an H-chain from P to $\Pi = (7, 6, 4, 3, 2, 1)$; its length is $w_H(8, 6, 5, 3, 1) + w_V(8, 6, 5, 3, 1) - w_V(7, 6, 4, 3, 2, 1) = 29 - 0 + 85 - 82 = 32$.

named $L(n, 3)$ in that article. To describe P and $w_V((n), P)$, first write n in the form $n = k(k+1) + \ell(k+1) + h$, where $0 \le \ell \le 1$, $0 \le h \le k$. The integers k, ℓ, h are all uniquely determined from n. We have

$$P = (\ell + 2k, \ell + 2(k-1), \dots, \ell + 2h, \ell + 2(h-1) + 1, \dots, \ell + 2 + 1, \ell + 1), \tag{3.3}$$

$$\text{and} \quad w_V(P) = \frac{(k-1)k(k+1)}{3} + \ell\frac{k(k+1)}{2} + h\frac{2k-h+1}{2}. \tag{3.4}$$

By using some calculus of two functions w_V and w_H, we can now state the following result (see Fig. 2 for an example):

Proposition 3.9. *Let p, q be unique integers such that $n = \frac{1}{2}p(p+1) + q$, $0 \le q \le p$, and let k, ℓ, h be unique integers such that $n = k(k+1) + \ell(k+1) + h$, $0 \le \ell \le 1$, $0 \le h \le k$. We have the following formula for the length L of longest chains in $\mathcal{SP}(n)$:*

$$L = \frac{k(k+1)(8k-5)}{6} + 2\ell k(k+1) + (2k+\ell)h - \frac{(p-1)p(p+1)}{3} - qp.$$

4. Infinite extension of $\mathcal{SP}(n)$

It is natural to ask whether one can construct the lattice $\mathcal{SP}(n+1)$ from $\mathcal{SP}(n)$ and, more generally, what the precise relationship between the lattices $\mathcal{SP}(n)$ is for various n. Our solution to the second question is to assemble $\mathcal{SP}(n)$ together into a lattice $\mathcal{SP}(\infty)$ called the lattice of strict partitions of infinity. More precisely, $\mathcal{SP}(\infty)$ is the lattice obtained from the dynamical system with two transition rules as those for $\mathcal{SP}(n)$, and the initial configuration is infinity. A *strict partition of infinity* is just a sequence of finitely many strictly decreasing positive integers, except the first entry: $(\infty, a_2, a_3, \dots a_k)$. The partial order is defined by declaring that $a \ge_\infty b$ if $\sum_{i \ge j} a_i \le \sum_{i \ge j} b_i$ for all $j \ge 2$.

Many results presented in this section are obtained initially in the case of normal partitions in [8]. However, the proofs are not completely similar since we must be careful that our operations are within the set of strict partitions. In fact, even though $\mathcal{SP}(n)$ can be embedded in a $\mathcal{P}(n)$, the structure of the infinite lattice is different.

This section is organized as follows. Theorem 4.2 gives a decomposition of $\mathcal{SP}(n+1)$ as a set and Proposition 4.3 describes explicitly the set of successors of each element. We conclude with two useful interpretations of $\mathcal{SP}(\infty)$: it is both the limit of $\mathcal{SP}(n)$ when n goes to ∞ and can also be seen as a disjoint union of $\mathcal{SP}(n)$.

4.1. Notation and definitions

If $a = (a_1, a_2, \dots, a_k)$ is a strict partition, then the partition obtained from a by adding one grain on its j-th column is denoted by $a^{\downarrow j}$. Notice that $a^{\downarrow j}$ is not necessarily a strict partition. If S is a set of strict partitions, then $S^{\downarrow j}$ denotes the set $\{a^{\downarrow j} | a \in S\}$. We write $a \xrightarrow{j} b$ if b is obtained from a by applying a transition at position j and denote by $\mathrm{Succ}(a)$ the set of successors of a (configurations directly reachable from a).

Write $d_i(a) = a_i - a_{i+1}$. We say that a has a *cliff* at position i if $d_i(a) \geq 3$. If there exists an $\ell \geq i$ such that $d_j(a) = 1$ for all $i \leq j < \ell$ and $d_\ell(a) = 2$, then we say that a has a *slippery plateau* at i. Likewise, a has a *non-slippery plateau* at i if $d_j(a) = 1$ for all $i \leq j < \ell$ and $d_\ell(a) \geq 3$. The integer $\ell - i + 1$ is called the *length* of the plateau at i. Note that in the special case $\ell = i$, the plateau is of length 1. The set of elements of $\mathcal{SP}(n)$ that begin with a cliff, a slippery plateau of length ℓ and a non-slippery plateau of length ℓ are denoted by C, SP_ℓ, nSP_ℓ respectively.

4.2. Constructing $\mathcal{SP}(n+1)$ from $\mathcal{SP}(n)$

Let $a = (a_1, a_2, \ldots, a_k)$ be a strict partition. It is clear that $a^{\downarrow 1}$ is again a strict partition. This defines an embedding $\pi : \mathcal{SP}(n) \to \mathcal{SP}(n)^{\downarrow 1} \subset \mathcal{SP}(n+1)$ which can be proved, by using the infimum formula of $\mathcal{SP}(n)$ and $\mathcal{SP}(n+1)$, as a lattice map.

Lemma 4.1. *$\mathcal{SP}(n)^{\downarrow 1}$ is a sublattice of $\mathcal{SP}(n+1)$.*

Our next result characterizes the remaining elements of $\mathcal{SP}(n+1)$ that are not in $\mathcal{SP}(n)^{\downarrow 1}$.

Theorem 4.2. *For all $n \geq 1$, we have $\mathcal{SP}(n+1) = \mathcal{SP}(n)^{\downarrow 1} \sqcup_{\ell \geq 1} SP_\ell^{\downarrow \ell+1}$.*

Proof. It is easy to check that each element in one of the sets $\mathcal{SP}(n)^{\downarrow 1}$ and $SP_\ell^{\downarrow \ell+1}$ is an element of $\mathcal{SP}(n+1)$, and that these sets are disjoint. Now let us consider an element b of $\mathcal{SP}(n+1)$. If $d_1(b) \geq 2$ then b is in $\mathcal{SP}(n)^{\downarrow 1}$. If b begins with a plateau of length $\ell + 1$, $\ell \geq 1$, then b is in $SP_\ell^{\downarrow \ell+1}$. $\square$

Finally, we describe an algorithm for computing the successors of any given element of $\mathcal{SP}(n+1)$, thus giving a complete construction of $\mathcal{SP}(n+1)$ from $\mathcal{SP}(n)$.

Proposition 4.3. *Let x be an element of $\mathcal{SP}(n+1)$.*

(1) *If $x = a^{\downarrow 1} \in \mathcal{SP}(n)^{\downarrow 1}$:*
  (a) *If a is in C or nSP then* $\mathrm{Succ}(x) = \mathrm{Succ}(a)^{\downarrow 1}$.
  (b) *If a is in SP_ℓ ($\ell \geq 1$) then* $\mathrm{Succ}(x) = \mathrm{Succ}(a)^{\downarrow 1} \cup \{a^{\downarrow \ell+1}\}$.

(2) *If $x = a^{\downarrow \ell+1} \in SP_\ell^{\downarrow \ell+1}$ for some $a \in SP_\ell$, ($\ell \geq 1$), then*
  (a) *If a has a cliff at $\ell + 1$ or a non-slippery plateau at $\ell + 1$, then* $\mathrm{Succ}(x) = \mathrm{Succ}(a)^{\downarrow \ell+1}$.
  (b) *If a has a slippery plateau at $\ell+1$, let b be such that $a \xrightarrow{\ell} b$ in $\mathcal{SP}(n)$; then* $\mathrm{Succ}(x) = (\mathrm{Succ}(a) \setminus \{b\})^{\downarrow \ell+1} \cup \{b^{\downarrow \ell}\}$.

Proof. We will give the proof for the two most difficult cases (1a) and (2b); others cases are similar. Consider the first case, $x = a^{\downarrow 1}$ where $a \in C$: notice first that the transitions possible from a on columns other than the first one are still possible from $a^{\downarrow 1}$, and on the other hand the addition of one grain on a cliff does not allow any new transition from the first column, since such a transition was already possible.

In the case, $x = a^{\downarrow \ell+1}$ where $a \in SP_\ell$ and a has a slippery plateau of length ℓ' at $\ell + 1$. Then, $a \xrightarrow{\ell} b$ in $\mathcal{SP}(n)$. The possible transitions from $a^{\downarrow \ell+1}$ are the same as the possible ones from a, except the transition on the column ℓ. All the elements directly reachable from a except b have a slippery plateau at 1; therefore the elements of $(\mathrm{Succ}(a) \setminus \{b\})^{\downarrow \ell+1} \in \mathrm{Succ}(a^{\downarrow \ell+1})$. The only one missing transition is $a^{\downarrow \ell+1} \xrightarrow{\ell+1} a^{\downarrow \ell+\ell'+1}$. But we can verify that $a^{\downarrow \ell+\ell'+1} = b^{\downarrow \ell}$. $\square$

Proposition 4.3 makes it possible to write an algorithm for constructing the lattice $\mathcal{SP}(n)$ in linear time (with respect to its size).

4.3. The infinite lattice $\mathcal{SP}(\infty)$

Imagine that (∞) is the initial configuration where the first column contains infinitely many grains and all the other columns contain no grains. Then the transitions V and H defined in the first section can be performed on (∞) just as if it is finite, and we name as $\mathcal{SP}(\infty)$ the set of all the configurations reachable from (∞). A typical element

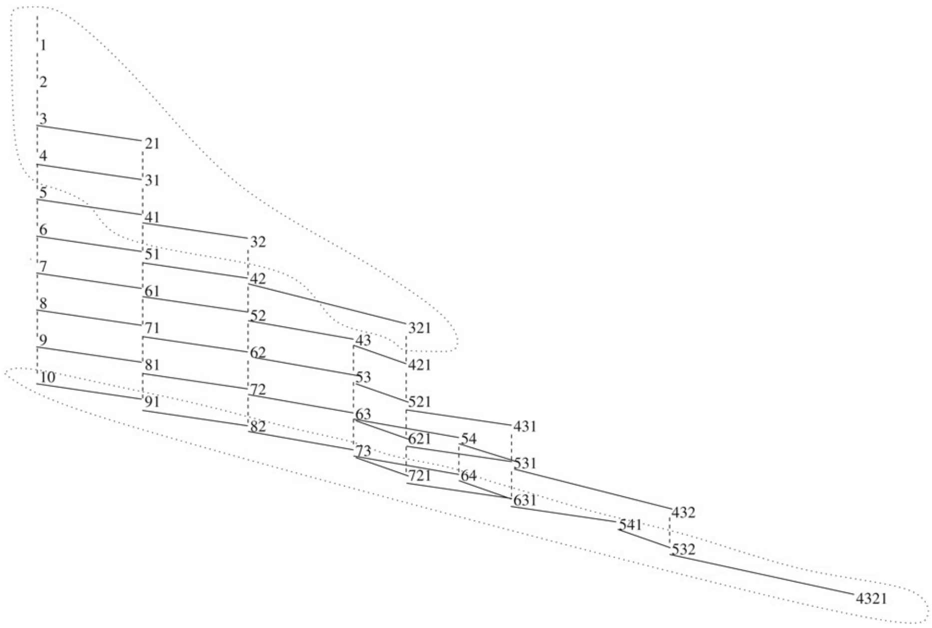

Fig. 3. First elements and transitions of $\mathcal{SP}(\infty)$. As shown on this figure for $n = 10$, we will see two ways of finding parts of $\mathcal{SP}(\infty)$ isomorphic to $\mathcal{SP}(n)$ for any n.

a of $\mathcal{SP}(\infty)$ has the form $(\infty, a_2, a_3, \dots, a_k)$. As in the previous section, we find that the dominance ordering on $\mathcal{SP}(\infty)$ (when the first part is ignored) is equivalent to the order induced by the dynamical model. The first partitions in $\mathcal{SP}(\infty)$ are given in Fig. 3 along with their covering relations (the first part, equal to ∞, is not represented on this diagram).

For any two elements $a = (\infty, a_2, \dots, a_k)$ and $b = (\infty, b_2, \dots, b_\ell)$ of $\mathcal{SP}(\infty)$, we define c by $c_i = \max(\sum_{j \geq i} a_j, \sum_{j \geq i} b_j) - \sum_{j > i} c_j$ for all i such that $2 \leq i \leq \max(k, \ell)$. One can check that c is an element of $\mathcal{SP}(\infty)$, *i.e.* $c_1 = \infty$ and $c_i > c_{i+1}$ for all $i > 1$, and then $c = a \wedge b$. This implies that:

Theorem 4.4. *The set $\mathcal{SP}(\infty)$ is a lattice.*

Now for any $n > 1$, there are two canonical embeddings of $\mathcal{SP}(n)$ in $\mathcal{SP}(\infty)$, defined by

$$\begin{array}{llcl} \pi : & \mathcal{SP}(n) & \longrightarrow & \mathcal{SP}(\infty) \\ & a = (a_1, a_2, \dots, a_k) & \mapsto & \pi(a) = (\infty, a_2, \dots, a_k) \\ \chi : & \mathcal{SP}(n) & \longrightarrow & \mathcal{SP}(\infty) \\ & a = (a_1, a_2, \dots, a_k) & \mapsto & \chi(a) = (\infty, a_1, a_2, \dots, a_k). \end{array}$$

The following result is straightforward:

Proposition 4.5. *Both π and χ are embeddings of lattices.*

By using the embedding χ, one can consider $\mathcal{SP}(\infty)$ as the disjoint union of $\mathcal{SP}(n)$ for all n, $\mathcal{SP}(\infty) = \bigsqcup_{n \geq 0} \mathcal{SP}(n)$.

References

[1] A. Bjorner, L. Lovász, Chip firing games on directed graphs, J. Algebraic Combin. 1 (1992) 305–328.
[2] A. Bjorner, L. Lovász, W. Shor, Chip-firing games on graphs, European J. Combin. 12 (1991) 283–291.
[3] T. Brylawski, The lattice of integer partitions, Discrete Math. 6 (1973) 201–219.
[4] E. Goles, M.A. Kiwi, Games on line graphs and sand piles, Theoret. Comput. Sci. 115 (1993) 321–349.
[5] M. Latapy, H.D. Phan, The lattice of integer partitions and its infinite extension (long version), (submitted for publication).
[6] E. Goles, M. Morvan, H.D. Phan, The structure of linear chip firing game and related models, Theoret. Comput. Sci. 270 (2002) 827–841.
[7] C. Greene, D.J. Kleiman, Longest chains in the lattice of integer partitions ordered by majorization, European J. Combin. 7 (1986) 1–10.

[8] E. Goles, M. Latapy, C. Magnien, M. Movan, H.D. Phan, Sandpile models and lattices: A comprehensive survey, Theoret. Comput. Sci. 322 (2004) 383–407.
[9] M. Latapy, H.D. Phan, The lattice structure of chip firing games, Physica D 115 (2001) 69–82.