Discrete Mathematics 309 (2009) 2934–2936

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Two graph isomorphism polytopes

Shmuel Onn

Technion - Israel Institute of Technology, 32000 Haifa, Israel

ARTICLE INFO

Article history:
Received 16 January 2008
Accepted 1 July 2008
Available online 29 July 2008

Keywords:
Graph isomorphism
Computational complexity
Linear programming
Polyhedral combinatorics
Integer programming
Combinatorial optimization

ABSTRACT

The convex hull $\psi_{n,n}$ of certain $(n!)^2$ tensors was considered recently in connection with graph isomorphism. We consider the convex hull ψ_n of the $n!$ diagonals among these tensors. We show: 1. The polytope ψ_n is a face of $\psi_{n,n}$. 2. Deciding if a graph G has a subgraph isomorphic to H reduces to optimization over ψ_n. 3. Optimization over ψ_n reduces to optimization over $\psi_{n,n}$. In particular, this implies that the subgraph isomorphism problem reduces to optimization over $\psi_{n,n}$.

1. Introduction

Let $\mathcal{P}_n$ be the set of $n \times n$ permutation matrices and consider the following two polytopes,

$$\psi_n := \text{conv}\{P \otimes P : P \in \mathcal{P}_n\}, \qquad \psi_{n,n} := \text{conv}\{P \otimes Q : P, Q \in \mathcal{P}_n\}.$$

The polytope $\psi_{n,n}$ was considered recently in [1] in connection with the graph isomorphism problem, of deciding if two given graphs are isomorphic, whose computational complexity is unknown to date. Note that ψ_n and $\psi_{n,n}$ have $n!$ and $(n!)^2$ vertices, respectively. In this short note we show:

1. The polytope ψ_n is a face of the polytope $\psi_{n,n}$.
2. Deciding if a graph G has a subgraph isomorphic to a graph H reduces to optimization over ψ_n.
3. Optimization over ψ_n reduces to optimization over $\psi_{n,n}$.

In particular, this implies a result of [1] that subgraph isomorphism reduces to optimization over $\psi_{n,n}$.

To make the computational complexity consequences of the last two statements above precise, note that each of the polytopes ψ_n and $\psi_{n,n}$ is uniquely determined by n, and so the input for either optimization problem $\max\{\langle W, X\rangle : X \in \psi_n\}$ or $\max\{\langle W, X\rangle : X \in \psi_{n,n}\}$ consists of n and an integer tensor W (see next section for the precise definition of the bilinear form). Recall that the subgraph isomorphism problem, of deciding if a given graph G has a subgraph isomorphic to a given graph H, which includes the graph isomorphism problem as a special case, is known to be complete for NP. Therefore, if $P \neq NP$ then optimization and separation over ψ_n and hence over $\psi_{n,n}$ cannot be done in polynomial time and a compact inequality description of ψ_n and hence of $\psi_{n,n}$ cannot be determined.

Deciding if G has a subgraph which is isomorphic to H can also be reduced to optimization over a related polytope ϕ_n defined as follows. Each permutation σ of the vertices of the complete graph K_n naturally induces a permutation Σ of its edges by $\Sigma(\{i,j\}) := \{\sigma(i), \sigma(j)\}$. Then ϕ_n is defined as the convex hull of all $\binom{n}{2} \times \binom{n}{2}$ permutation matrices of induced permutations Σ. This polytope and a broader class of so-called *Young polytopes* have been studied in [2]. In particular, therein it was shown that the graph of ϕ_n is complete, so pivoting algorithms cannot be exploited for optimization over this polytope. It is an interesting question whether ψ_n and ϕ_n, having $n!$ vertices each, are isomorphic.

E-mail address: onn@ie.technion.ac.il.
URL: http://ie.technion.ac.il/~onn.

0012-365X/$ – see front matter
doi:10.1016/j.disc.2008.07.001

2. Statements

Define bilinear forms on $\mathbb{R}^{n\times n}$ and on $\mathbb{R}^{n\times n}\otimes\mathbb{R}^{n\times n}$ (note the shuffled indexation on the right) by

$$\langle A,B\rangle := \sum_{i,j} A_{i,j}B_{i,j}, \qquad \langle X,Y\rangle := \sum_{i,j,s,t} X_{i,s,j,t}Y_{i,j,s,t}.$$

Let I be the $n\times n$ identity matrix and for a graph G let A_G be its adjacency matrix. We show:

Theorem 2.1. *The polytope ψ_n is a face of $\psi_{n,n}$ given by*

$$\psi_n = \psi_{n,n}\cap\{X : \langle I\otimes I, X\rangle = n\}.$$

Theorem 2.2. *Let G and H be two graphs on n vertices with m the number of edges of H. Then*

$$\max\{\langle A_G\otimes A_H, X\rangle : X\in\psi_n\}\le 2m$$

with equality if and only if G has a subgraph which is isomorphic to H.

Theorem 2.3. *Let $W=(W_{i,s,j,t})$ be any tensor and let $w := 2n^2\max|W_{i,s,j,t}|$. Then*

$$\max\{\langle W,X\rangle : X\in\psi_n\} = \max\{\langle W+wI\otimes I, X\rangle : X\in\psi_{n,n}\} - nw.$$

Combining Theorems 2.2 and 2.3 with $W = A_G\otimes A_H$ and $w = n^2$ (sufficing since $W\ge 0$, as is clear from the proof of Theorem 2.3 below), we get the following somewhat tighter form of a result of [1].

Corollary 2.4. *Let G and H be two graphs on n vertices with m the number of edges of H. Then*

$$\max\{\langle A_G\otimes A_H + nI\otimes nI, X\rangle : X\in\psi_{n,n}\}\le 2m + n^3$$

with equality if and only if G has a subgraph which is isomorphic to H.

3. Proofs

We record the following statement that follows directly from the definitions of the bilinear forms above.

Proposition 3.1. *For any two simple tensors $X = A\otimes B$ and $Y = P\otimes Q$ we have*

$$\langle X,Y\rangle = \langle A\otimes B, P\otimes Q\rangle = \sum_{i,j,s,t} A_{i,s}B_{j,t}P_{i,j}Q_{s,t} = \langle PBQ^{\intercal}, A\rangle.$$

Proof of Theorem 2.1. For every $P,Q\in\mathcal{P}_n$, the matrix $PIQ^{\intercal}$ is a permutation matrix, with $PIQ^{\intercal}=I$ if and only if $P=Q$. It follows that for every two distinct $P,Q\in\mathcal{P}_n$ we have

$$\langle I\otimes I, P\otimes Q\rangle = \langle PIQ^{\intercal}, I\rangle \le n-1 < n = \langle PIP^{\intercal}, I\rangle = \langle I\otimes I, P\otimes P\rangle. \quad\blacksquare \tag{1}$$

Proof of Theorem 2.2. For any $P\in\mathcal{P}_n$, the matrix $PA_HP^{\intercal}$ is the adjacency matrix of the permutation of H by P. So $\langle PA_HP^{\intercal}, A_G\rangle\le 2m$ with equality if and only if H is isomorphic via P to a subgraph of G. Since the maximum of a linear form over a polytope is attained at a vertex we get

$$\begin{aligned}\max\{\langle A_G\otimes A_H, X\rangle : X\in\psi_n\} &= \max\{\langle A_G\otimes A_H, P\otimes P\rangle : P\in\mathcal{P}_n\}\\ &= \max\{\langle PA_HP^{\intercal}, A_G\rangle : P\in\mathcal{P}_n\}\le 2m\end{aligned}$$

with the last inequality holding with equality if and only if G has a subgraph isomorphic to H. ■

Proof of Theorem 2.3. For every $P,Q\in\mathcal{P}_n$, the tensor $P\otimes Q = (P_{i,j}Q_{s,t})$ has n^2 entries which are equal to 1 and all other entries equal to 0, and therefore $-\frac{1}{2}w\le\langle W, P\otimes Q\rangle\le\frac{1}{2}w$. Combining this with inequality (1) we see that for every two distinct $P,Q\in\mathcal{P}_n$ we have

$$\begin{aligned}\langle W+wI\otimes I, P\otimes Q\rangle &= \langle W, P\otimes Q\rangle + w\langle I\otimes I, P\otimes Q\rangle\\ &\le \frac{1}{2}w + (n-1)w = -\frac{1}{2}w + nw\\ &\le \langle W, P\otimes P\rangle + w\langle I\otimes I, P\otimes P\rangle = \langle W+wI\otimes I, P\otimes P\rangle.\end{aligned}$$

Since the maximum of a linear form over a polytope is attained at a vertex we obtain the equality

$$\begin{aligned}\max\{\langle W + wI \otimes I, X\rangle : X \in \psi_{n,n}\} &= \max\{\langle W + wI \otimes I, P \otimes Q\rangle : P, Q \in \mathcal{P}_n\} \\ &= \max\{\langle W + wI \otimes I, P \otimes P\rangle : P \in \mathcal{P}_n\} \\ &= \max\{\langle W, P \otimes P\rangle : P \in \mathcal{P}_n\} + nw \\ &= \max\{\langle W, X\rangle : X \in \psi_n\} + nw. \quad \blacksquare\end{aligned}$$

References

[1] S. Friedland, On the graph isomorphism problem. e-print: arXiv:0801.0398.
[2] S. Onn, Geometry, complexity and combinatorics of permutation polytopes, Journal of Combinatorial Theory Series A 64 (1993) 31–49.