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Hyperanalytic Riemann boundary value problem on d-summable closed curves

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ABSTRACT

We are interested in finding solvability conditions for the Riemann boundary value problems for hyperanalytic functions in a simply connected bounded open subset of the complex plane whose boundary is merely required to be a d-summable closed curve.

1. Introduction

The classical Riemann boundary value problem with Hölder continuous coefficients was discussed in different classes of domains. Many well-known results on its solvability, mainly concern the case of a piece-smooth boundaries, was given by Gakhov in [5] and by Lu in [12].

Further extensions of the problem are treated and have led to numerous important results not only for non-smoothly bounded domain, which differs with the former, but for general assumptions on the data of the problem, such as generalized Hölder coefficients or special subspaces of it. Moreover, the Riemann boundary value problem was studied for generalized analytic functions, as well as for many other linear and nonlinear elliptic systems in the plane. The best general references here are [2,6].

During the last decades, some results about the analytic Riemann boundary value problems on non-smooth or non-rectifiable curves have arisen (see [10,11]).

The hyperanalytic Riemann boundary value problems on rectifiable curves were studied in [1,7,14] and in [15] a direct generalization for non-rectifiable framework is given.

Our purpose is to get a sufficiently complete picture of solvability of the hyperanalytic Riemann boundary value problem for a great generality dealing directly with the d-summability of the boundary as essential hypothesis for integration. In the process of this study we find that basic results obtained in the aforementioned references are extended or improved to a more suitable approach.

Throughout the paper we assume Ω to be a simply connected bounded open subset of $\mathbb{C}$ and γ is the boundary curve of Ω. When necessary we shall use the temporary notation $\Omega_+ := \Omega$, $\Omega_- := \mathbb{C} \setminus \overline{\Omega}$.

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The hyperanalytic Riemann boundary value problem considered here is that of finding all functions $\Phi(z)$ hyperanalytic in $\overline{\mathbb{C}} \setminus \gamma$ satisfying the boundary condition

$$\Phi^+(t) = F(t)\Phi^-(t) + f(t), \quad t \in \gamma, \tag{1}$$

where F and f are two given continuous functions defined on γ, and $\Phi^\pm(t)$ are the limit values of the desired function $\Phi(z)$ at a point t as this point is approached from $\Omega_\pm$ respectively.

A simplest particular case of (1) is the so-called jump problem:

$$\Phi^+(t) - \Phi^-(t) = f(t), \quad t \in \gamma. \tag{2}$$

When investigating the above-stated problem (1), it is required that the unknown functions are continuous up to the boundary, for what is usually called the continuous hyperanalytic Riemann boundary value problem.

2. Preliminaries

2.1. Douglis algebras and hyperanalytic functions

A Douglis algebra is a class of functions which is generated by the elements i and e, it is a generalization of the complex-valued function in the complex plane $\mathbb{C}$. For details about this topic we refer the reader to [2,6,14]. The Douglis analysis is then the study of the Douglis algebra-valued functions.

Let $\mathbb{D}$ be the Douglis algebra, the multiplication in $\mathbb{D}$ is governed by the rules:

$$i^2 = -1, \qquad ie = ei, \qquad e^r = 0, \qquad e^0 = 1,$$

where r is a positive integer.

Any arbitrary element $a \in \mathbb{D}$ may be written as a hypercomplex number in the form

$$a = \sum_{k=0}^{r-1} a_k e^k,$$

where each a_k is a complex number, a_0 is the complex part of a, meanwhile $A = \sum_{k=1}^{r-1} a_k e^k$ its nilpotent part.

Let a be any hypercomplex number then its conjugation $\bar{a}$ is defined as $\bar{a} = \sum_{k=0}^{r-1} \overline{a_k} e^k$.

The algebraic norm in $\mathbb{D}$ is defined by $|a| := \sum_{k=0}^{r-1} |a_k|$.

If the complex part a_0 of a hypercomplex a is not null then the multiplicative inverse a^{-1} of a is given by

$$a^{-1} = a_0^{-1} \sum_{k=0}^{r-1} (-1)^k \left(\frac{A}{a_0} \right)^k.$$

If $a_0 = 0$ then a is called nilpotent and it does not have multiplicative inverse.

Let f be a $\mathbb{D}$-valued function then f may be written as $f = \sum_{k=0}^{r-1} f_k e^k$, where f_k are complex-valued functions.

The Douglis operator $\partial_{\bar{z}}^q$ is given by

$$\partial_{\bar{z}}^q := \partial_{\bar{z}} + q(z)\partial_z, \quad z = x + iy,$$

where $q(z)$ is a known nilpotent hypercomplex function and

$$\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y), \qquad \partial_z := \frac{1}{2}(\partial_x - i\partial_y).$$

Suppose $\Omega \subset \mathbb{C}$ to be a domain, a smooth hypercomplex function f defined in Ω is said to be hyperanalytic in Ω if $\partial_{\bar{z}}^q f = 0$ in Ω.

As an example for hyperanalytic function we take the generating solution of the Douglis operator given by

$$W(z) = z + \sum_{k=1}^{r-1} W_k(z) e^k,$$

where its nilpotent part posses bounded and continuous derivate up to order two in $\mathbb{C}$.

The following properties concerning with the generating solution will be used frequently:

$$\left| W(z_1) - W(z_2) \right| \leqslant c|z_1 - z_2|, \qquad \frac{1}{|W(z_1) - W(z_2)|} \leqslant c|z_1 - z_2|^{-1}, \quad z_1 \neq z_2.$$

Notation c will be used for constants which may vary from one occurrence to the next, in general these constants only depend on q.

Other important example for hyperanalytic function is the so-called hypercomplex Cauchy kernel, i.e., the fundamental solution of the Douglis operator, given by

$$e_z(\zeta) := \frac{1}{2\pi}\frac{\partial_\zeta W(\zeta)}{W(\zeta) - W(z)}, \quad \zeta \neq z.$$

The nature of the singularity of $e_z(\zeta)$ is the same as that which the complex Cauchy kernel $\frac{1}{\zeta - z}$ has at $\zeta = z$.

We will denote by $\mathcal{C}(\gamma)$ the set of all continuous hypercomplex functions defined on γ. Moreover, for $f \in \mathcal{C}(\gamma)$ we introduce the modulus of continuity for f being defined by

$$\omega_f(\tau) := \tau \sup_{\rho \geqslant \tau} \rho^{-1} \max_{t_1, t_2 \in \gamma,\, |t_1 - t_2| \leqslant \rho} \big| f(t_1) - f(t_2) \big|, \quad \tau > 0.$$

Let us consider also the subclass $\mathcal{H}_\nu(\gamma) \subset \mathcal{C}(\gamma)$ of all functions f satisfying a Hölder condition $\omega_f(\delta) \leqslant c\delta^\nu$, $\delta \in (0, |\gamma|]$, with exponent ν, $0 < \nu \leqslant 1$.

Here and subsequently, $|\mathbf{E}|$ denotes the diameter of $\mathbf{E} \subset \mathbb{C}$.

It is worth pointing out that a very successful tool in the theory of Riemann boundary value problems both for analytic and hyperanalytic functions are the corresponding Cauchy type integral. Hence, it is not surprising the necessity to discuss this concept in full generality concerning the geometric properties of the contour of integration.

If γ is a Jordan closed rectifiable curve, then for any $f \in C(\gamma)$, the customary hypercomplex Cauchy type integral

$$(\mathbf{C}_\gamma f)(z) := \int_\gamma e_z(\zeta) n_q(\zeta) f(\zeta)\, ds, \quad z \notin \gamma, \tag{3}$$

where $n_q(\zeta) := n(\zeta) + \bar{n}(\zeta)q(\zeta)$ being $n(\zeta)$ the exterior unit normal vector at the point ζ on γ in Federer's sense (see [4]), and ds denotes the arclength differential, exists and represents a hyperanalytic function in $\overline{\mathbb{C}} \setminus \gamma$.

At almost all (with respect to ds) points $t \in \gamma$ this function has non-tangential boundary limit values from both sides, and these values almost everywhere satisfy the relation

$$\big(\mathbf{C}_\gamma^+ f\big)(t) - \big(\mathbf{C}_\gamma^- f\big)(t) = f(t), \quad t \in \gamma. \tag{4}$$

If $f \in \mathcal{H}_\nu(\gamma)$ and $\nu > \frac{1}{2}$, then the function $\mathbf{C}_\gamma f$ has continuous boundary values on the whole γ (see [15]). If the curve is Ahlfors David regular, then these properties are valid for any $\nu \in (0, 1]$ (see [1]).

When assuming a much more pathological situation, e.g., γ is assumed to be a fractal then the definition (3) of the Cauchy type integral falls, but the hyperanalytic Riemann boundary value problem is still suitable and the influence of the geometry of the boundary on the solvability of the problem is necessarily revelled. In Section 3 we will look more closely at this phenomenon.

We end this section by introducing some important facts of fractal geometry.

2.2. Box dimension and d-summable sets in $\mathbb{C}$

The standard approach consider a fractal to be a set with a non-integer Hausdorff dimension. However, frequently the box dimension is more appropriated dimension than those of Hausdorff to measure the roughness of a bounded set.

By definition the box dimension of a bounded set $\mathbf{E} \subset \mathbb{C}$ is equal to

$$\alpha(\mathbf{E}) := \overline{\lim_{\epsilon \to 0}} \frac{\log N_{\mathbf{E}}(\epsilon)}{-\log \epsilon}, \tag{5}$$

where $N_{\mathbf{E}}(\epsilon)$ stands for the least number of ϵ-balls needed to cover $\mathbf{E}$.

The limit in (5) is unchanged if $N_{\mathbf{E}}(\epsilon)$ is taking as the number of squares needed to cover $\mathbf{E}$ with $2^{-k} \leqslant \epsilon < 2^{-k+1}$ intersecting $\mathbf{E}$.

The set $\mathbf{E}$ is said to be d-summable if the improper integral

$$\int_0^1 N_{\mathbf{E}}(x) x^{d-1}\, dx$$

converges. This geometric notion was introduced by Harrison and Norton in [8].

It is easy to check that any d-summable set $\mathbf{E}$ has box dimension $\alpha(\mathbf{E}) \leqslant d$. Moreover, the assumption $\alpha(\mathbf{E}) < d$ implies the d-summability of $\mathbf{E}$.

To deal with appropriated extension for hypercomplex functions f defined on γ to the whole complex plane $\mathbb{C}$ we will consider the Whitney extension operator denoted by $\mathcal{E}_0$, see [13,15]. Indeed, if $f \in H_\nu(\gamma)$, then its Whitney extension $\mathcal{E}_0(f)$

belongs to $H_\nu(\mathbb{C})$ and has partial derivatives of all orders at any point $z \in \mathbb{C} \setminus \gamma$. Moreover, there exists a constant $c > 0$ such that

$$\left|\partial_{\bar z}^q \mathcal{E}_0(f)(z)\right| \leqslant c\big(\operatorname{dist}(z,\gamma)\big)^{\nu-1}, \quad z \in \mathbb{C} \setminus \gamma. \tag{6}$$

In particular, for $\nu = 1$ the function $\partial_{\bar z}^q \mathcal{E}_0(f)$ is bounded.

If $\mathcal{X}(z)$ denotes the characteristic function of the set $\overline{\Omega}$, we shall write $f^\omega(z) = \mathcal{X}(z)\mathcal{E}_0(f)(z)$.

3. The Cauchy type integral on *d*-summable curves

This section is aimed to present an alternative definition of the hypercomplex Cauchy type integral when the contour is allowed to be a fractal.

Definition 1. Let $d \in (1,2]$ and let Ω be a domain with d-summable boundary γ and suppose $\nu > d-1$. We define the hypercomplex Cauchy type integral of $f \in \mathcal{H}_\nu(\gamma)$ by the formula

$$\big(\mathbf{C}_\gamma^* f\big)(z) = f^\omega(z) - \int_\Omega e_z(\zeta)\partial_{\bar\zeta}^q \mathcal{E}_0(f)(\zeta)\,d\xi\,d\eta, \quad z \in \mathbb{C} \setminus \gamma, \tag{7}$$

with $\zeta = \xi + i\eta$.

The following proposition makes this definition legitimate.

Proposition 1. *The hypercomplex function* (7) *is correctly defined for any* $z \in \mathbb{C} \setminus \gamma$ *and its value does not depend on the particular choice of* $\mathcal{E}_0(f)$.

Proof. It is enough to prove that

$$\int_\Omega \left|\partial_{\bar\zeta}^q \mathcal{E}_0(f)(\zeta)\right| d\xi\,d\eta < \infty.$$

We follow [13] in considering the Whitney decomposition of Ω, $\mathcal{W} = \bigcup_k \mathcal{W}^k$, which consists of disjoint squares Q satisfying

$$|Q| \leqslant \operatorname{dist}(Q,\gamma) \leqslant 4|Q|.$$

Then we have

$$\begin{aligned}\int_\Omega \left|\partial_{\bar\zeta}^q \mathcal{E}_0(f)(\zeta)\right| d\xi\,d\eta &= \sum_{Q\in\mathcal{W}} \int_Q \left|\partial_{\bar\zeta}^q \mathcal{E}_0(f)(\zeta)\right| d\xi\,d\eta \\ &\leqslant c \sum_{Q\in\mathcal{W}} \int_Q \big(\operatorname{dist}(\zeta,\gamma)\big)^{\nu-1} d\xi\,d\eta,\end{aligned}$$

being the last inequality a consequence of (6).

Consequently

$$\int_\Omega \left|\partial_{\bar\zeta}^q \mathcal{E}_0(f)(\zeta)\right| d\xi\,d\eta \leqslant c \sum_{Q\in\mathcal{W}} |Q|^{\nu+1}.$$

At this stage we exploit Lemma 2 in [8] to obtain the finiteness of the last sum by using the d-sumability of γ together with the fact that $\nu + 1 > d$.

Now suppose that $\mathcal{E}_0^1(f)$ and $\mathcal{E}_0^2(f)$ are two different Whitney extensions of f. Then $\mathcal{E}_0(g) := \mathcal{E}_0^1(f) - \mathcal{E}_0^2(f)$, is also an extension of Whitney type, but of the null function and hence $\mathcal{E}_0(g)|_\gamma = 0$.

If we prove that the hypercomplex function

$$g^\omega(z) - \int_\Omega e_z(\zeta)\partial_{\bar\zeta}^q \mathcal{E}_0(g)(\zeta)\,d\xi\,d\eta \tag{8}$$

vanishes in $\mathbb{C} \setminus \gamma$, the assertion follows.

Define

$$\Omega_k = \big\{\underline{x} \in Q:\ Q \in \mathcal{W}^j \text{ for some } j \leqslant k\big\}$$

and $\Delta_k = \Omega \setminus \Omega_k$. The boundary of Ω_k, denoted by γ_k, is actually composed by certain sides of some squares $Q \in \mathcal{W}^k$.

We have

$$\int_{\Omega} e_z(\zeta)\partial_{\bar\zeta}^q \mathcal{E}_0(g)(\zeta)\,d\xi\,d\eta = \lim_{k\to\infty}\left(\int_{\Omega_k} + \int_{\Delta_k}\right) e_z(\zeta)\partial_{\bar\zeta}^q \mathcal{E}_0(g)(\zeta)\,d\xi\,d\eta. \tag{9}$$

Let $z \in \Omega$ and let k_0 be so large chosen that $z \in \Omega_{k_0}$ and $\operatorname{dist}(z, \gamma_k) \geqslant |Q_0|$ for $k > k_0$, where Q_0 is a square of $\mathcal{W}^{k_0}$.

By Borel–Pompeiu formula we deduce

$$g^{\omega}(z) - \int_{\Omega_k} e_z(\zeta)\partial_{\bar\zeta}^q \mathcal{E}_0(g)(\zeta)\,d\xi\,d\eta = \int_{\gamma_k} e_z(\zeta) n_q(\zeta) g^{\omega}(\zeta)\,ds, \quad z \in \Omega_k. \tag{10}$$

Next, let $\zeta \in \gamma_k$, $Q \in \mathcal{W}^k$ a square containing ζ, and $\zeta_* \in \gamma$ such that $|\zeta - \zeta_*| = \operatorname{dist}(\zeta, \gamma)$.

Since $\mathcal{E}_0(g)|_{\gamma} = 0$, we have

$$\left|\mathcal{E}_0(g)(\zeta)\right| = \left|\mathcal{E}_0(g)(\zeta) - \mathcal{E}_0(g)(\zeta_*)\right| \leqslant c|\zeta - \zeta_*|^{\nu} \leqslant c|Q|^{\nu}.$$

If Σ denotes a side of γ_k and $Q \in \mathcal{W}^k$ is the k-square containing Σ, we have for $k > k_0$

$$\left|\int_{\Sigma} e_z(\zeta) n_q(\zeta) g^{\omega}(\zeta)\,ds\right| \leqslant \frac{c}{|Q_0|}\int_{\Sigma}\left|g^{\omega}(\zeta)\right| ds \leqslant \frac{c}{|Q_0|}|Q|^{\nu+1}.$$

As it was noticed before, each side of γ_k is one of those 4 of some $Q \in \mathcal{W}^k$. Therefore, for $k > k_0$

$$\left|\int_{\gamma_k} e_z(\zeta) n_q(\zeta) g^{\omega}(\zeta)\,ds\right| \leqslant \frac{c}{|Q_0|}\sum_{Q\in\mathcal{W}^k}|Q|^{\nu+1}.$$

The finiteness of $\sum_{Q\in\mathcal{W}}|Q|^{\nu+1}$ implies

$$\lim_{k\to\infty}\int_{\gamma_k} e_z(\zeta) n_q(\zeta) g^{\omega}(\zeta)\,ds = 0.$$

Combining (9) with (10) yields that (8) vanishes for $z \in \Omega$.

The same conclusion can be drawn for $z \in \mathbb{C} \setminus \overline{\Omega}$. The only point now is to note that $\operatorname{dist}(z, \gamma_k) \geqslant \operatorname{dist}(z, \gamma)$ for $z \in \mathbb{C} \setminus \overline{\Omega}$. □

Remark 1. After some necessary modifications due to the presence of the hypercomplex Cauchy kernel the proof of Proposition 1 could be given more directly, since the previously introduced Cauchy transform $\mathbf{C}_{\gamma}^{*} f$ could be rewritten by using the well-defined contour integration according to [9]. However, this topic exceeds the scope of this paper.

We will conclude the section with two theorems concerning the question on the resolvability of the problem on reconstruction of a hyperanalytic function in $\overline{\mathbb{C}} \setminus \gamma$ by its jump (i.e., by the boundary condition (2)) on a d-summable curve γ, to be used mainly in Section 4.

Theorem 1. *Let γ be a d-summable curve and $f \in \mathcal{H}_{\nu}(\gamma)$. If $\nu > \frac{d}{2}$ then the Cauchy type integral* (7) *has continuous limits values on γ from both domains $\Omega_{\pm}$.*

Proof. The function $\partial_{\bar z}^q \mathcal{E}_0(f)$ is integrable in $\mathbb{C}$ with any degree not exceeding $\frac{2-d}{1-\nu}$, i.e., under the assumption $\nu > \frac{d}{2}$ then $\frac{2-d}{1-\nu} > 2$ and $\partial_{\bar z}^q \mathcal{E}_0(f)$ is integrable with some exponent $p > 2$. From this it follows that the integral term in (7) represents a continuous hypercomplex function in $\mathbb{C}$ satisfying there a Hölder condition with exponent $1 - \frac{2}{p}$ (see Theorem 1.25 in [6]).

Consequently, $(\mathbf{C}_{\gamma}^{*} f)(z)$ represents a hyperanalytic function in $\overline{\mathbb{C}} \setminus \gamma$ whose restrictions $(\mathbf{C}_{\gamma}^{*} f)|_{\Omega_+}$ and $(\mathbf{C}_{\gamma}^{*} f)|_{\Omega_-}$ are continuous in $\overline{\Omega}_+$ and $\overline{\Omega}_-$, respectively. The boundary values of these restrictions $(\mathbf{C}_{\gamma}^{*\pm} f)$, thought of the usual continuous limit values, are given by

$$\left(\mathbf{C}_{\gamma}^{*+} f\right)(t) = f(t) - \int_{\Omega} e_t(\zeta)\partial_{\bar\zeta}^q \mathcal{E}_0(f)(\zeta)\,d\xi\,d\eta, \quad t \in \gamma,$$

$$\left(\mathbf{C}_{\gamma}^{*-} f\right)(t) = -\int_{\Omega} e_t(\zeta)\partial_{\bar\zeta}^q \mathcal{E}_0(f)(\zeta)\,d\xi\,d\eta, \quad t \in \gamma.$$

Thus, the proof is complete. □

Note that the proof above gives more, namely $(\mathbf{C}^*_\gamma f)(z)$ is a solution of the jump problem (2)

$$\big(\mathbf{C}^{*\,+}_\gamma f\big)(t) - \big(\mathbf{C}^{*\,-}_\gamma f\big)(t) = f(t), \quad t \in \gamma.$$

To ensure uniqueness of the solution of (2) we need to introduce some additional requirements.

The function Φ, hyperanalytic in $\mathbb{C} \setminus \gamma$ must satisfy a Hölder condition with exponent μ, $0 < \mu < 1$ on each of the sets $\overline{\Omega_\pm}$, i.e. the restrictions $\Phi|_{\Omega_+}$ and $\Phi|_{\Omega_-}$ must be μ-Hölder continuous in the closed domains respectively, and the boundary vales of these restrictions $\Phi^\pm$ are the usual continuous limit values.

It is essential to point out that, $\int_\Omega e_t(\zeta)\partial^q_{\bar z}\mathcal{E}_0(f)(\zeta)\,d\xi\,d\eta \in \mathcal{H}_\mu(\mathbb{C})$ with

$$\mu < \frac{2\nu - d}{2 - d}. \tag{11}$$

At the same time the desired uniqueness of (2) follows from the removability of the curve γ under the condition that

$$\mu > \alpha_H(\gamma) - 1, \tag{12}$$

where $\alpha_H(\gamma)$ denotes the Hausdorff dimension of γ.

To be specific, let us to state the analogue of Dolzhenko's theorem (see [3]) for hyperanalytic functions, which may be proved in much the same way as the former: If $U \supset \gamma$ is a domain in $\mathbb{C}$ and a function $\Psi \in \mathcal{H}_\mu(U)$, $\mu > \alpha_H(\gamma) - 1$, is hyperanalytic in $U \setminus \gamma$, then it is hyperanalytic in U.

A function Φ, being μ-Hölder continuous on $\overline{\Omega_\pm}$ whenever μ satisfies the conditions (11) and (12), is said to be of class $\mathcal{H}_\mu$.

Summarizing, we have

Theorem 2. *Under the hypotheses of Theorem* 1, *if moreover*

$$\alpha_H(\gamma) - 1 < \mu < \frac{2\nu - d}{2 - d},$$

then there exists a unique solution of the jump problem (2) *in the class* $\mathcal{H}_\mu$.

4. Solvability of the hyperanalytic Riemann boundary value problem

It is our purpose in this section to develop a theory of the well-posed continuous Riemann boundary value problem for hyperanalytic functions by assuming that the given continuous coefficients defined on γ have to agree with the desired boundary behaviour of the solutions, i.e., the solutions including their boundary values on γ are continuous functions too.

The curve γ is assumed to be d-summable and let us consider the problem (1), requiring $F, f \in \mathcal{H}_\nu(\gamma)$. Without loss of generality we can assume that $0 \in \Omega_+$. Moreover, F will be regarded as a hypercomplex function with complex part F_0 never vanishing on γ. Then the integer

$$\kappa := \frac{1}{2\pi}\big[\arg F_0(\zeta)\big]_\gamma$$

has significant importance, and is called the index of F with respect to γ, also called index of the Riemann problem. Note that the index of the function W^κ with respect to γ is κ, and hence the index of $W^{-\kappa}F$ is zero.

We may verify directly that the function

$$X(z) := \begin{cases} X^+(z) = \exp(\mathbf{C}^*_\gamma \ln[W^{-\kappa}F])(z), & z \in \Omega_+, \\ X^-(z) = W(z)^{-\kappa}\exp(\mathbf{C}^*_\gamma \ln[W^{-\kappa}F])(z), & z \in \Omega_- \end{cases}$$

is a hyperanalytic function in $\mathbb{C} \setminus \gamma$ which satisfies (1) for $f \equiv 0$, if $1 > \nu > \frac{d}{2}$ or $\nu = 1$. Hence

$$X^+(t) = F(t)X^-(t), \quad t \in \gamma.$$

More details about the hypercomplex exponential and logarithmic functions can be found for instance in [6].

By a hypercomplex polynomial we mean $a_0 + a_1 W(z) + \cdots + a_s W^s(z)$, $s \geqslant 0$.

It follows that $X \in \mathcal{H}_\mu$, being assumed $\nu > \frac{d}{2}$ and $\mu < \frac{2\nu - d}{2-d}$. If Φ is an arbitrary solution of (1) for $f \equiv 0$ in the class $\mathcal{H}_\mu$ then so $\Psi = \frac{\Phi}{X} \in \mathcal{H}_\mu$.

For $\mu > \alpha_H(\gamma) - 1$ the hyperanalicity of Ψ in $\mathbb{C}$ follows from the hypercomplex version of the Dolzhenko's theorem mentioned above. From the hypercomplex version of the general Liouville theorem Ψ is seen to be a hypercomplex polynomial of degree at most κ, if $\kappa \geqslant 0$ and $\Psi \equiv 0$ for $\kappa < 0$.

Now we consider the general case, i.e., $f \neq 0$. Standard transformations bring the problem (1) to the form

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{f(t)}{X^+(t)}, \quad t \in \gamma.$$

Although this can be thought of as a jump problem we are not in a position to use the approach showed before for the solvability of such problems. Note that the Hölder index of the function $X^+(t)$ and thus, that of $\frac{f(t)}{X^+(t)}$ is less that ν, and, in general, $\partial_{\bar{z}}^q(\frac{f}{X^+})$ is not integrable in a power greater than two.

The function

$$\Psi(z) = f^\omega(z) - X(z) \int\limits_{\Omega_+} e_z(\zeta)\big(\partial_{\bar{\zeta}}^q f^\omega\big)(\zeta) X^{-1}(\zeta)\, d\xi\, d\eta$$

can easily be shown to be a solution of the problem (1) if $\kappa \geqslant -1$ and $\nu > \frac{d}{2}$ or $\nu = 1$.

Rewriting Ψ in the form $\Psi = X(z)\Psi_0$, where

$$\Psi_0(z) = \psi(z) - \int\limits_{\Omega_+} e_z(\zeta)\partial_{\bar{\zeta}}^q \psi(\zeta)\, d\xi\, d\eta,$$

$\psi(z) = f^\omega(z)X^{-1}(z)$ we check at one that $\Psi \in \mathcal{H}_\mu$ satisfies (1) and is hyperanalytic in $\mathbb{C} \setminus \gamma$, besides for $\kappa \geqslant -1$ also in $\overline{\mathbb{C}} \setminus \gamma$. Consequently, we can deduce the general form of a solution for $\kappa \geqslant -1$.

In order that Ψ for $\kappa < -1$ behaves hyperanalytic at infinity it is necessary and sufficient that it has a zero of order κ, i.e., the first $-\kappa - 1$ coefficients of the series representation near infinity $\Psi_0(z) = \sum_{k=1}^{\infty} c_k W^{-k}(z)$ vanish.

Regarding a radius R sufficiently large

$$c_k = \frac{1}{2\pi i} \int\limits_{|\zeta|=R} \Psi_0(\zeta) W^{k-1}(\zeta)\, dW(\zeta),$$

thus the solvability conditions are seen to be

$$\int\limits_{\Omega_+} \partial_\zeta W(\zeta) \frac{(\partial_{\bar{\zeta}}^q f^\omega)(\zeta)}{X^+(\zeta)} W^{k-1}(\zeta)\, d\xi\, d\eta = 0, \quad k = 1, \ldots, -\kappa - 1. \tag{13}$$

In case when solutions vanishing at infinity are looked for moreover c_{-k} has to vanish.

We can now formulate the results proved.

Theorem 3. *Suppose that $\nu > \frac{d}{2}$ and $\alpha_H(\gamma) - 1 < \mu < \frac{2\nu-d}{2-d}$ or $\nu = 1$ and $\alpha_H(\gamma) - 1 < \mu < 1$.*

1. *Let $f \equiv 0$. If $\kappa \geqslant 0$ then the general solution of the problem* (1) *is given by*

   $$\Phi(z) = X(z)P_\kappa(z),$$

   where P_κ is an arbitrary hypercomplex polynomial of degree κ. However, for $\kappa < 0$ this problem has no solutions except 0 *in the class $\mathcal{H}_\mu$.*
2. *Let $f \neq 0$. For $\kappa \geqslant 0$ the general solution of the problem* (1) *in the class $\mathcal{H}_\mu$ has the form $\Phi(z) = \Psi(z) + X(z)P_\kappa(z)$, where P_κ is an arbitrary hypercomplex polynomial of degree κ. When $\kappa = -1$ the function Ψ is the unique solution of* (1) *in this class. Under the above assumptions and if $\kappa < -1$ the problem* (1) *is solvable if and only if the conditions* (13) *are satisfied.*

Note that the index κ counts the number of linear independent solutions to the problem (1)*, $f \equiv 0$.*

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