On Compact Perturbations of Certain Nonlinear Equations in Banach Spaces

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INTRODUCTION

Let $X$ be a reflexive Banach space, $A$ a linear maximal monotone mapping with domain $D(A)$ in $X$ to $X^*$, and $N$ a nonlinear mapping from $X$ to $X^*$. The purpose of this paper is to establish the existence of solutions of a nonlinear equation of the form $Au + Nu = w$ for a given $w$ in $X^*$ and equations obtained by compact perturbations of the same, namely, equations of the form $Au + Nu + C_tu = w$, where $C_t$, $t \in [0, 1]$, is a continuous family of compact mappings from $X$ into $X^*$. Our result on the existence of solutions of equation $Au + Nu = w$ is similar to a result of Brézis [3] and Browder-Hess [5]. It contains a result of Kenmochi [12] as a special case. The result on the perturbed equation $Au + Nu + C_tu = w$ is new, although the proof is similar to an earlier result of the author on the perturbed equation $u + ANu + C_tu = v$ (see [11, Theorem 6]). We may, however, mention that our result on the equation $Au + Nu + C_tu = w$ cannot be obtained as a corollary of corresponding results for $u + ANu + C_tu = v$ since the mapping $A$ in general is unbounded. Finally, the results of this paper can be extended easily to the case of multivalued mappings $N$, but we stick to the single-valued case for simplicity sake.

MAIN RESULTS

Let $X$ be a reflexive Banach space and $X^*$ denote the dual Banach space of $X$. Let $(w, x)$ denote the duality pairing between the elements $w$ in $X^*$ and $x$ in $X$. A (multivalued) mapping $T$ from $X$ into $2^{X^*}$ is said to be monotone if its graph $G(T) = \{(x, u) \mid u \in Tx\}$ is a monotone subset of $X \times X^*$ in the sense that $(u - v, x - y) \geq 0$ for all $(x, u)$, $(y, v) \in G(T)$, and $T$ is said to be maximal monotone if $G(T)$ is not a proper subset of any other monotone subset.
of $X \times X^*$. Observe that a mapping $T$ from $X$ into $2^{X^*}$ is maximal monotone iff the inverse mapping $T^{-1}$ from $X^*$ into $2^X$ is maximal monotone. A maximal monotone mapping $T$ from $X$ into $2^{X^*}$ is said to be linear maximal monotone if its graph $G(T)$ is a vector subspace of $X \times X^*$. Again, it is clear that $T$ from $X$ into $2^{X^*}$ is linear maximal monotone iff $T^{-1}$ from $X^*$ into $2^X$ is linear maximal monotone. For a multivalued mapping $T$ from $X$ into $2^{X^*}$, we define the effective domain $D(T)$ of $T$ by $D(T) = \{x \in X \mid Tx \neq \emptyset \}$. A mapping $T$ from $X$ into $2^{X^*}$ is said to be coercive if

$$\lim_{\|x\| \to \infty} \frac{\langle w, x \rangle}{\|x\|} = \infty.$$ 

A single-valued mapping defined everywhere in $X$ is said to be of type (M) if it satisfies the following two conditions:

(M1) For any sequence $\{x_n\}$ in $X$ such that $x_n \rightharpoonup x_0 \in X$ weakly with the sequence $Tx_n - w_0 \in X^*$ (weakly) and $\limsup_{n \to \infty} (Tx_n, x_n) \leq (w_0, x_0)$, we have $Tx_0 = w_0$;

(M2) $T$ is continuous from finite-dimensional subspaces of $X$ to $X^*$ endowed with weak topology;

A single-valued mapping $T$ defined everywhere in $X$ to $X^*$ is said to satisfy condition (S+) if for any sequence $\{x_n\}$ in $X$ such that $x_n \rightharpoonup x_0 \in X$ (weakly) and $\limsup_{n \to \infty} (T(x_n), x_n) < (w_0, x_0)$ then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x_0$ (strongly) and $Tx_{n_k} \rightharpoonup Tx_0$ (weakly).

**Theorem 1.** Let $X$ be a reflexive Banach space, $A$ a linear maximal monotone mapping from $X$ into $2^{X^*}$, and $N$ a single-valued bounded mapping of type (M) from $X$ into $X^*$. Suppose further that there exists an $r > 0$ such that $(Nx, x) > 0$ for $\|x\| \leq r$. Then there exists an $x \in X$ with $\|x\| \leq r$ such that $x \in D(A)$ and $-N(x) \in A(x)$ or equivalently $0 \in A(x) + N(x)$. Further, the set $\{x \in X \mid 0 \in Ax + Nx\}$ is weakly compact.

**Proof.** We may assume using a result of Asplund [1] that $X$ is endowed with a norm in which both $X$ and $X^*$ are strictly convex.

Now, let $J$ be the duality mapping from $X^*$ into $X$ defined for a given $w \in X^*$ by $Jw = u$, where $(w, Ju) = \|w\|^2$ and $\|Ju\| = \|w\|$. Such a $u$ exists by Hahn–Banach Theorem and is unique since $X$ is strictly convex. Further, for $\omega_1, \omega_2 \in X^*$, we have

$$(\omega_1 - \omega_2, J\omega_1 - J\omega_2) \geq (\|\omega_1\| - \|\omega_2\|)^2.$$ 

Now, for $\epsilon > 0$, set $A_\epsilon = (A^{-1} + \epsilon J)^{-1}$. Then it is easy to see (e.g., from Lemma 3 [12]) that $A_\epsilon$ is a single-valued everywhere defined bounded, maxi-
mal monotone, demicontinuous (i.e., continuous from norm topology of $X$ to $X^*$ endowed with weak topology) mapping from $X$ into $X^*$. Now, let $A$ denote the family of finitedimensional subspaces of $X$ and suppose that $A$ is partially ordered by inclusion. For $F \in A$, let $j_F: F \to X$ denote the inclusion mapping and $j_F^*: X^* \to F^*$ be the corresponding projection mapping. Then it follows from above that the mapping $B_F = j_F^*(A_\epsilon + N)j_F: F \to F^*$ is continuous and is such that $(B_Fu, u) > 0$ for $u \in F$ and $\|u\| \geq r$. Indeed, $(B_Fu, u) = (j_F^*(A_\epsilon + N)j_Fu, u) = ((A_\epsilon + N)u, u) \geq (Nu, u) > 0$ for $u \in F$ and $\|u\| \geq r$ since $A_\epsilon(0) = 0$. So, it follows (e.g., from [2, Theorem 10]) that there is an $u_F \in F$ with $\|u_F\| \leq r$ such that $B_Fu_F = 0$ or equivalently $y_F = (A_\epsilon + N)u_F \in F^\perp$ (the annihilator of $F$ in $X^*$).

Clearly, $y_F \to 0$ following the filter $A$, in $X^*$. For $F_0 \in A$, let

$$V_{F_0} = \{u_{F_0}, Nu_{F_0} | F \supset F_0\}.$$ 

Since $\|u_{F_0}\| \leq r$ for every $F \in A$ and $N$ is a bounded mapping, we see that $V_{F_0}$ is contained in some closed ball $S$ in $X \times X^*$ which is weakly compact since $X$ is reflexive. Let $\overline{V_{F_0}}$ denote the weak closure of $V_{F_0}$ in $X \times X^*$.

The family $\{\overline{V_{F_0}}\}_{F \in A}$ of weakly closed subsets of the weakly compact set $S$ clearly has finite intersection property and so $\bigcap_{F \in A} \overline{V_{F_0}} \neq \emptyset$. Now, let $[u_\epsilon, g] \in \bigcap_{F \in A} \overline{V_{F_0}}$. We assert that $g = Nu_\epsilon$. Indeed, let $F_0 \in A$ be such that $u_\epsilon \in F_0$. Since $[u_\epsilon, g] \in \overline{V_{F_0}}$, we see that there exists a sequence $\{F_n\}$ in $A$ such that $F_n \supset F_0$ for every $n$ and $u_{F_n} \to u_\epsilon$, $Nu_{F_n} \to g$. Using the monotonicity of $A$ and the fact that $y_{F_n} \in F_n^\perp \subset F_0^\perp$, we see from

$$0 \leq (A_\epsilon u_{F_n} - A_\epsilon u_\epsilon, u_{F_n} - u_\epsilon) = (y_{F_n} - Nu_{F_n} - A_\epsilon u_\epsilon, u_{F_n} - u_\epsilon)$$

that

$$\limsup_{n \to \infty} (Nu_{F_n}, u_{F_n}) \leq (g, u_\epsilon),$$

which implies that $g = Nu_\epsilon$ since $N$ is of type (M).

So we see that $[u_\epsilon, Nu_\epsilon] \in \bigcap_{F \in A} \overline{V_{F_0}}$. This gives that there is an ultrafilter $A'$ of $A$ such that $u_{F_0} \to u_\epsilon$, $Nu_{F_0} \to Nu_\epsilon$ following $A'$. Also, since $A'$ is an ultrafilter of $A$, we have $y_{F_0} \to 0$ following $A'$. Now for $F \in A'$, we have

$$u_{F_0} \in A^{-1}(y_{F_0} - Nu_{F_0}) + \epsilon j(y_{F_0} - Nu_{F_0}),$$

and so there exists $v_{F_0} \in A^{-1}(y_{F_0} - Nu_{F_0})$ such that

$$u_{F_0} = v_{F_0} + \epsilon j(y_{F_0} - Nu_{F_0}).$$

Using the fact that $A$ is linear so that $0 \in A^{-1}(0)$, we get that

$$(y_{F_0} - Nu_{F_0}, u_{F_0}) \geq \epsilon \|y_{F_0} - Nu_{F_0}\|^2.$$
Since now \( \| u_{F\epsilon} \| \leq r \) for every \( F \in \mathcal{A}' \) and \( \epsilon > 0 \), we see from the boundedness of the mapping, \( N \), and the fact that \( y_{F\epsilon} \in F_\epsilon \) that there exists a constant \( C \) independent of \( F \in \mathcal{A}' \) and \( \epsilon > 0 \) such that
\[
\epsilon \| u_{F\epsilon} - Nu_{F\epsilon} \|^2 \leq C^2, \quad \text{i.e.,} \quad \epsilon^{1/2} \| y_{F\epsilon} - Nu_{F\epsilon} \| \leq C. \quad (1)
\]
It then follows that
\[
\| u_{F\epsilon} - v_{F\epsilon} \| = \epsilon \| y_{F\epsilon} - Nu_{F\epsilon} \| \leq \epsilon^{1/2} C \quad (2)
\]
for every \( F \in \mathcal{A}' \) and \( \epsilon > 0 \). From (1) and (2), by passing to another ultrafilter (if necessary), we may assume that there are elements \( h_\epsilon \in X \) and \( k_\epsilon \in X^* \) such that \( y_{F\epsilon} - Nu_{F\epsilon} \to k_\epsilon \) for \( F \in \mathcal{A}' \) and \( v_{F\epsilon} \in A^{-1}(y_{F\epsilon} - Nu_{F\epsilon}) \to h_\epsilon \) for \( F \in \mathcal{A}' \). Since \( A^{-1} \) is linear maximal monotone, \( G(A^{-1}) \) is weakly closed in \( X^* \times X \) and \( k_\epsilon \in D(A^{-1}) \) and \( h_\epsilon \in A^{-1}(k_\epsilon) \) or equivalently \( k_\epsilon \in A(h_\epsilon) \).

Also, our work above shows that \( k_\epsilon = -Nu_\epsilon \). Now, using the weak lower semicontinuity of the norm in \( X \), we get from (2) that \( \| u_\epsilon - h_\epsilon \| \leq \epsilon^{1/2} C \) for every \( \epsilon > 0 \). So \( \lambda_\epsilon = u_\epsilon - h_\epsilon \to 0 \) strongly in \( X \) as \( \epsilon \to 0 \).

Let \( \{\epsilon_n\} \) be a sequence of positive numbers such that \( \lim_{n \to \infty} \epsilon_n = 0 \). We may assume that there exist \( u \in X \) and \( g \in X^* \) such that \( u_{\epsilon_n} \to u \) and \( Nu_{\epsilon_n} \to g \). Now \( h_{\epsilon_n} = u_{\epsilon_n} - \lambda_{\epsilon_n} \to u \) and \( A(h_{\epsilon_n}) \ni k_{\epsilon_n} = -Nu_{\epsilon_n} \to g \) imply, using the fact that \( A \) is linear maximal monotone, that \( u \in D(A) \) with \( -g \in A(u) \). It then suffices to prove that \( g = Nu \) to conclude that \( 0 \in A(u) + N(u) \).

Now, using the monotonicity of \( A \) and the fact that \(-g \in A(u)\), we get from
\[
0 \leq (k_{\epsilon_n} + g, h_{\epsilon_n} - u) - (-Nu_{\epsilon_n} + g, u_{\epsilon_n} - \lambda_{\epsilon_n} - u)
\]
that
\[
\lim_{n \to \infty} \sup (Nu_{\epsilon_n}, u_{\epsilon_n}) \leq (g, u),
\]
which gives that \( g = Nu \) since \( N \) is of type (M). Finally, the weak-compactness of the set \( \{x \in X \mid 0 \in Ax + Nx\} \) follows from the facts that the set \( \{x \in X \mid 0 \in Ax + Nx\} \) is a bounded subset of \( X \), the weak-closure of a bounded subset of a reflexive Banach space consists of limits of weakly convergent sequences in the set, the linear maximal monotonicity of \( A \), and that \( N \) is a mapping of type (M).

This completes a proof of the theorem. Q.E.D.

**Theorem 2.** Let \( X \) be a reflexive Banach space, \( A \) a linear maximal monotone mapping from \( X \) into \( 2^{X^*} \), and \( N \) a single-valued bounded coercive mapping of type (M) from \( X \) into \( X^* \). Then the range \( R(A + N) = X^* \).

**Proof.** Since \( N \) is a coercive mapping, we see that for each \( w \) in \( X^* \) there exists an \( r > 0 \) such that \( (Nu - w, u) > 0 \) for \( \| u \| \geq r \). It then follows
from Theorem 1 that there exists an \( x \in X \) such that \( 0 \in A(x) + N(x) - w \), i.e., \( w \in Ax + Nx \). Thus \( R(A + N) = X^* \). Hence the theorem. Q.E.D.

Remark 1. Theorems 1 and 2 can be extended easily to the case of multivalued bounded mappings \( N \) of type \((M)\). We have studied and presented the single-valued case only because of simplicity and our interest in studying the compact perturbations of equations of the form \( w = Au + Nu \).

We may also remark that Theorem 2 was proved earlier by Kenmochi [12] for mappings \( N \) of type \((M)\) which were defined via the use of filters instead of sequences as we do here. As was observed earlier by Brézis [2], the definition of mappings of type \((M)\) using filters is equivalent to mappings of type \((M)\) defined using sequences iff the Banach space is separable and the mapping is bounded. So our theorem is valid for nonseparable Banach spaces, in contrast to Kenmochi's theorem which is valid only for separable Banach spaces if the definition of type \((M)\) mappings is made using sequences.

Remark 2. It will be interesting to prove Theorem 2 in case of nonlinear maximal monotone mapping \( A \) which question is open at this time.

Theorem 3. Let \( X \) be a reflexive Banach space, \( A \) a linear maximal monotone mapping from \( X \) into \( 2^{X^*} \), and \( N \) a single-valued bounded mapping of type \((M)\) from \( X^* \) into \( X \) such that \( \lim_{\epsilon \to 0} (u, N(u + \epsilon)) > 0 \) for all \( u \in X^* \). Then the range \( R(I + AN) \) of the mapping \( I + AN \) is all of \( X^* \).

Proof. We first observe that it suffices to show that \( 0 \in R(I + AN) \). Indeed, for any \( v \in X^* \), to show that \( v \in R(I + AN) \) is equivalent to showing that \( 0 \in R(I + AN_v) \), where \( N_v \) is a mapping from \( X^* \) to \( X \) defined by \( N_v(u) = N(u + v) \) for all \( u \in X^* \), and \( N_v \) satisfies the conditions of the theorem iff \( N \) docs. Now \( 0 \in R(I + AN) \) iff \( 0 \in R(A^{-1} + N) \) and then the theorem follows from Theorem 1. Hence, the theorem. Q.E.D.

Remark 3. We may remark that Theorem 3 does not in general imply Theorems 1 or 2.

Proposition 1. Let \( X \) be a reflexive Banach space, \( A \) a linear maximal monotone mapping of \( X \) into \( 2^{X^*} \), and \( N \) a single-valued bounded mapping of type \((M)\) from \( X \) into \( X^* \). Then:

(i) For any sequence \( \{u_j\} \) in \( (A) \) with \( u_j \to u \) in \( X \) and \( w_j \in Au_j + Nu_j \) so that \( w_j \to w \), we have \( u \in D(A) \) and \( w \in Au + Nu \).

(ii) For any closed bounded convex subset \( C \) of \( X \) (or more generally any sequentially weakly compact subset), \((A + N)(C)\) is closed in \( X^* \).

Proof. (i) Since \( N \) is bounded and \( X \) is reflexive, we may assume (by going to a subsequence if necessary) that \( v_j = Nu_j \to v \in X^* \). Now, by
hypothesis, $w_j \to w$, so $w_j - Nu_j \in Au_j \to w - v$. It then follows from the facts $u_j \to u$ and $A$ is linear maximal monotone that $u \in D(A)$ and $w - v \in A(u)$. It will now suffice to show that $v = Nu$ which will follow from $\limsup_j (N u_j, u_j) \leq (v, u)$. Indeed, we have

$$0 \leq (w_j - Nu_j - w + v, u_j - u) = (w_j - w, u_j - u) + (v, u_j - u) - (N u_j, u_j) + (N u_j, u),$$

which implies that

$$\limsup_j (N u_j, u_j) \leq (v, u)$$

and so $Nu = v$. This completes the proof of (i).

(ii) The proof of (ii) now follows from (i) because if $\{u_j\}$ is a sequence in $C$ with $w_j \in Au_j + Nu_j$ such that $w_j \to w$, we can assume (again by choosing a subsequence, if necessary) that $u_j \to u \in C$ and then (i) implies that $u \in D(A)$ and $w \in Au + Nu$. Hence the proposition. Q.E.D.

**Proposition 2.** Let $X$ be a reflexive Banach space, $A$ a linear maximal monotone mapping of $X$ into $2^{X^*}$, and $N$ a single-valued bounded mapping which satisfies condition $(S^+)$. Then:

(i) $A + N$ is a proper mapping from bounded closed subsets of $X$ into $X^*$ (i.e., for each compact subset $K$ of $X^*$ and each closed ball $B$ of $X$, $(A + N)^{-1}(K) \cap B$ is a compact subset of $X$).

(ii) For each bounded closed subset $C$ of $X$, $(A + N)(C)$ is closed in $X^*$.

**Proof.** (i) Let $\{u_j\}$ be a bounded sequence in $X$ such that $Au_j + Nu_j$ is contained in a compact subset $K$ of $X^*$. We want to show in view of Proposition 1 that we may extract an infinite subsequence from the sequence $\{u_j\}$ which is strongly convergent in $X$. Letting $w_j \in Au_j + Nu_j$, we may assume that $u_j \to u \in X$ and $w_j \to w \in X^*$. It then follows from Proposition 1 that $u \in D(A)$ and $w \in Au + Nu$. Also, using the monotonicity of $A$, we obtain that $\limsup(N u_j, u_j - u) \leq 0$. Now, since $N$ satisfies condition $S^+$, we see that $u_j \to u$. This proves (i) of the proposition.

(ii) Let $w \in \text{closure of } (A + N)(C)$ in $X^*$. Then there exists a sequence $\{w_j\}$ in $(A + N)(C)$ such that $w_j \to w$. Let $u_j \in C$ be such that $w_j \in Au_j + Nu_j$. Now the subset $K = \{w\} \cup \bigcup_{j \geq k} \{w_j\}$ is a compact subset of $X^*$, we see from (i) that there is an infinite subsequence of $\{u_j\}$ (which we denote by $u_j$ itself) such that $u_j \to u \in C$. Since the sequence $\{u_j\}$ eventually lies in

$$(A + N)^{-1}\left(\{w\} \cup \bigcup_{j \geq k} \{w_j\}\right)$$
and the latter set is compact and hence closed in $X$, it follows that for each $k$,
\[ u \in (A + N)^{-1}\left(\{w\} \cup \bigcup_{j \neq k} \{w_j\}\right). \]

Hence $w \in Au + Nu$, proving that $(A + N)(C)$ is closed in $X^*$.

This completes the proof of the proposition. Q.E.D.

**Theorem 4.** Let $X$ be a reflexive Banach space, $N$ a bounded monotone coercive mapping of $X$ into $X^*$ satisfying condition $(S^+)$, and let $A : X \to 2^{X^*}$ be a linear maximal monotone mapping. Suppose that we are given a continuous family $\{C_t\}$ of compact mappings of $X$ into $X^*$ such that $C_0 = 0$. Let $G$ be a bounded open subset of $X$, $w$ an element of $X^*$ such that the equation $w \in Au + Nu$ has a solution in $G$. Suppose that for each $t \in [0, 1]$, the equation $w \in Au + Nu + C_t u$ has no solution on the boundary of $G$.

Then, for each $t \in [0, 1]$, the equation $w \in Au + Nu + C_t u$ has a solution $u$ in $G$.

We need the following proposition from the theory of generalized Leray–Schauder degree for multivalued compact mappings in Banach spaces, which we state here without proof and refer the reader to [6–10] for the proof of this proposition. We also need a variant of Theorem 3.10 of Browder [4] for the case of nonlinear mappings $T$ which may not be defined everywhere and which may not be continuous.

**Proposition 3.** Let $X$ be a Banach space, $S$ an upper semicontinuous mapping of $X \times [0, 1]$ into $2^X$ such that for each $u \in X$ and $t \in [0, 1]$, $S(u, t)$ is a closed convex nonempty subset of $X$. Suppose that $S$ is compact in the sense that for each bounded subset $B$ of $X$, $S(B \times [0, 1])$ is relatively compact in $X$. For each $t$ in $[0, 1]$, let $S_t$ be the mapping of $X$ into $X$ given by $S_t(x) = S(x, t)$ and let $G$ be a bounded open subset of $X$. Suppose that $S_0$ is a constant map of the closure of $G$ into an element of $2^X$ which intersects $G$, while for each $t$ in $[0, 1]$, there exists no point $x$ of the boundary of $G$ such that $x$ lies in $S_t(x)$. Then, for each $t$ in $[0, 1]$, there exists $x_t \in G$ such that $x_t \in S_t(x_t)$.

**Theorem 5.** Let $X$ and $Y$ be Banach spaces, $T$ with domain $D(T)$ in $X$ to $2^Y$ is a proper mapping, $\{C_t\}$ a continuous family of compact mappings from $X$ into $Y$, with $C_0 = 0$. Let $G$ be a bounded open subset of $X$ and $y_0 \in Y$. Suppose that all the following conditions hold:

1. For each $y \in Y$, $T^{-1}(y)$ is a closed convex nonempty subset of $X$.
2. There exists $u_0 \in G$ such that $y_0 \in T(u_0)$.
(3) There exists no point \( x \) in the boundary of \( G \) and no value of \( t \) in \([0, 1]\) such that \( y_0 \in (T + C_t)(x) \).

(4) For any sequence of elements \( \{y_n\} \) in \( Y \) with \( y_n \rightarrow y_0 \) and \( u_n \in T^{-1}(y_n) \) such that \( u_n \rightarrow u_0 \), we have \( u_0 \in T^{-1}(y_0) \).

Then for each \( t \in [0, 1] \), there exists \( u_t \in U \) such that \( y_0 \in (T + C_t)(u_t) \).

**Proof.** We apply Proposition 3 to the mapping \( S \) from \( X \times [0, 1] \) into \( 2^X \) defined by

\[
S(x, t) = T^{-1}(y_0 - C_t(x)).
\]

It follows obviously from this definition that \( x \) lies in \( S_t(x) = S(x, t) \) iff \( y_0 \in (T + C_t)(x) \). To apply Proposition 3, we first note by condition (1) of the theorem that \( S(x, t) \) is a closed, convex, nonempty subset of \( X \). If \( B \) is a bounded subset of \( X \), then by the definition of a continuous family of compact operators, \( C(B \times [0, 1]) \) is a relatively compact subset of \( Y \). Since \( T \) is proper, \( T^{-1}(C(B \times [0, 1])) \) is relatively compact in \( X \). Hence, \( S(B \times [0, 1]) \) is relatively compact in \( X \). For \( t = 0 \), \( y_0 - C_0(x) = y_0 \), so that \( S_0(x) = T^{-1}(y_0) \), and thus \( S_0(x) \) intersects \( G \) by hypothesis. To complete the proof of the theorem, it therefore suffices to prove that \( S \) is an upper semicontinuous mapping of \( X \times [0, 1] \) into \( 2^X \). Let \( x_0 \in X \), \( t_0 \in [0, 1] \) and let \( U \) be a neighborhood in \( X \) of \( S(x_0, t_0) \). To show that \( S \) is upper semicontinuous, we must show that there exists \( \epsilon > 0 \) such that if \( || x - x_0 || < \epsilon \) and \( | t - t_0 | < \epsilon \), then \( S(x, t) \subset U \). Suppose that this were not true. Then there would exist a sequence \( \{x_n\} \) in \( X \) converging to \( x_0 \), a sequence \( \{t_n\} \) in \([0, 1]\) converging to \( t_0 \), and for each \( n \) an element \( u_n \) in \( S(x_n, t_n) \) \( \subset U \). By the compactness of the mapping \( S_t \), we may assume without loss of generality that \( u_n \) converges to some element \( u_0 \in X \setminus U \). For each \( n \), we know that \( y_0 - C_{t_n}(x_n) \in T(u_n) \) which converges strongly to \( y_0 - C_{t_0}(x_0) \). So by condition (4), we have that \( u_0 \in T^{-1}(y_0 - C_{t_0}x_0) \) which contradicts the fact that \( u_0 \) lies outside of \( U \). Hence, the theorem. Q.E.D.

**Proof of Theorem 4.** We apply Theorem 5 to the mapping \( T = A + N \) of \( X \) into \( X^* \) which is maximal monotone and satisfies the hypothesis of Theorem 5 in view of Theorem 1, the standard results on monotone operators, and Propositions 1 and 2. Q.E.D.

**Theorem 6.** Let \( X \) be a reflexive Banach space, \( N: X \rightarrow X^* \) a bounded hemicontinuous strongly monotone mapping [i.e., there exists a constant \( \alpha > 0 \) such that \( (Nu - Nv, u - v) > \alpha \| u - v \|^2 \) for all \( u, v \) in \( X \)] and let \( A: X \rightarrow 2^{X^*} \) be a linear maximal monotone mapping. Let \( \{C_t\} \) be a continuous family of compact mappings from \( X \) into \( X^* \) with \( C_0 = 0 \). Suppose that \( G \) is a bounded open subset of \( X \), \( w \) an element of \( X^* \) such that the equa-
tion \( w \in Au + Nu \) has a solution in \( G \). Further, suppose that for each \( t \in [0, 1] \), the equation \( w \in Au + Nu + C_t u \) has no solution on the boundary of \( G \).

Then for each \( t \in [0, 1] \), there exist \( u_t \in G \) such that \( w \in Au_t + Nu_t + C_t u_t \).

Proof. It is immediate to verify that \( N \) is a monotone bounded coercive mapping satisfying condition \((S^+)\) from \( X \) into \( X^* \). The theorem then follows from Theorem 4. Q.E.D.

REFERENCES