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Variational q-calculus ☆

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Abstract

We propose q-versions of some basic concepts of continuous variational calculus such as the Euler–Lagrange equation and its applications to the isoperimetric, Lagrange and optimal control problems ("the maximum principle"), and also to the Hamilton systems and commutation equations.

1. Introduction

In [3], Cadzow proposed a discrete version of some basic concepts of continuous variational calculus such as the Euler–Lagrange equation and its applications to the isoperimetric, Lagrange and optimal control problems. In the time to follow, most of researches in the area were mainly directed to the study of the complete integrability of the discrete Euler–Lagrange equation (see, e.g., [6–10,12]). That is to say that at our best knowledge, the question of the generalization of the continuous (differential) variational calculus, to the calculus of variation on lattices more general than the linear one (treated in [3]), had never been considered. In this work we propose an extension of the continuous variational calculus to the variational calculus on the q-linear lattice $x = Aq^s + B$, $s \in \mathbf{Z}$, A, B some constants. More precisely, we are concerned in the extremum problem for the functional

$$J\big(y(x)\big) = \int_{q^\alpha}^{q^\beta} F\big(x, y(x), D_q y(x), \ldots, D_q^k y(x)\big)\, d_q x$$

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$$\stackrel{\text{def}}{=} (1-q)\sum_{q^\alpha}^{q^\beta} xF\bigl(x, y(x), D_q y(x), \ldots, D_q^k y(x)\bigr) \tag{1}$$

under the boundary constraints

$$\begin{aligned} &y(q^\alpha) = y(q^{\beta+1}) = c_0,\\ &D_q y(q^\alpha) = D_q y(q^{\beta+1}) = c_1,\\ &\quad\vdots\\ &D_q^{k-1} y(q^\alpha) = D_q^{k-1} y(q^{\beta+1}) = c_{k-1}, \end{aligned} \tag{2}$$

where

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}, \quad 0 < q < 1,\ k \in \mathbf{Z}^+, \tag{3}$$

while the summation is performed by x on the set (we shall sometimes write simply $\sum_{q^\beta}^{q^\beta}$ or $\sum_L$)

$$L = \{q^\beta, q^{\beta-1}, \ldots, q^{\alpha+1}, q^\alpha\}, \quad 0 \leqslant \alpha < \beta \leqslant +\infty. \tag{4}$$

For $\alpha \rightsquigarrow 0$, $\beta \rightsquigarrow +\infty$, (1) and (2) read

$$\begin{aligned} J\bigl(y(x)\bigr) &= \int_0^1 F\bigl(x, y(x), D_q y(x), \ldots, D_q^k y(x)\bigr)\, d_q x\\ &\stackrel{\text{def}}{=} (1-q)\sum_0^1 xF\bigl(x, y(x), D_q y(x), \ldots, D_q^k y(x)\bigr) \end{aligned} \tag{5}$$

and

$$D_q^i y(0) = D_q^i y(1), \quad i = 0, \ldots, k-1, \tag{6}$$

respectively. If the function $\tilde{F}(x) = F(x, y(x), Dy(x), \ldots, D^k y(x))$ is Riemann-integrable on the interval [0, 1], then it is easily seen that for $q \rightsquigarrow 1$, the q-integral in Eq. (5) and the constraints in Eq. (6) tends to the continuous integral

$$J\bigl(y(x)\bigr) = \int_0^1 F\bigl(x, y(x), Dy(x), \ldots, D^k y(x)\bigr)\, dx, \tag{7}$$

where $Df(x) = (d/dx) f(x)$, and the boundary constraints

$$\begin{aligned} &y(0) = y(1) = c_0,\\ &Dy(0) = Dy(1) = c_1,\\ &\quad\vdots\\ &D^{k-1} y(0) = D^{k-1} y(1) = c_{k-1}, \end{aligned} \tag{8}$$

respectively. Hence the functional in Eq. (5) can be considered as a natural q-version of the one in Eq. (7).

Remark 1. By carrying out in (1) the linear change of variable

$$t(s) = a + x(s)(b-a) = a + q^s(b-a) \tag{9}$$

(a, b finite for simplicity), we obtain a q-version of the integral obtained from (7) by the linear change of variable

$$t = a + x(b-a), \tag{10}$$

and both the two new integrals have now a and b as boundaries of integration. Clearly the converse to (9) and (10) transformations are also valid. Hence in that sense, there is no lost of generalities considering in this work integrals of type (5) or (7) or even the little bit more general integral in (1). This allows to avoid cumbersome treatments unessential in addition in the reasoning.

In the following, we derive a q-version of the Euler–Lagrange equation, deriving the Euler–Lagrange equation of the functional in Eq. (1) and showing that for $q \to 1$ ($\alpha \rightsquigarrow 0$, $\beta \rightsquigarrow +\infty$ in the boundary constraints), it tends to the Euler–Lagrange equation of the functional in Eq. (7). Next, we apply it to the continuous variational calculus, q-versions of the isoperimetric, Lagrange and optimal control problems. Q-versions of some interconnections between the Euler–Lagrange equation of variational calculus, Hamilton and Hamilton–Pontriaguine systems are also sketched. Equally as an application, a q-version of the commutation equations is also discussed. The reader will note that most of ideas used here are simply q-versions of similar ideas used in continuous or discrete variational calculus. But as these ideas work, it means probably that this generalization of the classical variational calculus is a natural one.

2. The q-Euler–Lagrange equation

We consider the q-integral functional

$$J\big(y(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} xF\big(x, y(x), D_q y(x), \ldots, D_q^k y(x)\big). \tag{11}$$

Here the function $F(x, y_0(x), \ldots, y_k(x))$ is defined on A as a function of x, together with its first partial derivatives relatively to all its arguments. Let E be the linear space of functions $y(x)$ ($q^\alpha \leqslant x \leqslant q^\beta$) in which is defined the norm

$$\|y\| = \max_{0\leqslant i\leqslant k}\Big(\max_{x\in L}\big|D_q^i y(x)\big|\Big), \tag{12}$$

and let E' be the linear manifold of functions belonging in E and satisfying to the constraints in (2). We study the extremum problem for the functional J on the manifold E'. We first calculate the first variation of the functional J on the linear manifold E':

$$
\begin{aligned}
\delta J\big(y(x),h(x)\big) &= \frac{d}{dt}J\big(y(x)+th(x)\big)\Big|_{t=0} \\
&= (1-q)\frac{d}{dt}\sum_{q^\alpha}^{q^\beta}\big[xF\big(x,y(x)+th(x),\dots,D_q^k y(x)+tD_q^k h(x)\big)\big]\Big|_{t=0} \\
&= (1-q)\sum_{q^\alpha}^{q^\beta}\sum_{i=0}^{k}\big[xF_i\big(x,y(x),D_q y(x),\dots,D_q^k y(x)\big)D_q^i h(x)\big],
\end{aligned}
\tag{13}
$$

where

$$
F_i=\frac{\partial F}{\partial y_i}\quad \big(F=F(x,y_0,y_1,\dots,y_k)\big),\ i=0,\dots,k. \tag{14}
$$

The variation is dependent on an arbitrary function $h(x)$. Since the variation is performed on the linear manifold E', $h(x)$ is such that $y(x)+th(x)$ belongs also to the linear manifold E' and in particular satisfies the constraints (2). A direct consequence of this is that the function $h(x)$ satisfies the constraints

$$
\begin{aligned}
&h(q^\alpha)=h(q^{\beta+1})=0,\\
&D_q h(q^\alpha)=D_q h(q^{\beta+1})=0,\\
&\quad\vdots\\
&D_q^{k-1}h(q^\alpha)=D_q^{k-1}h(q^{\beta+1})=0.
\end{aligned}
\tag{15}
$$

From the relation $D_q(fg)(x)=f(qx)D_q g(x)+g(x)D_q f(x)$, one obtains the formula of the q-integration by parts:

$$
(1-q)\sum_{q^\alpha}^{q^\beta}xf(qx)D_q g(x)=(1-q)\sum_{q^\alpha}^{q^\beta}xD_q(fg)-(1-q)\sum_{q^\alpha}^{q^\beta}xg(x)D_q f(x). \tag{16}
$$

Using (15), and (16), (13) gives

$$
\begin{aligned}
\delta J\big(y(x),h(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta}x\sum_{0}^{k}(-1)^i q^{\frac{i-1}{2}i}D_q^i\big[F_i\big(q^{-i}x,y(q^{-i}x),D_q y(q^{-i}x),\\
\dots,D_q^k y(q^{-i}x)\big)\big]h(x).
\end{aligned}
\tag{17}
$$

(Very important to distinguish $D_q f(kx)$ which means here $[D_q f](kx)$ with $D_q[f(kx)]$ meaning $D_q g(x)$ for $g(x)=f(kx)$.) Next, it is necessary to note that the boundary constraints in Eq. (15) are equivalents to the following:

$$
h(q^{\alpha+i})=h(q^{\beta+1+i})=0,\quad i=0,1,\dots,k-1. \tag{18}
$$

Consequently, (17) gives

$$\delta J\big(y(x), h(x)\big) = (1-q) \sum_{q^{\alpha+k}}^{q^\beta} x \sum_{0}^{k} (-1)^i q^{\frac{i-1}{2}i} D_q^i \big[F_i\big(q^{-i}x, y(q^{-i}x), D_q y(q^{-i}x), \dots, D_q^k y(q^{-i}x)\big)\big] h(x). \tag{19}$$

For deriving the corresponding q-Euler–Lagrange equation, we need the following lemma, which constitutes a q-version of what is called "fundamental lemma of variational calculus" (for the continuous version, see, e.g., [5]).

Lemma 2.1. *Consider the functional*

$$I(\hat{f}) = (1-q) \sum_{B} x \hat{f}(x) h(x), \tag{20}$$

where $B = \{q^r, q^{r+1}, \dots, q^s\}$. *If* $I(\hat{f}) = 0$, *for all* h *defined on* B, *then* $\hat{f}(x) \equiv 0$ *on* B.

Proof. As $I(\hat{f}) = 0$, $\forall h$ defined on B, we have that

$$\begin{aligned} &q^r \hat{f}(q^r) h_1(q^r) + \dots + q^s \hat{f}(q^s) h_1(q^s) = 0, \\ &q^r \hat{f}(q^r) h_2(q^r) + \dots + q^s \hat{f}(q^s) h_2(q^s) = 0, \\ &\quad\vdots \\ &q^r \hat{f}(q^r) h_{s-r+1}(q^r) + \dots + q^s \hat{f}(q^s) h_{s-r+1}(q^s) = 0 \end{aligned} \tag{21}$$

for any choice of the $(s-r+1)^2$ numbers

$$a_{ij} = h_i(q^{j+r-1}), \quad i, j = 1, \dots, s-r+1. \tag{22}$$

This is a linear homogeneous system with the matrix

$$(a_{ij})_{i,j=1}^{s-r+1} \tag{23}$$

and the vector $[T_j = q^{j+r-1} \hat{f}(q^{j+r-1})]_{j=1}^{s-r+1}$. Choosing the numbers

$$h_i(q^{j+r-1}), \quad i, j = 1, \dots, s-r+1, \tag{24}$$

in such a way that the corresponding matrix in (23) does not be singular, (21) gives $T_j = 0$, $j = 1, \dots, s-r+1$, or equivalently, $\hat{f}(q^{j+r-1}) = 0$, $j = 1, \dots, s-r+1$, which proves the lemma. □

Next, remark that (19) is written under the form

$$\delta J\big(y(x), h(x)\big) = I(\hat{f}) = (1-q) \sum_{q^{\alpha+k}}^{q^\beta} x \hat{f}(x) h(x), \tag{25}$$

where $\hat{f}$ represents the expression within the external brackets. Hence the necessary condition for the extremum problem (1)–(4) can be written as

$$I(\hat{f}) = 0 \tag{26}$$

and this for all $h(x)$ defined on

$$B = \{q^r, q^{r+1}, \dots, q^s\}, \quad r = \alpha + k,\ \beta = s. \tag{27}$$

By the fundamental lemma of the variational q-calculus (see Lemma 2.1), this leads to

$$\hat{f}(x) \equiv 0. \tag{28}$$

Thus the necessary condition for the extremum problem (1)–(4) reads

$$\begin{aligned} &\sum_0^k (-1)^i q^{\frac{i-1}{2}i} D_q^i \big[F_i\big(q^{-i}x, y(q^{-i}x), D_q y(q^{-i}x), \dots, D_q^k y(q^{-i}x)\big)\big] = 0, \\ &D_q^i y(q^\alpha) = D_q^i y(q^{\beta+1}) = c_i, \quad i = 0, \dots, k-1. \end{aligned} \tag{29}$$

For $k = 1$ and $k = 2$, for example, we have respectively

$$\begin{aligned} &F_0\big(x, y(x), D_q y(x)\big) - D_q\big[F_1\big(q^{-1}x, y(q^{-1}x), D_q y(q^{-1}x)\big)\big] = 0, \\ &y(q^\alpha) = y(q^{\beta+1}) = c_0 \end{aligned} \tag{30}$$

and

$$\begin{aligned} &F_0\big(x, y(x), D_q y(x), D_q^2 y(x)\big) - D_q\big[F_1\big(q^{-1}x, y(q^{-1}x), D_q y(q^{-1}x), D_q^2 y(q^{-1}x)\big)\big] \\ &\quad + q D_q^2\big[F_2\big(q^{-2}x, y(q^{-2}x), D_q y(q^{-2}x), D_q^2 y(q^{-2}x)\big)\big] = 0, \\ &y(q^\alpha) = y(q^{\beta+1}) = c_0, \qquad D_q y(q^\alpha) = D_q y(q^{\beta+1}) = c_1. \end{aligned} \tag{31}$$

Let us note that while the q-integral (1) tends to the continuous integral (7) for $q \rightsquigarrow 1$, $\alpha \rightsquigarrow 0$, $\beta \rightsquigarrow +\infty$, the q-equation in (29) tends to the corresponding to (7) differential Euler–Lagrange equation:

$$\begin{aligned} &\sum_0^k (-1)^i D^i F_i\big(x, y(x), Dy(x), \dots, D^k y(x)\big) = 0, \\ &D^i y(0) = D^i y(1) = c_i, \quad i = 0, \dots, k-1. \end{aligned} \tag{32}$$

That is why it is convenient to call (29) the *q-Euler–Lagrange equation* corresponding to the q-integral (1). Equation (29) is a q-difference equation of degree $2k$ which is in principle solved uniquely under the $2k$ boundary constraints.

Remark 2. If the functional in (11) is dependent on more than one variable i.e., $J = J(y_1, \dots, y_n)$, then the necessary extremum condition leads to type (29) n q-Euler–Lagrange equations with y replaced by y_i, $i = 1, \dots, n$.

3. Applications

3.1. On the continuous variational calculus

The direct application of the variational q-calculus is its application on the continuous (differential) variational calculus: Instead of solving the Euler–Lagrange equation (32) for finding the extremum of the functional (7), it suffices to solve the q-Euler–Lagrange equation (29) and then pass to the limit while $q \rightsquigarrow 1$. Remark that thought this can appear at the first glad as a contradiction (by the fact of the phenomenon of discretization), the variational q-calculus is a generalization of the continuous variational calculus due to the presence of the extra-parameter q (which may be physical, economical or another) in the first and its absence in the second.

Example. Suppose it is desirable to find the extremum of the integration functional

$$J\bigl(y(x)\bigr) = \int_0^1 \left(x^\nu y + \frac{1}{2}(Dy)^2\right) dx, \quad \nu > 0, \tag{33}$$

under the boundary constraints $y(0) = c$, $y(1) = \tilde{c}$. The q-version of the problem consists in finding the extremum of the q-integration functional

$$J\bigl(y(x)\bigr) = (1-q)\sum_0^1 x\left[x^\nu y + \frac{1}{2}(D_q y)^2\right], \quad \nu > 0, \tag{34}$$

under the same boundary constraints. According to (30), the q-Euler–Lagrange equation of the latter problem reads

$$x^\nu - D_q\bigl[D_q y(q^{-1}x)\bigr] = 0, \tag{35}$$

which solution is

$$\begin{aligned} y(x) = x^{\nu+2}&\left[\frac{(1-q)^2 q^{\nu+1}}{(1-q^{\nu+1})(1-q^{\nu+2})}\right] \\ &+ \left[y(1) - y(0) - \frac{(1-q)^2 q^{\nu+1}}{(1-q^{\nu+1})(1-q^{\nu+2})}\right]x + y(0). \end{aligned} \tag{36}$$

As it can be verified, for $q \rightsquigarrow 1$, the function in (36) tends to the function

$$y(x) = \frac{x^{\nu+2}}{(\nu+1)(\nu+2)} + \left[y(1) - y(0) - \frac{1}{(\nu+1)(\nu+2)}\right]x + y(0), \tag{37}$$

solution of the Euler–Lagrange equation of the functional in (33).

3.2. The q-isoperimetric problem

Suppose that it is required to find the extremum of the functional

$$\begin{aligned} &J\bigl(y(x)\bigr) = (1-q)\sum_{q^\alpha}^{q^\beta} x f\bigl(x, y(x), D_q y(x), \dots, D_q^k y(x)\bigr), \\ &D_q^i y(q^\alpha) = D_q^i y(q^{\beta+1}) = c_i, \quad i = 0, 1, \dots, k-1, \end{aligned} \tag{38}$$

under the constraints

$$\tilde{J}_i\big(y(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} x f^i\big(x, y(x), D_q y(x), \ldots, D_q^k y(x)\big) = C_i,$$
$$i = 1, \ldots, m. \tag{39}$$

To solve this problem we need to consider the following generalities. Let $J(y)$ and $\tilde{J}_1(y), \ldots, \tilde{J}_m$ be some differentiable functionals on the normed space E, or on its manifold E'. We have the following theorem (see, e.g., [5]).

Theorem 3.1. *If a functional $J(y)$ attains its extremum in the point $\bar{y}$ under the additional conditions $\tilde{J}_i(y) = C_i$, $i = 1, \ldots, m$, and $\bar{y}$ is not a stationary point for any one of the functionals $\tilde{J}_i$ ($\delta\tilde{J}_i(\bar{y}, h) \neq 0$, $i = 1, \ldots, m$, identically) while the functionals $\delta\tilde{J}_i$ ($i = 1, \ldots, m$) are linearly independent, then $\bar{y}$ is a stationary point for the functional $J - \sum_{i=1}^m \lambda_i \tilde{J}_i$ where the λ_i are some constants.*

Thus by this theorem, the necessary extremum condition for the functional $J(y)$ under the additional constraints $\tilde{J}_i(y) = C_i$, $i = 1, \ldots, m$, verifying the conditions of the theorem (let us note that considering the formula (17), a type (11) functional, i.e., satisfying the same definition conditions, is differentiable on E'), is given by Eq. (29) with

$$F = f - \sum_{i=1}^m \lambda_i f^i. \tag{40}$$

It is a q-difference equation of order $2k$ containing m unknown parameters. It is in principle solved uniquely under the $2k$ boundary constraints and the additional m conditions.

Example. Suppose it is required to solve the problem of finding the extremum of the q-integration functional

$$J\big(y(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} x\big[ax^2\big(D_q^2 y\big)^2 + b(D_q y)^2\big], \quad a, b > 0, \tag{41}$$

under the boundary constraints

$$D_q^i y(q^\alpha) = D_q^i y(q^{\beta+1}) = c_i, \quad i = 0, 1, \tag{42}$$

and an additional condition that $J_1(y(x)) = c$, c some constant, where J_1 is a q-integration functional given by

$$J_1\big(y(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} x^2 y. \tag{43}$$

According to Theorem 3.1, the problem is equivalent to that of finding the extremum of the q-integration functional

$$J\big(y(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} x\big[ax^2\big(D_q^2 y\big)^2 + b(D_q y)^2 - \lambda x y\big], \tag{44}$$

for some constant λ, under the same boundary constraints (42). The corresponding q-Euler–Lagrange equation reads

$$-\lambda x - 2bD_q\big[D_q y(q^{-1}x)\big] + 2aq^{-3}D_q^2\big[x^2 D_q^2 y(q^{-2}x)\big] = 0, \tag{45}$$

or equivalently after reduction and integration (c_1, c_2 constants of integration)

$$\begin{aligned} &y(x) - \big[q(q-1)^2 b/a + q + 1\big]y(q^{-1}x) + qy(q^{-2}x) \\ &\quad = \frac{(1-q)^2}{2a}\left(c_1 x + c_2 + \frac{\lambda x^3}{(q+1)(q^2+q+1)}\right). \end{aligned} \tag{46}$$

This is a constant coefficients linear nonhomogeneous second-order q-difference equation which can be solved uniquely (under the constraints (42)) by methods similar to that of analogous differential or difference equations.

3.3. *The q-Lagrange problem*

Suppose now that it is required to find the extremum of the functional

$$\begin{aligned} J\big(y_1(x), \dots, y_n(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} xf\big(x, y_1(x), \dots, y_n(x), \\ D_q y_1(x), \dots, D_q y_n(x)\big) \end{aligned} \tag{47}$$

under the constraints

$$\begin{aligned} &f^i\big(x, y_1(x), \dots, y_n(x), D_q y_1(x), \dots, D_q y_n(x)\big) = 0, \quad i = 1, \dots, m,\ m < n, \\ &y_i(q^\alpha) = y_i(q^{\beta+1}) = c_i, \quad i = 1, \dots, n. \end{aligned} \tag{48}$$

This problem can be transformed in the q-isoperimetric one as follows: First, multiply every ith equation in (48) by an arbitrary function $\lambda_i(x)$ defined as all the remaining on $L = \{q^\beta, \dots, q^\alpha\}$ and then apply the q-integration on L on the result:

$$\begin{aligned} \tilde{J}_i\big(y_1(x), \dots, y_n(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} x\lambda_i(x) f^i\big(x, y_1(x), \dots, y_n(x), \\ D_q y_1(x), \dots, D_q y_n(x)\big) = 0, \\ i = 1, \dots, m. \end{aligned} \tag{49}$$

The remaining question is that of knowing if the two constraints (48) and (49) are equivalent. The answer is yes since obviously from (48) follows (49). Finally, it is by the fundamental lemma of the variational q-calculus (see Lemma 2.1) that (48) follows from (49).

Example. Suppose that the problem consists in finding the extremum of the functional

$$J\big(x(t), u(t)\big) = \frac{1}{2}(1-q)\sum_{q^\alpha}^{q^\beta} t\big[u^2(t) - x^2(t)\big] \tag{50}$$

under the constraints

$$D_q^2 x = u, \qquad x(q^\alpha) = x(q^{\beta+1}) = c, \qquad D_q x(q^\alpha) = D_q x(q^{\beta+1}) = \tilde{c}. \tag{51}$$

The problem is equivalent to the q-Lagrange problem of finding the extremum of the functional

$$J\big(x(t), y(t), z(t)\big) = \frac{1}{2}(1-q)\sum_{q^\alpha}^{q^\beta} t\big[z^2(t) - x^2(t)\big] \tag{52}$$

under the constraints

$$\begin{aligned} &D_q x = y, \qquad D_q y = z, \\ &x(q^\alpha) = x(q^{\beta+1}) = c, \qquad y(q^\alpha) = y(q^{\beta+1}) = \tilde{c}. \end{aligned} \tag{53}$$

Hence the problem is equivalent to that of finding the extremum of the functional

$$J(x, y, z, \lambda_1, \lambda_2) = (1-q)\sum_{q^\alpha}^{q^\beta} tF\big(x(t), y(t), z(t), \lambda_1(t), \lambda_2(t)\big), \tag{54}$$

where

$$\begin{aligned} &F\big(x(t), y(t), z(t), \lambda_1(t), \lambda_2(t)\big) \\ &\quad = \frac{1}{2}\big(z^2(t) - x^2(t)\big) + \lambda_1(t)\big(D_q x(t) - y(t)\big) + \lambda_2(t)\big(D_q y(t) - z(t)\big) \end{aligned} \tag{55}$$

under the boundary constraints

$$x(q^\alpha) = x(q^{\beta+1}) = c, \qquad y(q^\alpha) = y(q^{\beta+1}) = \tilde{c}. \tag{56}$$

The corresponding q-Euler–Lagrange equations give

$$y(t) = D_q x(t), \qquad z(t) = \lambda_2(t) = D_q^2 x(t), \qquad \lambda_1(t) = -q^2 D_q^3\big[x(q^{-1}t)\big], \tag{57}$$

$$-x(t) + q^5 D_q^4\big[x(q^{-2}t)\big] = 0. \tag{58}$$

Hence it is sufficient to solve Eq. (58). Searching its solution as an integer power series $x(t) = \sum_0^\infty C_n t^n$, one is led to the following fourth-order difference equation for the coefficient c_n:

$$C_n = q^{2n-5}\left(\frac{1-q}{1-q^n}\right)\left(\frac{1-q}{1-q^{n-1}}\right)\left(\frac{1-q}{1-q^{n-2}}\right)\left(\frac{1-q}{1-q^{n-3}}\right)C_{n-4}, \tag{59}$$

with the coefficients C_0, C_1, C_2, C_3 determined by the four boundary constraints (56). The solution of (59) reads

$$C_n = \prod_{i=n_c}^{n}\left(\frac{1-q}{1-q^i}\right)\prod_{i=1}^{(n-n_c)/4} q^{2(n_c+4i)-5} C_{n_c}, \tag{60}$$

where $n \equiv n_c \bmod 4$, $0 \leqslant n_c \leqslant 3$.

To obtain the four basic elements for the space of solutions of (58), one can make the following four independent choices for the constants C_0, C_1, C_2, C_3: Choosing (a) $C_n =$

$1/n!$ for $n = 0, \dots, 3$ leads to $x(t) = e_q^t$; (b) $C_n = (-1)^n/n!$ for $n = 0, \dots, 3$ leads to $x(t) = e_q^{-t}$; (c) $C_n = (-1)^{n/2}[(1)^n + (-1)^n]/2n!$ for $n = 0, \dots, 3$ leads to $x(t) = \cos_q t$; (d) $C_n = (-1)^{(n-1)/2}[(1)^n - (-1)^n]/2n!$ for $n = 0, \dots, 3$ leads to $x(t) = \sin_q t$.

The functions e_q^t, e_q^{-t}, $\cos_q t$ and $\sin_q t$ have in the integer power series, the indicated coefficients for $n = 0, \dots, 3$ and the coefficients in (60) for $n > 3$. As it can be verified, for $q \rightsquigarrow 1$, these functions have as limits the functions e^t, e^{-t}, $\cos t$ and $\sin t$, respectively. The latter are nothing else than a basis of the space of solutions of a similar to (58) differential equation for the corresponding continuous problem.

3.4. The q-optimal control problem

Suppose that it is given a k-order q-difference equation of the type

$$f^0\big(x, y(x), D_q y(x), \dots, D_q^k y(x), u(x)\big) = 0. \tag{61}$$

The equation is said to be controlled, $u(x)$ and $y(x)$ the control function and control trajectory, respectively. Let $J(y(x), u(x))$ be a controlled q-integral functional in the sense that it depends on the control function $u(x)$:

$$J\big(y(x), u(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} x f\big(x, y(x), D_q y(x), \dots, D_q^k y(x), u(x)\big). \tag{62}$$

The optimal control problem consists in that among all admissible control functions $u(x)$ for which the corresponding solution of the q-difference equation in (61) satisfies the boundary constraints

$$D_q^i y(q^\alpha) = D_q^i y(q^{\beta+1}) = c_i, \quad i = 0, 1, \dots, k-1, \tag{63}$$

find that for which the solution in question is an extremum for the functional in (62). For that it is convenient to reduce the q-difference equation (61) in a first-order q-difference system of range k (supposing that Eq. (61) is solvable in rapport with $D_q^k y(x)$): Letting $z_1 = y(x)$, $z_2 = D_q y(x), \dots, z_k = D_q^{k-1} y(x)$, and

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix},$$

(61) and (63) can be written simply

$$\begin{aligned} &D_q z(x) = \tilde{f}^0\big(x, z(x), u(x)\big), \\ &z(q^\alpha) = z(q^\beta) = C, \end{aligned} \tag{64}$$

and the functional in (62) takes the form

$$\tilde{J}\big(z(x), u(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} x \tilde{f}\big(x, z(x), u(x)\big). \tag{65}$$

We note by passing that the algorithms for the evaluation of $\tilde{f}^0$ and $\tilde{f}$ are elementary ones. Thus following the q-Lagrange problem, our extremum problem consists in finding the extremum of the functional under the constraints below (remark that as there is no any derivative of $u(x)$, no boundary constraints for it are needed):

$$\hat{J}\big(y(x), u(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} x\big\{\tilde{f}(x,z,u) - \lambda(x)\big[\tilde{f}^0(x,z,u) - D_q z\big]\big\},$$
$$z(q^\alpha) = z(q^\beta) = C. \tag{66}$$

According to (30), the corresponding q-Euler–Lagrange system reads

$$\big(\tilde{f}_z - \lambda(x)\tilde{f}_z^0\big) - D_q\big[\lambda(q^{-1}x)\big] = 0,$$
$$\tilde{f}_u - \lambda(x)\tilde{f}_u^0 = 0. \tag{67}$$

Combining (67) with the first equation in (64), we conclude that the solution of the problem satisfies the system

$$D_q z = +H_\lambda,$$
$$D_q\big[\lambda(q^{-1}x)\big] = -H_z,$$
$$0 = H_u, \tag{68}$$

where

$$H(x,z,\lambda,u) = -\tilde{f}(x,z,u) + \lambda(x)\tilde{f}^0(x,z,u). \tag{69}$$

Seeing the similarities of the problem posed and the formula obtained (Eqs. (68)–(69)), with their analogs in the continuous optimal control, one can say that we were dealing with a q-version of one of the version of the "*maximum principle*" (see [11] or [5], for example). Hence we can refer to H in (69) as the q-Hamilton–Pontriaguine function, (68) as the q-Hamilton–Pontriaguine system. Recall that the reference to Pontriaguine is linked to the "maximum principle" in [11], the one to Hamilton is linked to the fact that in the case of pure calculus of variation (the control function and system are not present explicitly), the Hamilton and Hamilton–Pontriaguine systems are equivalent (see the following section for the q-situation).

Example (*q-Linear–quadratic problem*). Suppose that the problem is that of finding a control function $u(x)$ such that the corresponding solution of the controlled system

$$D_q y = -ay(x) + u(x), \quad a > 0, \tag{70}$$

satisfying the boundary conditions $y(q^\alpha) = y(q^{\beta+1}) = c$, is an extremum element for the q-integral functional (*q-quadratic cost functional*)

$$J\big(y(x), u(x)\big) = \frac{1}{2}(1-q)\sum_{q^\alpha}^{q^\beta} x\big(y^2(x) + u^2(x)\big). \tag{71}$$

According to (68) and (69), the solution of the problem satisfies

$$\begin{aligned} &D_q y = H_\lambda,\\ &D_q\big[\lambda(q^{-1}x)\big] = -H_y,\\ &H_u = 0, \end{aligned} \tag{72}$$

where

$$H(y,\lambda,u) = -\frac{1}{2}(y^2+u^2) + (-ay+u)\lambda(x). \tag{73}$$

(72) and (73) give

$$\begin{aligned} &D_q y = -ay+u,\\ &D_q\lambda(x) = qy(qx) + aq\lambda(qx),\\ &\lambda = u. \end{aligned} \tag{74}$$

In term of $y(x)$, this system can be simplified in the following:

$$D_q^2 y(x) + aD_q y(x) = (a^2+1)qy(qx) + aqD_q y(qx). \tag{75}$$

Searching the solution of (75) under the form of an integer power series

$$y(x) = \sum_0^\infty c_n x^n \tag{76}$$

one is led to a variable coefficient linear homogeneous second-order difference equation for c_n:

$$c_n = a(q-1)c_{n-1} + q(a^2+1)\frac{(1-q)^2}{(1-q^{n-1})(1-q^n)}c_{n-2}. \tag{77}$$

This difference equation can naturally be solved recursively starting from the initial data c_0 and c_1.

However, even without solving it, we can search for what gives the corresponding function in (76), in the limiting case when $q \rightsquigarrow 1$. In (77), for $q \rightsquigarrow 1$, the factor of c_{n-1} gives zero, while that of c_{n-2} gives $(a^2+1)/n(n-1)$. Hence for $q \rightsquigarrow 1$, (77) gives

$$c_n = \frac{a^2+1}{n(n-1)}c_{n-2}, \quad n = 2, \ldots. \tag{78}$$

Choosing c_0 and c_1 (this is equivalent to that choosing $y(q^\alpha)$ and $y(q^{\beta+1})$) as $c_0 = 1$ and $c_1 = \sqrt{a^2+1}$ or $c_1 = -\sqrt{a^2+1}$, (78) gives as solutions

$$c_n = \frac{(a^2+1)^{n/2}}{n!} \quad \text{or} \quad c_n = (-1)^n\frac{(a^2+1)^{n/2}}{n!}$$

and the corresponding power series gives

$$y(x) = \exp\big(\sqrt{a^2+1}x\big) \quad \text{or} \quad y(x) = \exp\big(-\sqrt{a^2+1}x\big),$$

respectively. As it can be verified, the latter are the solutions for $y(x)$ in the corresponding continuous problem.

3.5. Interconnection between the variational q-calculus, the q-optimal control and the q-Hamilton system

Here, we want to show that for the simplest case of finding the extremum of the functional

$$J\big(y(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} xF\big(y(x), D_q y(x)\big),$$
$$y(q^\alpha) = y(q^{\beta+1}) = c_0, \tag{79}$$

the three kinds of problems are equivalents, i.e., are equivalent the q-Euler–Lagrange equation, the q-Hamilton–Pontriaguine and the q-Hamilton systems. We show this in three steps:

(a) We first show how to obtain the q-Hamilton system from the q-Euler–Lagrange equation. For the functional in (79), the q-Euler–Lagrange equation reads

$$F_0\big(y(x), D_q y(x)\big) - D_q\big[F_1\big(y(q^{-1}x), D_q y(q^{-1}x)\big)\big] = 0. \tag{80}$$

Letting

$$\lambda(x) = F_1\big(y(x), D_q y(x)\big) \tag{81}$$

and

$$H = -F + \lambda(x) D_q y, \tag{82}$$

then we get from (80), (81) and (82) the q-Hamilton system

$$D_q y = +H_\lambda\big(y(x), \lambda, D_q y\big),$$
$$D_q\big[\lambda(q^{-1}x)\big] = -H_y\big(y(x), \lambda, D_q y\big). \tag{83}$$

(b) To get the q-Hamilton–Pontriaguine system from q-Hamilton system (83), it suffices to suppose $u(x) = D_q y(x)$ to be the control q-equation for the given initial noncontrolled extremum problem. In that case, (83) gives

$$D_q y = +H_\lambda\big(y(x), \lambda, u(x)\big),$$
$$D_q\big[\lambda(q^{-1}x)\big] = -H_y\big(y(x), \lambda, u(x)\big), \tag{84}$$

with

$$H\big(y(x), \lambda(x), u(x)\big) = -F\big(y(x), u(x)\big) + \lambda(x)u(x), \tag{85}$$

the q-Hamilton–Pontriaguine function, and from (81) we get the third equation in (68):

$$H_u = 0. \tag{86}$$

(c) Finally we show how to obtain the q-Euler–Lagrange equation (80) from the q-Hamilton–Pontriaguine system (84)–(86). From (85) and (86), we have

$$\lambda(x) = F_1\big(y(x), u(x)\big) = F_1\big(y(x), D_q y(x)\big), \tag{87}$$

while from (84) we get

$$D_q\big[\lambda(q^{-1}x)\big] = F_0\big(y(x), u(x)\big) = F_0\big(y(x), D_q y(x)\big). \tag{88}$$

Finally, (87) and (88) give the q-Euler–Lagrange equation (80).

3.6. A q-version of the commutation equations

Let $L = -D^2 + y(x)$, where $Df(x) = df(x)/dx = f'(x)$, be the Schrödinger operator and let A_m be a sequence of differential operators of order $2m+1$, $m = 0, 1, 2, \ldots,$ which coefficients are arbitrary differential polynomials of the potential $y(x)$. By commutation equations, one understands the equations $[L, A_m] = LA_m - A_m L = 0$ in the coefficients of the operators. It is known since [1,2] that for any m, $m = 0, 1, 2, \ldots,$ there exists such an operator A_m of order $2m+1$, such that the operator $[L, A_m] = LA_m - A_m L$ is an operator of multiplication by a scalar function $f_m(y, y', y'', \ldots)$: $[L, A_m] = f_m(y, y', y'', \ldots)$. The corresponding commutation equations then read

$$[L, A_m] = f_m(y, y', y'', \ldots) = 0. \tag{89}$$

Its nontrivial solutions are elliptic or hyperelliptic (or their degenerate cases) functions for $m = 1$ and $m > 1$, respectively (see [1,2]). Since the 70s of the last century (see, e.g., [4, §30], it is known that the commutation equations (89) are equivalent to type (32) Euler–Lagrange equations for the functionals

$$J_m\big(y(x)\big) = \int_a^b L_m\big(y(x), y'(x), \ldots, y^{(k)}(x)\big)\,dx \tag{90}$$

with L_m related to A_m in a known way (see, e.g., [4]).

If $m = 1$, for example, $L_1(y, y') = y'^2/2 + y^3 + c_1 y^2 + c_2 y$ (c_1, c_2: constants) and the corresponding Euler–Lagrange equation (commutation equation) reads:

$$y'' = 3y^2 + 2c_1 y + c_2. \tag{91}$$

Up to a linear transformation $y \to c_3 y + c_4$, its solution is the well known Weierstrass function $\mathcal{P}(x)$.

Considering now the q-functional

$$J_m\big(y(x)\big) = (1-q)\sum_{q^\alpha}^{q^\beta} x L_m\big(y(x), D_q y(x), \ldots, D_q^k y(x)\big) \tag{92}$$

we obtain that the corresponding to type (29) q-Euler–Lagrange equations are *q-versions of the commutation equations* (89). For example, for $m = 1$, we have $L_1(y(x), D_q y(x)) = [D_q y]^2/2 + y^3 + c_1 y^2 + c_2 y$ and the corresponding q-Euler–Lagrange equation reads

$$3y^2 + 2c_1 y + c_2 - q D_q^2\big[y(q^{-1}x)\big] = 0, \tag{93}$$

or equivalently

$$y(qx) = (q+1)y(x) + (qx - x)^2\big(3y^2(x) + 2c_1 y(x) + c_2\big) - q y(q^{-1}x). \tag{94}$$

Obviously, the q-Euler–Lagrange equation (93) (or (94)) tends to the Euler–Lagrange one in (91), while $q \rightsquigarrow 1$. One will note that thought we up to now do not know an analytical resolution of this equation, its solution satisfying given boundary constraints can be found recursively. Here is naturally the main advantage of the analysis on lattices.

Remark 3. What we done in this section is to give a q-version of the commutation equations in terms of the q-Euler–Lagrange equations of q-integration functionals. One may ask why do not give q-versions of commutation equations in terms of commutation equations of q-difference operators, i.e., operators obtained from differential ones replacing D by D_q. The situation is that this line of attack is not hopeful especially because of the absence of symmetries in most of operations with the q-derivative. For example, the simple fact that the formula $D_q fg = f(qx)D_q g + g(x)D_q f$ is not symmetric in rapport with f and g is tedious in classical q-analysis. Clearly, a study of the q-commutation equations using the q-variational method needs an independent consecration.

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