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On a stochastic singular diffusion equation in R^d

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Abstract

We establish the existence and uniqueness of a strong solution to the Cauchy problem for a singular diffusion equation with random noise in R^d with initial data in $L^2(R^d)$ with bounded variation or in $H^1(R^d)$. We also prove the existence of an invariant measure and extinction of a solution in finite time.

Keywords: Singular diffusion; Random noise; Cauchy problem; Torus; Existence of a strong solution; Invariant measure; Extinction of a solution

0. Introduction

In this paper, we will discuss three issues concerned with a singular diffusion equation in R^d, $d \geqslant 2$, with random noise.

$$\frac{\partial u}{\partial t} - \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) = g_0 + \sum_{j=1}^{\infty} g_j(u) \frac{dB_j}{dt}, \quad (t, x) \in (0, \infty) \times R^d, \tag{0.1}$$

$$u(0, x) = u_0(x), \quad x \in R^d, \tag{0.2}$$

where u is a scalar-valued function, and $\{B_j\}_{j=1}^{\infty}$ is a sequence of mutually independent standard Brownian motions. Each $g_j(\cdot)$ is a given map. For the deterministic case where $g_j \equiv 0$, for all j, Eq. (0.1) is associated with some mathematical models in image processing and facet growth of

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crystals; see [12,13,15,18–20]. For the deterministic case $g_j \equiv 0$, for all j, an initial boundary value problem on a bounded space domain was discussed in [2,3,5], where the existence and uniqueness of an entropy solution and a strong solution was established with initial data in L^1 and L^2, respectively. For the zero Dirichlet boundary condition and initial datum in L^2, the solution becomes extinct in finite time, and for the zero Neumann boundary condition, the solution reaches the average of the initial datum in finite time. This was proved in [4]. The Cauchy problem in the whole space R^d was discussed in [10]; see also [5]. They proved the existence and uniqueness of a solution when the initial conditions are given in $L^2(R^d)$. For the initial condition in $L^1_{loc}(R^d)$, they proved the existence and uniqueness of an entropy solution. In [2,3,5,10], the main tool for the existence of a solution is the nonlinear semigroup generated by maximal monotone operators.

In a random environment, necessary information for mathematical modelling is obtained statistically. This gives rise to stochastic model equations. For any evolution equation, it is natural to ask how random perturbation can affect solutions from the statistical viewpoint. This often poses challenges in analysis. At present, [8] seems to be the only work for the equation perturbed by random noise. When the space domain is a bounded subset of R^d, $d \leqslant 2$, with the zero Dirichlet or Neumann boundary condition, the existence of a solution with initial datum in L^2 was proved in [8]. The definition of a solution is given in the form of an inequality which involves random test functions satisfying a certain stochastic evolution equation. For additive noise satisfying some conditions, the uniqueness of a solution and existence of an invariant measure was also proved. For linear multiplicative noise, the uniqueness of a solution was left open. The basic idea for the existence of a solution in [8] was to regularize the singular term by means of the Yosida approximation.

We now point out a major hurdle in the stochastic equation (0.1). For the deterministic case, the singular term can be represented by a function in L^2 for almost all time if the initial data are in L^2. But this is not true in the stochastic case, and the definition of a solution of the deterministic equation cannot be adapted to the stochastic case in a natural manner. This makes it necessary to employ a definition of a solution which looks quite different from that of the deterministic case.

The goal of this paper is to obtain solutions of the Cauchy problem in R^d according to the definition of a solution which is a natural adaptation of that for the deterministic equation. Hence, we require more regularity on the initial data. For this, there are two different directions. On the one hand, application to image processing suggests the use of the function class of bounded variation. So we will consider initial data in $L^2(R^d)$ with bounded variation. This class includes the initial data which are the characteristic functions of a bounded set with finite perimeter, which can generate explicit solutions. These explicit solutions were studied in the above references for the deterministic equation. We will show that if the initial data are in this class and the random noise is purely multiplicative under some conditions, the singular term can be represented by a function in $L^2(R^d)$, and hence, the definition of a solution for the deterministic equation can be adapted. In this setting, we will also prove the existence of a unique invariant measure for any $d \geqslant 2$, and finite time extinction of a solution in probability when $d = 2$ and $g_0 \equiv 0$ in (0.1). For the deterministic equation on a bounded space domain, finite time extinction was proved in [4] without restriction on d. However, the method in [4] does not seem to be extended to the stochastic equation.

On the other hand, $H^1(R^d)$ is a standard candidate next to $L^2(R^d)$ if more regularity is needed. For a related problem on p-harmonic map heat flow, initial data in H^1 were used in [9]. We will show that if the initial data are in $H^1(R^d)$, the gradient of a solution lies in both $L^2(R^d)$ and $L^1(R^d)$, for almost all time. Furthermore, we can also include additive random noise. However, the existence of an invariant measure and finite time extinction in the H^1-setting are still open questions; see Remark 5.2 below.

Our results on the existence of a solution to the Cauchy problem are new even for the deterministic equation. But, we do not know whether our existence results for R^d are valid for a bounded domain with the Dirichlet or Neumann boundary conditions.

The general strategy for our results is to use the method of expanding torus. We first obtain a solution on a torus with period L. By virtue of periodic property and compactness of a torus, we can obtain basic estimates uniformly in the period L. More specifically, we can regularize the singular term in a straightforward manner because the domain is bounded, and integration by parts with respect to the space variables is easy because the domain has no boundary. Then, we obtain a solution in R^d as a limit by passing $L \to \infty$. Our procedure is different from those of all previous works mentioned above.

The paper is organized as follows. In Section 1, we explain some notation, and state the main results. In Section 2, we present some technical preliminaries. The Cauchy problem on a torus is discussed in Section 3, which provides basic estimates for the existence of a solution in R^d. The remaining sections are devoted to the proof of the main results.

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1. Notation and statement of the main results

A stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is given throughout this paper, where $\{\mathcal{F}_t\}$ is a filtration over the probability space $(\Omega, \mathcal{F}, P)$ such that it satisfies the usual condition, i.e., it is right continuous, and $\mathcal{F}_0$ contains all P-negligible sets in $\mathcal{F}$. $\{B_j\}_{j=1}^{\infty}$ is a sequence of mutually independent standard Brownian motions over this stochastic basis. All stochastic integrals are defined in the sense of Ito. For a topological space $\mathcal{X}$, $\mathcal{B}(\mathcal{X})$ denotes the set of all Borel subsets of $\mathcal{X}$.

Let $T > 0$ be given and consider a collection of sets defined by

$$\mathcal{G} = \bigl\{A \in \mathcal{F}_T \otimes \mathcal{B}\bigl([0,T]\bigr) \bigm| A \cap \bigl(\Omega \times [0,t]\bigr) \in \mathcal{F}_t \otimes \mathcal{B}\bigl([0,t]\bigr) \text{ for each } t \in [0,T]\bigr\}.$$

Then, $(\Omega \times [0,T], \mathcal{G}, dP \otimes dt)$ is a finite measure space. Let $\mathcal{X}$ be a topological space. Consider the set of all functions from $\Omega \times [0,T]$ to $\mathcal{X}$. In this set, we identify $dP \otimes dt$-equivalent functions as the same function, and a function is said to be $\mathcal{X}$-valued progressively measurable if it has a $dP \otimes dt$-equivalent function f such that $f^{-1}(Q) \in \mathcal{G}$ for each $Q \in \mathcal{B}(\mathcal{X})$.

When $\mathcal{X}$ is a Banach space, $L^r(\Omega \times [0,T]; \mathcal{G}; \mathcal{X})$, $1 \leqslant r \leqslant \infty$, is defined in the usual sense of Bochner with respect to the measure $dP \otimes dt$.

For general references on stochastic analysis relevant to the present paper, see [11,14,16].

Let $\mathcal{M}(R^d)$ be the set of all finite Radon measures on R^d. Then, it is a Banach space under the norm of total variation $\|\cdot\|_{\mathcal{M}(R^d)}$. When $\nu = (\nu_1, \ldots, \nu_d) \in (\mathcal{M}(R^d))^d$, we employ the norm

$$\|\nu\|_{(\mathcal{M}(R^d))^d} = \sup\left\{ \sum_{j=1}^{d} \int_{R^d} \psi_j \, d\nu_j \biggm| \psi = (\psi_1, \ldots, \psi_d) \in \bigl(C_0\bigl(R^d\bigr)\bigr)^d,\ \bigl\||\psi|\bigr\|_{L^\infty(R^d)} \leqslant 1 \right\}$$

where $C_0(R^d)$ stands for the set of all bounded continuous functions on R^d which vanish at infinity.

Let us write

$$\mathcal{Y} = L^2(R^d) \times (\mathcal{M}(R^d))^d$$

where $L^2(R^d)$ is equipped with norm topology and $(\mathcal{M}(R^d))^d$ is equipped with weak star topology. Let

$$\mathcal{S} = \{ f \in L^2(R^d) \mid \nabla f \in (\mathcal{M}(R^d))^d \},$$

and $\Lambda : f \mapsto (f, \nabla f)$. Then, Λ is a one-to-one linear mapping from $\mathcal{S}$ into $\mathcal{Y}$, and $\Lambda(\mathcal{S})$ is a linear subspace of $\mathcal{Y}$. We can equip $\mathcal{S}$ with the topology generated by $\Lambda^{-1}(\Theta)$ for all open subset Θ of $\mathcal{Y}$. It holds that $\mathcal{B}(\mathcal{S}) \subset \mathcal{B}(L^2(R^d))$; see Lemma 2.6 below.

Throughout this paper, we avoid using the notation $BV(R^d)$, because its definition requires that $BV(R^d) \subset L^1(R^d)$. The notation $\sup_{0 \leqslant t \leqslant T}\{\cdots\}$ will be used to denote the essential supremum of $\{\cdots\}$ on the interval $[0, T]$.

1.1. Existence I

Let $T > 0$ be given. We assume

$$g_0 \in L^2(0, T; H^1(R^d)), \qquad g_j \in C([0, T] \times L^2(R^d); L^2(R^d)), \quad \forall j \geqslant 1, \tag{1.1}$$

such that for all $t \in [0, T]$, all $v, w \in L^2(R^d)$,

$$\|g_j(t, v)\|_{L^2(R^d)} \leqslant c_j + d_j \|v\|_{L^2(R^d)}, \quad \forall j \geqslant 1, \tag{1.2}$$

$$\|g_j(t, v) - g_j(t, w)\|_{L^2(R^d)} \leqslant d_j \|v - w\|_{L^2(R^d)}, \quad \forall j \geqslant 1, \tag{1.3}$$

where c_j's and d_j's are nonnegative constants, and

$$\sum_{j=1}^{\infty} (c_j^2 + d_j^2) < \infty. \tag{1.4}$$

We also assume that for each t and $v \in H^1(R^d)$, $g_j(t, v) \in H^1(R^d)$, and

$$\|g_j(t, v)\|_{H^1(R^d)} \leqslant c_j + d_j \|v\|_{H^1(R^d)}, \quad \forall j \geqslant 1. \tag{1.5}$$

Definition 1.1. u is said to be a solution of (0.1)–(0.2) if the following conditions are satisfied.

(i) u is $H^1(R^d)$-valued progressively measurable,
(ii) $u \in C([0, T]; L^2(R^d))$, P-almost surely,
(iii) $E(\sup_{0 \leqslant t \leqslant T} \|u\|^2_{H^1(R^d)}) < \infty$,
(iv) $\nabla u \in L^1(\Omega \times [0, T]; \mathcal{G}; (L^1(R^d))^d)$,

(v) there is $\Pi \in L^\infty(\Omega \times [0,T] \times R^d; \mathcal{G} \otimes \mathcal{B}(R^d); R^d)$ such that

$$|\Pi| \leqslant 1, \quad dP \otimes dt \otimes dx\text{-almost all } (\omega, t, x),$$
$$\Pi \cdot \nabla u = |\nabla u|, \quad dP \otimes dt \otimes dx\text{-almost all } (\omega, t, x),$$

and for each $\psi \in C_0^\infty(R^d)$, it holds that

$$\psi u(t) = \psi u_0 + \int_0^t \psi \nabla \cdot \Pi(s)\, ds + \int_0^t \psi g_0(s)\, ds + \sum_{j=1}^\infty \int_0^t \psi g_j\big(u(s)\big)\, dB_j(s) \quad \text{in } H^{-1}\big(R^d\big) \tag{1.6}$$

for all $t \in [0,T]$, P-almost surely.

Theorem 1.2. *Suppose that* $u_0 \in L^2(\Omega; \mathcal{F}_0; H^1(R^d))$. *Under the conditions* (1.1)–(1.5)*, there is a unique solution of* (0.1)–(0.2).

1.2. Existence II

We assume

$$g_0 \in L^2\big(0,T; L^2\big(R^d\big)\big), \qquad \nabla g_0 \in \big(L^1\big([0,T] \times R^d\big)\big)^d,$$
$$g_j \in C\big([0,T] \times R\big), \quad \forall j \geqslant 1, \tag{1.7}$$

such that for all $t \in [0,T]$, and all $z, w \in R$,

$$g_j(t, 0) = 0, \quad \forall j \geqslant 1, \tag{1.8}$$

and

$$\big|g_j(t,z) - g_j(t,w)\big| \leqslant d_j |z - w|, \quad \forall j \geqslant 1, \tag{1.9}$$

for nonnegative constants d_j's such that

$$\sum_{j=1}^\infty d_j^2 < \infty. \tag{1.10}$$

Definition 1.3. u is said to be a solution of (0.1)–(0.2) if the following conditions are satisfied.

(i) u is $\mathcal{S}$-valued progressively measurable,
(ii) $u \in C([0,T]; \mathcal{S})$, P-almost surely,
(iii) $E(\sup_{0 \leqslant t \leqslant T} \|u(t)\|^2_{L^2(R^d)}) + E(\sup_{0 \leqslant t \leqslant T} \|\nabla u(t)\|_{(\mathcal{M}(R^d))^d}) < \infty$,

(iv) there is $\Pi \in L^\infty(\Omega \times [0,T] \times R^d; \mathcal{G} \otimes \mathcal{B}(R^d); R^d)$ such that

$$|\Pi| \leqslant 1, \quad dP \otimes dt \otimes dx\text{-almost all } (\omega, t, x),$$
$$\nabla \cdot \Pi \in L^2\big(\Omega \times [0,T]; \mathcal{G}; L^2\big(R^d\big)\big),$$
$$-\langle \nabla \cdot \Pi, u\rangle_{L^2(R^d)} = \|\nabla u\|_{(\mathcal{M}(R^d))^d}, \quad dP \otimes dt\text{-almost all } (\omega, t)$$

and it holds P-almost surely that

$$u(t) = u_0 + \int_0^t \nabla \cdot \Pi(s)\, ds + \int_0^t g_0(s)\, ds + \sum_{j=1}^\infty \int_0^t g_j\big(u(s)\big)\, dB_j \tag{1.11}$$

in $L^2(R^d)$, for all $t \in [0,T]$.

Theorem 1.4. *Let $u_0 \in L^2(\Omega; \mathcal{F}_0; L^2(R^d))$ such that $\|\nabla u_0\|_{(\mathcal{M}(R^d))^d} \in L^1(\Omega; \mathcal{F}_0)$. Under the conditions* (1.7)–(1.10)*, there is a unique solution of* (0.1)–(0.2)*.*

1.3. Invariant measures

We assume

$$g_0 \in L^2\big(R^d\big), \qquad \nabla g_0 \in \big(L^1\big(R^d\big)\big)^d, \qquad g_j = \xi_j(\cdot) \in W^{1,\infty}(R), \quad \forall j \geqslant 1, \tag{1.12}$$
$$\xi_j(0) = 0, \qquad \big|\xi_j(z) - \xi_j(w)\big| \leqslant \beta_j |z - w|, \quad \forall z, w \in R, \tag{1.13}$$
$$\xi_j'(z) \geqslant \alpha_j, \quad \text{for almost all } z \in R, \quad \text{or} \quad \xi_j'(z) \leqslant -\alpha_j, \quad \text{for almost all } z \in R, \tag{1.14}$$

where α_j's and β_j's are nonnegative constants satisfying the condition

$$\sum_{j=1}^\infty \beta_j^2 < 2 \sum_{j=1}^\infty \alpha_j^2 < \infty. \tag{1.15}$$

Under (1.12)–(1.15), the conditions (1.7)–(1.10) are satisfied for all $0 < T < \infty$.

Definition 1.5. Let $X(\cdot; 0, y)$ denote the solution of (0.1) with $X(0; 0, y) = y \in \mathcal{S}$ according to Definition 1.3. A probability measure μ on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ is called an invariant measure if

$$\mu(G) = \int_{\mathcal{S}} E\big(\chi_G\big(X(t; 0, y)\big)\big)\, d\mu(y)$$

for all $G \in \mathcal{B}(\mathcal{S})$ and all $t \geqslant 0$.

Theorem 1.6. *Under the assumptions* (1.12)–(1.15)*, there is a unique invariant measure over* $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$.

1.4. Extinction of a solution in probability

We assume

$$g_0 \equiv 0, \qquad g_j = \xi_j(\cdot) \in W^{1,\infty}(R), \quad \forall j \geqslant 1, \tag{1.16}$$

$$\sum_{j=1}^{\infty} \beta_j^2 < \frac{3}{2} \sum_{j=1}^{\infty} \alpha_j^2 < \infty. \tag{1.17}$$

Under (1.13)–(1.14) and (1.16)–(1.17), all conditions for Theorem 1.4 are satisfied for all $0 < T < \infty$.

Let u be a solution of (0.1)–(0.2) for $(t, x) \in (0, \infty) \times R^d$. We define a stopping time

$$\tau = \begin{cases} \inf\{t > 0 \mid \|u(t)\|_{L^2(R^d)} = 0\}, \\ \infty, \quad \text{if the set } \{\cdots\} \text{ is empty.} \end{cases} \tag{1.18}$$

Theorem 1.7. *Suppose $d = 2$ and $u_0 \in L^2(\Omega; \mathcal{F}_0; L^2(R^2))$ such that $\|\nabla u_0\|_{(\mathcal{M}(R^2))^2} \in L^1(\Omega; \mathcal{F}_0)$. Under the assumptions* (1.13)–(1.14)*, and* (1.16)–(1.17)*,*

$$u(t) = 0, \quad \forall t \geqslant \tau(\omega), \ P\text{-almost surely}, \tag{1.19}$$

and

$$P\{\tau \leqslant t\} \geqslant 1 - \frac{C}{e^{\sigma t} - 1} E\big(\|u_0\|_{L^2(R^2)}^{\frac{1}{2}}\big), \quad \forall t > 0, \tag{1.20}$$

where C and σ are positive constants independent of u_0.

2. Technical lemmas

Let G_L be the building block of the d-dimensional torus $\mathcal{T}_L$ with period L, i.e.,

$$G_L = \left\{ x = (x_1, \ldots, x_d) \,\middle|\, -\frac{L}{2} \leqslant x_j < \frac{L}{2}, \ j = 1, \ldots, d \right\}.$$

For a real number s, the Sobolev spaces H_L^s on $\mathcal{T}_L$ is characterized in terms of the Fourier series.

$$H_L^s = \left\{ f \,\middle|\, f = \sum_{k \in \mathcal{Z}^d} c_k e^{i\frac{2\pi k \cdot x}{L}}, \ c_{-k} = \overline{c_k}, \ \sum_{k \in \mathcal{Z}^d} \big(1 + |k|^2\big)^s |c_k|^2 < \infty \right\}$$

where $\mathcal{Z}$ is the set of all integers, and $k = (k_1, \ldots, k_d)$. $L^2(G_L)$ can be identified with H_L^0. We equip H_L^0 and H_L^1 with the following norms, respectively.

$$\|f\|_{H_L^0} = \|f\|_{L^2(G_L)}, \qquad \|f\|_{H_L^1} = \big(\|f\|_{L^2(G_L)}^2 + \big\||\nabla f|\big\|_{L^2(G_L)}^2\big)^{\frac{1}{2}}.$$

Lemma 2.1. *Let $h \in H_L^2$. It holds that for each positive constant ϵ,*

$$\left\langle \nabla \cdot \left(\frac{\nabla h}{\sqrt{\epsilon + |\nabla h|^2}} \right), \Delta h \right\rangle_{L^2(G_L)} \geqslant 0.$$

Proof. Let $\{e_j\}_{j=1}^d$ denote the standard orthonormal basis for R^d. Then, for each $1 \leqslant j \leqslant d$,

$$\frac{h(\cdot + \eta e_j) - 2h(\cdot) + h(\cdot - \eta e_j)}{\eta^2} \to \frac{\partial^2 h}{\partial x_j^2}(\cdot)$$

as $\eta \to 0$, strongly in $L^2(G_L)$. Thus,

$$\begin{aligned}
&\left\langle \nabla \cdot \left(\frac{\nabla h}{\sqrt{\epsilon + |\nabla h|^2}} \right), \frac{\partial^2 h}{\partial x_j^2} \right\rangle_{L^2(G_L)} \\
&= \lim_{\eta \to 0} \left\langle \nabla \cdot \left(\frac{\nabla h}{\sqrt{\epsilon + |\nabla h|^2}} \right), \frac{h(\cdot + \eta e_j) - h(\cdot)}{\eta^2} - \frac{h(\cdot) - h(\cdot - \eta e_j)}{\eta^2} \right\rangle_{L^2(G_L)} \\
&= \lim_{\eta \to 0} \left\langle \nabla \cdot \left(\frac{\nabla h(\cdot)}{\sqrt{\epsilon + |\nabla h(\cdot)|^2}} \right) - \nabla \cdot \left(\frac{\nabla h(\cdot + \eta e_j)}{\sqrt{\epsilon + \left|\nabla h(\cdot + \eta e_j)\right|^2}} \right), \frac{h(\cdot + \eta e_j) - h(\cdot)}{\eta^2} \right\rangle_{L^2(G_L)} \\
&\geqslant 0,
\end{aligned}$$

where the last inequality follows from the convexity of the functional

$$h \mapsto \int_{G_L} \sqrt{\epsilon + |\nabla h|^2}\, dx. \qquad \square$$

If $f \in BV(R^d)$, i.e., $f \in L^1(R^d)$ with $\nabla f \in (\mathcal{M}(R^d))^d$, the following fact follows from results in [6]; see also [5]. But for $f \in \mathcal{S}$, we need extra effort to find an approximating sequence in $\mathcal{S} \cap L^1(R^d)$.

Lemma 2.2. *Let $\Psi \in (L^\infty(R^d))^d$ be such that $\nabla \cdot \Psi \in L^2(R^d)$. Then, for all $f \in \mathcal{S}$,*

$$-\langle \nabla \cdot \Psi, f \rangle_{L^2(R^d)} \leqslant \big\| |\Psi| \big\|_{L^\infty(R^d)} \|\nabla f\|_{(\mathcal{M}(R^d))^d}. \tag{2.1}$$

Proof. First we assume that $f \in \mathcal{S}$ has compact support in R^d. Let ρ_ϵ be the Friedrichs mollifier. Then,

$$-\langle \Psi * \rho_\epsilon, \nabla f * \rho_\epsilon \rangle_{(L^\infty(R^d))^d, (L^1(R^d))^d} = \big\langle (\nabla \cdot \Psi) * \rho_\epsilon, f * \rho_\epsilon \big\rangle_{L^2(R^d)} \to \langle \nabla \cdot \Psi, f \rangle_{L^2(R^d)}$$

and

$$\big| \langle \Psi * \rho_\epsilon, \nabla f * \rho_\epsilon \rangle_{(L^\infty(R^d))^d, (L^1(R^d))^d} \big| \leqslant \big\| |\Psi| \big\|_{L^\infty(R^d)} \|\nabla f\|_{(\mathcal{M}(R^d))^d}, \quad \forall \epsilon > 0.$$

Thus, (2.1) holds. Next we assume that $f \in \mathcal{S} \cap L^1(R^d)$. Choose a function $\varphi \in C_0^\infty(R^d)$ such that

$$\varphi(x) = \begin{cases} 1, & \text{for } |x| \leqslant 1, \\ 0, & \text{for } |x| \geqslant 2, \end{cases} \quad 0 \leqslant \varphi(x) \leqslant 1, \ \forall x. \tag{2.2}$$

Set

$$\varphi_R(x) = \varphi\left(\frac{x}{R}\right), \quad \text{for each } x. \tag{2.3}$$

Then,

$$\left\| \nabla(\varphi_R f)\right\|_{(\mathcal{M}(R^d))^d} \leqslant \|\nabla f\|_{(\mathcal{M}(R^d))^d} + \frac{C}{R}\|f\|_{L^1(R^d)}$$

for some positive constant C depending only on φ. Since (2.1) holds for $\varphi_R f$ and

$$\varphi_R f \to f \quad \text{strongly in } L^2\big(R^d\big),$$

(2.1) is valid for f by passing $R \to \infty$. Next we drop the assumption $f \in L^1(R^d)$. Let us define

$$h_\delta(z) = \begin{cases} \frac{z^2}{\delta} sgn(z), & -\delta < z < \delta, \\ z, & \text{otherwise}. \end{cases} \tag{2.4}$$

If $f \in \mathcal{S}$, then, $h_\delta(f * \rho_\epsilon) \in \mathcal{S} \cap L^1(R^d)$, and we can apply the previous result to $h_\delta(f * \rho_\epsilon)$. In the meantime, for fixed $\epsilon > 0$,

$$h_\delta(f * \rho_\epsilon) \to f * \rho_\epsilon \quad \text{strongly in } L^2\big(R^d\big), \text{ as } \delta \to 0.$$

Since

$$\nabla(f * \rho_\epsilon)(x) = 0, \quad \text{for almost all } x, \text{ on the set } \big\{x \bigm| (f * \rho_\epsilon)(x) = 0\big\}$$

we find that

$$h'_\delta\big((f * \rho_\epsilon)(x)\big)\nabla(f * \rho_\epsilon)(x) \to \nabla(f * \rho_\epsilon)(x) \quad \text{as } \delta \to 0, \text{ for almost all } x.$$

It follows that (2.1) is valid for $f * \rho_\epsilon$ and hence, for $f \in \mathcal{S}$. □

The following inequality is well known for $f \in BV(R^d)$. For $f \in \mathcal{S}$, it can be proved by means of $h_\delta(\cdot)$ defined by (2.4). But it also follows as a special case of Theorem 3.47 in [1].

Lemma 2.3. *Let $v \in L^2(R^d)$ be such that $\nabla v \in (\mathcal{M}(R^d))^d$. Then, it holds that*

$$\|v\|_{L^{\frac{d}{d-1}}(R^d)} \leqslant C\|\nabla v\|_{(\mathcal{M}(R^d))^d} \tag{2.5}$$

for some positive constant C independent of v.

Lemma 2.4. *Suppose $f \in L^2(\Omega; \mathcal{F}_0; L^2(R^d))$ such that $\|\nabla f\|_{(\mathcal{M}(R^d))^d} \in L^1(\Omega; \mathcal{F}_0)$.*
Then, there is a sequence $\{f_k\}_{k=1}^{\infty}$ such that

$$f_k \to f \quad \text{in } L^2\big(\Omega; \mathcal{F}_0; L^2\big(R^d\big)\big), \text{ as } k \to \infty,$$
$$f_k \in W^{1,1}\big(R^d\big) \cap H^1\big(R^d\big), \quad P\text{-almost surely}, \ \forall k,$$
$$\text{support of } f_k \text{ is contained in } \{x \mid |x| < R_k\}, \ P\text{-almost surely, for some } 0 < R_k < \infty,$$
$$E\big(\|\nabla f_k\|_{(L^1(R^d))^d}\big) \leqslant 1 + E\big(\|\nabla f\|_{(\mathcal{M}(R^d))^d}\big), \quad \forall k.$$

Proof. Let $h_\delta(\cdot)$ and ρ_ϵ be the same as in Lemma 2.2, and let φ_R be defined by (2.2)–(2.3). It is enough to note the following facts.

$$\big\|\nabla(f * \rho_\epsilon)\big\|_{(L^1(R^d))^d} \leqslant \|\nabla f\|_{(\mathcal{M}(R^d))^d}, \quad \forall \epsilon > 0, \ P\text{-almost surely},$$
$$\lim_{\epsilon \to 0} E\big(\|f * \rho_\epsilon - f\|^2_{L^2(R^d)}\big) = 0,$$
$$h_\delta(f * \rho_\epsilon) \in W^{1,1}\big(R^d\big), \quad \forall \delta > 0, \ \forall \epsilon > 0, \ P\text{-almost surely},$$
$$\big\|h_\delta(f * \rho_\epsilon)\big\|_{L^1(R^d)} \leqslant \frac{1}{\delta}\|f * \rho_\epsilon\|^2_{L^2(R^d)}, \quad P\text{-almost surely},$$
$$\lim_{\delta \to 0} E\big(\big\|\nabla h_\delta(f * \rho_\epsilon) - \nabla(f * \rho_\epsilon)\big\|_{(L^1(R^d))^d}\big) = 0, \quad \text{for each fixed } \epsilon > 0,$$
$$\lim_{\delta \to 0} E\big(\big\|h_\delta(f * \rho_\epsilon) - (f * \rho_\epsilon)\big\|^2_{L^2(R^d)}\big) = 0, \quad \text{for each fixed } \epsilon > 0,$$
$$\lim_{R \to \infty} E\big(\big\|\varphi_R h_\delta(f * \rho_\epsilon) - h_\delta(f * \rho_\epsilon)\big\|^2_{L^2(R^d)}\big) = 0, \quad \text{for each fixed } \delta > 0 \text{ and } \epsilon > 0,$$
$$\lim_{R \to \infty} E\big(\big\|\nabla\big(\varphi_R h_\delta(f * \rho_\epsilon)\big)\big\|_{(L^1(R^d))^d}\big) = E\big(\big\|\nabla h_\delta(f * \rho_\epsilon)\big\|_{(L^1(R^d))^d}\big),$$

for each fixed $\delta > 0$ and $\epsilon > 0$. □

Lemma 2.5. *Suppose that $v \in L^2(\Omega \times [0,T]; \mathcal{G}; L^2(R^d))$ such that*

$$E\Big(\sup_{0 \leqslant t \leqslant T}\big\|v(t)\big\|^2_{L^2(R^d)}\Big) < \infty, \tag{2.6}$$
$$\nabla v \in \big(L^1\big((0,T) \times R^d\big)\big)^d, \quad P\text{-almost surely.} \tag{2.7}$$

Let $\Xi \in L^\infty(\Omega \times [0,T] \times R^d; \mathcal{G} \otimes \mathcal{B}(R^d); R^d)$, $v_0 \in L^2(\Omega; \mathcal{F}_0; L^2(R^d))$, and $\psi_j \in L^2(\Omega \times [0,T]; \mathcal{G}; L^2(R^d))$, $\forall j \geqslant 1$, such that

$$\sum_{j=1}^{\infty} E\left(\int_0^T \big\|\psi_j(t)\big\|^2_{L^2(R^d)}\,dt\right) < \infty. \tag{2.8}$$

If it holds that for each $R > 0$,

$$\varphi_R v(t)=\varphi_R v_0+\int_0^t \varphi_R\nabla\cdot\Xi(s)\,ds+\sum_{j=1}^{\infty}\int_0^t \varphi_R\psi_j(s)\,dB_j(s)\quad in\ H^{-1}\big(R^d\big),\tag{2.9}$$

for all $t\in[0,T]$, P-almost surely, where φ_R is defined by (2.2)–(2.3)*, then $v\in C([0,T];L^2(R^d))$, P-almost surely, and*

$$\begin{aligned}\big\|v(t)\big\|^2_{L^2(R^d)}&=\|v_0\|^2_{L^2(R^d)}-2\int_0^t\int_{R^d}\Xi(s)\cdot\nabla v(s)\,ds\\&\quad+2\sum_{j=1}^{\infty}\int_0^t\big\langle\psi_j(s),v(s)\big\rangle_{L^2(R^d)}\,dB_j(s)+\sum_{j=1}^{\infty}\int_0^t\big\|\psi_j(s)\big\|^2_{L^2(R^d)}\,ds\end{aligned}\tag{2.10}$$

for all $t\in[0,T]$, P-almost surely.

Proof. (2.6) and (2.9) imply that v is $L^2(R^d)$-weakly continuous in $t\in[0,T]$, P-almost surely, and that for each $R\geqslant 1$ and $\epsilon>0$,

$$(\varphi_R v)*\rho_\epsilon\in C\big([0,T];L^2\big(R^d\big)\big),\quad P\text{-almost surely.}$$

As above, the convolution is taken with respect to the space variables.

Choose sequences $\{R_k\}\uparrow\infty$, and $\{\epsilon_n\}\downarrow 0$.

It follows from (2.9) and Ito's formula that

$$\begin{aligned}\big\|\big(\varphi_R v(t)\big)*\rho_\epsilon\big\|^2_{L^2(R^d)}&=\big\|(\varphi_R v_0)*\rho_\epsilon\big\|^2_{L^2(R^d)}\\&\quad+2\int_0^t\big\langle\big(\varphi_R\nabla\cdot\Xi(s)\big)*\rho_\epsilon,\big(\varphi_R v(s)\big)*\rho_\epsilon\big\rangle_{L^2(R^d)}\,ds\\&\quad+2\sum_{j=1}^{\infty}\int_0^t\big\langle\big(\varphi_R v(s)\big)*\rho_\epsilon,\big(\varphi_R\psi_j(s)\big)*\rho_\epsilon\big\rangle_{L^2(R^d)}\,dB_j(s)\\&\quad+\sum_{j=1}^{\infty}\int_0^t\int_{R^d}\big|\big(\varphi_R\psi_j(s)\big)*\rho_\epsilon\big|^2\,dx\,ds,\quad\forall t\in[0,T],\end{aligned}\tag{2.11}$$

for all $R=R_k$ and all $\epsilon=\epsilon_n$, P-almost surely. Fix $\epsilon=\epsilon_n$, and $t\in[0,T]$. Then, P-almost surely,

$$\begin{aligned}&\int_0^t\big\langle\big(\varphi_R\nabla\cdot\Xi(s)\big)*\rho_\epsilon,\big(\varphi_R v(s)\big)*\rho_\epsilon\big\rangle_{L^2(R^d)}\,ds\\&\quad=\int_0^t\big\langle\nabla\cdot\big(\varphi_R\Xi(s)\big)*\rho_\epsilon,\big(\varphi_R v(s)\big)*\rho_\epsilon\big\rangle_{L^2(R^d)}\,ds\end{aligned}$$

$$-\int_0^t \big\langle \big(\nabla\varphi_R \cdot \Xi(s)\big) * \rho_\epsilon, \big(\varphi_R v(s)\big) * \rho_\epsilon \big\rangle_{L^2(R^d)}\, ds.$$

Meanwhile, by (2.2)–(2.3) and the fact that $\rho_\epsilon(x) = 0$, for $|x| > \epsilon$, it holds that

$$\int_0^T \big|\big\langle \big(\nabla\varphi_R \cdot \Xi(s)\big) * \rho_\epsilon, \big(\varphi_R v(s)\big) * \rho_\epsilon \big\rangle_{L^2(R^d)}\big|\, ds$$

$$\leqslant \big\| |\Xi| \big\|_{L^\infty([0,T]\times R^d)} \big\| |\nabla\varphi_R| \big\|_{L^d(R^d)} \int_0^T \bigg(\int_{R-\epsilon\leqslant|x|\leqslant 2R+\epsilon} \big|\big(\varphi_R v(s)\big) * \rho_\epsilon\big|^{\frac{d}{d-1}}\, dx \bigg)^{\frac{d-1}{d}} ds$$

and

$$\big\| |\nabla\varphi_R| \big\|_{L^d(R^d)} \leqslant C, \quad \forall R > 1.$$

Let $\chi_{R,\epsilon}$ stand for the characteristic function of the set $\{x \mid R-\epsilon \leqslant |x| \leqslant 2R+\epsilon\}$. It follows from Lemma 2.3 that

$$\begin{aligned}\big\| \chi_{R,\epsilon}\big((\varphi_R v) * \rho_\epsilon\big) \big\|_{L^{\frac{d}{d-1}}(R^d)} &\leqslant \big\| \chi_{R,\epsilon}\big(|v| * \rho_\epsilon\big) \big\|_{L^{\frac{d}{d-1}}(R^d)} \\ &\leqslant \|v\|_{L^{\frac{d}{d-1}}(R^d)} \leqslant C\|\nabla v\|_{(L^1(R^d))^d}\end{aligned}$$

and

$$\lim_{R\to\infty} \big\| \chi_{R,\epsilon}\big(|v| * \rho_\epsilon\big) \big\|_{L^{\frac{d}{d-1}}(R^d)} = 0, \quad dP \otimes dt\text{-almost all } (\omega, t).$$

Hence, by (2.7) and Lebesgue's dominated convergence theorem,

$$\lim_{R\to\infty} \int_0^T \bigg(\int_{R^d} \chi_{R,\epsilon} \big|\big(\varphi_R v(s)\big) * \rho_\epsilon\big|^{\frac{d}{d-1}}\, dx \bigg)^{\frac{d-1}{d}} ds = 0, \quad P\text{-almost surely},$$

and thus,

$$\lim_{R\to\infty} \int_0^T \big|\big\langle \big(\nabla\varphi_R \cdot \Xi(s)\big) * \rho_\epsilon, \big(\varphi_R v(s)\big) * \rho_\epsilon \big\rangle_{L^2(R^d)}\big|\, ds = 0, \quad P\text{-almost surely}. \tag{2.12}$$

Next we see that

$$\int_0^t \langle \nabla \cdot (\varphi_R \Xi(s)) * \rho_\epsilon, (\varphi_R v(s)) * \rho_\epsilon \rangle_{L^2(R^d)} \, ds$$
$$= -\int_0^t \langle (\varphi_R \Xi(s)) * \rho_\epsilon, \nabla(\varphi_R v(s)) * \rho_\epsilon \rangle_{(L^2(R^d))^d} \, ds$$

and, by using (2.7) and the same estimate as for (2.12),

$$\lim_{R\to\infty} \int_0^t \langle (\varphi_R \Xi(s)) * \rho_\epsilon, \nabla(\varphi_R v(s)) * \rho_\epsilon \rangle_{(L^2(R^d))^d} \, ds = \int_0^t \int_{R^d} (\Xi(s) * \rho_\epsilon) \cdot (\nabla v(s) * \rho_\epsilon) \, dx \, ds$$

for all $t \in [0, T]$, P-almost surely. We then pass $\epsilon = \epsilon_n \to 0$ to obtain the third term in (2.10). Next we handle the martingale term in (2.11). Let us set

$$M_{R,\epsilon}(t) = \sum_{j=1}^\infty \int_0^t \langle (\varphi_R v(s)) * \rho_\epsilon, (\varphi_R \psi_j(s)) * \rho_\epsilon \rangle_{L^2(R^d)} \, dB_j(s),$$
$$M_\epsilon(t) = \sum_{j=1}^\infty \int_0^t \langle v(s), \psi_j(s) * \rho_\epsilon * \rho_\epsilon \rangle_{L^2(R^d)} \, dB_j(s)$$

and

$$M(t) = \sum_{j=1}^\infty \int_0^t \langle v(s), \psi_j(s) \rangle_{L^2(R^d)} \, dB_j(s).$$

We can rewrite $M_{R,\epsilon}(t)$ as

$$M_{R,\epsilon}(t) = \sum_{j=1}^\infty \int_0^t \langle v(s), \varphi_R((\varphi_R \psi_j(s)) * \rho_\epsilon * \rho_\epsilon) \rangle_{L^2(R^d)} \, dB_j(s).$$

By the Burkholder–Davis–Gundy inequality and (2.6),

$$E\Big(\sup_{0\leqslant t\leqslant T} |M_{R,\epsilon}(t) - M_\epsilon(t)|\Big)$$
$$\leqslant CE\left(\sum_{j=1}^\infty \int_0^T |\langle v(t), \varphi_R((\varphi_R \psi_j(t)) * \rho_\epsilon * \rho_\epsilon) - \psi_j(t) * \rho_\epsilon * \rho_\epsilon \rangle_{L^2(R^d)}|^2 \, dt\right)^{\frac{1}{2}}$$

$$\leqslant CE\left(\sup_{0\leqslant t\leqslant T}\|v(t)\|_{L^2(R^d)}\left(\sum_{j=1}^{\infty}\int_0^T\|\varphi_R\big((\varphi_R\psi_j(t))*\rho_\epsilon*\rho_\epsilon\big)\right.\right.$$

$$\left.\left.-\psi_j(t)*\rho_\epsilon*\rho_\epsilon\|^2_{L^2(R^d)}\,dt\right)^{\frac{1}{2}}\right)$$

$$\leqslant C\left(E\left(\sum_{j=1}^{\infty}\int_0^T\|\varphi_R\big((\varphi_R\psi_j(t))*\rho_\epsilon*\rho_\epsilon\big)-\psi_j(t)*\rho_\epsilon*\rho_\epsilon\|^2_{L^2(R^d)}\,dt\right)\right)^{\frac{1}{2}}.$$

It holds that

$$\begin{aligned}&\|\varphi_R\big((\varphi_R\psi_j)*\rho_\epsilon*\rho_\epsilon\big)-\psi_j*\rho_\epsilon*\rho_\epsilon\|_{L^2(R^d)}\\ &\quad\leqslant\|\varphi_R\big((\varphi_R\psi_j)*\rho_\epsilon*\rho_\epsilon\big)-(\varphi_R\psi_j)*\rho_\epsilon*\rho_\epsilon\|_{L^2(R^d)}\\ &\qquad+\|(\varphi_R\psi_j)*\rho_\epsilon*\rho_\epsilon-\psi_j*\rho_\epsilon*\rho_\epsilon\|_{L^2(R^d)}\\ &\quad\leqslant 2\|(1-\varphi_R)\big(|\psi_j|*\rho_\epsilon*\rho_\epsilon\big)\|_{L^2(R^d)}\leqslant 2\|\psi_j\|_{L^2(R^d)}.\end{aligned}$$

By (2.8) and Lebesgue's dominated convergence theorem,

$$\lim_{R\to\infty}E\Big(\sup_{0\leqslant t\leqslant T}\big|M_{R,\epsilon}(t)-M_\epsilon(t)\big|\Big)=0$$

and hence, there is a subsequence still denoted by $\{R_k\}$ such that

$$\lim_{R_k\to\infty}M_{R_k,\epsilon}(t)=M_\epsilon(t)$$

for all $t\in[0,T]$, P-almost surely. In the same way,

$$\lim_{\epsilon_n\to 0}M_{\epsilon_n}(t)=M(t)$$

for all $t\in[0,T]$, P-almost surely.

The other terms in (2.11) are easy to handle first by passing $R=R_k\to\infty$ and then, by passing $\epsilon=\epsilon_n\to 0$. By (2.10), $v\in C([0,T];L^2(R^d))$, P-almost surely. □

Lemma 2.6. $\mathcal{B}(\mathcal{S})\subset\mathcal{B}(L^2(R^d))$.

Proof. Let

$$\mathcal{Y}_N=\big\{(f,m)\in\mathcal{Y}\;\big|\;\|m\|_{(\mathcal{M}(R^d))^d}\leqslant N\big\}$$

for each positive integer N, equipped with topology induced by $\mathcal{Y}$. Then, $\mathcal{Y}_N$ is a Polish space, and so is its closed subset $\Lambda(\mathcal{S})\cap\mathcal{Y}_N$. We write

$$\mathcal{S}_N = \Lambda^{-1}(\mathcal{Y}_N).$$

Then, $\mathcal{S}_N$ is a Polish space, and

$$\mathcal{S} = \bigcup_{N=1}^{\infty} \mathcal{S}_N.$$

If $\mathcal{O}$ is an open subset of $\mathcal{S}$, then $\mathcal{O} \cap \mathcal{S}_N$ is an open subset of $\mathcal{S}_N$. Since the identity map $\mathcal{S}_N \to L^2(R^d)$ is one-to-one and continuous, every Borel subset of $\mathcal{S}_N$ is also a Borel subset of $L^2(R^d)$: see [7, p. 67]. Hence, every Borel subset of $\mathcal{S}$ is a Borel subset of $L^2(R^d)$. □

Lemma 2.7. *Let $f_k \to f$ weakly in $L^2(\Omega \times [0, T]; \mathcal{G}; H_L^0)$. Suppose that*

$$E\Big(\sup_{0\leqslant t\leqslant T} \|\nabla f_k\|^2_{(H_L^0)^d}\Big) \leqslant C_1, \quad \forall k,$$

and

$$E\Big(\sup_{0\leqslant t\leqslant T} \|\nabla f_k\|_{(L^1(G_L))^d}\Big) \leqslant C_2, \quad \forall k,$$

for positive constants C_1 and C_2. Then, it holds that

$$E\Big(\sup_{0\leqslant t\leqslant T} \|\nabla f\|^2_{(H_L^0)^d}\Big) \leqslant C_1, \tag{2.13}$$

and

$$E\Big(\sup_{0\leqslant t\leqslant T} \|\nabla f\|_{(L^1(G_L))^d}\Big) \leqslant C_2. \tag{2.14}$$

Proof. There is a sequence $\{\hat{f}_k\}$ such that each $\hat{f}_k$ is a convex combination of finite f_k's and

$$\hat{f}_k \to f \quad \text{strongly in } L^2\big(\Omega \times [0, T]; \mathcal{G}; H_L^0\big).$$

Hence there is a subsequence still denoted by $\{\hat{f}_k\}$ such that

$$\hat{f}_k(\omega, t) \to f(\omega, t) \quad \text{strongly in } H_L^0,$$

for $dP \otimes dt$ almost all (ω, t). It follows that

$$\|\nabla f\|_{(H_L^0)^d} \leqslant \varliminf_{k\to\infty} \|\nabla \hat{f}_k\|_{(H_L^0)^d}$$

and

$$\|\nabla f\|_{(L^1(G_L))^d} \leqslant \varliminf_{k\to\infty} \|\nabla \hat{f}_k\|_{(L^1(G_L))^d}$$

for $dP \otimes dt$ almost all (ω, t). Thus,

$$E\Big(\sup_{0\leqslant t\leqslant T}\|\nabla f\|_{(L^1(R^d))^d}\Big) \leqslant E\Big(\sup_{0\leqslant t\leqslant T}\varliminf_{k\to\infty}\|\nabla \hat{f}_k\|_{(L^1(R^d))^d}\Big)$$
$$\leqslant E\Big(\varliminf_{k\to\infty}\sup_{0\leqslant t\leqslant T}\|\nabla \hat{f}_k\|_{(L^1(R^d))^d}\Big)$$
$$\leqslant \varliminf_{k\to\infty}E\Big(\sup_{0\leqslant t\leqslant T}\|\nabla \hat{f}_k\|_{(L^1(R^d))^d}\Big) \leqslant C_2$$

which gives (2.14). In the same way, (2.13) can be shown. □

3. The Cauchy problem on $\mathcal{T}_L$

3.1. Modification of coefficients of the noise

Case I. Assume (1.1)–(1.5). Let Λ_L be the operator from $L^2(R^d)$ into H_L^0 such that

$$\Lambda_L(w)(x) = w(x), \quad \forall x \in G_L. \tag{3.1}$$

Set $R = L/4$, and

$$g_{0,L} = \Lambda_L(\varphi_R g_0), \quad L \geqslant 1, \tag{3.2}$$
$$g_{j,L}(w) = \Lambda_L\big(\varphi_R(\cdot) g_j\big(\varphi_R(\cdot) w\big)\big), \quad L \geqslant 1,\ \forall j \geqslant 1, \tag{3.3}$$

where φ_R is defined by (2.2)–(2.3). Then, it is easy to see that $g_{0,L}$ and $g_{j,L}$ have the following properties.

$$g_{0,L} \in L^2\big(0,T; H_L^1\big), \qquad g_{j,L} \in C\big([0,T]\times H_L^0; H_L^0\big), \quad \forall j \geqslant 1,\ \forall L \geqslant 1, \tag{3.4}$$
$$\varphi_R g_{0,L} \to g_0 \quad \text{strongly in } L^2\big(0,T; H^1\big(R^d\big)\big), \tag{3.5}$$

and, for all $t \in [0,T]$,

$$\big\|g_{j,L}(t,v)\big\|_{H_L^0} \leqslant c_j + d_j\|v\|_{H_L^0}, \quad \forall v \in H_L^0,\ \forall j \geqslant 1, \tag{3.6}$$
$$\big\|g_{j,L}(v) - g_{j,L}(w)\big\|_{H_L^0} \leqslant d_j\|v - w\|_{H_L^0}, \quad \forall v, w \in H_L^0,\ \forall j \geqslant 1, \tag{3.7}$$

and

$$\big\|g_{j,L}(t,v)\big\|_{H_L^1} \leqslant Kc_j + Kd_j\|v\|_{H_L^1}, \quad \forall v \in H_L^1,\ \forall j \geqslant 1, \tag{3.8}$$

where c_j's and d_j's are the same as in (1.2)–(1.3), and K denotes positive constants independent of $j \geqslant 1$ and $L \geqslant 1$.

Case II. Assume (1.7)–(1.10). Following the same argument as in Lemma 2.4, we set

$$g_{0,L_k} = \Lambda_{L_k}\big(\varphi_{R_k} h_{\delta_k}(g_0 * \rho_{\epsilon_k})\big) \tag{3.9}$$

where $L_k = 4R_k$. We can find sequences

$$\{R_k\} \uparrow \infty, \qquad \{\delta_k\} \downarrow 0, \qquad \{\epsilon_k\} \downarrow 0$$

so that

$$g_{0,L_k} \in L^2\big(0,T; H^1_{L_k}\big), \quad \forall k, \tag{3.10}$$

$$\|\nabla g_{0,L_k}\|_{(L^1([0,T]\times G_L))^d} \leqslant 1 + \|\nabla g_0\|_{(L^1([0,T]\times R^d))^d}, \quad \forall k, \tag{3.11}$$

$$\varphi_{R_k} g_{0,L_k} \to g_0 \quad \text{strongly in } L^2\big(0,T; L^2\big(R^d\big)\big), \text{ as } k \to \infty. \tag{3.12}$$

We also set

$$g_{j,L_k}(t,w) = g_j(t,w), \quad \forall j \geqslant 1,\ \forall k, \tag{3.13}$$

where g_j satisfies (1.8)–(1.10).

We note that (3.13) implies (3.6)–(3.8).

We consider the initial value problem on a d-dimensional torus $\mathcal{T}_L$ with period L.

$$\frac{\partial u}{\partial t} = \nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right) + g_{0,L} + \sum_{j=1}^{\infty} g_{j,L}(u)\frac{dB_j}{dt}, \tag{3.14}$$

$$u(0) = u_{0,L}. \tag{3.15}$$

Definition 3.1. u is said to be a solution of (3.14)–(3.15) on $\mathcal{T}_L$ if the following conditions are satisfied.

(i) u is H^1_L-valued progressively measurable such that

$$E\Big(\sup_{t\in[0,T]}\big\|u(t)\big\|^2_{H^1_L}\Big) < \infty$$

and

$$u \in C\big([0,T]; H^0_L\big), \quad P\text{-almost surely},$$

(ii) there is some $\Pi_L \in L^\infty(\Omega\times[0,T]\times\mathcal{T}_L; \mathcal{G}\otimes\mathcal{B}(\mathcal{T}_L); R^d)$ such that

$$|\Pi_L| \leqslant 1, \quad dP\otimes dt\otimes dx\text{-almost all } (\omega,t,x),$$

$$\Pi_L\cdot\nabla u = |\nabla u|, \quad dP\otimes dt\otimes dx\text{-almost all } (\omega,t,x),$$

and

$$u(t) = u_{0,L} + \int_0^t \nabla \cdot \Pi_L(s)\,ds + \int_0^t g_{0,L}(s)\,ds + \sum_{j=1}^{\infty} \int_0^t g_{j,L}\big(u(s)\big)\,dB_j(s) \tag{3.16}$$

holds in H_L^{-1} for all $t \in [0, T]$, P-almost surely.

Theorem 3.2. *Assume* (3.4) *and* (3.6)–(3.8). *Let* $u_{0,L} \in L^2(\Omega; \mathcal{F}_0; H_L^1)$. *Then, there is a unique solution* u_L *of* (3.14)–(3.15). *Furthermore, it holds that*

$$E\Big(\sup_{0 \leqslant t \leqslant T} \|u_L\|_{H_L^1}^2\Big) \leqslant CE\big(\|u_{0,L}\|_{H_L^1}^2\big) + C\int_0^T \|g_{0,L}\|_{H_L^1}^2\,dt + C \tag{3.17}$$

and

$$E\Big(\sup_{0 \leqslant t \leqslant T} \|u_L\|_{H_L^0}^2\Big) + E\left(\int_0^T \|\nabla u_L\|_{(L^1(G_L))^d}\,dt\right) \leqslant CE\big(\|u_{0,L}\|_{L^2(G_L)}^2\big) + C\int_0^T \|g_{0,L}\|_{L^2(G_L)}^2\,dt + C \tag{3.18}$$

where C *denotes various positive constants independent of* $u_{0,L}$, $g_{0,L}$, *and* $L \geqslant 1$.

If g_{j,L_k}, $j \geqslant 0$, *satisfies* (3.10), (3.11) *and* (3.13), *we also have*

$$E\Big(\sup_{0 \leqslant t \leqslant T} \|\nabla u_{L_k}\|_{(L^1(G_{L_k}))^d}\Big) \leqslant CE\big(\|\nabla u_{0,L_k}\|_{(L^1(G_{L_k}))^d}\big) + C\int_0^T \|\nabla g_{0,L_k}\|_{(L^1(G_{L_k}))^d}\,dt \tag{3.19}$$

and

$$E\left(\int_0^T \|\nabla \cdot \Pi_{L_k}\|_{L^2(G_{L_k})}^2\,dt\right) \leqslant CE\big(\|\nabla u_{0,L_k}\|_{(L^1(G_{L_k}))^d}\big) + C\int_0^T \|\nabla g_{0,L_k}\|_{(L^1(G_{L_k}))^d}\,dt \tag{3.20}$$

for some positive constants C *independent of* u_{0,L_k}, g_{0,L_k}, *and* $L_k \geqslant 1$.

As mentioned above, the conditions (3.13) implies (3.6)–(3.8).

The remainder of this section is devoted to the proof. The details are presented through three steps.

Step 1. Throughout this step, we fix $L > 1$, $\eta > 0$, and $\epsilon > 0$, and consider the initial value problem on $\mathcal{T}_L$.

$$\frac{\partial u}{\partial t} + \eta\Delta^2 u - \epsilon\Delta u - \nabla\cdot\left(\frac{\nabla u}{\sqrt{\epsilon+|\nabla u|^2}}\right) = g_{0,L} + \sum_{j=1}^{\infty} g_{j,L}(u)\frac{dB_j}{dt}, \tag{3.21}$$

$$u(0) = u_{0,L} \tag{3.22}$$

where $u_{0,L}$ is the same as in Theorem 3.2.

For each $v \in H_L^2$ and $t \in [0,T]$, we set

$$A(t,v) = -\eta\Delta^2 v + \epsilon\Delta v + \nabla\cdot\left(\frac{\nabla v}{\sqrt{\epsilon+|\nabla v|^2}}\right) + g_{0,L}(t).$$

It is easy to see the following properties.

(i) For each $v_1, v_2, v_3 \in H_L^2$, and $t \in [0,T]$, the map

$$\lambda \mapsto \big\langle A(t, v_1 + \lambda v_2), v_3\big\rangle_{H_L^{-2}, H_L^2}$$

is continuous on R.

(ii) For each $v, w \in H_L^2$ and $t \in [0,T]$,

$$2\big\langle A(t,v) - A(t,w), v - w\big\rangle_{H_L^{-2}, H_L^2} + \sum_{j=1}^{\infty}\big\|g_{j,L}(t,v) - g_{j,L}(t,w)\big\|_{H_L^0}^2 \leqslant \gamma_1\|v-w\|_{H_L^0}^2$$

where γ_1 is a constant satisfying

$$\gamma_1 \geqslant \sum_{j=1}^{\infty} d_j^2, \quad d_j\text{'s are the constants in (3.7).} \tag{3.23}$$

(iii) For each $v \in H_L^2$ and $t \in [0,T]$,

$$2\big\langle A(t,v), v\big\rangle_{H_L^{-2}, H_L^2} + \sum_{j=1}^{\infty}\big\|g_{j,L}(t,v)\big\|_{H_L^0}^2 \leqslant \gamma_2\|v\|_{H_L^0}^2 - \eta\|v\|_{H_L^2}^2 + \big\|g_{0,L}(t)\big\|_{H_L^0}^2$$

for some positive constant γ_2 independent of t and v.

(iv) For each $v \in H_L^2$ and $t \in [0,T]$,

$$\big\|A(t,v)\big\|_{H_L^{-2}} \leqslant \gamma_3\|v\|_{H_L^2} + \big\|g_{0,L}(t)\big\|_{H_L^0}$$

for some constant γ_3 independent of t and v.

By virtue of the properties (i)–(iv), we can apply Theorem 4.2.4 in [16] to obtain a unique solution u of (3.21)–(3.22) such that $u \in C([0,T]; H_L^0)$, P-almost surely, and

$$u \in L^2\big(\Omega \times [0,T]; \mathcal{G}; H_L^2\big), \quad E\Big(\sup_{0\leqslant t\leqslant T} \big\|u(t)\big\|_{H_L^0}^2\Big) < \infty. \tag{3.24}$$

Furthermore, it holds that

$$\begin{aligned}
&\big\|u(t)\big\|_{L^2(G_L)}^2 + 2\eta\int_0^t \big\|\Delta u(s)\big\|_{L^2(G_L)}^2\,ds + 2\epsilon\int_0^t \big\|\nabla u(s)\big\|_{(L^2(G_L))^d}^2\,ds \\
&\quad + 2\int_0^t\int_{G_L} \frac{|\nabla u(s)|^2}{\sqrt{\epsilon + |\nabla u(s)|^2}}\,dx\,ds = \|u_{0,L}\|_{L^2(G_L)}^2 + 2\int_0^t \big\langle g_{0,L}(s), u(s)\big\rangle_{L^2(G_L)}\,ds \\
&\quad + 2\sum_{j=1}^\infty \int_0^t \big\langle g_{j,L}\big(u(s)\big), u(s)\big\rangle_{L^2(G_L)}\,dB_j + \sum_{j=1}^\infty\int_0^t \big\|g_{j,L}\big(u(s)\big)\big\|_{L^2(G_L)}^2\,ds
\end{aligned} \tag{3.25}$$

for all $t \in [0,T]$, P-almost surely. For each $\kappa = \frac{1}{n}$, $n = 1,2,\ldots,$ it holds that

$$\begin{aligned}
(u * \rho_\kappa)(t) &= u_{0,L} * \rho_\kappa - \eta\int_0^t \Delta^2\big(u(s) * \rho_\kappa\big)\,ds + \epsilon\int_0^t \Delta\big(u(s) * \rho_\kappa\big)\,ds \\
&\quad + \int_0^t \nabla\cdot\bigg(\frac{\nabla u(s)}{\sqrt{\epsilon + |\nabla u(s)|^2}}\bigg) * \rho_\kappa\,ds + \int_0^t g_{0,L}(s) * \rho_\kappa\,ds \\
&\quad + \sum_{j=1}^\infty\int_0^t g_{j,L}\big(u(s)\big) * \rho_\kappa\,dB_j
\end{aligned}$$

in H_L^1, for all $t \in [0,T]$, P-almost surely. Here ρ_κ denotes the Friedrichs mollifier, and the convolution is taken with respect to the space variables.

By Ito's formula, we see that

$$\begin{aligned}
&\big\|\nabla u(t) * \rho_\kappa\big\|_{(L^2(G_L))^d}^2 + 2\eta\int_0^t \big\|\nabla\big(\Delta u(s) * \rho_\kappa\big)\big\|_{(L^2(G_L))^d}^2\,ds \\
&\quad + 2\epsilon\int_0^t \big\|\Delta u(s) * \rho_\kappa\big\|_{L^2(G_L)}^2\,ds \\
&\quad + 2\int_0^t \bigg\langle \nabla\cdot\bigg(\frac{\nabla u(s)}{\sqrt{\epsilon + |\nabla u(s)|^2}}\bigg) * \rho_\kappa, \Delta u(s) * \rho_\kappa\bigg\rangle_{L^2(G_L)}\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \|\nabla u_{0,L} * \rho_\kappa\|^2_{(L^2(G_L))^d} + 2\int_0^t \langle \nabla g_{0,L}(s) * \rho_\kappa, \nabla u(s) * \rho_\kappa\rangle_{(L^2(G_L))^d}\, ds \\
&\quad + 2\sum_{j=1}^\infty \int_0^t \langle \nabla g_{j,L}\big(u(s)\big) * \rho_\kappa, \nabla u(s) * \rho_\kappa\rangle_{(L^2(G_L))^d}\, dB_j \\
&\quad + \sum_{j=1}^\infty \int_0^t \big\|\nabla g_j\big(u(s)\big) * \rho_\kappa\big\|^2_{(L^2(G_L))^d}\, ds
\end{aligned} \tag{3.26}
$$

for all $t \in [0, T]$, P-almost surely. By (3.24), we have

$$
E\left(\int_0^T \left|\left\langle \nabla \cdot \left(\frac{\nabla u(t)}{\sqrt{\epsilon + |\nabla u(t)|^2}}\right) * \rho_\kappa, \Delta u(t) * \rho_\kappa\right\rangle_{L^2(G_L)}\right| dt\right) \leqslant C \tag{3.27}
$$

for some constant C independent of κ. But it may depend on $\epsilon > 0$. By means of the Burkholder–Davis–Gundy inequality together with (3.4)–(3.8), (3.24) and (3.27), we can derive from (3.26) that

$$
E\Big(\sup_{0\leqslant t\leqslant T} \big\|\nabla u(t) * \rho_\kappa\big\|^2_{(L^2(G_L))^d}\Big) \leqslant C \tag{3.28}
$$

for some constant C independent of κ. Since

$$
\big\|\nabla u(\omega, t)\big\|_{(L^2(G_L))^d} = \lim_{\kappa\to 0}\big\|\nabla u(\omega, t) * \rho_\kappa\big\|_{(L^2(G_L))^d}
$$

for $dP \otimes dt$-almost all (ω, t), it holds that

$$
\begin{aligned}
E\Big(\sup_{t\in[0,T]}\big\|\nabla u(t)\big\|_{(L^2(G_L))^d}\Big) &\leqslant E\Big(\varliminf_{\kappa\to 0}\sup_{t\in[0,T]}\big\|\nabla u(t) * \rho_\kappa\big\|_{(L^2(G_L))^d}\Big) \\
&\leqslant \varliminf_{\kappa\to 0} E\Big(\sup_{t\in[0,T]}\big\|\nabla u(t) * \rho_\kappa\big\|_{(L^2(G_L))^d}\Big) \leqslant C.
\end{aligned} \tag{3.29}
$$

Since $u \in C([0, T]; H_L^0)$, and $\nabla u \in L^\infty(0, T; (H_L^0)^d)$, P-almost surely, it follows that $\nabla u(t)$ is $(H_L^0)^d$-weakly continuous in $t \in [0, T]$, P-almost surely. Thus,

$$
\big\|\nabla u(t)\big\|_{(L^2(G_L))^d} = \lim_{\kappa\to 0}\big\|\nabla u(t) * \rho_\kappa\big\|_{(L^2(G_L))^d}
$$

for all $t \in [0, T]$, P-almost surely. Also, by (3.24),

$$
\lim_{\kappa\to 0}\int_0^t \left\langle \nabla \cdot \left(\frac{\nabla u(s)}{\sqrt{\epsilon + |\nabla u(s)|^2}}\right) * \rho_\kappa, \Delta u(s) * \rho_\kappa\right\rangle_{L^2(G_L)} ds
$$

$$= \int_0^t \left\langle \nabla \cdot \left(\frac{\nabla u(s)}{\sqrt{\epsilon + |\nabla u(s)|^2}} \right), \Delta u(s) \right\rangle_{L^2(G_L)} ds$$

for all $t \in [0, T]$, P-almost surely. Hence, we have

$$\left\| \nabla u(t) \right\|^2_{(L^2(G_L))^d} + 2\epsilon \int_0^t \left\| \Delta u(s) \right\|^2_{L^2(G_L)} ds + 2 \int_0^t \left\langle \nabla \cdot \left(\frac{\nabla u(s)}{\sqrt{\epsilon + |\nabla u(s)|^2}} \right), \Delta u(s) \right\rangle_{L^2(G_L)} ds$$

$$\leqslant \|\nabla u_{0,L}\|^2_{(L^2(G_L))^d} + 2 \int_0^t \left\langle \nabla g_{0,L}(s), \nabla u(s) \right\rangle_{(L^2(G_L))^d} ds$$

$$+ 2 \sum_{j=1}^{\infty} \int_0^t \left\langle \nabla g_{j,L}\big(u(s)\big), \nabla u(s) \right\rangle_{(L^2(G_L))^d} dB_j + \sum_{j=1}^{\infty} \int_0^t \left\| \nabla g_{j,L}\big(u(s)\big) \right\|^2_{(L^2(G_L))^d} ds \tag{3.30}$$

for all $t \in [0, T]$, P-almost surely. It follows from Lemma 2.1, (3.25) and (3.30) that

$$E\Big(\sup_{t \in [0,T]} \big(\left\| u(t) \right\|^2_{L^2(G_L)} + \left\| \nabla u(t) \right\|^2_{(L^2(G_L))^d} \big) \Big) \leqslant C, \tag{3.31}$$

$$E\left(\int_0^T \int_{G_L} \frac{|\nabla u|^2}{\sqrt{\epsilon + |\nabla u|^2}} \, dx \, dt \right) \leqslant C, \tag{3.32}$$

$$E\left(\int_0^T \|\Delta u\|^2_{L^2(G_L)} dt \right) \leqslant C_\epsilon, \tag{3.33}$$

where C is a positive constant independent of $\eta > 0$ and $\epsilon > 0$, and C_ϵ denotes a positive constant independent of $\eta > 0$.

Step 2. Throughout this step, $L > 0$ and $\epsilon > 0$ are fixed. Let u_η be the solution of (3.21)–(3.22) obtained above. Following the argument in [16, pp. 88–90], we pass $\eta \to 0$ to obtain a solution of

$$\frac{\partial u}{\partial t} - \epsilon \Delta u - \nabla \cdot \left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}} \right) = g_{0,L} + \sum_{j=1}^{\infty} g_{j,L}(u) \frac{dB_j}{dt}, \tag{3.34}$$

$$u(0) = u_{0,L}, \tag{3.35}$$

such that the solution u satisfies (3.31), (3.33), and $u \in C([0, T]; H^0_L)$, P-almost surely. Also, $u(t)$ is H^1_L-weakly continuous in $t \in [0, T]$, P-almost surely, and it holds that

$$\left\| u(t) \right\|^2_{L^2(G_L)} + 2\epsilon \int_0^t \left\| \nabla u(s) \right\|^2_{(L^2(G_L))^d} ds + 2 \int_0^t \int_{G_L} \frac{|\nabla u(s)|^2}{\sqrt{\epsilon + |\nabla u(s)|^2}} \, dx \, ds$$

$$= \|u_{0,L}\|^2_{L^2(G_L)} + 2\int_0^t \langle g_{0,L}(s), u(s)\rangle_{L^2(G_L)}\,ds$$
$$+ 2\sum_{j=1}^{\infty}\int_0^t \langle g_{j,L}(u(s)), u(s)\rangle_{L^2(G_L)}\,dB_j + \sum_{j=1}^{\infty}\int_0^t \|g_{j,L}(u(s))\|^2_{L^2(G_L)}\,ds \tag{3.36}$$

for all $t \in [0, T]$, P-almost surely. We now consider

$$d(u * \rho_\kappa) = \epsilon\Delta u * \rho_\kappa\,dt + \nabla\cdot\left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}}\right) * \rho_\kappa\,dt$$
$$+ g_{0,L} * \rho_\kappa\,dt + \sum_{j=1}^{\infty}\big(g_{j,L}(u) * \rho_\kappa\big)\,dB_j$$

where ρ_κ is the Friedrichs mollifier. Let us set

$$J_\epsilon(v) = \int_{G_L}\sqrt{\epsilon + |\nabla v|^2}\,dx.$$

By Ito's formula, it holds P-almost surely that

$$J_\epsilon\big(u(t) * \rho_\kappa\big) = J_\epsilon(u_{0,L} * \rho_\kappa) + \epsilon\int_0^t \left\langle \Delta u(s) * \rho_\kappa, -\nabla\cdot\left(\frac{\nabla u(s) * \rho_\kappa}{\sqrt{\epsilon + |\nabla u(s) * \rho_\kappa|^2}}\right)\right\rangle_{L^2(G_L)} ds$$
$$+ \int_0^t \left\langle \nabla\cdot\left(\frac{\nabla u(s)}{\sqrt{\epsilon + |\nabla u(s)|^2}}\right) * \rho_\kappa, -\nabla\cdot\left(\frac{\nabla u(s) * \rho_\kappa}{\sqrt{\epsilon + |\nabla u(s) * \rho_\kappa|^2}}\right)\right\rangle_{L^2(G_L)} ds$$
$$+ \int_0^t\int_{G_L} \frac{(\nabla u(s) * \rho_\kappa)\cdot(\nabla g_{0,L}(s) * \rho_\kappa)}{\sqrt{\epsilon + |\nabla u(s) * \rho_\kappa|^2}}\,dx\,ds$$
$$+ \sum_{j=1}^{\infty}\int_0^t\int_{G_L} \frac{(\nabla u(s) * \rho_\kappa)\cdot(\nabla g_{j,L}(u(s)) * \rho_\kappa)}{\sqrt{\epsilon + |\nabla u(s) * \rho_\kappa|^2}}\,dx\,dB_j$$
$$+ \frac{1}{2}\sum_{j=1}^{\infty}\int_0^t\int_{G_L}\left(\frac{|\nabla g_{j,L}(u(s)) * \rho_\kappa|^2}{\sqrt{\epsilon + |\nabla u(s) * \rho_\kappa|^2}}\right.$$
$$\left. - \frac{|(\nabla u(s) * \rho_\kappa)\cdot(\nabla g_{j,L}(u(s)) * \rho_\kappa)|^2}{(\sqrt{\epsilon + |\nabla u(s) * \rho_\kappa|^2})^3}\right)dx\,ds$$

for all $t \in [0, T]$, and $\kappa = \frac{1}{n}$, $n = 1, 2, \ldots.$

Since $u \in H_L^2$, $dP \otimes dt$-almost all (ω, t),

$$\left(\nabla \cdot \frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}}\right) * \rho_\kappa \to \nabla \cdot \frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}}$$

strongly in $L^2(G_L)$, as $\kappa \to 0$, $dP \otimes dt$-almost all (ω, t). In the meantime, it holds $dP \otimes dt$-almost all (ω, t) that

$$\left\| \nabla \cdot \frac{\nabla u * \rho_\kappa}{\sqrt{\epsilon + |\nabla u * \rho_\kappa|^2}} \right\|_{L^2(G_L)} \leqslant \frac{C}{\sqrt{\epsilon}} \|\Delta u * \rho_\kappa\|_{L^2(G_L)} \leqslant \frac{C}{\sqrt{\epsilon}} \|\Delta u\|_{L^2(G_L)}$$

for all κ, and

$$\int_{G_L} \frac{\nabla(u * \rho_\kappa) \cdot \nabla \phi}{\sqrt{\epsilon + |\nabla u * \rho_\kappa|^2}}\, dx \to \int_{G_L} \frac{\nabla u \cdot \nabla \phi}{\sqrt{\epsilon + |\nabla u|^2}}\, dx$$

as $\kappa \to 0$, for all $\phi \in H_L^1$. Hence,

$$\nabla \cdot \left(\frac{\nabla u * \rho_\kappa}{\sqrt{\epsilon + |\nabla u * \rho_\kappa|^2}} \right) \to \nabla \cdot \left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}} \right)$$

weakly in $L^2(G_L)$, as $\kappa \to 0$, $dP \otimes dt$-almost all (ω, t). Thus,

$$\begin{aligned} &\left\langle \nabla \cdot \left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}} \right) * \rho_\kappa, -\nabla \cdot \left(\frac{\nabla u * \rho_\kappa}{\sqrt{\epsilon + |\nabla u * \rho_\kappa|^2}} \right) \right\rangle_{L^2(G_L)} \\ &\quad \to -\left\| \nabla \cdot \left(\frac{\nabla u}{\sqrt{\epsilon + |\nabla u|^2}} \right) \right\|_{L^2(G_L)}^2 \end{aligned}$$

and, for each $j \geqslant 1$,

$$\int_{G_L} \frac{(\nabla u * \rho_\kappa) \cdot (\nabla g_{j,L}(u) * \rho_\kappa)}{\sqrt{\epsilon + |\nabla u * \rho_\kappa|^2}}\, dx \to \int_{G_L} \frac{\nabla u \cdot \nabla g_{j,L}(u)}{\sqrt{\epsilon + |\nabla u|^2}}\, dx,$$

as $\kappa \to 0$, $dP \otimes dt$-almost all (ω, t). Also,

$$\left| \int_{G_L} \frac{(\nabla u * \rho_\kappa) \cdot (\nabla g_{j,L}(u) * \rho_\kappa)}{\sqrt{\epsilon + |\nabla u * \rho_\kappa|^2}}\, dx \right|^2 \leqslant L^d \|\nabla g_{j,L}(u)\|_{(L^2(G_L))^d}^2$$

$dP \otimes dt$-almost all (ω, t). Thus,

$$\sum_{j=1}^{\infty}\int_0^t\int_{G_L}\frac{(\nabla u*\rho_\kappa)\cdot(\nabla g_{j,L}(u)*\rho_\kappa)}{\sqrt{\epsilon+|\nabla u*\rho_\kappa|^2}}\,dx\,dB_j$$
$$\to\sum_{j=1}^{\infty}\int_0^t\int_{G_L}\frac{\nabla u\cdot\nabla g_{j,L}(u)}{\sqrt{\epsilon+|\nabla u|^2}}\,dx\,dB_j$$

for all $t\in[0,T]$, P-almost surely.

By these convergence results and the fact that $\nabla u(t)\in(H_L^0)^d$, for all $t\in[0,T]$, P-almost surely, we can pass $\kappa\to 0$ to arrive at

$$\begin{aligned}
J_\epsilon\bigl(u(t)\bigr)&=J_\epsilon(u_{0,L})+\epsilon\int_0^t\left\langle\Delta u(s),-\nabla\cdot\left(\frac{\nabla u(s)}{\sqrt{\epsilon+|\nabla u(s)|^2}}\right)\right\rangle_{L^2(G_L)}ds\\
&\quad-\int_0^t\left\|\nabla\cdot\left(\frac{\nabla u}{\sqrt{\epsilon+|\nabla u|^2}}\right)\right\|_{L^2(G_L)}^2 ds+\int_0^t\int_{G_L}\frac{(\nabla u(s))\cdot(\nabla g_{0,L}(s))}{\sqrt{\epsilon+|\nabla u(s)|^2}}\,dx\,ds\\
&\quad+\sum_{j=1}^{\infty}\int_0^t\int_{G_L}\frac{(\nabla u(s))\cdot(\nabla g_{j,L}(u(s)))}{\sqrt{\epsilon+|\nabla u(s)|^2}}\,dx\,dB_j\\
&\quad+\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t\int_{G_L}\left(\frac{|\nabla g_{j,L}(u(s))|^2}{\sqrt{\epsilon+|\nabla u(s)|^2}}-\frac{|(\nabla u(s))\cdot(\nabla g_{j,L}(u(s)))|^2}{(\sqrt{\epsilon+|\nabla u(s)|^2})^3}\right)dx\,ds
\end{aligned}\tag{3.37}$$

for all $t\in[0,T]$, P-almost surely.

We also have

$$\begin{aligned}
&\bigl\|\nabla\bigl(u(t)*\rho_\kappa\bigr)\bigr\|_{(L^2(G_L))^d}^2+2\epsilon\int_0^t\|\Delta u(s)*\rho_\kappa\|_{L^2(G_L)}^2\,ds\\
&\qquad+2\int_0^t\left\langle\nabla\cdot\left(\frac{\nabla u(s)}{\sqrt{\epsilon+|\nabla u(s)|^2}}\right)*\rho_\kappa,\Delta\bigl(u(s)*\rho_\kappa\bigr)\right\rangle_{L^2(G_L)}ds\\
&\quad=\|\nabla u_{0,L}*\rho_\kappa\|_{(L^2(G_L))^d}^2+2\int_0^t\bigl\langle\nabla g_{0,L}(s)*\rho_\kappa,\nabla u(s)*\rho_\kappa\bigr\rangle_{(L^2(G_L))^d}\,ds\\
&\qquad+2\sum_{j=1}^{\infty}\int_0^t\bigl\langle\nabla g_{j,L}\bigl(u(s)\bigr)*\rho_\kappa,\nabla u(s)*\rho_\kappa\bigr\rangle_{(L^2(G_L))^d}\,dB_j\\
&\qquad+\sum_{j=1}^{\infty}\int_0^t\bigl\|\nabla g_{j,L}\bigl(u(s)\bigr)*\rho_\kappa\bigr\|_{(L^2(G_L))^d}^2\,ds
\end{aligned}$$

for all $t \in [0,T]$, P-almost surely. Since $\nabla u(t) \in (H_L^0)^d$, for all $t \in [0,T]$, and $\Delta u \in L^2(0,T;H_L^0)$, P-almost surely, we can pass $\kappa \to 0$ to arrive at

$$\begin{aligned}
&\|\nabla u(t)\|^2_{(L^2(G_L))^d} + 2\epsilon \int_0^t \|\Delta u(s)\|^2_{L^2(G_L)}\,ds + 2\int_0^t \left\langle \nabla\cdot\left(\frac{\nabla u(s)}{\sqrt{\epsilon+|\nabla u(s)|^2}}\right), \Delta u(s)\right\rangle_{L^2(G_L)} ds \\
&\quad = \|\nabla u_{0,L}\|^2_{(L^2(G_L))^d} + 2\int_0^t \langle \nabla g_{0,L}(s), \nabla u(s)\rangle_{(L^2(G_L))^d}\,ds \\
&\qquad + 2\sum_{j=1}^\infty \int_0^t \langle \nabla g_{j,L}(u(s)), \nabla u(s)\rangle_{(L^2(G_L))^d}\,dB_j \\
&\qquad + \sum_{j=1}^\infty \int_0^t \|\nabla g_{j,L}(u(s))\|^2_{(L^2(G_L))^d}\,ds \qquad (3.38)
\end{aligned}$$

for all $t \in [0,T]$, P-almost surely.

Now we denote by u_ϵ the above solution of (3.34)–(3.35) with given $\epsilon > 0$. Under the conditions (3.4)–(3.8), we use Lemma 2.1 and the Burkholder–Davis–Gundy inequality to derive from (3.36) and (3.38)

$$E\Big(\sup_{0\leqslant t\leqslant T} \|u_\epsilon\|^2_{H_L^1}\Big) \leqslant CE\big(\|u_{0,L}\|^2_{H_L^1}\big) + CE\left(\int_0^T \|g_{0,L}\|^2_{H_L^1}\,dt\right) + C \qquad (3.39)$$

and, by the inequality $\frac{|z|^2}{\sqrt{\epsilon+|z|^2}} \geqslant |z| - \sqrt{\epsilon}$, $\forall z \in R^d$,

$$\begin{aligned}
E\left(\int_0^T \|\nabla u_\epsilon\|_{L^1(G_L)}\,dt\right) &\leqslant CE\big(\|u_{0,L}\|^2_{L^2(G_L)}\big) + CE\left(\int_0^T \|g_{0,L}\|^2_{L^2(G_L)}\,dt\right) + C \\
&\quad + \sqrt{\epsilon}\,TL^d \qquad (3.40)
\end{aligned}$$

where C denotes positive constants independent of $u_{0,L}$, $g_{0,L}$, ϵ, and L.

Step 3. Throughout this step, $L > 0$ is fixed. Let u_ϵ be the solution of (3.34)–(3.35) obtained in Step 2 above. Our goal is to pass $\epsilon \to 0$ to arrive at a solution of (3.14)–(3.15).

Set

$$\Pi_{L,\epsilon} = \frac{\nabla u_\epsilon}{\sqrt{\epsilon + |\nabla u_\epsilon|^2}}.$$

Then, we can extract subsequences still in the same notation such that

$$\Pi_{L,\epsilon} \to \Pi_L \quad \text{weak star in } L^\infty\big(\Omega \times [0,T] \times \mathcal{T}_L; \mathcal{G} \otimes \mathcal{B}(\mathcal{T}_L); R^d\big),$$
$$u_\epsilon \to u \quad \text{weakly in } L^2\big(\Omega \times [0,T]; \mathcal{G}; H_L^1\big),$$

and

$$g_{j,L}(u_\epsilon) \to p_{j,L} \quad \text{weakly in } L^2\big(\Omega \times [0,T]; \mathcal{G}; H_L^1\big).$$

It follows that

$$|\Pi_L| \leqslant 1, \quad dP \otimes dt \otimes dx\text{-almost all } (\omega, t, x), \tag{3.41}$$

and, by Lemma 2.7,

$$E\Big(\sup_{0\leqslant t\leqslant T} \|u\|_{H_L^1}^2\Big) \leqslant CE\big(\|u_{0,L}\|_{H_L^1}^2\big) + CE\left(\int_0^T \|g_{0,L}\|_{H_L^1}^2 \, dt\right) + C, \tag{3.42}$$

$$E\left(\int_0^T \|\nabla u\|_{L^1(G_L)} \, dt\right) \leqslant CE\big(\|u_{0,L}\|_{L^2(G_L)}^2\big) + CE\left(\int_0^T \|g_{0,L}\|_{L^2(G_L)}^2 \, dt\right) + C, \tag{3.43}$$

where C denotes positive constants independent of $u_{0,L}$, $g_{0,L}$, and L. It holds that

$$u(t) = u_{0,L} + \int_0^t \nabla \cdot \Pi_L(s) \, ds + \int_0^t g_{0,L}(s) \, ds + \sum_{j=1}^\infty \int_0^t p_{j,L}(s) \, dB_j \tag{3.44}$$

in H_L^{-1}, for all $t \in [0,T]$, P-almost surely. We still use the argument presented in [16] to show that u is a solution of (3.14)–(3.15). Since we are handling a singular limit at this stage, we will provide the technical details. By a result of Krylov–Rozovskii (see [16, p. 75]), it holds that

$$u \in C\big([0,T]; H_L^0\big), \quad P\text{-almost surely}, \tag{3.45}$$

and

$$\begin{aligned} e^{-ct}\big\|u(t)\big\|_{H_L^0}^2 = {} & \|u_{0,L}\|_{H_L^0}^2 - c\int_0^t e^{-cs}\big\|u(s)\big\|_{H_L^0}^2 \, ds \\ & - 2\int_0^t e^{-cs}\big\langle \Pi_L, \nabla u(s)\big\rangle_{(H_L^0)^d} \, ds + 2\int_0^t e^{-cs}\big\langle g_{0,L}(s), u(s)\big\rangle_{H_L^0} \, ds \\ & + 2\sum_{j=1}^\infty \int_0^t e^{-cs}\big\langle p_{j,L}(s), u(s)\big\rangle_{H_L^0} \, dB_j + \sum_{j=1}^\infty \int_0^t e^{-cs}\big\|p_{j,L}(s)\big\|_{H_L^0}^2 \, ds \end{aligned} \tag{3.46}$$

for all $t \in [0, T]$, P-almost surely, where $c = \gamma_1$ is a positive constant which is the same as in (3.23).

Meanwhile, for every $\phi \in L^2(\Omega \times [0, T]; \mathcal{G}; H_L^0)$, the solution u_ϵ of (3.34)–(3.35) satisfies

$$\begin{aligned}
e^{-ct} \|u_\epsilon(t)\|_{H_L^0}^2 \leqslant{}& \|u_{0,L}\|_{H_L^0}^2 - c \int_0^t e^{-cs} \|u_\epsilon(s) - \phi(s)\|_{H_L^0}^2 \, ds \\
& - 2c \int_0^t e^{-cs} \langle \phi(s), u_\epsilon(s) \rangle_{H_L^0} \, ds + c \int_0^t e^{-cs} \|\phi(s)\|_{H_L^0}^2 \, ds \\
& - 2 \int_0^t \int_{G_L} e^{-cs} \frac{|\nabla u_\epsilon(s)|^2}{\sqrt{\epsilon + |\nabla u_\epsilon(s)|^2}} \, dx \, ds + 2 \int_0^t e^{-cs} \langle g_{0,L}(s), u_\epsilon(s) \rangle_{H_L^0} \, ds \\
& + 2 \sum_{j=1}^\infty \int_0^t e^{-cs} \langle g_{j,L}(u_\epsilon(s)), u_\epsilon(s) \rangle_{H_L^0} \, dB_j(s) \\
& + \sum_{j=1}^\infty \int_0^t e^{-cs} \|g_{j,L}(u_\epsilon(s)) - g_{j,L}(\phi(s))\|_{H_L^0}^2 \, ds \\
& + 2 \sum_{j=1}^\infty \int_0^t e^{-cs} \langle g_{j,L}(u_\epsilon(s)), g_{j,L}(\phi(s)) \rangle_{H_L^0} \, ds \\
& - \sum_{j=1}^\infty \int_0^t e^{-cs} \|g_{j,L}(\phi(s))\|_{H_L^0}^2 \, ds
\end{aligned} \tag{3.47}$$

for all $t \in [0, T]$, P-almost surely. Let us choose any bounded nonnegative Borel function ψ on $[0, T]$. Using (3.23) and the inequality

$$\int_0^t \int_{G_L} e^{-cs} \frac{|\nabla u_\epsilon(s)|^2}{\sqrt{\epsilon + |\nabla u_\epsilon(s)|^2}} \, dx \, ds \geqslant \int_0^t \int_{G_L} e^{-cs} |\nabla u_\epsilon(s)| \, dx \, ds - \sqrt{\epsilon} \int_0^t \int_{G_L} e^{-cs} \, dx \, ds,$$

we combine (3.46) and (3.47) to obtain

$$\begin{aligned}
& -cE\left(\int_0^T \psi(t) \int_0^t e^{-cs} \|u(s)\|_{H_L^0}^2 \, ds \, dt \right) - 2E\left(\int_0^T \psi(t) \int_0^t e^{-cs} \langle \Pi_L(s), \nabla u(s) \rangle_{(H_L^0)^d} \, ds \, dt \right) \\
& \quad + \sum_{j=1}^\infty E\left(\int_0^T \psi(t) \int_0^t e^{-cs} \|p_{j,L}(s)\|_{H_L^0}^2 \, ds \, dt \right)
\end{aligned}$$

$$\leqslant -2cE\left(\int_0^T \psi(t)\int_0^t e^{-cs}\langle u(s),\phi(s)\rangle_{H_L^0}\,ds\,dt\right)+cE\left(\int_0^T \psi(t)\int_0^t e^{-cs}\|\phi(s)\|_{H_L^0}^2\,ds\,dt\right)$$
$$-2\varliminf_{\epsilon\to 0} E\left(\int_0^T \psi(t)\int_0^t\int_{G_L} e^{-cs}\left|\nabla u_\epsilon(s)\right|dx\,ds\,dt\right)$$
$$+\sum_{j=1}^\infty 2E\left(\int_0^T \psi(t)\sum_{j=1}^\infty\int_0^t e^{-cs}\langle p_{j,L}(s), g_{j,L}\big(\phi(s)\big)\rangle_{H_L^0}\,ds\,dt\right)$$
$$-\sum_{j=1}^\infty E\left(\int_0^T \psi(t)\sum_{j=1}^\infty\int_0^t e^{-cs}\big\|g_{j,L}\big(\phi(s)\big)\big\|_{H_L^0}^2\,ds\,dt\right)$$

which yields

$$-cE\left(\int_0^T \psi(t)\int_0^t e^{-cs}\|u(s)-\phi(s)\|_{H_L^0}^2\,ds\,dt\right)$$
$$+2\varliminf_{\epsilon\to 0} E\left(\int_0^T \psi(t)\int_0^t\int_{G_L} e^{-cs}\left|\nabla u_\epsilon(s)\right|dx\,ds\,dt\right)$$
$$-2E\left(\int_0^T \psi(t)\int_0^t\int_{G_L} e^{-cs}\Pi_L(s)\cdot\nabla u(s)\,dx\,ds\,dt\right)$$
$$+\sum_{j=1}^\infty E\left(\int_0^T \psi(t)\sum_{j=1}^\infty\int_0^t e^{-cs}\big\|p_{j,L}(s)-g_{j,L}\big(\phi(s)\big)\big\|_{H_L^0}^2\,ds\,dt\right)\leqslant 0. \tag{3.48}$$

Set $\phi=u$ in (3.48). Since ∇u_ϵ converges to ∇u weakly in $L^2(\Omega\times[0,T];\mathcal{G};(H_L^0)^d)$, it holds that

$$E\left(\int_0^T \psi(t)\int_0^t\int_{G_L} e^{-cs}\left|\nabla u(s)\right|dx\,ds\,dt\right)\leqslant \varliminf_{\epsilon\to 0} E\left(\int_0^T \psi(t)\int_0^t\int_{G_L} e^{-cs}\left|\nabla u_\epsilon(s)\right|dx\,ds\,dt\right).$$

Also, by (3.41),

$$E\left(\int_0^T \psi(t)\int_0^t\int_{G_L} e^{-cs}\Pi_L(s)\cdot\nabla u(s)\,dx\,ds\,dt\right)\leqslant E\left(\int_0^T \psi(t)\int_0^t\int_{G_L} e^{-cs}\left|\nabla u(s)\right|dx\,ds\,dt\right).$$

Hence, we can derive from (3.48) that

$$p_{j,L} = g_{j,L}(u), \quad dP \otimes dt \otimes dx\text{-almost all } (\omega, t, x),\ \forall j \geqslant 1, \tag{3.49}$$

and

$$E\left(\int_0^T \psi(t) \int_0^t \int_{G_L} e^{-cs} \left|\nabla u(s)\right| dx\, ds\, dt\right) \leqslant \varliminf_{\epsilon \to 0} E\left(\int_0^T \psi(t) \int_0^t \int_{G_L} e^{-cs} \left|\nabla u_\epsilon(s)\right| dx\, ds\, dt\right)$$
$$= E\left(\int_0^T \psi(t) \int_0^t \int_{G_L} e^{-cs} \Pi_L(s) \cdot \nabla u(s)\, dx\, ds\, dt\right).$$

Since

$$\Pi_L \cdot \nabla u \leqslant |\nabla u|, \quad dP \otimes dt \otimes dx\text{-almost all } (\omega, t, x),$$

it holds that

$$\Pi_L \cdot \nabla u = |\nabla u|, \quad dP \otimes dt \otimes dx\text{-almost all } (\omega, t, x). \tag{3.50}$$

By virtue of (3.41)–(3.45) and (3.49)–(3.50), u is a solution of (3.14)–(3.15) according to Definition 3.1.

For uniqueness, let u and $\hat{u}$ be two solutions of (3.14)–(3.15). Let Π_L and $\hat{\Pi}_L$ correspond to u and $\hat{u}$, respectively. Then,

$$\langle \Pi_L - \hat{\Pi}_L, \nabla u - \nabla \hat{u} \rangle_{(H_L^0)^d} = \|\nabla u\|_{(L^1(G_L))^d} + \|\nabla \hat{u}\|_{(L^1(G_L))^d}$$
$$- \langle \hat{\Pi}_L, \nabla u \rangle_{(H_L^0)^d} - \langle \Pi_L, \nabla \hat{u} \rangle_{(H_L^0)^d} \geqslant 0, \quad \text{by (3.41)},$$

for $dP \otimes dt$-almost all (ω, t). We use this for the estimate of $\|u(t) - \hat{u}(t)\|_{H_L^0}^2$ to derive $u \equiv \hat{u}$.

If (3.10)–(3.13) hold, (3.37) yields

$$E\Big(\sup_{0 \leqslant t \leqslant T} J_\epsilon\big(u_\epsilon(t)\big)\Big) \leqslant CE\big(J_\epsilon(u_{0,L})\big) + C \int_0^T \|\nabla g_{0,L}\|_{(L^1(G_L))^d}\, dt \tag{3.51}$$

and

$$E\left(\int_0^T \|\nabla \cdot \Pi_{L,\epsilon}\|_{H_L^0}^2\, dt\right) \leqslant CE\big(J_\epsilon(u_{0,L})\big) + C \int_0^T \|\nabla g_{0,L}\|_{(L^1(G_L))^d}\, dt \tag{3.52}$$

where C denotes positive constants independent of $u_{0,L}$, $g_{0,L}$, ϵ and L. By Lemma 2.7, we can pass $\epsilon \to 0$ to obtain (3.19) and (3.20). The proof of Theorem 3.2 is complete.

Remark 3.3. The purpose of the term $\eta\Delta^2 u$ in (3.21) is to obtain H_L^2-regularity directly from the known result in the existing literature. This regularity is used to apply Lemma 2.1. However, this term is a technical obstacle in obtaining (3.51)–(3.52). This is the reason why we passed $\eta \to 0$ with $\epsilon > 0$ fixed. We also note that a deterministic version of Eq. (3.34) was discussed in [20] over a unit square with the Neumann boundary condition. It was also discussed in [2].

4. Proof of Theorems 1.2 and 1.4

The basic idea for the existence of a solution of (0.1)–(0.2) in R^d is to obtain a solution as the limit of the sequence $\{u_k\}_{k=1}^{\infty}$, where each u_k is a solution in Theorem 3.2 above with $L = L_k \uparrow \infty$, as $k \to \infty$.

We start with the following fact which is an adaptation of Lemma 2.7.

Lemma 4.1. *Suppose that $\{u_{L_k}\}_{k=1}^{\infty}$ is a sequence such that*

$$L_k \uparrow \infty, \quad \text{as } k \to \infty,$$
$$u_{L_k} \in L^2\big(\Omega \times [0,T]; \mathcal{G}; H^0_{L_k}\big), \quad \forall k,$$
$$\chi_k u_{L_k} \to u_\infty,$$

weakly in $L^2(\Omega \times [0,T]; \mathcal{G}; L^2(R^d))$, where χ_k denotes the characteristic function of the set G_{L_k}.

(i) *if $E(\sup_{0\leqslant t\leqslant T} \|u_{L_k}(t)\|^2_{H^0_{L_k}}) \leqslant C_1$, for all k, then $E(\sup_{0\leqslant t\leqslant T} \|u_\infty(t)\|^2_{L^2(R^d)}) \leqslant C_1$,*

(ii) *if $E(\sup_{0\leqslant t\leqslant T} \|u_{L_k}(t)\|^2_{H^1_{L_k}}) \leqslant C_2$, for all k, then $E(\sup_{0\leqslant t\leqslant T} \|u_\infty(t)\|^2_{H^1(R^d)}) \leqslant C_2$,*

(iii) *if $E(\sup_{0\leqslant t\leqslant T} \|\nabla u_{L_k}(t)\|_{(L^1(G_{L_k}))^d}) \leqslant C_3$, for all k, then*

$$E\Big(\sup_{0\leqslant t\leqslant T} \big\|\nabla u_\infty(t)\big\|_{(\mathcal{M}(R^d))^d}\Big) \leqslant C_3,$$

(iv) *if $E(\int_0^T \|\nabla u_{L_k}(t)\|^2_{(L^2(G_{L_k}))^d}\,dt) \leqslant C_4$, and $E(\int_0^T \|\nabla u_{L_k}(t)\|_{(L^1(G_{L_k}))^d}\,dt) \leqslant C_5$, for all k, then $E(\int_0^T \|\nabla u_\infty(t)\|^2_{(L^2(R^d))^d}\,dt) \leqslant C_4$, and*

$$E\left(\int_0^T \big\|\nabla u_\infty(t)\big\|_{(L^1(R^d))^d}\,dt\right) \leqslant C_5.$$

Proof. We will show (iii). Since u_∞ is $L^2(R^d)$-valued $\mathcal{G}$-measurable and for each nonnegative number r, the set $\{f \in L^2(R^d) \mid \|\nabla f\|_{(\mathcal{M}(R^d))^d} \leqslant r\}$ is a closed subset of $L^2(R^d)$, $\|\nabla u_\infty\|_{(\mathcal{M}(R^d))^d}$ is $\mathcal{G}$-measurable.

Choose any $1 \leqslant R < \infty$. There is a sequence $\{v_m\}_{m=1}^{\infty}$ such that each v_m is a convex combination of finitely many $\chi_k u_{L_k}$, $L_k > 4R$, and

$$v_m \to u_\infty \quad \text{strongly in } L^2\big(\Omega \times [0,T]; \mathcal{G}; L^2\big(R^d\big)\big),$$
$$E\Big(\sup_{0\leqslant t\leqslant T} \big\|\varphi_R \nabla v_m(t)\big\|_{(L^1(R^d))^d}\Big) \leqslant C_3, \quad \forall m, \tag{4.1}$$

where φ_R is defined by (2.2)–(2.3).

There is a subsequence still denoted by $\{v_m\}$ and $Q \in \mathcal{G}$ such that

$$dP \otimes dt\big(\Omega \times [0,T] \setminus Q\big) = 0,$$
$$v_m \to u_\infty \quad \text{strongly in } L^2\big(R^d\big), \text{ for each } (\omega,t) \in Q.$$

Thus, $\nabla v_m \to \nabla u_\infty$ in the sense of distributions over R^d, for each $(\omega,t) \in Q$. It follows that

$$\begin{aligned} E\Big(\sup_{0\leqslant t\leqslant T} \big\|\varphi_R \nabla u_\infty(t)\big\|_{(\mathcal{M}(R^d))^d}\Big) &\leqslant E\Big(\sup_{0\leqslant t\leqslant T} \varliminf_{m\to\infty} \big\|\varphi_R \nabla v_m(t)\big\|_{(L^1(R^d))^d}\Big) \\ &\leqslant E\Big(\varliminf_{m\to\infty} \sup_{0\leqslant t\leqslant T} \big\|\varphi_R \nabla v_m(t)\big\|_{(L^1(R^d))^d}\Big) \\ &\leqslant \varliminf_{m\to\infty} E\Big(\sup_{0\leqslant t\leqslant T} \big\|\varphi_R \nabla v_m(t)\big\|_{(L^1(R^d))^d}\Big) \leqslant C_3, \quad \text{by (4.1)}, \end{aligned}$$

and thus,

$$\begin{aligned} E\Big(\sup_{0\leqslant t\leqslant T} \big\|\nabla u_\infty(t)\big\|_{(\mathcal{M}(R^d))^d}\Big) &\leqslant E\Big(\sup_{0\leqslant t\leqslant T} \varliminf_{R\to\infty} \big\|\varphi_R \nabla u_\infty(t)\big\|_{(\mathcal{M}(R^d))^d}\Big) \\ &\leqslant E\Big(\varliminf_{R\to\infty} \sup_{0\leqslant t\leqslant T} \big\|\varphi_R \nabla u_\infty(t)\big\|_{(\mathcal{M}(R^d))^d}\Big) \\ &\leqslant \varliminf_{R\to\infty} E\Big(\sup_{0\leqslant t\leqslant T} \big\|\varphi_R \nabla u_\infty(t)\big\|_{(\mathcal{M}(R^d))^d}\Big) \leqslant C_3. \end{aligned}$$

In the same way, (i) and (ii) follow. For (iv), we use the same argument to obtain first $E(\int_0^T \|\nabla u_\infty(t)\|^2_{(L^2(R^d))^d}\,dt) \leqslant C_4$. For each $R > 1$, we use uniform integrability to see that

$$\varphi_R \nabla u_k \to \varphi_R \nabla u_\infty \quad \text{weakly in } \big(L^1\big(\Omega \times [0,T] \times R^d; \mathcal{G} \otimes \mathcal{B}\big(R^d\big)\big)\big)^d,$$

and thus, $E(\int_0^T \|\varphi_R \nabla u_\infty(t)\|_{(L^1(R^d))^d}\,dt) \leqslant C_5$. By passing $R \to \infty$, we see that

$$E\left(\int_0^T \big\|\nabla u_\infty(t)\big\|_{(L^1(R^d))^d}\,dt\right) \leqslant C_5. \qquad \square$$

4.1. Proof of Theorem 1.2

Let $u_0 \in L^2(\Omega; \mathcal{F}_0; H^1(R^d))$ be given. Define

$$u_{0,k} = \Lambda_{L_k}(\varphi_{R_k} u_0), \quad L_k = 4R_k$$

where $R_k \uparrow \infty$, and Λ_L was defined by (3.1). Then, $u_{0,k} \in L^2(\Omega; \mathcal{F}_0; H^1_{L_k})$ and

$$\varphi_{R_k} u_{0,k} \to u_0 \quad \text{strongly in } L^2\bigl(\Omega; \mathcal{F}_0; H^1\bigl(R^d\bigr)\bigr).$$

Also, $g_{0,L}$ and $g_{j,L}$ are defined by (3.2) and (3.3), respectively.

Let u_k be the solution in Theorem 3.2 above with $L = L_k$ and $u_{0,L} = u_{0,k}$.

We then have

$$E\Bigl(\sup_{0 \leqslant t \leqslant T} \bigl\|u_k(t)\bigr\|^2_{H^1_{L_k}}\Bigr) \leqslant C, \quad \forall k,$$

$$E\left(\int_0^T \int_{G_{L_k}} |\nabla u_k|\,dx\,dt\right) \leqslant C, \quad \forall k,$$

for some positive constants C. Hence, there is a subsequence still denoted by $\{u_k\}_{k=1}^{\infty}$ such that

$$\chi_k u_k \to u_\infty, \quad \text{weakly in } L^2\bigl(\Omega \times [0,T]; \mathcal{G}; L^2\bigl(R^d\bigr)\bigr), \tag{4.2}$$

where χ_k is the characteristic function of the set G_{L_k}, and, by Lemma 4.1,

$$E\Bigl(\sup_{0 \leqslant t \leqslant T} \bigl\|u_\infty(t)\bigr\|^2_{H^1(R^d)}\Bigr) \leqslant C, \tag{4.3}$$

$$E\left(\int_0^T \int_{R^d} |\nabla u_\infty|\,dx\,dt\right) \leqslant C. \tag{4.4}$$

If necessary, we can further extract a subsequence $\{u_k\}_{k=1}^{\infty}$ so that the corresponding sequences $\{\Pi_{L_k}\}_{k=1}^{\infty}$ and $\{g_{j,L_k}(u_k)\}_{k=1}^{\infty}$ be convergent as follows.

$$\chi_k g_{j,L_k}(u_k) \to g_{j,\infty} \quad \text{weakly in } L^2\bigl(\Omega \times [0,T]; \mathcal{G}; L^2\bigl(R^d\bigr)\bigr),\ j = 1, 2, \ldots, \tag{4.5}$$

and

$$\chi_k \Pi_{L_k} \to \Pi_\infty \quad \text{weak star in } L^\infty\bigl(\Omega \times [0,T] \times R^d; \mathcal{G} \otimes \mathcal{B}\bigl(R^d\bigr); R^d\bigr), \tag{4.6}$$

which yields

$$|\Pi_\infty| \leqslant 1, \quad dP \otimes dt \otimes dx\text{-almost all } (\omega, t, x). \tag{4.7}$$

Choose any $R > 1$. Then, it holds that

$$\varphi_R u_\infty(t) = \varphi_R u_0 + \int_0^t \varphi_R \nabla \cdot \Pi_\infty(s)\,ds + \int_0^t \varphi_R g_0(s)\,ds + \sum_{j=1}^\infty \int_0^t \varphi_R g_{j,\infty}(s)\,dB_j(s) \tag{4.8}$$

in $H^{-1}(R^d)$, for all $t \in [0,T]$, P-almost surely.

It follows from Lemma 2.5 that

$$u_\infty \in C\big([0,T]; L^2\big(R^d\big)\big), \quad P\text{-almost surely,} \tag{4.9}$$

and

$$\begin{aligned} e^{-ct}\big\|u_\infty(t)\big\|^2_{L^2(R^d)} &= \|u_0\|^2_{L^2(R^d)} - c\int_0^t e^{-cs}\big\|u_\infty(s)\big\|^2_{L^2(R^d)}\,ds \\ &\quad - 2\int_0^t\int_{R^d} e^{-cs}\Pi_\infty(s)\cdot\nabla u_\infty(s)\,dx\,ds + 2\int_0^t e^{-cs}\big\langle g_0(s), u_\infty(s)\big\rangle_{L^2(R^d)}\,ds \\ &\quad + 2\sum_{j=1}^\infty \int_0^t e^{-cs}\big\langle g_{j,\infty}(s), u_\infty(s)\big\rangle_{L^2(R^d)}\,dB_j(s) \\ &\quad + \sum_{j=1}^\infty \int_0^t e^{-cs}\big\|g_{j,\infty}(s)\big\|^2_{L^2(R^d)}\,ds \end{aligned} \tag{4.10}$$

for all $t \in [0,T]$, P-almost surely, where $c = \gamma_1$ is the positive constant satisfying (3.23). Let us choose any arbitrary $\phi \in L^2(\Omega \times [0,T]; \mathcal{G}; L^2(R^d))$ such that $\phi \equiv 0$, for $|x| \geqslant 2R > 2$.

We take $L_k \geqslant 4R$. Since u_k is the solution of (3.14)–(3.15) with $L = L_k$ and $u_{0,L} = u_{0,k}$, it follows from (3.46), (3.49) and (3.50) that

$$\begin{aligned} e^{-ct}\big\|u_k(t)\big\|^2_{L^2(G_{L_k})} &= \|u_{0,k}\|^2_{L^2(G_{L_k})} - c\int_0^t e^{-cs}\big\|u_k(s)-\phi(s)\big\|^2_{L^2(G_{L_k})}\,ds \\ &\quad - 2c\int_0^t e^{-cs}\big\langle \phi(s), u_k(s)\big\rangle_{L^2(G_{L_k})}\,ds + c\int_0^t e^{-cs}\big\|\phi(s)\big\|^2_{L^2(G_{L_k})}\,ds \\ &\quad - 2\int_0^t\int_{G_{L_k}} e^{-cs}\big|\nabla u_k(s)\big|\,dx\,ds + 2\int_0^t e^{-cs}\big\langle g_{0,L_k}(s), u_k(s)\big\rangle_{L^2(G_{L_k})}\,ds \\ &\quad + 2\sum_{j=1}^\infty \int_0^t e^{-cs}\big\langle g_{j,L_k}\big(u_k(s)\big), u_k(s)\big\rangle_{L^2(G_{L_k})}\,dB_j(s) \end{aligned}$$

$$+\sum_{j=1}^{\infty}\int_0^t e^{-cs}\big\|g_{j,L_k}\big(u_k(s)\big)-g_{j,L_k}\big(\phi(s)\big)\big\|^2_{L^2(G_{L_k})}\,ds$$
$$+2\sum_{j=1}^{\infty}\int_0^t e^{-cs}\big\langle g_{j,L_k}\big(u_k(s)\big),g_{j,L_k}\big(\phi(s)\big)\big\rangle_{L^2(G_{L_k})}\,ds$$
$$-\sum_{j=1}^{\infty}\int_0^t e^{-cs}\big\|g_{j,L_k}\big(\phi(s)\big)\big\|^2_{L^2(G_{L_k})}\,ds \tag{4.11}$$

for all $t\in[0,T]$, P-almost surely.

Next we choose any bounded nonnegative Borel function ψ on $[0,T]$.

It follows from (3.23), (4.2)–(4.6), and (4.10)–(4.11) that

$$-cE\left(\int_0^T\psi(t)\int_0^t e^{-cs}\big\|u_\infty(s)\big\|^2_{L^2(R^d)}\,ds\,dt\right)$$
$$-2E\left(\int_0^T\psi(t)\int_0^t\int_{R^d}e^{-cs}\Pi_\infty(s)\cdot\nabla u_\infty(s)\,dx\,ds\,dt\right)$$
$$+\sum_{j=1}^{\infty}E\left(\int_0^T\psi(t)\int_0^t e^{-cs}\big\|g_{j,\infty}(s)\big\|^2_{L^2(R^d)}\,ds\,dt\right)$$
$$\leqslant -2cE\left(\int_0^T\psi(t)\int_0^t e^{-cs}\big\langle u_\infty(s),\phi(s)\big\rangle_{L^2(R^d)}\,ds\,dt\right)$$
$$+cE\left(\int_0^T\psi(t)\int_0^t e^{-cs}\big\|\phi(s)\big\|^2_{L^2(R^d)}\,ds\,dt\right)$$
$$-2\varliminf_{k\to\infty}E\left(\int_0^T\psi(t)\int_0^t\int_{G_{L_k}}e^{-cs}\big|\nabla u_k(s)\big|\,dx\,ds\,dt\right)$$
$$+2E\left(\int_0^T\psi(t)\sum_{j=1}^{\infty}\int_0^t e^{-cs}\big\langle g_{j,\infty}(s),g_j\big(\phi(s)\big)\big\rangle_{L^2(R^d)}\,ds\,dt\right)$$
$$-E\left(\int_0^T\psi(t)\sum_{j=1}^{\infty}\int_0^t e^{-cs}\big\|g_j\big(\phi(s)\big)\big\|^2_{L^2(R^d)}\,ds\,dt\right). \tag{4.12}$$

We rearrange terms in (4.12) to arrive at

$$-cE\left(\int_0^T \psi(t)\int_0^t e^{-cs}\left\|u_\infty(s)-\phi(s)\right\|^2_{L^2(R^d)}\,ds\,dt\right)$$

$$+2\varliminf_{k\to\infty} E\left(\int_0^T \psi(t)\int_0^t\int_{G_{L_k}} e^{-cs}\left|\nabla u_k(s)\right|dx\,ds\,dt\right)$$

$$-2E\left(\int_0^T \psi(t)\int_0^t\int_{G_{L_k}} e^{-cs}\Pi_\infty(s)\cdot\nabla u_\infty(s)\,dx\,ds\,dt\right)$$

$$+E\left(\int_0^T \psi(t)\sum_{j=1}^\infty\int_0^t e^{-cs}\left\|g_{j,\infty}(s)-g_j\big(\phi(s)\big)\right\|^2_{L^2(R^d)}\,ds\,dt\right)\leqslant 0. \tag{4.13}$$

We can take $\phi=\varphi_R u_\infty$ in (4.13). Then, pass $R\to\infty$ to arrive at

$$2\varliminf_{k\to\infty} E\left(\int_0^T \psi(t)\int_0^t\int_{G_{L_k}} e^{-cs}\left|\nabla u_k(s)\right|dx\,ds\,dt\right)$$

$$-2E\left(\int_0^T \psi(t)\int_0^t\int_{R^d} e^{-cs}\Pi_\infty(s)\cdot\nabla u_\infty(s)\,dx\,ds\,dt\right)$$

$$+E\left(\int_0^T \psi(t)\sum_{j=1}^\infty\int_0^t e^{-cs}\left\|g_{j,\infty}(s)-g_j\big(u_\infty(s)\big)\right\|^2_{L^2(R^d)}\,ds\,dt\right)\leqslant 0. \tag{4.14}$$

By virtue of Lemma 4.1 and (4.7), we have

$$E\left(\int_0^T \psi(t)\int_0^t\int_{R^d} e^{-cs}\Pi_\infty(s)\cdot\nabla u_\infty(s)\,dx\,ds\,dt\right)$$

$$\leqslant E\left(\int_0^T \psi(t)\int_0^t\int_{R^d} e^{-cs}\left|\nabla u_\infty(s)\right|dx\,ds\,dt\right)$$

$$\leqslant \varliminf_{k\to\infty} E\left(\int_0^T \psi(t)\int_0^t\int_{G_{L_k}} e^{-cs}\left|\nabla u_k(s)\right|dx\,ds\,dt\right).$$

Thus, it follows from (4.14) that

$$\left\|g_{j,\infty}-g_j(u_\infty)\right\|_{L^2(R^d)}=0,\quad dP\otimes dt\text{-almost all }(\omega,t),\ \forall j\geqslant 1, \tag{4.15}$$

and

$$E\left(\int_0^T \psi(t)\int_0^t\int_{R^d} e^{-cs}\Pi_\infty(s)\cdot\nabla u_\infty(s)\,dx\,ds\,dt\right) = E\left(\int_0^T \psi(t)\int_0^t\int_{R^d} e^{-cs}\big|\nabla u_\infty(s)\big|\,dx\,ds\,dt\right).$$

Since

$$\Pi_\infty\cdot\nabla u_\infty \leqslant |\nabla u_\infty|,\quad dP\otimes dt\otimes dx\text{-almost all } (\omega,t,x),$$

it holds that

$$\Pi_\infty\cdot\nabla u_\infty = |\nabla u_\infty|,\quad dP\otimes dt\otimes dx\text{-almost all } (\omega,t,x). \tag{4.16}$$

By virtue of (4.3)–(4.4), (4.7)–(4.9), and (4.15)–(4.16), u_∞ is a solution according to Definition 1.1. For uniqueness of a solution, let u and $\hat{u}$ be two solutions. By means of the property (v) in Definition 1.1 and Lemma 2.5, we can easily derive

$$E\Big(\sup_{0\leqslant t\leqslant T}\big\|u(t)-\hat{u}(t)\big\|^2_{L^2(R^d)}\Big) = 0.$$

The proof of Theorem 1.2 is complete.

4.2. Proof of Theorem 1.4

Recall conditions (1.7)–(1.10) and (3.9)–(3.13).
Suppose $u_0 \in L^2(\Omega;\mathcal{F}_0;L^2(R^d))$ and $\|\nabla u_0\|_{(\mathcal{M}(R^d))^d} \in L^1(\Omega;\mathcal{F}_0)$.
Then, by Lemma 2.4, there is a sequence $\{f_k\}_{k=1}^\infty$ such that

$$f_k \to u_0 \quad \text{in } L^2\big(\Omega;\mathcal{F}_0;L^2\big(R^d\big)\big), \text{ as } k\to\infty,$$
$$f_k \in W^{1,1}\big(R^d\big)\cap H^1\big(R^d\big),\quad P\text{-almost surely},\ \forall k,$$
$$\text{supp } f_k \text{ is contained in } \{x \mid |x|<R_k\},\ P\text{-almost surely},$$

where $R_k \uparrow \infty$, as $k\to\infty$

$$E\big(\|\nabla f_k\|_{(L^1(R^d))^d}\big) \leqslant 1 + E\big(\|\nabla u_0\|_{(\mathcal{M}(R^d))^d}\big),\quad \forall k.$$

We set

$$u_{0,k} = \Lambda_{L_k}(f_k)$$

where $L_k = 4R_k$, for each k. Let u_k be the solution in Theorem 3.2 with $L = L_k$ and $u_{0,L} = u_{0,k}$. We utilize the estimates (3.18)–(3.20).

Through the same procedure as above, we find u_∞ and Π_∞, and arrive at

$$\begin{aligned}
&2 \varliminf_{k\to\infty} E\left(\int_0^T \psi(t)\int_0^t \int_{G_{L_k}} e^{-cs}\left|\nabla u_k(s)\right| dx\,ds\,dt\right)\\
&\quad + 2E\left(\int_0^T \psi(t)\int_0^t \int_{R^d} e^{-cs}\big(\nabla\cdot \Pi_\infty(s)\big)u_\infty(s)\,dx\,ds\,dt\right)\\
&\quad + \sum_{j=1}^\infty E\left(\int_0^T \psi(t)\sum_{j=1}^\infty\int_0^t e^{-cs}\left\|g_{j,\infty}(s)-g_j\big(u_\infty(s)\big)\right\|^2_{L^2(R^d)} ds\,dt\right)\leqslant 0. \qquad (4.17)
\end{aligned}$$

Fix any $\epsilon > 0$ and $R > 1$. Since $\|\varphi_R(\nabla u_k * \rho_\epsilon)\|_{(L^2(R^d))^d} \leqslant \|u_k\|_{L^2(G_{L_k})}\|\nabla\rho_\epsilon\|_{(L^1(R^d))^d}$, for $L_k \geqslant 4R$, we have

$$E\left(\int_0^T \left\|\varphi_R(\nabla u_k * \rho_\epsilon)\right\|^2_{L^2(R^d)} dt\right)\leqslant C_\epsilon$$

for some constant C_ϵ independent of $L_k \geqslant 4R$. Hence,

$$\varphi_R(\nabla u_k * \rho_\epsilon)\to\varphi_R(\nabla u_\infty * \rho_\epsilon)$$

weakly in $(L^1(\Omega\times[0,T]\times R^d))^d$, and

$$\begin{aligned}
&E\left(\int_0^T \psi(t)\int_0^t \int_{R^d} e^{-cs}\left|\varphi_R\big(\nabla u_\infty(s)*\rho_\epsilon\big)\right| dx\,ds\,dt\right)\\
&\quad\leqslant \varliminf_{k\to\infty} E\left(\int_0^T \psi(t)\int_0^t \int_{R^d} e^{-cs}\left|\varphi_R\big(\nabla u_k(s)*\rho_\epsilon\big)\right| dx\,ds\,dt\right)\\
&\quad\leqslant \varliminf_{k\to\infty} E\left(\int_0^T \psi(t)\int_0^t \int_{G_{L_k}} e^{-cs}\left|\nabla u_k(s)\right| dx\,ds\,dt\right).
\end{aligned}$$

We pass $R\to\infty$, and then, pass $\epsilon\to 0$ to arrive at

$$\begin{aligned}
&E\left(\int_0^T \psi(t)\int_0^t e^{-cs}\left\|\nabla u_\infty(s)\right\|_{(\mathcal{M}(R^d))^d} ds\,dt\right)\\
&\quad\leqslant \varliminf_{k\to\infty} E\left(\int_0^T \psi(t)\int_0^t \int_{G_{L_k}} e^{-cs}\left|\nabla u_k(s)\right| dx\,ds\,dt\right). \qquad (4.18)
\end{aligned}$$

It follows from Lemma 2.2 and (4.17)–(4.18) that

$$\big\|g_{j,\infty}-g_j(u_\infty)\big\|_{L^2(R^d)}=0,\quad dP\otimes dt\text{-almost all }(\omega,t),\ \forall j\geqslant 1,\tag{4.19}$$

and

$$-\langle\nabla\cdot\Pi_\infty,u_\infty\rangle_{L^2(R^d)}=\|\nabla u_\infty\|_{(\mathcal{M}(R^d))^d},\quad dP\otimes dt\text{-almost all }(\omega,t).\tag{4.20}$$

By means of (3.18)–(3.20), (4.19)–(4.20), and Lemma 4.1, we can argue as in the proof of Theorem 1.2 to conclude that u_∞ is a solution according to Definition 1.3. Uniqueness of a solution also follows in the same way as above.

5. Proof of Theorem 1.6

First of all, we note that $0<T<\infty$ was arbitrarily given in Theorem 1.4. By the pathwise uniqueness of a solution, we may assume that the solution exists on the interval $[0,\infty)$, P-almost surely. We will derive some necessary estimates.

Let u be a solution of (0.1)–(0.2) according to Definition 1.3. Then, it holds that

$$\begin{aligned}\big\|u(t)\big\|^2_{L^2(R^d)}&=\big\|u(0)\big\|^2_{L^2(R^d)}+2\int_0^t\big\langle\nabla\cdot\Pi(s),u(s)\big\rangle_{L^2(R^d)}\,ds\\&\quad+2\int_0^t\big\langle g_0,u(s)\big\rangle_{L^2(R^d)}\,ds+2\sum_{j=1}^\infty\int_0^t\big\langle\xi_j\big(u(s)\big),u(s)\big\rangle_{L^2(R^d)}\,dB_j\\&\quad+\sum_{j=1}^\infty\int_0^t\big\|\xi_j\big(u(s)\big)\big\|^2_{L^2(R^d)}\,ds\end{aligned}\tag{5.1}$$

for all $t\geqslant 0$, P-almost surely.

Let us set $X(t)=\|u(t)\|^2_{L^2(R^d)}$. By Ito's formula, it follows from (5.1) that

$$\begin{aligned}&E\big(e^{\lambda t}\big(1+X(t)\big)^r\big)\\&\quad=E\big(\big(1+X(0)\big)^r\big)+\lambda\int_0^t e^{\lambda s}E\big(\big(1+X(s)\big)^r\big)\,ds\\&\qquad-2r\int_0^t e^{\lambda s}E\big(\big(1+X(s)\big)^{r-1}\big\|\nabla u(s)\big\|_{(\mathcal{M}(R^d))^d}\big)\,ds\\&\qquad+2r\int_0^t e^{\lambda s}E\big(\big(1+X(s)\big)^{r-1}\big\langle g_0,u(s)\big\rangle_{L^2(R^d)}\big)\,ds\end{aligned}$$

$$+ r\sum_{j=1}^{\infty}\int_0^t e^{\lambda s}E\big(\big(1+X(s)\big)^{r-1}\big\|\xi_j\big(u(s)\big)\big\|^2_{L^2(R^d)}\big)\,ds$$

$$+ 2r(r-1)\sum_{j=1}^{\infty}\int_0^t e^{\lambda s}E\big(\big(1+X(s)\big)^{r-2}\big|\big\langle\xi_j\big(u(s)\big),u(s)\big\rangle_{L^2(R^d)}\big|^2\big)\,ds \tag{5.2}$$

where λ and r are constants. By (1.15), we can choose $\lambda > 0$, $\delta > 0$, and $0 < r < \frac{1}{2}$ such that

$$\lambda + r\delta + r\sum_{j=1}^{\infty}\beta_j^2 - 2r(1-r)(1-\delta)\sum_{j=1}^{\infty}\alpha_j^2 \leqslant 0. \tag{5.3}$$

We estimate the right-hand side of (5.2). It holds that

$$(1+X)^{r-1}\langle g_0,u\rangle_{L^2(R^d)} \leqslant \frac{1}{2\delta}\|g_0\|^2_{L^2(R^d)} + \frac{\delta}{2}(1+X)^r. \tag{5.4}$$

By the inequality

$$x^2 \geqslant (1-\delta)(1+x)^2 - C_\delta, \quad \forall x \in R,$$

for some positive constant C_δ, we see that

$$\big(1+X(s)\big)^{r-2}\big|\big\langle\xi_j\big(u(s)\big),u(s)\big\rangle_{L^2(R^d)}\big|^2 \geqslant \alpha_j^2(1-\delta)\big(1+X(s)\big)^r - \alpha_j^2 C_\delta. \tag{5.5}$$

By virtue of (1.12)–(1.15) and (5.3)–(5.5), we find that

$$\begin{aligned} E\big(e^{\lambda t}\big(1+X(t)\big)^r\big) \leqslant{}& E\big(\big(1+X(0)\big)^r\big) \\ &- 2r\int_0^t e^{\lambda s}E\big(\big(1+X(s)\big)^{r-1}\big\|\nabla u(s)\big\|_{(\mathcal{M}(R^d))^d}\big)\,ds \\ &+ \int_0^t e^{\lambda s}\left(\frac{r}{\delta}\|g_0\|^2_{L^2(R^d)} + 2r(1-r)C_\delta\sum_{j=1}^{\infty}\alpha_j^2\right)ds \end{aligned} \tag{5.6}$$

which yields

$$E\big(\big(1+X(t)\big)^r\big) \leqslant e^{-\lambda t}E\big(\big(1+X(0)\big)^r\big) + \frac{1-e^{-\lambda t}}{\lambda}C \leqslant C, \quad \forall t \geqslant 0, \tag{5.7}$$

where C denotes various positive constants. If we take $\lambda = 0$ in (5.3), then (5.6) is still valid with $\lambda = 0$, and thus,

$$\frac{1}{T}E\left(\int_0^T \left(1+X(t)\right)^{r-1}\left\|\nabla u(t)\right\|_{(\mathcal{M}(R^d))^d}\,dt\right)$$
$$\leqslant \frac{1}{2rT}E\left(\left(1+X(0)\right)^r\right)+C\leqslant C, \quad \forall T>1.$$

It follows that

$$\frac{1}{T}E\left(\int_0^T \left\|\nabla u(t)\right\|^r_{(\mathcal{M}(R^d))^d}\,dt\right)$$
$$=\frac{1}{T}E\left(\int_0^T \left(1+X(t)\right)^{r(r-1)}\left\|\nabla u(t)\right\|^r_{(\mathcal{M}(R^d))^d}\left(1+X(t)\right)^{r(1-r)}\,dt\right)$$
$$\leqslant \frac{r}{T}E\left(\int_0^T \left(1+X(t)\right)^{r-1}\left\|\nabla u(t)\right\|_{(\mathcal{M}(R^d))^d}\,dt\right)+\frac{1-r}{T}E\left(\int_0^T \left(1+X(t)\right)^r\,dt\right)$$
$$\leqslant C, \quad \forall T>1. \tag{5.8}$$

Next let u_1 and u_2 be two solutions of (0.1) according to Definition 1.3. It holds that

$$\begin{aligned}\left\|u_1(t)-u_2(t)\right\|^2_{L^2(R^d)} &= \left\|u_1(0)-u_2(0)\right\|^2_{L^2(R^d)}\\ &\quad+2\int_0^t \left\langle \nabla\cdot\Pi_1(s)-\nabla\cdot\Pi_2(s), u_1(s)-u_2(s)\right\rangle_{L^2(R^d)}\,ds\\ &\quad+2\sum_{j=1}^\infty\int_0^t \left\langle \xi_j\left(u_1(s)\right)-\xi_j\left(u_2(s)\right), u_1(s)-u_2(s)\right\rangle_{L^2(R^d)}\,dB_j\\ &\quad+\sum_{j=1}^\infty\int_0^t \left\|\xi_j\left(u_1(s)\right)-\xi_j\left(u_2(s)\right)\right\|^2_{L^2(R^d)}\,ds\end{aligned} \tag{5.9}$$

for all $t\geqslant 0$, P-almost surely.

Let

$$Y(t)=\left\|u_1(t)-u_2(t)\right\|^2_{L^2(R^d)}.$$

By applying Ito's formula to (5.9), we have

$$\begin{aligned}E\left(e^{\lambda t}\left(\eta+Y(t)\right)^r\right) &= E\left(\left(\eta+Y(0)\right)^r\right)+\lambda\int_0^t e^{\lambda s}E\left(\left(\eta+Y(s)\right)^r\right)ds\\ &\quad+2r\int_0^t e^{\lambda s}\left(\eta+Y(s)\right)^{r-1}\left\langle \nabla\cdot\Pi_1(s)-\nabla\cdot\Pi_2(s), u_1(s)-u_2(s)\right\rangle_{L^2(R^d)}\,ds\end{aligned}$$

$$+ r \sum_{j=1}^{\infty} \int_0^t e^{\lambda s} E\big((\eta + Y(s))^{r-1} \| \xi_j(u_1(s)) - \xi_j(u_2(s)) \|_{L^2(R^d)}^2 \big) \, ds$$

$$+ 2r(r-1) \sum_{j=1}^{\infty} \int_0^t e^{\lambda s} E\big((\eta + Y(s))^{r-2}$$

$$\times \big| \langle \xi_j(u_1(s)) - \xi_j(u_2(s)), u_1(s) - u_2(s) \rangle_{L^2(R^d)} \big|^2 \big) \, ds. \tag{5.10}$$

By virtue of Definition 1.3 and Lemma 2.2, we have the inequality

$$\langle \nabla \cdot \Pi_1 - \nabla \cdot \Pi_2, u_1 - u_2 \rangle_{L^2(R^d)} \leqslant 0$$

and thus, it follows from (1.13), (1.14) and (5.10) that

$$E\big(e^{\lambda t} (\eta + Y(t))^r\big) \leqslant E\big((\eta + Y(0))^r\big) + \lambda \int_0^t e^{\lambda s} E\big((\eta + Y(s))^r\big) \, ds$$

$$+ r \sum_{j=1}^{\infty} \int_0^t e^{\lambda s} \beta_j^2 E\big((\eta + Y(s))^{r-1} Y(s)\big) \, ds$$

$$+ 2r(r-1) \sum_{j=1}^{\infty} \int_0^t e^{\lambda s} \alpha_j^2 E\big((\eta + Y(s))^{r-2} Y(s)^2\big) \, ds. \tag{5.11}$$

Obviously, the same $\lambda > 0$ and $0 < r < \frac{1}{2}$ in (5.3) satisfy

$$\lambda + r \sum_{j=1}^{\infty} \beta_j^2 + 2r(r-1) \sum_{j=1}^{\infty} \alpha_j^2 < 0. \tag{5.12}$$

Thus, by passing $\eta \downarrow 0$ in (5.11), we obtain

$$E\big(e^{\lambda t} Y(t)^r\big) \leqslant E\big(Y(0)^r\big), \quad \forall t \geqslant 0. \tag{5.13}$$

We now follow the procedure discussed in [11] and [16].

Let us extend the stochastic basis and the sequence of the Brownian motions to the whole real line, and let $X(\cdot; s, y)$ denote the solution of (0.1) for $-\infty < s < \infty$ such that $X(s; s, y) = y \in \mathcal{S}$. For $-\infty < s < t < \infty$, it follows from (5.9) that

$$E\big(\| X(t; s, \xi_1) - X(t; s, \xi_2) \|_{L^2(R^d)}^2\big) \leqslant C(t, s) E\big(\|\xi_1 - \xi_2\|_{L^2(R^d)}^2\big), \tag{5.14}$$

for all $\xi_1, \xi_2 \in L^2(\Omega; \mathcal{F}_s; L^2(R^d))$ such that $\|\nabla \xi_i\|_{(\mathcal{M}(R^d))^d} \in L^1(\Omega; \mathcal{F}_s)$, $i = 1, 2$, where $C(t, s)$ denotes some positive constant which depends only on $t - s$.

For $-\infty < r \leqslant s \leqslant t < \infty$, we use (5.14) and approximation of $X(s; r, z)$ by $\mathcal{S}$-valued simple functions with respect to the norm of $L^2(\Omega; \mathcal{F}_s; L^2(R^d))$ to show

$$E\bigl(\phi\bigl(X(t; r, z)\bigr)\big|\mathcal{F}_s\bigr) = E\bigl(\phi\bigl(X(t; s, y)\bigr)\bigr)\big|_{y=X(s;r,z)}, \quad P\text{-almost surely,} \tag{5.15}$$

if ϕ is a bounded Lipschitz continuous function on $L^2(R^d)$. Thus, (5.15) is valid when $\phi = \chi_\Gamma$, $\forall \Gamma \in \mathcal{B}(L^2(R^d))$, where χ_Γ is the characteristic function of Γ. Since $\mathcal{B}(\mathcal{S}) \subset \mathcal{B}(L^2(R^d))$, (5.15) shows that $X(\cdot : \cdot, \cdot)$ is a Markov process over $\mathcal{S}$. It is also time homogeneous in the following sense.

Lemma 5.1. *Let $\mathcal{L}\{\cdots\}$ denote the probability law of $\{\cdots\}$.*

$$\mathcal{L}\bigl\{X(t; s, y)\bigr\} = \mathcal{L}\bigl\{X(t-s; 0, y)\bigr\}, \quad \forall -\infty < s \leqslant t < \infty,\ \forall y \in \mathcal{S}.$$

Proof. Typically, this follows from the Yamada–Watanabe theorem in infinite dimensions. The result worked out in [17] is quite general, but it cannot be directly applied to our case. Because of the singular term $\nabla \cdot \Pi$, the equation cannot be put in the required functional setting of [17]. So we will get around this hurdle. Suppose that we are given two stochastic bases.

$$\Theta_1 \stackrel{\text{def}}{=} \bigl(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, \{B_j\}_{j=1}^{\infty}\bigr), \qquad \Theta_2 \stackrel{\text{def}}{=} \bigl(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \{\hat{\mathcal{F}}_t\}, \{\hat{B}_j\}_{j=1}^{\infty}\bigr).$$

Let u and $\hat{u}$ be solutions of (0.1) with the initial condition $u(0) = \hat{u}(0) = y \in \mathcal{S}$ over Θ_1 and Θ_2, respectively. It is enough to show that

$$\mathcal{L}\bigl\{u(t)\bigr\} = \mathcal{L}\bigl\{\hat{u}(t)\bigr\}, \quad \forall t > 0.$$

Fix any $0 < T < \infty$ and $y \in \mathcal{S}$. By the pathwise uniqueness of a solution, we may assume that both u and $\hat{u}$ have been constructed according to the procedure in Section 4.2. There is a sequence of positive numbers $\{L_k\}_{k=1}^{\infty} \uparrow \infty$, and a sequence $\{y_k\}_{k=1}^{\infty}$ such that $y_k \in H^1_{L_k}$, $\forall k$, and

$$\chi_k y_k \to y \quad \text{in } L^2\bigl(R^d\bigr), \quad \text{and} \quad \|\nabla y_k\|_{(L^1(G_{L_k}))^d} \leqslant 1 + \|\nabla y\|_{(\mathcal{M}(R^d))^d}, \quad \forall k,$$

where χ_k is the characteristic function of the set G_{L_k}. Choose a sequence of positive numbers $\{\epsilon_j\}_{j=1}^{\infty} \downarrow 0$. Let u_{L_k,ϵ_j} and $\hat{u}_{L_k,\epsilon_j}$ be solutions of (3.34) with $L = L_k$, $\epsilon = \epsilon_j$, and $u_{L_k,\epsilon_j}(0) = \hat{u}_{L_k,\epsilon_j}(0) = y_k$ over Θ_1 and Θ_2, respectively. Also, let u_{L_k} and $\hat{u}_{L_k}$ be solutions of (3.14) with $L = L_k$, and $u_{L_k}(0) = \hat{u}_{L_k}(0) = y_k$ over Θ_1 and Θ_2, respectively. Let $\mathcal{E}$ be the set of all $(\chi_k u_{L_k,\epsilon_j}, \chi_k \hat{u}_{L_k,\epsilon_j})$, $k, j = 1, 2, \ldots$. Let $\mathcal{K}$ denote the weak closure of the convex hull of $\mathcal{E}$ in

$$L^2\bigl(\Omega \times [0,T]; \mathcal{G}; dP \otimes dt; L^2\bigl(R^d\bigr)\bigr) \times L^2\bigl(\hat{\Omega} \times [0,T]; \hat{\mathcal{G}}; d\hat{P} \otimes dt; L^2\bigl(R^d\bigr)\bigr).$$

Then, $(\chi_k u_{L_k}, \chi_k \hat{u}_{L_k}) \in \mathcal{K}$ for each k. Let u and $\hat{u}$ be solutions of (0.1) with $u(0) = \hat{u}(0) = y$, over Θ_1 and Θ_2, respectively. According to the procedure to construct a solution of (0.1)–(0.2), we see that $(u, \hat{u}) \in \mathcal{K}$. Since $\mathcal{K}$ is also strongly closed in $L^2(\Omega \times [0,T]; \mathcal{G}; dP \otimes dt; L^2(R^d)) \times L^2(\hat{\Omega} \times [0,T]; \hat{\mathcal{G}}; d\hat{P} \otimes dt; L^2(R^d))$, there is a sequence $\{(\zeta_i, \hat{\zeta}_i)\}_{i=1}^{\infty}$ in the convex hull of $\mathcal{E}$ which converges to $(u, \hat{u})$ strongly in $L^2(\Omega \times [0,T]; \mathcal{G}; dP \otimes dt; L^2(R^d)) \times$

$L^2(\hat{\Omega} \times [0,T]; \hat{\mathcal{G}}; d\hat{P} \otimes dt; L^2(R^d))$. We can extract a subsequence still denoted by $\{(\zeta_i, \hat{\zeta}_i)\}_{i=1}^{\infty}$ such that

$$\int_{\Omega} \phi\big(u(t)\big)\,dP = \lim_{i\to\infty} \int_{\Omega} \phi\big(\zeta_i(t)\big)\,dP, \tag{5.16}$$

and

$$\int_{\hat{\Omega}} \phi\big(\hat{u}(t)\big)\,d\hat{P} = \lim_{i\to\infty} \int_{\hat{\Omega}} \phi\big(\hat{\zeta}_i(t)\big)\,d\hat{P}, \tag{5.17}$$

for any bounded Lipschitz continuous function ϕ on $L^2(R^d)$, and all $t \in Q$, where $dt([0,T] \setminus Q) = 0$. We will show that ζ_i and $\hat{\zeta}_i$ have the same probability law for each i.

Choose any two finite collections $\{u_{L_k,\epsilon_j}\}$, and $\{\hat{u}_{L_k,\epsilon_j}\}$, $k = 1,\dots,m$, $j = 1,\dots,n$. We may interpret these two collections as solutions of a system of equations over Θ_1 and Θ_2, respectively, which can be put in the framework of [17]. Hence, for each $t \in [0,T]$, the joint probability law of $\{\chi_k u_{L_k,\epsilon_j}(t)\}$, $k = 1,\dots,m$, $j = 1,\dots,n$, and the joint probability law of $\{\chi_k \hat{u}_{L_k,\epsilon_j}(t)\}$, $k = 1,\dots,m$, $j = 1,\dots,n$, are the same over $(L^2(R^d))^{mn}$. Thus, for any given set of constants $c_{k,j}$, $k = 1,\dots,m$, $j = 1,\dots,n$,

$$\sum_{k=1}^{m}\sum_{j=1}^{n} c_{k,j}\chi_k u_{L_k,\epsilon_j}(t) \quad \text{and} \quad \sum_{k=1}^{m}\sum_{j=1}^{n} c_{k,j}\chi_k \hat{u}_{L_k,\epsilon_j}(t)$$

have the same probability law, which yields

$$\int_{\Omega} \phi\big(\zeta_i(t)\big)\,dP = \int_{\hat{\Omega}} \phi\big(\hat{\zeta}_i(t)\big)\,d\hat{P}, \quad \forall i,\ \forall t. \tag{5.18}$$

Hence, it follows from (5.16)–(5.18) that

$$\int_{\Omega} \phi\big(u(t)\big)\,dP = \int_{\hat{\Omega}} \phi\big(\hat{u}(t)\big)\,d\hat{P} \tag{5.19}$$

for any bounded Lipschitz continuous function ϕ on $L^2(R^d)$, and all $t \in Q$. Since $u \in C([0,T]; L^2(R^d))$, P-almost surely, and $\hat{u} \in C([0,T]; L^2(R^d))$, $\hat{P}$-almost surely, (5.19) yields that

$$\mathcal{L}\{u(t)\} = \mathcal{L}\{\hat{u}(t)\}, \quad \forall t. \qquad \square$$

We now proceed to construct an invariant measure. By virtue of (5.7) and (5.13), we find that

$$E\big(\|X(0;-T_1,y) - X(0;-T_2,y)\|_{L^2(R^d)}^{2r}\big) \leqslant Ce^{-\lambda T_1}, \quad \forall T_2 > T_1 > 0,$$

where the positive constant C depends only on y, g_0, and ξ_j's.

The space $L^{2r}(\Omega; \mathcal{F}; L^2(R^d))$ is an F-space under the metric

$$\rho(y_1, y_2) = E\big(\|y_1 - y_2\|^{2r}_{L^2(R^d)}\big).$$

So there exists $X_\infty \in L^{2r}(\Omega; \mathcal{F}; L^2(R^d))$ such that

$$\lim_{T\to\infty} E\big(\big\|X_\infty - X(0; -T, 0)\big\|^{2r}_{L^2(R^d)}\big) = 0. \tag{5.20}$$

Let μ_T and μ_∞ be probability measures over $L^2(R^d)$ defined by

$$\mu_T = \mathcal{L}\big\{X(0; -T, 0)\big\}, \qquad \mu_\infty = \mathcal{L}\{X_\infty\}.$$

Then, by (5.20), as $T \to \infty$,

$$\mu_T \to \mu_\infty \quad \text{weak star.}$$

We also define a probability measure over $L^2(R^d)$ by

$$\nu_T(\Gamma) = \frac{1}{T}\int_0^T E\big(\chi_\Gamma\big(X(0; -t, 0)\big)\big)\,dt, \quad \forall\Gamma \in \mathcal{B}\big(L^2\big(R^d\big)\big).$$

It is easy to see that $\nu_T \to \mu_\infty$, weak star as $T \to \infty$.

For each $N \geqslant 1$, $\mathcal{S}_N = \{f \in L^2(R^d) \mid \|\nabla f\|_{(\mathcal{M}(R^d))^d} \leqslant N\}$ is a closed subset of $L^2(R^d)$, and

$$\begin{aligned}\frac{1}{T}\int_0^T P\big\{X(t; 0, 0) \in \mathcal{S}_N\big\}\,dt &= 1 - \frac{1}{T}\int_0^T P\big\{\big\|\nabla X(t; 0, 0)\big\|_{(\mathcal{M}(R^d))^d} > N\big\}\,dt \\ &\geqslant 1 - \frac{1}{T}\int_0^T \frac{1}{N^r} E\big(\big\|\nabla X(t; 0, 0)\big\|^r_{(\mathcal{M}(R^d))^d}\big)\,dt \\ &\geqslant 1 - \frac{C}{N^r}, \quad \text{by (5.8)},\end{aligned}$$

and

$$\mu_\infty(\mathcal{S}_N) \geqslant \overline{\lim_{T\to\infty}}\, \nu_T(\mathcal{S}_N) \geqslant 1 - \frac{C}{N^r}, \quad \forall N > 1.$$

Thus,

$$\mu_\infty(\mathcal{S}) = 1. \tag{5.21}$$

Next choose any bounded Lipschitz continuous function ϕ on $L^2(R^d)$, and fix $t > 0$.

Let us define

$$q(y) = E\big(\phi\big(X(t;0,y)\big)\big), \quad \forall y \in \mathcal{S}.$$

By (5.14), $q(\cdot)$ is continuous on $\mathcal{S}$ with respect to the $L^2(R^d)$-norm. Let $\hat{q}(\cdot)$ be the unique extension of $q(\cdot)$ to $L^2(R^d)$. Since $\hat{q}$ is a bounded continuous function on $L^2(R^d)$,

$$\int\limits_{L^2(R^d)} \hat{q}(y)\, d\mu_T(y) \to \int\limits_{L^2(R^d)} \hat{q}(y)\, d\mu_\infty(y). \tag{5.22}$$

In the meantime, by (5.20) and Lemma 5.1,

$$\begin{aligned}\int\limits_{L^2(R^d)} \hat{q}(y)\, d\mu_T(y) &= \int\limits_{\mathcal{S}} q(y)\, d\mu_T(y) = E\big(E\big(\phi\big(X(t;0,y)\big)\big)\big|_{y=X(0;-T,0)}\big) \\ &= E\big(\phi\big(X(0;-t-T,0)\big)\big) \to E\big(\phi(X_\infty)\big) = \int\limits_{L^2(R^d)} \phi(y)\, d\mu_\infty(y)\end{aligned}$$

which, combined with (5.21) and (5.22), yields

$$\int\limits_{\mathcal{S}} q(y)\, d\mu_\infty(y) = \int\limits_{L^2(R^d)} \hat{q}(y)\, d\mu_\infty(y) = \int\limits_{L^2(R^d)} \phi(y)\, d\mu_\infty(y). \tag{5.23}$$

Consequently, it holds that

$$\int\limits_{\mathcal{S}} E\big(\chi_\Gamma\big(X(t;0,y)\big)\big)\, d\mu_\infty(y) = \mu_\infty(\Gamma), \quad \forall \Gamma \in \mathcal{B}\big(L^2\big(R^d\big)\big).$$

Hence, the restriction of μ_∞ to $\mathcal{S}$ is an invariant measure.

For uniqueness, suppose μ_1 and μ_2 are invariant measures over $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$. We can extend them to $(L^2(R^d), \mathcal{B}(L^2(R^d)))$ such that

$$\hat{\mu}_i(G) = \mu_i(G \cap \mathcal{S}), \quad \forall G \in \mathcal{B}\big(L^2\big(R^d\big)\big),\ i = 1, 2.$$

So it is enough to show $\hat{\mu}_1 = \hat{\mu}_2$. For this, it is enough to show that for any bounded Lipschitz continuous function ϕ on $L^2(R^d)$,

$$\begin{aligned}\lim_{t\to\infty} E\big(\phi\big(X(t;0,z)\big)\big) &= \lim_{t\to\infty} \int\limits_{L^2(R^d)} E\big(\phi\big(X(t;0,y)\big)\big)\, d\hat{\mu}_i(y) \\ &= \int\limits_{L^2(R^d)} \phi(y)\, d\hat{\mu}_i(y), \quad \forall z \in \mathcal{S},\ i = 1, 2.\end{aligned} \tag{5.24}$$

Fix any $y, z \in \mathcal{S}$. It is known that for some $0 < r < \frac{1}{2}$,

$$E\left(\left\|X(t;0,y)-X(t;0,z)\right\|_{L^2(R^d)}^{2r}\right)\to 0$$

as $t\to\infty$. So

$$\left\|X(t;0,y)-X(t;0,z)\right\|_{L^2(R^d)}\to 0$$

in probability as $t\to\infty$, and thus,

$$\left|\phi\left(X(t;0,y)\right)-\phi\left(X(t;0,z)\right)\right|\to 0$$

in probability as $t\to\infty$. It follows that

$$E\left(\left|\phi\left(X(t;0,y)\right)-\phi\left(X(t;0,z)\right)\right|\right)\to 0$$

as $t\to\infty$ for each $y,z\in\mathcal{S}$. This shows the first equality in (5.24), because $\hat{\mu}_i(\mathcal{S})=1$, $i=1,2$. The second inequality is valid because μ_i is an invariant measure. The proof of Theorem 1.6 is complete.

Remark 5.2. In the H^1-setting, we can proceed in the same way as above to obtain a probability measure μ_∞ over $L^2(R^d)$. However, it is not known whether $\mu_\infty(H^1(R^d))=1$, even though $\mu_T(H^1(R^d))=1$, $\forall T>0$. We need further estimates for this. In fact, we can obtain a uniform bound of $E(\|u_{L,\epsilon}(t)\|_{H_L^1}^\sigma)$ independent of $L>4$, $\epsilon>0$, and $t>0$, for some $0<\sigma<1$, where $u_{L,\epsilon}$ is a solution of (3.34). But this estimate does not carry over to a solution u_L of (3.14) or to a solution u of (0.1), because u_L and u are obtained as weak limits, and the functional $v\mapsto E(\|v\|_{H_L^1}^\sigma)$ is not convex.

6. Proof of Theorem 1.7

Let $d=2$, and $u_0\in L^2(\Omega;\mathcal{F}_0;L^2(R^2))$ such that $\|\nabla u_0\|_{(\mathcal{M}(R^2))^2}\in L^1(\Omega;\mathcal{F}_0)$. As above, we set $X(t)=\|u(t)\|_{L^2(R^2)}^2$, and apply Ito's formula to (5.1) with $g_0\equiv 0$ to derive

$$\begin{aligned}
&E\left(e^{\sigma(\tau\wedge t)}\left(\eta+X(\tau\wedge t)\right)^r\right)+2rE\left(\int_0^{\tau\wedge t}e^{\sigma s}\left(\eta+X(s)\right)^{r-1}\left\|\nabla u(s)\right\|_{(\mathcal{M}(R^2))^2}\,ds\right)\\
&\quad=E\left(\left(\eta+X(0)\right)^r\right)+\sigma E\left(\int_0^{\tau\wedge t}e^{\sigma s}\left(\eta+X(s)\right)^r\,ds\right)\\
&\qquad+rE\left(\sum_{j=1}^\infty\int_0^{\tau\wedge t}e^{\sigma s}\left(\eta+X(s)\right)^{r-1}\left\|\xi_j\left(u(s)\right)\right\|_{L^2(R^2)}^2\,ds\right)\\
&\qquad+2r(r-1)E\left(\sum_{j=1}^\infty\int_0^{\tau\wedge t}e^{\sigma s}\left(\eta+X(s)\right)^{r-2}\left|\left\langle\xi_j\left(u(s)\right),u(s)\right\rangle_{L^2(R^2)}\right|^2\,ds\right)
\end{aligned}\tag{6.1}$$

where τ is the stopping time defined by (1.18), and $\sigma > 0$, $\eta > 0$, and $0 < r < \frac{1}{2}$ are constants such that

$$-\delta \overset{\text{def}}{=} \sigma + r \sum_{j=1}^{\infty} \beta_j^2 - 2r(1-r) \sum_{j=1}^{\infty} \alpha_j^2 < 0. \tag{6.2}$$

By (1.17), we may take $r = \frac{1}{4}$ and some small $\sigma > 0$ to satisfy (6.2), and pass $\eta \downarrow 0$ in (6.1) to find

$$\delta E\left(\int_0^{\tau \wedge t} e^{\sigma s} \|u(s)\|_{L^2(R^2)}^{\frac{1}{2}} \, ds \right) \leqslant E\left(\|u_0\|_{L^2(R^2)}^{\frac{1}{2}}\right), \quad \forall t > 0, \tag{6.3}$$

and

$$E\left(\int_0^{\tau \wedge t} e^{\sigma s} \|u(s)\|_{L^2(R^2)}^{-\frac{3}{2}} \|\nabla u(s)\|_{(\mathcal{M}(R^2))^2} \, ds \right) \leqslant C E\left(\|u_0\|_{L^2(R^2)}^{\frac{1}{2}}\right), \quad \forall t > 0. \tag{6.4}$$

By virtue of Lemma 2.3, (6.4) yields

$$E\left(\int_0^{\tau \wedge t} e^{\sigma s} \|u(s)\|_{L^2(R^2)}^{-\frac{1}{2}} \, ds \right) \leqslant C E\left(\|u_0\|_{L^2(R^2)}^{\frac{1}{2}}\right), \quad \forall t > 0. \tag{6.5}$$

It follows that

$$\begin{aligned} E\left(\int_0^{\tau \wedge t} e^{\sigma s} \, ds \right) &= E\left(\int_0^{\tau \wedge t} e^{\sigma s} \|u(s)\|_{L^2(R^2)}^{-\frac{1}{4}} \|u(s)\|_{L^2(R^2)}^{\frac{1}{4}} \, ds \right) \\ &\leqslant \frac{1}{2} E\left(\int_0^{\tau \wedge t} e^{\sigma s} \|u(s)\|_{L^2(R^2)}^{-\frac{1}{2}} \, ds \right) + \frac{1}{2} E\left(\int_0^{\tau \wedge t} e^{\sigma s} \|u(s)\|_{L^2(R^2)}^{\frac{1}{2}} \, ds \right) \\ &\leqslant C E\left(\|u_0\|_{L^2(R^2)}^{\frac{1}{2}}\right), \quad \forall t > 0. \end{aligned}$$

By passing $t \to \infty$,

$$E\left(e^{\sigma \tau} - 1\right) \leqslant C \sigma E\left(\|u_0\|_{L^2(R^2)}^{\frac{1}{2}}\right)$$

which yields (1.20).

Next we will show (1.19).

Choose any $0 \leqslant t_1 < t_2 < \infty$, and any $\mathcal{O} \in \mathcal{F}_{t_1}$. As in (6.1) with $r = \frac{1}{4}$ and $\sigma = 0$, we can derive

$$E\left(\chi_{\mathcal{O}}\left(\eta + \|u(t_2)\|_{L^2(R^2)}^2\right)^{\frac{1}{4}}\right) \leqslant E\left(\chi_{\mathcal{O}}\left(\eta + \|u(t_1)\|_{L^2(R^2)}^2\right)^{\frac{1}{4}}\right).$$

By passing $\eta \to 0$,

$$E\big(\chi_{\mathcal{O}}\big\|u(t_2)\big\|^{\frac{1}{2}}_{L^2(R^2)}\big) \leqslant E\big(\chi_{\mathcal{O}}\big\|u(t_1)\big\|^{\frac{1}{2}}_{L^2(R^2)}\big)$$

which shows that $\|u(t)\|^{\frac{1}{2}}_{L^2(R^2)}$ is a supermartingale. Thus,

$$E\big(\big\|u(s+\tau)\big\|^{\frac{1}{2}}_{L^2(R^2)}\big) \leqslant E\big(\big\|u(\tau)\big\|^{\frac{1}{2}}_{L^2(R^2)}\big) = 0, \quad \forall s > 0.$$

Hence, it follows that

$$\big\|u(s+\tau)\big\|_{L^2(R^2)} = 0, \quad \text{for any positive rational number } s, \ P\text{-almost surely.}$$

Since $\|u(t)\|^{\frac{1}{2}}_{L^2(R^2)}$ is continuous in t, P-almost surely, (1.19) follows. Proof of Theorem 1.7 is complete.

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