Generalized Fixed Point Algebras and Square-Integrable Group Actions

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We analyze Rieffel’s construction of generalized fixed point algebras in the setting of group actions on Hilbert modules. Let $G$ be a locally compact group acting on a $C^*$-algebra $B$. We construct a Hilbert module $F$ over the reduced crossed product of $G$ and $B$, using a pair $(E, R)$, where $E$ is an equivariant Hilbert module over $B$ and $R$ is a dense subspace of $E$ with certain properties. The generalized fixed point algebra is the $C^*$-algebra of compact operators on $F$. Any Hilbert module over the reduced crossed product arises by this construction for a pair $(E, R)$ that is unique up to isomorphism. A necessary condition for the existence of $R$ is that $E$ be square-integrable. The consideration of square-integrable representations of Abelian groups on Hilbert space shows that this condition is not sufficient and that different choices for $R$ may yield different generalized fixed point algebras. If $B$ is proper in Kasparov’s sense, there is a unique $R$ with the required properties. Thus the generalized fixed point algebra only depends on $E$.

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1. INTRODUCTION

Let $(G, X, \alpha)$ be a dynamical system, consisting of a locally compact group $G$, a locally compact space $X$, and a continuous left action $\alpha: G \times X \to X$. The action is called proper iff for all compact subsets $K, L \subseteq X$, the set of $g \in G$ with $\alpha_g(K) \cap L \neq \emptyset$ is (relatively) compact. Proper actions have many nice properties. For instance, the orbit space $G \setminus X$ is again a locally compact space. Rieffel [11] has initiated a program to extend the notions of proper action and orbit space to noncommutative dynamical systems, that is, group actions on $C^*$-algebras.
Suppose that the group $G$ is compact. Then all actions of $G$ on $C^*$-algebras are proper. The role of the orbit space is played by the fixed point algebra

$$A^G := \{ a \in A \mid \sigma_g(a) = a \text{ for all } g \in G \}.$$ 

This is reasonable because $A^G \cong C_0(G \setminus X)$ if $A = C_0(X)$.

If $G$ fails to be compact, there are several ways to define “proper” actions on $C^*$-algebras. The weakest reasonable notion is square-integrability. It has interesting applications in equivariant Kasparov theory [7] but is not enough to construct an “orbit space”, that is, a generalized fixed point algebra. A slightly more restrictive assumption is continuous square-integrability, which is exactly what is needed to construct a generalized fixed point algebra. Another much more restrictive notion of properness is due to Kasparov (see below). To avoid a conflict of notation, we only use the word “proper” in Kasparov’s sense.

We are going to explain square-integrability and a variant of Rieffel’s construction of generalized fixed point algebras. For both purposes, it is very illuminating to allow group actions on Hilbert modules, not just on $C^*$-algebras. Hilbert modules are more flexible because there are always plenty of adjointable operators between them, whereas there tend to be few $*$-homomorphisms between $C^*$-algebras. Let $B$ be a $G$-$C^*$-algebra, let $\mathcal{E}$ be a $G$-equivariant Hilbert module over $B$, and let $\xi \in \mathcal{E}$. Denote the action of $G$ on $\mathcal{E}$ by $\gamma$. Define

$$\langle \xi \rangle : \mathcal{E} \to C_c(G, B), \quad \langle \xi \rangle (g)(f) := \langle \gamma_g(\xi) \mid f \rangle, \quad (1)$$

and

$$|\xi| : C_c(G, B) \to \mathcal{E}, \quad |\xi| f := \int_G \gamma_g(\xi) : f(g) \, dg. \quad (2)$$

The operators $|\xi|$ and $\langle \xi \rangle$ are $G$-equivariant and adjoint to each other with respect to the pairing $\langle f_1 \mid f_2 \rangle := \int_G f_1(g)^* f_2(g) \, dg$ between $C_c(G, B)$ and $C_c(G, B)$. We call $\xi$ square-integrable iff $\langle \xi \rangle \eta \in L^2(G, B)$ for all $\eta \in \mathcal{E}$. Let $\mathcal{E}_u \subseteq \mathcal{E}$ be the subspace of square-integrable elements. If $\xi \in \mathcal{E}_u$, then we may view $\langle \xi \rangle$ as an operator $\mathcal{E} \to L^2(G, B)$. The adjoint of $\langle \xi \rangle$ exists and extends $|\xi|$ to an operator $L^2(G, B) \to \mathcal{E}$. Let $B^G(L^2(G, B), \mathcal{E})$ be the space of equivariant, adjointable operators $L^2(G, B) \to \mathcal{E}$. Then

$$\mathcal{E} \supseteq \mathcal{E}_u \cong |\mathcal{E}_u| \subseteq B^G(L^2(G, B), \mathcal{E}).$$

If $G$ is compact, then $\mathcal{E} = \mathcal{E}_u$. If $G$ is discrete and $B$ is unital, then $|\mathcal{E}_u| = B^G(L^2(G, B), \mathcal{E})$. We examine the operators $|\xi|$ and $\langle \xi \rangle$ and the space $\mathcal{E}_u$ in detail in Section 4.
We call $\mathcal{E}$ square-integrable iff $\mathcal{E}_u$ is dense in $\mathcal{E}$. We call $B$ square-integrable iff it is square-integrable as a Hilbert module over itself. Square-integrable $C^*$-algebras are called “proper” in [10] and [7]. The name “square-integrable" is motivated by the relationship to square-integrable group representations observed by Rieffel [10]. Square-integrable Hilbert modules are characterized by the existence of many equivariant, adjointable operators to $L^2(G, B)$. This gives rise to an equivariant version of Kasparov’s Stabilization Theorem [7]: A countably generated Hilbert module is a direct summand of $L^2(G, B^*)$ iff it is square-integrable.

The basic example of a square-integrable Hilbert module is $L^2(G, B)$. All elements of $C^c(G, B)$ are square-integrable. The closure of $\left\{ | \mathcal{K} T | \mathcal{K} \in C^c(G, B) \right\}$ may be identified with the reduced crossed product $C_{g^r}(G, B)$. We always think of $C_{g^r}(G, B)$ as a subalgebra of $B(G, L^2(G, B))$ in this way.

Our notation emphasizes that $| \mathcal{T}$ is part of an inner product $S_T = \left\langle \mathcal{S}_T \right\rangle$ on $E_u$. Since we want Hilbert modules over $C_{g^r}(G, B)$, we need subsets $\mathcal{R} \subseteq E_u$ for which $\left\langle \mathcal{R} \right\rangle$ is contained in $C_{g}(G, B)$. Following Exel [4], we call such a subset relatively continuous. Let $\mathcal{F} = \mathcal{F}(\mathcal{E}, \mathcal{R}) \subseteq B^\infty(L^2(G, B), \mathcal{E})$ be the closed linear span of $| \mathcal{R} \circ \mathcal{F} \subseteq C_{g^r}(G, B)$.

Then $\mathcal{F} \circ C_{g^r}(G, B) \subseteq \mathcal{F}$, $\mathcal{F}^* \circ \mathcal{F} \subseteq C_{g^r}(G, B)$, (3) so that $\mathcal{F}$ becomes a Hilbert module over $C_{g^r}(G, B)$ with respect to the inner product $\langle \xi | \eta \rangle := \xi^* \circ \eta$ and right module structure $\xi \cdot x := \xi \circ x$. To exclude degenerate cases, we usually assume that $\mathcal{R}$ is dense in $\mathcal{E}$.

The closed linear span Fix$(\mathcal{E}, \mathcal{R})$ of $\mathcal{F} \circ \mathcal{F}^* \subseteq B^\infty(\mathcal{E})$ is the generalized fixed point algebra. There is a canonical isomorphism between Fix$(\mathcal{E}, \mathcal{R})$ and the $C^*$-algebra $\mathcal{K}(\mathcal{F})$ of compact operators on $\mathcal{F}$. Thus Fix$(\mathcal{E}, \mathcal{R})$ is Morita–Rieffel equivalent to an ideal in $C_{g^r}(G, B)$. For compact $G$, we get an ordinary fixed point algebra because Fix$(\mathcal{E}, \mathcal{E}) = \mathcal{K}(\mathcal{E})$.

There are two obvious questions: Is square-integrability enough to guarantee the existence of a dense, relatively continuous subspace $\mathcal{R} \subseteq \mathcal{E}$? Are $\mathcal{F}(\mathcal{E}, \mathcal{R})$ and Fix$(\mathcal{E}, \mathcal{R})$ independent of $\mathcal{R}$? Unfortunately, the answer to both questions is negative. Counterexamples come from square-integrable representations of Abelian groups on Hilbert space. This situation can be analyzed completely (Section 8). The problems are due to the subtle difference between continuous and measurable fields of Hilbert spaces.

Positive answers can be obtained if we require much more than square-integrability. Following Kasparov [5], we call $B$ proper iff there are a proper $G$-space $X$ and an essential, equivariant $*$-homomorphism from $C_0(X)$ into the center of the multiplier algebra $\mathcal{M}(B)$ of $B$. If $B$ is proper, then any Hilbert $B, G$-module $\mathcal{E}$ is square-integrable, and $\mathcal{F}(\mathcal{E}, \mathcal{R})$ and Fix$(\mathcal{E}, \mathcal{R})$ do not depend on $\mathcal{R}$. Actually, we can do with slightly less than
properness. We only need that the induced action of $G$ on the primitive ideal space of $B$ is proper.

Our main result is that the construction $(\mathcal{E}, \mathcal{R}) \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R})$ can be inverted. That is, all Hilbert modules over $C^*_r(G, B)$ are of the form $\mathcal{F}(\mathcal{E}, \mathcal{R})$ for suitable $(\mathcal{E}, \mathcal{R})$, and $(\mathcal{E}, \mathcal{R})$ is unique up to isomorphism if we impose further conditions on $\mathcal{R}$. Let $\mathcal{F}$ be a Hilbert module over $C^*_r(G, B)$. Define

$$\mathcal{E} := \mathcal{F} \otimes_{C^*_r(G, B)} L^2(G, B), \quad \mathcal{R} := \mathcal{F} \otimes C_r(G, B) \subseteq \mathcal{E}. \quad (4)$$

The action of $G$ on $\mathcal{E}$ comes from the trivial action on $\mathcal{F}$ and the usual action on $L^2(G, B)$. Then $\mathcal{R}$ is dense and relatively continuous, and $\mathcal{F}(\mathcal{E}, \mathcal{R}) \cong \mathcal{F}$.

Our work is based on a detailed analysis of the construction of $\mathcal{F}(\mathcal{E}, \mathcal{R})$. It splits into two parts. First, a relatively continuous subset $\mathcal{R} \subseteq \mathcal{E}$ yields a closed linear subspace $\mathcal{F} \subseteq \mathcal{F}(L^2(G, B), \mathcal{E})$ satisfying (3). The key idea here is the map $\xi \mapsto \xi$. Secondly, $\mathcal{F}$ is turned into a Hilbert module over $C^*_r(G, B)$. Only the first part uses special properties of groups. The second part should work equally well for coactions or actions of Hopf algebras. To simplify future extensions of this kind, we treat the second part in greater generality.

Namely, we replace $L^2(G, B)$ and $C^*_r(G, B)$ by $\mathcal{L}$ and $A$, where $\mathcal{L}$ is any Hilbert $B, G$-module and $A$ is an essential $C^*$-subalgebra of $\mathcal{B}(\mathcal{L})$. A closed linear subspace $\mathcal{F} \subseteq \mathcal{B}(\mathcal{L}, \mathcal{E})$ is called a concrete Hilbert $A$-module iff $\mathcal{F} \cdot A \subseteq \mathcal{F}$ and $\mathcal{L}^* \cdot \mathcal{F} \subseteq A$. We call $\mathcal{F}$ essential iff the linear span of $\mathcal{F}(\mathcal{L})$ is dense in $\mathcal{E}$. For example, the space $\mathcal{F}(\mathcal{E}, \mathcal{R}) \subseteq \mathcal{B}(L^2(G, B), \mathcal{E})$ is an essential, concrete Hilbert module over $C^*_r(G, B)$ if $\mathcal{R} \subseteq \mathcal{E}$ is dense and relatively continuous.

A concrete Hilbert $A$-module carries a canonical Hilbert $A$-module structure. We view the embedding $\mathcal{F} \subseteq \mathcal{B}(\mathcal{L}, \mathcal{E})$ as a representation of $\mathcal{F}$. The definition of a concrete Hilbert module is relative to the representation $A \subseteq \mathcal{B}(\mathcal{L})$. This has the consequence that all essential representations of $\mathcal{F}$ are isomorphic to a canonical representation

$$\mathcal{F} \cong \mathcal{H}(A, \mathcal{F}) \subseteq \mathcal{B}(A \otimes_A \mathcal{L}, \mathcal{F} \otimes_A \mathcal{L}) \cong \mathcal{B}(\mathcal{L}, \mathcal{F} \otimes_A \mathcal{L}).$$

In particular, $\mathcal{E} \cong \mathcal{F} \otimes_A \mathcal{L}$ if $\mathcal{F} \subseteq \mathcal{B}(\mathcal{L}, \mathcal{E})$ is an essential, concrete Hilbert module over $A$. This explains the first half of (4).

Let $\mathcal{F}$ be a concrete Hilbert module over $C^*_r(G, B)$ and let $\mathcal{R}_x$ be the set of all $\xi \in \mathcal{E}$ with $|\xi| \in \mathcal{F}$. Then $\mathcal{F}(\mathcal{E}, \mathcal{R}_x) = \mathcal{F}$. A subset $\mathcal{R} \subseteq \mathcal{E}$ is of the form $\mathcal{R}_x$ for some concrete Hilbert module $\mathcal{F}$ over $C^*_r(G, B)$ if and only if it is relatively continuous and complete in an appropriate sense. In addition, $\mathcal{F}$ is essential iff $\mathcal{R}_x$ is dense. A Hilbert module $\mathcal{E}$ equipped with a dense, complete, relatively continuous subspace $\mathcal{R} \subseteq \mathcal{E}$ is called a continuously square-integrable Hilbert module. This name is motivated by the case
$B = \mathbb{C}$ and $G$ Abelian, where $\mathcal{H}$ allows to recover a continuous field of Hilbert spaces from a measurable field. Our analysis shows that $(\mathcal{E}, \mathcal{H}) \mapsto \mathcal{F}(\mathcal{E}, \mathcal{H})$ yields a bijection between isomorphism classes of continuously square-integrable Hilbert $B, G$-modules and isomorphism classes of Hilbert modules over $C^*_r(G, B)$.

For trivial coefficients and groupoids instead of groups, the correspondence between Hilbert modules over $C^*_r(G)$ and continuously square-integrable representations of $G$ on Hilbert space has already been observed by Connes [3]. In order to do index theory on foliated manifolds, he has to deal with the reduced $C^*$-algebra of the holonomy groupoid $G$ of the foliation and the $KK$-theoretic description of its $K$-theory as

$$K_*(C^*_r G) \cong KK_*(\mathbb{C}, C^*_r G).$$

However, we know very little about Hilbert modules over $C^*_r(G)$. Already the determination of $K_*(C^*_r G)$ is a major problem. Therefore, Connes replaces a Hilbert module over $C^*_r G$ by a square-integrable representation on a Hilbert space, equipped with a dense subspace with suitable properties.

Although most of our results can be extended to groupoids $\textit{mutatis mutandis}$, we have decided not to cover groupoids. Otherwise, we would have to translate the basic theory of square-integrability, which so far has been written down only for groups, and to use more complicated notation, since groupoids do not act on $C^*$-algebras but on bundles of $C^*$-algebras. These changes would make the article more difficult to read without changing the content significantly.

A basic observation about square-integrable Hilbert modules is that $\mathcal{E}$ is square-integrable iff $\mathcal{K}(\mathcal{E})$ is square-integrable. This continues to hold for continuously square-integrable Hilbert modules. We can construct relatively continuous subspaces of $\mathcal{K}(\mathcal{E})$ from relatively continuous subspaces of $\mathcal{E}$ and vice versa. These constructions are inverse to each other if the group $G$ is $\textit{exact}$, that is, the functor $C^*_r(G, \cup)$ preserves short exact sequences. Otherwise, not all relatively continuous subspaces of $\mathcal{E}$ come from $\mathcal{K}(\mathcal{E})$. Since Abelian groups are exact, the counterexamples in Section 8—which involve group actions on Hilbert spaces to begin with—also yield counterexamples in the realm of group actions on $C^*$-algebras.

2. SOME CONVENTIONS

Throughout this article, $G$ is a locally compact group, $B$ is a $C^*$-algebra with a strongly continuous action $\beta: G \to \text{Aut}(B)$ of $G$ or, briefly, a $G$-$C^*$-algebra, and $\mathcal{E}$ is a $G$-equivariant Hilbert module over $B$ with
G-action γ or, briefly, a Hilbert B, G-module. We denote elements of G by g or g₂, and fix a left invariant Haar measure dg on G. Let \( L^2 G := L^2(G, dg) \), and let G act on \( L^2 G \) via the left regular representation. Let \( A: G \to \mathbb{R}_+^* \) be the modular function of G with the conventions \( d(g^{-1}) = \Delta(g^{-1}) dg \) and \( d(gg_2) = \Delta(g_2) dg \). We write \( \mathbb{B}(\mathcal{E}) \) and \( \mathbb{K}(\mathcal{E}) \) for the C*-algebras of adjointable and compact operators on \( \mathcal{E} \), and \( \mathbb{B}^0(\mathcal{E}) \) for the C*-algebra of G-equivariant, adjointable operators on \( \mathcal{E} \).

Since tensor products of Hilbert modules are very important for us, we recall the definition. Let A be another G-C*-algebra, let \( \mathcal{E}_1 \) be a Hilbert A, G-module and let \( \mathcal{E} \) be a Hilbert B, G-module. Let \( \phi: A \to \mathbb{B}(\mathcal{E}) \) be an equivariant *-homomorphism. Then \( \mathcal{E}_1 \otimes_A \mathcal{E} = \mathcal{E}_1 \otimes_B \mathcal{E} \) is a Hilbert B, G-module. It is the completion of the algebraic tensor product \( \mathcal{E}_1 \otimes_{alg} \mathcal{E} \) with respect to the inner product

\[
\langle x_1 \otimes \xi_1 | x_2 \otimes \xi_2 \rangle := \langle \xi_1 | \phi(\langle x_1 | x_2 \rangle_A) \xi_2 \rangle.
\]  

(5)

The group G acts diagonally on \( \mathcal{E}_1 \otimes_A \mathcal{E} \) by \( c g (x \otimes \xi) := c E_1 g(x) \xi \).

If \( A = \mathbb{C} \), we drop it from our notation. For \( \mathcal{E}_1 = L^2 G \), we get the Hilbert B, G-module \( L^2(G, B) := L^2 G \otimes_B B \).

The bra-ket notation is very useful in connection with Hilbert modules. For \( \zeta \in \mathcal{E} \) we define the operators \( |\zeta\rangle: B \to \mathcal{E} \) and \( \langle \zeta|: \mathcal{E} \to B \) by \( |\zeta\rangle(b) := \zeta \cdot b \) and \( \langle \zeta| (\eta) := \langle \zeta | \eta \rangle \), respectively. These operators are adjoints of one another. The composition \( \langle \zeta | \eta \rangle \) is the operator of multiplication with the inner product \( \langle \zeta | \eta \rangle \). The composition \( |\zeta\rangle \circ \langle \eta| \) is the "rank-one operator" \( |\zeta\rangle \langle \eta| (\zeta') := \zeta \cdot \langle \eta| \zeta' \).

The map \( \zeta \mapsto |\zeta\rangle \) is an isomorphism from \( \mathcal{E} \) onto \( \mathbb{K}(B, \mathcal{E}) \). The map \( \zeta \mapsto \langle \zeta| \) is an isomorphism from the dual \( \mathcal{E}^* \) of \( \mathcal{E} \), which is a Hilbert module over \( \mathbb{K}(\mathcal{E}) \) with \( \mathbb{K}(\mathcal{E}^*) \cong B \), onto \( \mathbb{K}(\mathcal{E}, B) \). The map

\[
\mathcal{E} \otimes_B \mathcal{E}^* \to \mathbb{K}(\mathcal{E}), \quad \xi \otimes \eta \mapsto |\xi\rangle \langle \eta|,
\]  

(6)

is an isomorphism of Hilbert modules over \( \mathbb{K}(\mathcal{E}) \).

3. THE REDUCED CROSSED PRODUCT

One of the basic observations of [11] is that \( C^*_r(G, B) \) arises as the generalized fixed point algebra of \( \mathbb{K}(L^2 G) \otimes B \). In order to make this isomorphism straightforward, we leave out the modular function in the adjoint. With this convention, modular functions do not show up in any of our formulas. Furthermore, it becomes easier to extend results to groupoids, where our convention is the standard one. In order to help readers who prefer the other convention, we explain how to modify formulas if the adjoint is defined differently.
Let \( C_c(G, B) \) be the space of continuous functions from \( G \) to \( B \) with compact support. We equip \( C_c(G, B) \) with the following \(*\)-algebra structure:

\[
K * L(g) := \int_G K(g_2) \cdot \beta_{g_2} (L(g_2^{-1} g)) \, dg_2,
\]

\[
K^*(g) := \beta_g (K(g^{-1}))^*.
\]

(7)

Usually, the adjoint is defined by

\[
K^*(g) := \beta_g (K(g^{-1}))^* \cdot A(g)^{-1}.
\]

(8)

Equations (8) and (9) yield isomorphic \(*\)-algebras. The map

\[
\mu: C_c(G, B) \to C_c(G, B), \quad (\mu K)(g) := A(g)^{1/2} K(g)
\]

is an isomorphism. Straightforward computations show

\[
\mu(K * L) = \mu(K) * \mu(L), \quad (\mu K)^* = \mu(K^*)
\]

for all \( K, L \in C_c(G, B) \). Define

\[
(\rho_K f)(g) := \int_G \beta_g (K(g_2^{-1} g_2)) \cdot f(g_2) \, dg_2
\]

(10)

for all \( g \in G \) and \( K, f \in C_c(G, B) \). Straightforward computations show that \( \rho_K \) extends to an adjointable operator on \( L^2(G, \mathbb{B}) \), and that \( \rho \) is a \(*\)-homomorphism from \( C_c(G, B) \) to \( \mathcal{B}(L^2(G, \mathbb{B})) \). If we define the adjoint by (9), we must replace \( \rho \) by \( \rho \circ \mu \). This yields an additional factor \( A(g^{-1}g_2)^{1/2} \) under the integral in (10).

The reduced crossed product \( C^*_r(G, B) \) is the closure of \( \rho(C_c(G, B)) \) with respect to the operator norm on \( \mathcal{B}(L^2(G, \mathbb{B})) \).

If an adjointable operator on \( L^2(G, \mathbb{B}) \) satisfies (10) for some not necessarily compactly supported continuous function \( K: G \to \mathbb{B} \), then we call it a Laurent operator with symbol \( K \) (following Exel’s notation for Abelian groups [4]). If we define the adjoint by (9), then \( \rho \) is replaced by \( \rho \circ \mu \). As a result, symbols are multiplied pointwise by \( A(g)^{-1/2} \).

We can also define \( \rho_K \) if \( K \) is only a distribution on \( G \) taking values in \( \mathcal{M}(\mathbb{B}) \). In particular, we consider the distributions \( b \cdot \delta_b \) for \( b \in \mathbb{B} \) and \( \delta_g \) for \( g \in G \) that are defined by \( \int_G b \cdot \delta_b (g) \cdot f(g) \, dg := bf(1) \) and \( \int_G \delta_b (g_2) \cdot f(g_2) \, dg_2 = f(g) \). If we plug them into (10), we get the operators

\[
\rho_b(f)(g_2) := \beta_{g_2} (b) \cdot f(g_2),
\]

\[
\rho_g(f)(g_2) := f(g_2 g).
\]

(11)

(12)
We have $\rho_k^* = \rho_k^{-1} \cdot A(g)^{-1}$, so that $\rho_k \cdot A(g)^{1/2}$ is unitary. It is elementary to verify that $\rho_k$ and $\rho_k^*$ multiply $\rho(C_c(G, B))$ and hence are contained in $\mathcal{A}(C^*_c(G, B))$. If we define the adjoint by (9) and replace $\rho$ by $\rho \circ \mu$, then $\rho_k$ remains unchanged and $\rho_k^*$ is replaced by $\rho_k^* \cdot A(g)^{1/2}$.

We can view $C_c(G, B)$ as the inductive limit of the Banach spaces of continuous functions $G \to B$ with support contained in $K \subseteq G$, where $K$ runs through the compact subsets of $G$. Hence $C_c(G, B)$ is a complete bornological vector space and thus a complete topological vector space in a canonical way.

The usual $L^1$-norm on $C_c(G, B)$ is not a $*$-algebra norm and does not control the operator norm on $C^*_c(G, B)$ because we left out the modular function in the adjoint. As a substitute, we define

$$\|K\|_I := \max \left\{ \int \|K(g)\| \, dg, \int \|K^*(g)\| \, dg \right\}.$$

This norm is submultiplicative and satisfies $\|K^*\|_I = \|K\|_I$ by definition. An application of the Cauchy–Schwarz Inequality shows that

$$\|\mu^{-1}(K)\|_I := \int \|\mu^{-1}(K)(g)\| \, dg = \int \|K(g)\| \cdot A(g)^{-1/2} \, dg \leq \left( \int \|K(g)\| \, dg \right)^{1/2} \cdot \left( \int \|K(g)\| \, d(g^{-1}) \right)^{1/2} \leq \|K\|_I.$$ 

Hence $\|\rho_k\| \leq \|K\|_I$. This estimate extends to groupoids if the norm $\|\|_I$ is defined appropriately. See Proposition 3.5 and Corollary 4.8 in [9].

Finally, we mention that $C_c(G, B)$ has approximate identities:

**Lemma 3.1.** There is a net $(u_j)_{j \in J}$ of elements of $C_c(G, B)$ such that:

- $u_j = u_j^*$ for all $j \in J$;
- $(u_j)$ is bounded with respect to the norm $\|\|$;
- $(u_j)$ is an approximate identity of $C_c(G, B)$ and of $C^*_c(G, B)$ with respect to the inductive limit bornology and the operator norm, respectively.

**Proof.** For groups, this is folklore, for groupoids the assertions follow from Lemma 3.2 of [9] and its proof.

4. SQUARE-INTEGRABLE HILBERT MODULES

In the introduction, we called $\xi \in \mathcal{E}$ square-integrable iff $\langle \xi | \eta \rangle \in L^2(G, \mathbb{B})$ for all $\eta \in \mathcal{E}$. We have to explain what $\langle \xi | \eta \rangle \in L^2(G, B)$ means. Let $(\chi_e)_{e \in I}$
be a net of continuous, compactly supported functions $G \rightarrow [0, 1]$ with $\chi_c(g) \rightarrow 1$ uniformly on compact subsets of $G$. Let $f \in C_b(G, B)$. We say that $f$ is square-integrable and write $f \in L^2(G, B)$ if the net $(\chi_c \cdot f)_{c \in I}$ converges in $L^2(G, B)$. We identify $f$ with the limit of this net, so that $f$ becomes an element of $L^2(G, B)$.

As a result, we may view $\langle \xi \rangle$ as an operator $\mathcal{E} \rightarrow L^2(G, B)$ if $\xi \in \mathcal{E}_a$. The Closed Graph Theorem implies that $\langle \xi \rangle$ is bounded as a map to $L^2(G, B)$. Since $\langle \xi \rangle$ is bounded as an operator to $C_b(G, B)$, its graph in $\mathcal{E} \times L^2(G, B)$ is closed. (Since we do not assume $G$ to be $\sigma$-compact as in [7], we cannot employ the Banach–Steinhaus Theorem as in the proof of [7, Lemma 8.1]). The same argument as in the proof of [7, Lemma 8.1] shows that $\langle \xi \rangle$ is adjointable and that its adjoint extends $|\xi|$ to an operator $L^2(G, B) \rightarrow \mathcal{E}$. Conversely, suppose that the operator $|\xi|$ defined in (2) extends to an adjointable operator $L^2(G, B) \rightarrow \mathcal{E}$. The computation that yields $|\xi| = |\xi|^*$ shows that $|\xi|^*(\eta) = \langle \xi \rangle \eta \in L^2(G, B)$ for all $\eta \in \mathcal{E}$. Hence $\xi \in \mathcal{E}_a$ iff $|\xi|$ extends to an adjointable operator $L^2(G, B) \rightarrow \mathcal{E}$.

It is clear that $\mathcal{E}_a$ is a vector space. It becomes a Banach space when equipped with the norm

$$|||\xi||| := ||\langle \xi | \eta \rangle||^{1/2} + ||\langle \eta | \xi \rangle||^{1/2},$$

The remainder of this section contains elementary computations with the operators $|\xi|$ and $\langle \xi \rangle$ that are needed later. It is convenient for reference purposes to collect these computation in a single section. Let $\xi \in \mathcal{E}$, then

$$[T(\xi)] = T \circ |\xi| \quad \forall T \in \mathcal{B}(\mathcal{E}, \mathcal{E}'), \tag{13}$$

$$|\xi \cdot b| = |\xi| \circ \rho_b \quad \forall b \in B, \tag{14}$$

$$|\gamma_\xi(\xi)| = |\xi| \circ \rho_\xi^* = |\xi| \circ \rho_\xi^{-1} \cdot A(g)^{-1} \quad \forall g \in G. \tag{15}$$

Equations (13)–(15) follow at once from the definitions (2), (11), and (12). If $\rho_\xi$ is replaced by $\rho_\xi \cdot A(g)^{-1/2}$, then (15) has to be modified accordingly. Therefore,

$$[T(\xi)]_a \leq |||\xi||| \cdot ||T|| \quad \forall T \in \mathcal{B}(\mathcal{E}, \mathcal{E}'), \xi \in \mathcal{E}_a, \tag{16}$$

$$||\xi \cdot b||_a \leq |||\xi||| \cdot ||b|| \quad \forall b \in B, \xi \in \mathcal{E}_a, \tag{17}$$

$$|||\gamma_\xi(\xi)|||_a \leq |||\xi||| \cdot \max\{1, A(g)^{-1/2}\} \quad \forall g \in G, \xi \in \mathcal{E}_a. \tag{18}$$

Thus $\mathcal{E}_a$ is $G$-invariant and a Banach bimodule over $\mathcal{B}(\mathcal{E}) \times B$. However, the action of $G$ on $\mathcal{E}_a$ need not be continuous, and $\mathcal{E}_a \cdot B$ need not be dense in $\mathcal{E}_a$.

Let $\xi, \eta \in \mathcal{E}_a$. We compute the compositions $|\xi| \langle \eta \rangle$ and $\langle \xi \rangle |\eta|$. Formally, we have $|\xi| \langle \eta \rangle = \int_T \gamma_\xi(\xi) \langle \gamma_\xi(\eta) |\xi\rangle dg$. To interpret this
integral, recall that $\langle \gamma_t(\eta) \mid \zeta \rangle$ is the limit of the net $(\chi_i(g) \cdot \langle \gamma_t(\eta) \mid \zeta \rangle)_{i \in I}$ in $L^2(G, B)$. Hence

$$\| \zeta \rangle \langle \eta \| = \int_G \gamma_t(\zeta) \langle \eta \rangle \, dg := \lim_{i \to \infty} \int_G \chi_i(g) \cdot \gamma_t(\zeta) \langle \eta \rangle \, dg$$

for all $\zeta, \eta \in \mathcal{E}_u$. The limit exists in the strict topology [7]. Moreover,

$$(\langle \xi \mid \eta \rangle \cdot f)(g) = \langle \gamma_t(\xi) \mid \eta \rangle \cdot f(g) = \int_G \langle \gamma_t(\xi) \mid \gamma_t(\eta) \rangle \cdot f(g) \, dg_2$$

for all $f \in C_c(G, B)$, $g \in G$. Comparing with (10), we see that $\langle \xi \mid \eta \rangle$ is a Laurent operator, whose symbol $\langle \xi \mid \eta \rangle \in C^*(G, B)$ is

$$\langle \xi \mid \eta \rangle(g) = \langle \xi \mid \gamma_t(\eta) \rangle \quad \forall g \in G, \xi, \eta \in \mathcal{E}_u. \quad (20)$$

It may happen that $\langle \xi \mid \eta \rangle \notin C^*(G, B)$. If we define the adjoint by (9), then (20) has to be replaced by $\langle \xi \mid \eta \rangle(g) = \langle \xi \mid \gamma_t(\eta) \rangle \cdot A(g)^{-1/2}$.

Consider the basic example $L^2(G, B)$. We claim that $C_c(G, B) \subseteq L^2(G, B)_u$. Hence $L^2(G, B)_u$ is dense in $L^2(G, B)$, that is, $L^2(G, B)$ is square-integrable. Let $K \in C_c(G, B)$. Then

$$(|K\rangle\langle f|)(g) = \int_G (\beta_{g_2}(K))(g) \cdot f(g_2) \, dg_2 = \int_G \beta_{g_2}(K(g_f^{-1})) \cdot f(g_2) \, dg_2.$$  

Comparing with (10), we see that

$$|K\rangle = \rho_K, \quad |\tilde{K}\rangle = \rho_{\tilde{K}}, \quad (21)$$

if

$$\tilde{K}(g) := \beta_{g_2}(K(g_f^{-1})). \quad (22)$$

The map $K \mapsto \tilde{K}$ is a bijection from $C_c(G, B)$ onto $C_c(G, B)$. As a result, $|K\rangle$ extends to an adjointable operator, so that $C_c(G, B) \subseteq L^2(G, B)_u$.

If we define the adjoint by (9), then $\rho$ has to be replaced by $\rho \circ \mu$. Hence we desire an equation $|K\rangle = \rho \circ \mu(\tilde{K})$ instead of (21) and put

$$\tilde{K}(g) := \tilde{K}(g) \cdot A(g)^{-1/2} = \beta_{g_2}(K(g_f^{-1})) \cdot A(g)^{-1/2}. \quad (23)$$

We turn $\mathcal{E}$ into a right module over the convolution algebra $C_c(G, B)$ by

$$\xi \ast K := |\xi\rangle\langle \tilde{K}|(g) = \int_G \gamma_t(\xi) \cdot \tilde{K}(g) \, dg = \int_G \gamma_t(\xi) \cdot K(g_f^{-1}) \, dg$$

(24)
for all \( \xi \in \mathfrak{d} \), \( K \in C_c(G, B) \). Since \(|\xi\rangle\rangle\) is equivariant, (13) and (21) yield

\[
|\xi \ast K\rangle = |\xi\rangle\rangle \circ \hat{K} = |\xi\rangle\rangle \circ \rho_K \quad \forall \xi \in \mathfrak{d}, \quad K \in C_c(G, B).
\]  

(25)

Hence

\[
|(\xi \ast K) \ast L\rangle = |\xi\rangle\rangle \circ \rho_K \circ \rho_L = |\xi\rangle\rangle \circ \rho_{K \ast L} = |\xi\rangle\rangle \circ (K \ast L)\rangle.
\]

Since the map \( \xi \mapsto |\xi\rangle\rangle \) is injective, \( \mathfrak{d} \) is a right module over \( C_c(G, B) \).

If we define the adjoint by (9), then we replace \( \hat{K} \) by \( \tilde{K} \) in (24). The same computation as above yields \(|\xi \ast K\rangle = |\xi\rangle\rangle \circ \mu(K) \) instead of (25).

Using \( \|\rho_K\| \leq \|K\|_I \) and (25), we obtain the following norm estimates:

\[
\|\xi \ast K\| \leq \|\xi\| \cdot \|K\|_I, \tag{26}
\]

\[
\|\xi \ast K\|_u \leq \|\xi\|_u \cdot \|K\|_I, \tag{27}
\]

\[
\|\xi \ast K\|_u \leq \|\xi\|_u \cdot \max\{\|\rho_K\|, \|\tilde{K}\|_{L^2(G, B)}\}. \tag{28}
\]

By Lemma 8.1 of [7], \( \xi \) is contained in the closure of \(|\xi\rangle\rangle(C_c(G, B))\). Hence \( \mathfrak{d} \ast C_c(G, B) \) is dense in \( \mathfrak{d} \). Using (26), we conclude that the approximate identities \((u_j)_j \) of Lemma 3.1 satisfy

\[
\lim \|\xi \ast u_j - \xi\| = 0 \quad \forall \xi \in \mathfrak{d}. \tag{29}
\]

However, \( \mathfrak{d}_u \ast C_c(G, B) \) need not be dense in \( \mathfrak{d}_u \) with respect to \( \|\cdot\|_u \).

5. REPRESENTATIONS OF HILBERT MODULES

Throughout this section, we let \( \mathcal{L} \) be a \( G \)-equivariant Hilbert module over a \( G \)-\( C^* \)-algebra \( B \), and we let \( A \subseteq \mathcal{B}^a(\mathcal{L}) \) be an essential \( C^* \)-subalgebra. That is, the closed linear span of \( A \cdot \mathcal{L} \) is dense in \( \mathcal{L} \). By Cohen’s Factorization Theorem, this implies \( A \cdot \mathcal{L} = \mathcal{L} \). We are particularly interested in the case \( A = C^*_e(G, B) \), \( \mathcal{L} = L^2(G, B) \). The group \( G \) is only there because we want to assert that our constructions are invariant with respect to a group action.

Definition 5.1. Let \( \mathfrak{d} \) be a Hilbert \( B, G \)-module. A concrete Hilbert \( A \)-module is a closed linear subspace \( \mathcal{F} \subseteq \mathcal{B}^a(\mathcal{L}, \mathfrak{d}) \) that satisfies \( \mathcal{F} \circ A \subseteq \mathcal{F} \) and \( \mathcal{F} \circ \mathcal{A} \subseteq \mathcal{A} \). We call \( \mathcal{F} \) essential iff the linear span of \( \mathcal{F}(\mathcal{L}) \) is dense in \( \mathfrak{d} \).
A concrete Hilbert $A$-module $\mathcal{F} \subseteq \mathcal{B}(\mathcal{L}, \mathcal{E})$ can be made essential by making $\mathcal{E}$ smaller. Let $\mathcal{E}' \subseteq \mathcal{E}$ be the closed linear span of $\mathcal{F}(\mathcal{L})$. Then $\mathcal{E}'$ is an invariant Hilbert submodule and $\mathcal{F} \subseteq \mathcal{B}(\mathcal{L}, \mathcal{E}')$ is an essential, concrete Hilbert $A$-module.

**Lemma 5.1.** A concrete Hilbert $A$-module $\mathcal{F} \subseteq \mathcal{B}(\mathcal{L}, \mathcal{E})$ becomes a Hilbert $A$-module when equipped with the right $A$-module structure

$$\zeta \cdot a := \zeta \circ a \quad \forall \zeta \in \mathcal{F}, \ a \in A$$

and the $A$-valued inner product

$$\langle \zeta | \eta \rangle := \zeta^* \circ \eta \quad \forall \zeta, \eta \in \mathcal{F}.$$ 

The Hilbert module norm and the operator norm on $\mathcal{F}$ coincide. We have

$$\mathcal{F} = \mathcal{F} \circ A = \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F}$$

(30)

and $\mathcal{F}(\mathcal{L}) = \mathcal{F} \circ \mathcal{F}^*(\mathcal{E}) = \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F}(\mathcal{L})$. Hence $\mathcal{F}$ is essential iff the linear span of $\mathcal{F} \circ \mathcal{F}^*(\mathcal{E})$ is dense in $\mathcal{E}$.

We always furnish a concrete Hilbert $A$-module with the Hilbert $A$-module structure defined above.

**Proof.** By assumption, $\zeta \cdot a \in \mathcal{F}$ for all $\zeta \in \mathcal{F}, a \in A$, and $\langle \zeta | \eta \rangle \in A$ for all $\zeta, \eta \in \mathcal{F}$. The conditions

$$\langle \zeta | \eta \cdot a \rangle = \langle \zeta | \eta \rangle \cdot a, \quad \langle \zeta | \eta \rangle = \langle \langle \eta | \zeta \rangle \rangle^*, \quad \langle \zeta | \zeta \rangle \geq 0$$

for a pre-Hilbert module are obviously satisfied. Since

$$\|\zeta\| = \|\zeta^* \|^{1/2} = \|\langle \zeta | \zeta \rangle\|^{1/2},$$

(31)

the norm that comes from the $A$-valued inner product equals the operator norm. Hence $\mathcal{F}$ is a Hilbert module. We have $\mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F} \subseteq \mathcal{F} \circ A \subseteq \mathcal{F}$. It is a general feature of Hilbert modules that any $\zeta \in \mathcal{F}$ may be written as $\eta \langle \eta | \zeta \rangle$ for some $\eta \in \mathcal{F}$. Hence $\mathcal{F} \subseteq \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F}$. Equation (30) follows. Since $\mathcal{F}(\mathcal{L}) = \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F}(\mathcal{L}) \subseteq \mathcal{F} \circ \mathcal{F}^*(\mathcal{E}) \subseteq \mathcal{F}(\mathcal{L})$, these three sets are equal.

Let $\mathcal{F}$ be a Hilbert $A$-module. We construct a canonical representation of $\mathcal{F}$ as a concrete Hilbert module. Equip $\mathcal{F}$ with the trivial action of $G$, so that $\mathcal{F} \otimes_A \mathcal{L}$ is a Hilbert $B, G$-module. The map $\mathcal{F} \mapsto \mathcal{F} \otimes_A \mathcal{L}$ is functorial, that is, an adjointable operator $x: \mathcal{F} \mapsto \mathcal{F}$ induces an equivariant, adjointable operator $x \otimes \text{id}_{\mathcal{L}}: \mathcal{F} \otimes_A \mathcal{L} \mapsto \mathcal{F} \otimes_A \mathcal{L}$. Using the isomorphism $A \otimes_A \mathcal{L} \cong A \cdot \mathcal{L} = \mathcal{L}$, we obtain a map

$$T: \mathcal{F} \cong \mathcal{K}(A, \mathcal{F}) \mapsto \mathcal{B}(A \otimes_A \mathcal{L}, \mathcal{F} \otimes_A \mathcal{L}) \cong \mathcal{B}(\mathcal{L}, \mathcal{F} \otimes_A \mathcal{L}).$$

(32)
More explicitly, we have $T(\xi)(f) := \xi \otimes f$ and $T(\xi)^* (\eta \otimes f) := \langle \xi | \eta \rangle (f)$ for all $\xi, \eta \in \mathcal{F}$ and $f \in \mathcal{L}$, where we view $\langle \xi | \eta \rangle \in A \subseteq B^0(\mathcal{L})$.

**Theorem 5.1.** Let $\mathcal{F}$ be a Hilbert $A$-module and define $T$ as in (32). Then $T(\mathcal{F})$ is an essential, concrete Hilbert $A$-module and $T: \mathcal{F} \to T(\mathcal{F})$ is an isomorphism of Hilbert $A$-modules. If $\mathcal{F} \subseteq B^0(\mathcal{L}, \mathcal{E})$ already is an essential, concrete Hilbert $A$-module, then

$$U: \mathcal{F} \otimes_A \mathcal{L} \to \mathcal{E}, \quad \xi \otimes f \mapsto \xi(f),$$

is an equivariant unitary that satisfies $U \circ (T(\xi)) = \xi$ for all $\xi \in \mathcal{F}$. That is, $\mathcal{F}$ and $T(\mathcal{F})$ are isomorphic as concrete Hilbert $A$-modules via $U$.

**Proof.** We have $T(\xi \cdot a) = T(\xi) \cdot a$ and $T(\xi)^* T(\eta) = \langle \xi | \eta \rangle$ for all $\xi, \eta \in \mathcal{F}$, $a \in A$. Equation (31) shows that $T$ is isometric, so that $T(\mathcal{F})$ is closed. Thus $T(\mathcal{F})$ is a concrete Hilbert $A$-module and $T: \mathcal{F} \to T(\mathcal{F})$ is an isomorphism with respect to the Hilbert $A$-module structure of Lemma 5.1. $T(\mathcal{F})$ is essential because $\mathcal{F} \otimes_A \mathcal{L}$ is generated by elementary tensors $\xi \otimes f = T(\xi)(f)$ with $\xi \in \mathcal{F}$ and $f \in \mathcal{L}$. Suppose that $\mathcal{F} \subseteq B^0(\mathcal{L}, \mathcal{E})$ is a concrete Hilbert $A$-module. The map $U$ is isometric (hence well-defined) by (5) and equivariant. If $\mathcal{F}$ is essential, then the range of $U$ is dense, so that $U$ is unitary. We compute $U(T(\xi)(f)) = U(\xi \otimes f) = \xi(f)$ for all $\xi \in \mathcal{F}$, $f \in \mathcal{L}$. That is, $U \circ (T(\xi)) = \xi$.

Put in a nutshell, any Hilbert $A$-module $\mathcal{F}$ can be represented as an essential, concrete Hilbert $A$-module, and this representation is unique up to isomorphism. The underlying Hilbert $B, G$-module $\mathcal{E}$ is canonically isomorphic to $\mathcal{F} \otimes_A \mathcal{L}$.

**Theorem 5.2.** Let $\mathcal{F} \subseteq B^0(\mathcal{L}, \mathcal{E})$ be a concrete Hilbert $A$-module. The map

$$\langle \xi | \eta \rangle \mapsto \xi \cdot \eta^* \in \mathcal{F} \circ \mathcal{F}^* \subseteq B(\mathcal{E})$$

extends to a $*$-isomorphism from $\mathcal{K}(\mathcal{F})$ onto the norm closure of $\mathcal{F} \circ \mathcal{F}^*$ in $B^0(\mathcal{E})$. This representation of $\mathcal{K}(\mathcal{F})$ is essential iff $\mathcal{F}$ is essential.

If $\mathcal{F}$ is essential, we may extend this representation of $\mathcal{K}(\mathcal{F})$ to a strictly continuous, injective, unital $*$-homomorphism $\phi: B(\mathcal{F}) \to B^0(\mathcal{E})$, whose range is

$$M := \{ x \in B(\mathcal{E}) \mid x \circ \mathcal{F} \subseteq \mathcal{F}, x^* \circ \mathcal{F} \subseteq \mathcal{F} \}.$$ 

**Proof.** It is clear that $M$ is a $C^*$-subalgebra of $B(\mathcal{E})$. Let $D \subseteq B(\mathcal{E})$ be the closed linear span of $\mathcal{F} \circ \mathcal{F}^*$. By construction, $D$ is closed and $D^* = D$. Equation (30) implies $D \circ \mathcal{F} \subseteq \mathcal{F}$ and hence $D \subseteq M$. If $x \circ \mathcal{F} \subseteq \mathcal{F}$, then $x \circ D \subseteq D$. Hence $D$ is a closed ideal in $M$. Conversely, if $x \circ D \subseteq D$, then

$$x \circ \mathcal{F} = x \circ \mathcal{F} \circ \mathcal{F}^* \circ \mathcal{F} \subseteq D \circ \mathcal{F} \subseteq \mathcal{F}.$$
by (30). Consequently, $x \in M$ iff $xD \subseteq D$ and $Dx \subseteq D$. We define a *-homomorphism $\psi: M \to B(\mathcal{F})$ by $\psi(x)(\xi) := x \circ \xi$ for $x \in M$, $\xi \in \mathcal{F}$. If $\mathcal{F}$ is essential, then $\psi$ is injective because if $\psi(x) = 0$, then $x$ vanishes on the dense subspace $\mathcal{F}(\mathcal{L}) \subseteq \mathcal{E}$, so that $x = 0$. In general, at least the restriction of $\psi$ to $D$ is injective because $xx^* = 0$ implies $x = 0$. We have $\psi(\xi \circ \eta^*) = |\xi\rangle\langle\eta|$ for all $\xi, \eta \in \mathcal{F}$. Hence $\psi(D) = K(\mathcal{F})$ and $|D|^{-1}$ equals the map $|\xi\rangle\langle\eta| \mapsto \xi \circ \eta^*$. Lemma 5.1 implies that $\psi|D^{-1}$ is essential iff $\mathcal{F}$ is essential.

Therefore, if $\mathcal{F}$ is essential, then $\psi|D^{-1}$ extends uniquely to a strictly continuous, unital, injective *-homomorphism $\phi$ from $B(\mathcal{F})$ to $B(\mathcal{E})$. Since $\phi \circ \psi = \text{id}_M$, the map $\phi$ is an isomorphism onto $M$. If $\mathcal{E} = \mathcal{F} \otimes_A \mathcal{L}$ and $\mathcal{F} \subseteq B^G(\mathcal{L}, \mathcal{E})$ is the standard representation (32), then $\phi(x) = x \otimes_A \text{id}_\mathcal{E}$ for all $x \in B(\mathcal{F})$.

**Corollary 5.1.** Let $\mathcal{F} \subseteq B^G(\mathcal{L}, \mathcal{E})$ be an essential, concrete Hilbert $A$-module. The following statements are equivalent:

1. $\phi: B(\mathcal{F}) \to B^G(\mathcal{E})$ is an isomorphism onto $B^G(\mathcal{E})$;
2. we have $u \circ \mathcal{F} = \mathcal{F}$ for all $u \in B^G(\mathcal{E})$;
3. the closed linear span of $\mathcal{F} \circ \mathcal{F}^*$ is an ideal in $B^G(\mathcal{E})$.

We call $\mathcal{F}$ ideal iff one of these assertions holds.

**Proof.** By Theorem 5.2, the first statement is equivalent to $M = B^G(\mathcal{E})$. The second statement asserts that all unitaries $u \in B^G(\mathcal{E})$ are contained in $M$. Since any element of $B^G(\mathcal{E})$ may be written as a sum of four unitaries, the first two statements are equivalent. In the proof of Theorem 5.2, we observed that $x \in B(\mathcal{E})$ satisfies $xD \subseteq D$ and $Dx \subseteq D$ iff $x \in M$. Hence the third statement is equivalent to $M = B^G(\mathcal{E})$ as well.

6. CONTINUOUSLY SQUARE-INTEGRABLE HILBERT MODULES

It is convenient to keep the abbreviations

$$A := C^*_r(G, B), \quad \mathcal{L} := L^2(G, B).$$

**Definition 6.1.** A subset $\mathcal{R} \subseteq \mathcal{E}$ is called relatively continuous iff $\mathcal{R} \subseteq \mathcal{E}_u$ and

$$\langle \mathcal{R} | \mathcal{R} \rangle := \{ |\xi\rangle\langle\eta| \mid \xi, \eta \in \mathcal{R} \}$$
is contained in $C^*_r(G, B) \subseteq B^G(L)$. If $\mathcal{A} \subseteq \mathcal{E}$ is relatively continuous, let $\mathcal{F}(\mathcal{E}, \mathcal{A}) \subseteq B^G(L, \mathcal{E})$ be the closed linear span of $\langle \mathcal{A} \rangle \circ C^*_r(G, B) \cup |\mathcal{A}|$.

Recall that $\langle \xi \mid \eta \rangle$ is a Laurent operator, whose symbol is given by (20). This often allows to verify relative continuity. In many interesting examples we have $\|\langle \xi \mid \eta \rangle\|_2 < \infty$ or even $\langle \xi \mid \eta \rangle \in C_\gamma(G, B)$ for all $\xi, \eta \in \mathcal{A}$.

**Proposition 6.1.** Let $\mathcal{A} \subseteq \mathcal{E}$ be relatively continuous. Then $\mathcal{F}(\mathcal{E}, \mathcal{A})$ is a concrete Hilbert $C^*_r(G, B)$-module. If $\mathcal{A}$ is dense in $\mathcal{E}$, then $\mathcal{F}(\mathcal{E}, \mathcal{A})$ is essential.

**Proof.** By construction, $\mathcal{F} := \mathcal{F}(\mathcal{E}, \mathcal{A})$ is a closed linear subspace and $\mathcal{F} \circ \mathcal{A} \subseteq \mathcal{F}$. The assumption $\langle \mathcal{A} \mid \mathcal{A} \rangle \subseteq \mathcal{A}$ implies $\mathcal{F}^* \circ \mathcal{F} \subseteq \mathcal{A}$. Suppose that $\mathcal{A}$ is dense in $\mathcal{E}$. Since $\mathcal{E} \subseteq \mathcal{A}$ is dense in $\mathcal{E}$, the subset $\mathcal{A} \ast C_\gamma(G, B) = |\mathcal{A}| \circ C_\gamma(G, B)$ is dense in $\mathcal{E}$. Therefore, $\mathcal{F}(\mathcal{E})$ is dense in $\mathcal{E}$. 

**Proposition 6.2.** Let $\mathcal{F} \subseteq B^G(L, \mathcal{E})$ be a concrete Hilbert $C^*_r(G, B)$-module. Define

$$\mathcal{A}_r := \{ x \in \mathcal{E} \mid |x \rangle \in \mathcal{F} \},$$

$$\mathcal{A}_r^0 := \{ \xi(K) \mid \xi \in \mathcal{F}, K \in C_\gamma(G, B) \}.$$

Then $\mathcal{A}_r^0 \subseteq \mathcal{A}_r$. Both $\mathcal{A}_r^0$ and $\mathcal{A}_r$ are relatively continuous, and $|\mathcal{A}_r^0 \rangle$ and $\langle \mathcal{A}_r^0 |$ are dense in $\mathcal{F}$. Thus

$$\mathcal{F}(\mathcal{E}, \mathcal{A}_r^0) = \mathcal{F}(\mathcal{E}, \mathcal{A}_r) = \mathcal{F}.$$

**Proof.** It is evident that $\mathcal{A}_r$ is relatively continuous. Let $\xi \in \mathcal{F}$ and $K \in C_\gamma(G, B)$. Since $\xi$ is equivariant, (13) and (21) yield

$$|\xi(K) \rangle = \xi \circ |K \rangle = \xi \circ \rho_K \in \mathcal{F} \circ C^*_r(G, B) \subseteq \mathcal{F}.$$

This implies $\mathcal{A}_r^0 \subseteq \mathcal{A}_r$. Thus $\mathcal{A}_r^0$ is relatively continuous. The above computation shows $|\mathcal{A}_r^0 \rangle \circ \mathcal{F} = \mathcal{F} \cdot C_\gamma(G, B)$. Since $C_\gamma(G, B)$ is dense in $\mathcal{A}$ and $\mathcal{F}$ is a Hilbert $\mathcal{A}$-module, $\mathcal{F} \cdot C_\gamma(G, B)$ is dense in $\mathcal{F}$. It follows that $|\mathcal{A}_r^0 \rangle$ and $\langle \mathcal{A}_r^0 |$ are dense subsets of $\mathcal{F}$. Therefore, $\mathcal{F}(\mathcal{E}, \mathcal{A}_r^0) = \mathcal{F}(\mathcal{E}, \mathcal{A}_r) = \mathcal{F}$.

The subspace $\mathcal{A} \subseteq \mathcal{F} \otimes_\mathcal{A} L$ that is defined in (4) equals $\mathcal{A}_r^0$.

**Definition 6.2.** We call $\mathcal{A} \subseteq \mathcal{E}$ complete iff $\mathcal{A}$ is a linear subspace of $\mathcal{E}_u$ that is closed with respect to the norm $\| \Box \|_u$ and satisfies $\mathcal{A} \ast C_\gamma(G, B) \subseteq \mathcal{A}$. The completion of a subset $\mathcal{A} \subseteq \mathcal{E}_u$ is the smallest complete subset that contains $\mathcal{A}$. That is, the completion of $\mathcal{A}$ is the $\| \Box \|_u$-closed linear span of $\mathcal{A} \cup \mathcal{A} \ast C_\gamma(G, B)$.
A continuously square-integrable Hilbert $B, G$-module is a Hilbert $B, G$-module together with a dense, complete, relatively continuous subspace.

If $\mathcal{R} \subseteq \mathcal{E}$ is a complete, relatively continuous subset, then the closure of $|\mathcal{R}|$ is already a right $A$-module by (25). Hence $\mathcal{F}(\mathcal{E}, \mathcal{R})$ is simply the closure of $|\mathcal{R}|$.

The last assertion of the following theorem is analogous to Connes’s description of Hilbert modules over the reduced $C^*$-algebra of a foliation [3, p. 579].

**Theorem 6.1.** The map $\mathcal{F} \mapsto \mathcal{F}_x$ is a bijection from the set of concrete Hilbert $C^*_f(G, B)$-modules $\mathcal{F} \subseteq \mathcal{B}(L^2(G, B), \mathcal{E})$ onto the set of complete, relatively continuous subspaces of $\mathcal{E}$. Its inverse is the map $\mathcal{R} \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R})$.

A concrete Hilbert module $\mathcal{F}$ is essential if and only if $\mathcal{R}_x$ is dense.

Isomorphism classes of Hilbert modules over $C^*_f(G, B)$ correspond bijectively to isomorphism classes of continuously square-integrable Hilbert $B, G$-modules.

**Proof.** Let $\mathcal{F}$ be a concrete Hilbert $A$-module. It is evident that $\mathcal{R}_x$ is complete and relatively continuous. Proposition 6.2 asserts $\mathcal{F}(\mathcal{E}, \mathcal{R}_x) = \mathcal{F}$. Conversely, let $\mathcal{R} \subseteq \mathcal{E}$ be complete and relatively continuous. Define $\mathcal{F} := \mathcal{F}(\mathcal{E}, \mathcal{R})$. Then $\mathcal{R} \subseteq \mathcal{R}_x$. We claim that $\mathcal{R} = \mathcal{R}_x$.

Let $x \in \mathcal{R}_x$, we want to show that $x \in \mathcal{R}$. Let $(u_j)_{j \in J}$ be an approximate identity as in Lemma 3.1. Since $\mathcal{F}$ is the closure of $|\mathcal{R}|$, there is a sequence $(x_n) \in \mathcal{R}$ with $\lim x_n = x$ in operator norm. Equation (28) implies

$$\lim_{n \to \infty} \|x_n \star u_j - x \star u_j\|_a = 0$$

for all $j \in J$. Hence $x \star u_j \in \mathcal{R}$ because $\mathcal{R}$ is complete. Since $(u_j)$ is an approximate identity for $A$, we have $\xi \cdot u_j \to \xi$ for all elements $\xi$ of a Hilbert $A$-module. In particular, $|x| : u_j = |x \star u_j|$ converges towards $|x|$. Together with (29), this means that $x \star u_j \to x$ in the norm $\|\cdot\|_a$. Hence $x \in \mathcal{R}$. This proves that $\mathcal{R}_x = \mathcal{R}$.

If $\mathcal{F}$ is essential, then $\mathcal{R}_{\mathcal{F}}$ is dense in $\mathcal{E}$. Hence $\mathcal{R}_x$ is dense in $\mathcal{E}$. Conversely, if $\mathcal{R}_x$ is dense, then $\mathcal{F}$ is essential by Proposition 6.1. The last assertion of the theorem follows from Theorem 5.1.

**Proposition 6.3.** Let $\mathcal{R} \subseteq \mathcal{E}$ be relatively continuous. Then the completion of $\mathcal{R}$ equals $\mathcal{R}_{\mathcal{F}(\mathcal{E}, \mathcal{R})}$. Thus the completion of $\mathcal{R}$ is still relatively continuous.

**Proof.** The assertion follows easily from Proposition 6.1 and Theorem 6.1 because the map $\mathcal{F} \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R})$ preserves inclusions. Hence $\mathcal{R}_{\mathcal{F}(\mathcal{E}, \mathcal{R})}$ is the smallest complete subspace containing $\mathcal{R}$.
Proposition 6.4. Let \( \mathcal{R} \subseteq \mathcal{E} \) be a complete, relatively continuous subspace. Equip \( \mathcal{R} \) with the norm \( \| \cdot \| \). The subspace \( \mathcal{R} \subseteq \mathcal{E} \) is \( G \)-invariant, the action of \( G \) on \( \mathcal{R} \) is continuous. Furthermore, \( \mathcal{R} \) is an essential right \( B \)-module, that is, \( \mathcal{R} \cdot B = \mathcal{R} \).

Proof. By Theorem 6.1, we have \( \mathcal{R} = \mathcal{R}_0 \) for a concrete Hilbert module \( \mathcal{F} \) over \( A \). Let \( x \in \mathcal{R} \). By Cohen's Factorization Theorem, the map \( a \mapsto |x\rangle \cdot a \) extends to a linear operator \( \mathcal{M}(A) \rightarrow \mathcal{F} \) that is continuous with respect to the strict topology on \( \mathcal{M}(A) \) and the norm topology on \( \mathcal{F} \). If there is \( y \in \mathcal{E}_0 \) with \( |y\rangle = |x\rangle \cdot a \), then automatically \( y \in \mathcal{R}_0 \). Using (14) and (15), we conclude that \( x \cdot b, \gamma (b) \in \mathcal{R} \) for all \( b \in B, \ g \in G \). Furthermore, we have norm estimates (17) and (18). Since the map \( g \mapsto \rho_g \) is strictly continuous, we have \( |x\rangle \rho_g \rightarrow |x\rangle \) for \( g \rightarrow 1 \). Therefore, the action of \( G \) on \( \mathcal{R} \) is continuous. Similarly, if \( (u_i) \) is an approximate identity of \( B \), then \( x \cdot u_i \rightarrow x \) in the norm \( \| \cdot \| \). Hence Cohen's Factorization Theorem yields \( \mathcal{R} \cdot B = \mathcal{R} \).

Since (20) describes \( \langle x | y \rangle \) explicitly, we can prove \( \langle x | y \rangle \in C^*_r(G, B) \) without showing \( x, y \in \mathcal{E}_0 \). For instance, it may happen that \( \langle x | y \rangle \in C_r(G, B) \) for all \( x, y \in \mathcal{R} \). If \( \mathcal{R} \) is dense in \( \mathcal{E} \), then this implies \( \mathcal{R} \subseteq \mathcal{E}_0 \):

Proposition 6.5. Let \( \mathcal{R} \subseteq \mathcal{E} \) be a dense subset such that \( \langle x | y \rangle \in C^*_r(G, B) \) for all \( x, y \in \mathcal{R} \). Then \( \mathcal{R} \subseteq \mathcal{E}_0 \), so that \( \mathcal{R} \) is relatively continuous.

Proof. Fix \( x \in \mathcal{R} \). If \( f \in C_r(G, B) \), then \( |x\rangle f \) is well-defined by (2) and

\[
\| |x\rangle f \| = \| \langle x | f \rangle |x\rangle \|^{1/2} = \| \langle f | \langle x | x \rangle f \rangle \|^{1/2} \leq \| f \| \| \langle x | x \rangle \|^{1/2}.
\]

Hence \( |x\rangle : C_r(G, B) \rightarrow \mathcal{E} \) extends to a bounded operator \( |x\rangle : L^2(G, B) \rightarrow \mathcal{E} \). The problem is to show that \( |x\rangle \) is adjointable. Let \( \mathcal{E}_0 \subseteq \mathcal{E} \) be the domain of \( |x\rangle^\ast \). That is, \( \xi \in \mathcal{E}_0 \) iff there is \( f \in L^2(G, B) \) with \( \langle \xi | |x\rangle f \rangle = \langle f | f \rangle \) for all \( f_1 \in L^2(G, B) \). It suffices to look at \( f_1 \in C_r(G, B) \). Since \( |x\rangle \) is bounded, \( \mathcal{E}_0 \) is closed. Hence the proof will be finished if we show that \( \mathcal{E}_0 \subseteq \mathcal{E} \) is dense. If \( y \in \mathcal{R} \), \( f_1, f_2 \in C_r(G, B) \), then \( \langle y \rangle f_1 \langle x | x \rangle f_2 \rangle = \langle f_1 | f_2 \rangle \) and hence \( |y\rangle f_1 \in \mathcal{E}_0 \). Elements of this form exhaust \( \mathcal{R} \ast C_r(G, B) \) by (24). Since \( \mathcal{R} \) is dense in \( \mathcal{E} \), (29) yields that \( \mathcal{R} \ast C_r(G, B) \subseteq \mathcal{E}_0 \) is dense in \( \mathcal{E} \), as desired.

Definition 6.3. Let \( (\mathcal{E}, \mathcal{R}) \) and \( (\mathcal{E'}, \mathcal{R'}) \) be continuously square-integrable Hilbert modules. An operator \( T \in B^G(\mathcal{E}, \mathcal{E'}) \) is called \( \mathcal{R} \)-continuous iff \( T(\mathcal{R}) \subseteq \mathcal{R'} \) and \( T^*(\mathcal{R'}) \subseteq \mathcal{R} \).
The generalized fixed point algebra $\text{Fix}(\mathcal{E}, \mathcal{R})$ is defined to be the closed linear span of $|\mathcal{R}\rangle\langle\mathcal{R}|$ in $B(\mathcal{E})$.

It follows immediately from (19) that $\text{Fix}(\mathcal{E}, \mathcal{R})$ is the closed linear span of the “averages” $\int_\mathcal{R} \gamma_\mathcal{R}(x) \, dg$ with $x = |\xi\rangle\langle\eta|$, $\xi, \eta \in \mathcal{R}$.

**Theorem 6.2.** Let $(\mathcal{E}, \mathcal{R})$ be a continuously square-integrable Hilbert $B, G$-module and let $\mathcal{F} := \mathcal{F}(\mathcal{E}, \mathcal{R})$. There is a canonical, injective, strictly continuous *-homomorphism $\phi: B(\mathcal{F}) \to B^G(\mathcal{E})$, whose range is the space of $\mathcal{R}$-continuous operators. It maps $\mathcal{K}(\mathcal{F})$ isometrically onto $\text{Fix}(\mathcal{E}, \mathcal{R})$.

$\text{Fix}(\mathcal{E}, \mathcal{R})$ is Morita–Rieffel equivalent to an ideal in $C^*_r(G, B)$, namely, the closed linear span of $|\mathcal{R}\rangle\langle\mathcal{R}|$ in $B^G(\mathcal{E})$.

**Proof.** Since $|\mathcal{R}\rangle\langle\mathcal{R}|$ is dense in $\mathcal{F}$, we conclude that $|\mathcal{R}\rangle\langle\mathcal{R}|$ is dense in $\mathcal{F}^*\mathcal{F}$ and that $|\mathcal{R}\rangle\langle\mathcal{R}|$ is dense in $\mathcal{F}^*\mathcal{F}^*$. Moreover, (13) yields that the space $M$ defined in Theorem 5.2 equals the space of $\mathcal{R}$-continuous operators. Hence the assertions of the first paragraph follow from Theorem 5.2 if we take the homomorphism $\phi$ defined there. Since $F_A \subseteq \mathcal{F}$, the closed linear span $J$ of $\mathcal{F}^*\mathcal{F}$ is an ideal in $A$. We may view $\mathcal{F}$ as an imprimitivity bimodule for $J$ and $\mathcal{K}(\mathcal{F}) \cong \text{Fix}(\mathcal{E}, \mathcal{R})$. That is, $\text{Fix}(\mathcal{E}, \mathcal{R})$ and $J$ are Morita–Rieffel equivalent. 

Theorem 6.2 implies that $(\mathcal{E}, \mathcal{R}) \mapsto \mathcal{F}(\mathcal{E}, \mathcal{R})$ is an equivalence between the $C^*$-categories of continuously square-integrable Hilbert $B, G$-modules and Hilbert modules over $C^*_r(G, B)$, if we take $\mathcal{R}$-continuous adjointable operators and adjointable operators as morphisms, respectively.

Let $\mathcal{E}$ be a square-integrable Hilbert $B, G$-module. It is an important question whether there is a canonical choice for a dense, complete, relatively continuous subset $\mathcal{R} \subseteq \mathcal{E}$. If $B$ is proper, then there is one and only one such $\mathcal{R}$. This is the strongest sense in which $\mathcal{R}$ may be canonical. More generally, canonical should mean that we can single out a specific subspace $\mathcal{R}$ using only that $\mathcal{E}$ is a Hilbert $B, G$-module. Hence if $u: \mathcal{E} \to \mathcal{E}$ is an equivariant unitary, then $u(\mathcal{R}) = \mathcal{R}$ because $u$ preserves the Hilbert $B, G$-module structure. This is equivalent to $u = \mathcal{F}(\mathcal{E}, \mathcal{R}) = \mathcal{F}(\mathcal{E}, \mathcal{R})$. By Corollary 5.1 this happens iff all operators in $B^G(\mathcal{E})$ are $\mathcal{R}$-continuous iff $\text{Fix}(\mathcal{E}, \mathcal{R})$ is an ideal in $B^G(\mathcal{E})$.

Unfortunately, $B^G(\mathcal{E})$ frequently is so big that no ideal of it qualifies as a generalized fixed point algebra. For instance, if $B = \mathbb{C}$, then $B^G(\mathcal{E})$ will be a commutant of a group action on a Hilbert space and thus a von Neumann algebra. Hence there can be no canonical choice for $\mathcal{R}$ in this case. However, this does not yet create a very serious lack of uniqueness. If $u \in B^G(\mathcal{E})$ is unitary, then $(\mathcal{E}, \mathcal{R})$ and $(\mathcal{E}, u(\mathcal{R}))$ correspond to two representations of the same abstract Hilbert module over $C^*_r(G, B)$ and hence give rise to isomorphic generalized fixed point algebras.
7. CONSTRUCTIONS WITH RELATIVELY CONTINUOUS SUBSETS

As a preparation for Theorem 7.1 and as an important special case, we discuss Hilbert modules over $C^*_r(G, B)$ of the form $C^*_r(G, E)$. Kasparov [5] defines $C^*_r(G, E)$ as a completion of $C_r(G, E)$ with respect to a certain pre-Hilbert module structure over $C_c(G, B)$. An equivalent definition is

$$C^*_r(G, E) := E \otimes_{B} C^*_r(G, B), \quad (33)$$

where we use the canonical map $B \to \mathcal{A}(C^*_r(G, B))$ to form the balanced tensor product. In our framework, $C^*_r(G, E)$ arises as follows. The subspace $C_c(G, E)$ is dense and relatively continuous. Equation (20) yields $S t_{\xi, \eta} \in C_c(G, B)$ for all $\xi, \eta \in C_r(G, E)$. Hence $C_c(G, E) \subseteq L^2(G, \mathcal{E})_0$ by Proposition 6.5. We claim that

$$\mathcal{F}(L^2(G, \mathcal{E}), C_r(G, E)) \cong C^*_r(G, E).$$

To verify this, we generalize (11) and define

$$\rho_\eta \in \mathcal{B}(L^2(G, B), L^2(G, E)), \quad (\rho_\eta f)(g) := \gamma_\eta(f(g), \quad (34)$$

for all $\eta \in \mathcal{E}$. Equations (21) and (13) yield

$$\rho_\eta \circ \rho_\kappa = \rho_\eta \circ |\mathcal{K}| = |\rho_\eta(\mathcal{K})|$$

for all $\eta \in \mathcal{E}, \mathcal{K} \in C_r(G, B)$. It follows that $\rho(\mathcal{E}) \subseteq \rho(C_r(G, B))$ and $|C_r(G, E)|$ have the same closed linear span in $\mathcal{B}(L^2(G, B), L^2(G, E))$. By (5), the map

$$\mathcal{E} \otimes_{B} C^*_r(G, B) \to \mathcal{F}(L^2(G, \mathcal{E}), C_r(G, E)), \quad \eta \otimes \mathcal{K} \mapsto \rho_\eta \circ \rho_\kappa,$$

is an isomorphism of Hilbert modules over $C^*_r(G, B)$.

Consider the following situation. Let $A$ and $B$ be $G$-$C^*$-algebras, let $\mathcal{E}_1$ and $\mathcal{E}_2$ be $G$-equivariant Hilbert modules over $A$ and $B$, respectively, and let $\phi: A \to \mathcal{B}(\mathcal{E}_2)$ be an equivariant, essential $*$-homomorphism. The map $\phi$ induces an essential $*$-homomorphism $C^*_r(G, A) \to \mathcal{B}(C^*_r(G, \mathcal{E}_2))$.

**Theorem 7.1.** Let $\mathcal{R}_1 \subseteq \mathcal{E}_1$ be a (dense) relatively continuous subspace. Let $\mathcal{R}_{12}$ be the image of $\mathcal{R}_1 \otimes_B \mathcal{E}_2 \subseteq \mathcal{E}_1 \otimes_B \mathcal{E}_2$ under the canonical map to $\mathcal{E}_{12} := \mathcal{E}_1 \otimes_A \mathcal{E}_2$. Then $\mathcal{R}_{12} \subseteq \mathcal{E}_{12}$ is (dense and) relatively continuous. We have

$$\mathcal{F}(\mathcal{E}_{12}, \mathcal{R}_{12}) \cong \mathcal{F}(\mathcal{E}_1, \mathcal{R}_1) \otimes_{C^*_r(G, A)} C^*_r(G, \mathcal{E}_2).$$

(35)
Let $t \in R_1$, $g \in E_2$. Since $f$ is essential, $L^2(G, A) \otimes_A E_2 \simeq L^2(G, E_2)$. Hence $\langle \xi \rangle \otimes_A \text{id}_{E_2} \in B^0(L^2(G, E_2), E_{12})$. The same simple computation that yields (14) shows that

$$\langle \xi \otimes \eta \rangle = (\langle \xi \rangle \otimes \text{id}_{E_2}) \circ \rho_\eta,$$

where $\rho_\eta$ is defined by (34). As a result, $R_{12} \subseteq (E_{12})_a$ and

$$\mathcal{A}_{12} = (\langle \mathcal{A}_1 \rangle \otimes \text{id}_{E_2}) \circ \rho(E_2).$$

By definition, $\mathcal{F}_{12} := \mathcal{F}(E_{12}, R_{12})$ is the closed linear span of $\langle \mathcal{A}_{12} \rangle \circ C^*_r(G, B)$. The discussion of $C^*_r(G, E_2)$ above shows that $C^*_r(G, E_2)$ is the closed linear span of $\rho(E_2) \circ C^*_r(G, B)$. Hence $\mathcal{F}_{12}$ is the closed linear span of $(\mathcal{F}_1 \otimes \text{id}_{E_2}) \circ C^*_r(G, E_2)$. Equation (5) yields that the map

$$\mathcal{F}_1 \otimes C^*_r(G, A) \rightarrow \mathcal{F}_{12}, \quad \xi \otimes \eta \mapsto (\xi \otimes_A \text{id}_{E_2}) \circ \eta$$

is an isometry of Hilbert modules over $C^*_r(G, B)$. Hence $\mathcal{F}_{12}$ is relatively continuous and satisfies (35).

A consequence of the proof (or of (35) and Proposition 6.3) is that the construction $R_1 \mapsto R_{12}$ is compatible with completions. That is, if the completions of $R_1$ and $R_1'$ are equal, then the same holds for $R_{12}$ and $R_{12}'$.

We consider some important special cases of Theorem 7.1.

**Corollary 7.1.** Let $(A, \mathcal{R})$ be a continuously square-integrable C*-algebra and let $\phi: A \rightarrow B(\mathcal{E})$ be an equivariant, essential *-homomorphism. Then $\mathcal{R}(\mathcal{E}) \subseteq \mathcal{E}$ is a dense, relatively continuous subset, and

$$\mathcal{F}(\mathcal{E}, \mathcal{R}(\mathcal{E})) \cong \mathcal{F}(A, \mathcal{R}) \otimes_{C^*_r(G, A)} C^*_r(G, \mathcal{E}).$$

**Proof.** The isomorphism $A \otimes_A \mathcal{E} \cong \mathcal{R} \otimes \mathcal{E}$ maps $\mathcal{R} \otimes \mathcal{E}$ onto the linear span of $\mathcal{R}(\mathcal{E})$.

In particular, if $\mathcal{R}(\mathcal{E})$ is continuously square-integrable, so is $\mathcal{E}$.

**Corollary 7.2.** Let $(\mathcal{E}, \mathcal{R})$ be continuously square-integrable. Then the linear span of

$$\mathcal{H} := \{ |\xi \rangle \langle \eta| : \xi \in \mathcal{R}, \eta \in \mathcal{E} \}$$

is a dense, relatively continuous subspace of $\mathcal{R}(\mathcal{E})$. We have

$$\mathcal{F}(\mathcal{R}(\mathcal{E}), \mathcal{H}) \cong \mathcal{F}(\mathcal{E}, \mathcal{R}) \otimes_{C^*_r(G, \mathcal{B})} C^*_r(G, \mathcal{E}^*).$$
Proof. The assertion follows from (6) and Theorem 7.1.

Let $S(\mathcal{E})$ and $S\mathcal{K}(\mathcal{E})$ be the sets of all dense, complete, relatively continuous subspaces of $\mathcal{E}$ and $\mathcal{K}(\mathcal{E})$, respectively. Corollaries 7.1 and 7.2 give rise to maps

$$i: S(\mathcal{E}) \to S\mathcal{K}(\mathcal{E}), \quad j: S\mathcal{K}(\mathcal{E}) \to S(\mathcal{E}).$$

We analyze whether these two maps are inverse to each other. Recall that a group $G$ is exact if and only if

$$C^*_g(G, I) = \ker(C^*_g(G, B) \to C^*_g(G, B/I))$$

whenever $I \subseteq B$ is an invariant closed ideal in a $G$-$C^*$-algebra $B$.

**Theorem 7.2.** For all $G$ and $B$ and all Hilbert $B, G$-modules $\mathcal{E}$, the composition $i \circ j$ is the identity map on $S\mathcal{K}(\mathcal{E})$ and $j \circ i(\mathcal{A}) \subseteq \mathcal{A}$ for all $\mathcal{A} \in S(\mathcal{E})$.

If the group $G$ is exact, then $j \circ i$ is the identity map on $S(\mathcal{E})$. Conversely, if $G$ is not exact, then there are Hilbert modules $\mathcal{E}$ for which $j \circ i$ is not the identity map.

**Proof.** Let $\mathcal{A} \in S\mathcal{K}(\mathcal{E})$. Then $j(\mathcal{A})$ is the completion of $\mathcal{A}(\mathcal{E}) \subseteq \mathcal{E}$. Since $i$ is compatible with completions, $i \circ j(\mathcal{A})$ is the completion of $|\mathcal{A}(\mathcal{E})\rangle = \mathcal{A} \circ |\mathcal{E}\rangle$. The linear span of $\mathcal{A} \circ |\mathcal{E}\rangle$ is a dense subspace of $\mathcal{A}$ by Proposition 6.4. Hence the completion of $\mathcal{A} \circ |\mathcal{E}\rangle$ equals $\mathcal{A}$. This proves $i \circ j = \text{id}$.

Conversely, let $\mathcal{A} \in S(\mathcal{E})$ and $\mathcal{F} := \mathcal{F}(\mathcal{E}, \mathcal{A})$. Then $j \circ i(\mathcal{A})$ is the completion of $|\mathcal{A}(\mathcal{E})\rangle = \mathcal{A} \cdot |\mathcal{E}\rangle$. Hence $\mathcal{F}(\mathcal{E}, j(\mathcal{A}))$ is the closed linear span of $\mathcal{A} \cdot |\mathcal{E}\rangle$. Let $I \subseteq B$ be the closed ideal generated by $|\mathcal{E}\rangle$. Cohen’s Factorization Theorem and Proposition 6.4 show that

$$j \circ i(\mathcal{A}) = \mathcal{A} \cdot I \subseteq \mathcal{A}, \quad \mathcal{F}(\mathcal{E}, j(\mathcal{A})) = \mathcal{A} \cdot I \subseteq \mathcal{F}.$$

Let $J \subseteq C^*_g(G, B)$ be the closed ideal generated by $\langle \mathcal{F} | \mathcal{F} \rangle$. If $J \subseteq C^*_g(G, I)$, then we may view $\mathcal{F}$ as a Hilbert module over $C^*_g(G, I)$. Hence $\mathcal{F} \cdot I = \mathcal{F}$. Conversely, if $\mathcal{F} \cdot I = \mathcal{F}$, then $J \cdot I = J$ and hence $J \subseteq C^*_g(G, B) \cdot I = C^*_g(G, I)$. Therefore,

$$j \circ i(\mathcal{A}) = \mathcal{A} \Leftrightarrow J \subseteq C^*_g(G, I).$$

If $\xi, \eta \in \mathcal{A}$, then $\langle \xi | \eta \rangle(g) \in I$ for all $g \in G$ by (20). Therefore, $\langle \xi | \eta \rangle$ is annihilated by the canonical map $C^*_g(G, B) \to C^*_g(G, B/I)$. If $G$ is exact, this implies that $\langle \xi | \eta \rangle \in C^*_g(G, I)$. Hence $J \subseteq C^*_g(G, I)$ and thus $j \circ i(\mathcal{A}) = \mathcal{A}$. 

Suppose that $G$ is not exact and that $I \subseteq B$ is an invariant ideal for which $C^*_r(G, I)$ is strictly smaller than the kernel $K$ of the map $C^*_r(G, B) \to C^*_r(G, B/I)$. View $K$ as a Hilbert module over $C^*_r(G, B)$ and let $(\mathcal{E}, \mathcal{R})$ be the associated continuously square-integrable Hilbert module (Theorem 6.1). Then $\langle \mathcal{R} | \mathcal{R} \rangle \subseteq K$. This implies $\langle \xi | \eta \rangle \in I$ for all $\xi, \eta \in \mathcal{R}$ by (20). Hence $\langle \mathcal{E} | \mathcal{E} \rangle \subseteq I$. However, $J = K$ is not contained in $C^*_r(G, I)$. Hence $\mathcal{R} \neq \mathcal{R} \cdot I$.

We remark that the identity $i \circ j = id$ is equivalent to the isomorphism

$$C^*_r(G, \mathcal{E}) \otimes_{C^*_r(G, B)} C^*_r(G, \mathcal{E}^*) \cong C^*_r(G, \mathcal{E} \otimes_B \mathcal{E}^*) \cong C^*_r(G, \mathcal{K}(\mathcal{E})).$$

8. SOME COUNTEREXAMPLES

In this section, we consider a simple special case in which a complete description of the square-integrable and continuously square-integrable Hilbert modules is possible. We assume that $B = C$ and that $G$ is Abelian, $\sigma$-compact, and metrizable, but not compact. Hence the Pontrjagin dual $\hat{G}$ of $G$ is not discrete. For instance, we may take $G = \mathbb{Z}^n$ for some $n \in \mathbb{N} \setminus \{0\}$.

Since $C^*_r(G, B) \cong C_0(\hat{G})$, we may view countably generated Hilbert modules over $C^*_r(G, B)$ as continuous fields of separable Hilbert spaces over $\hat{G}$. The Hilbert module over $C_0(\hat{G})$ associated to a continuous field of Hilbert spaces $(\mathcal{H}_x)_{x \in \hat{G}}$ is $C_0(\hat{G}, (\mathcal{H}_x))$, the space of continuous sections of $(\mathcal{H}_x)$ vanishing at infinity. The $C^*$-algebra of compact operators on this Hilbert module is isomorphic to $C_0(\hat{G}, \mathcal{K}(\mathcal{H}_x))$, where $(\mathcal{K}(\mathcal{H}_x))_{x \in \hat{G}}$ carries the canonical bundle structure.

A countably generated Hilbert $B, G$-module is nothing but a representation of $G$ on a separable Hilbert space. By the Equivariant Stabilization Theorem, a $G$-Hilbert space is square-integrable iff it is a direct summand in $(L^2G)^\sim \cong L^2(\hat{G}, dx)^\sim$, where $dx$ denotes the Haar measure on $\hat{G}$. Therefore, a $G$-Hilbert space is square-integrable iff it is equivalent to a Hilbert space of $dx$-square-integrable sections of some measurable field of Hilbert spaces over $\hat{G}$, equipped with the canonical representation of $G$ by pointwise multiplication. Two measurable fields yield equivalent representations of $G$ iff they are isomorphic outside a set of Haar measure zero.

Measurable fields of Hilbert spaces are classified by the dimension function $d : \hat{G} \to \mathbb{N} : x \mapsto d_x$ that associates to $x \in \hat{G}$ the dimension of the fiber over $x$. The function $d$ is measurable, and any measurable function arises as the dimension function of a measurable field. We say that an assertion holds a.e. (almost everywhere) iff it holds outside a set of measure zero. Two measurable fields are isomorphic a.e. if and only if the dimension
functions agree a.e.. Hence isomorphism classes of square-integrable, separable $G$-Hilbert spaces correspond to a.e.-equality classes of measurable functions $\hat{G} \to \mathbb{N}$.

Let $(\mathcal{H}_x)_{x \in \hat{G}}$ be a continuous field of Hilbert spaces over $\hat{G}$. If we view $(\mathcal{H}_x)$ as a Hilbert module over $C^*_\text{r}G$ and apply the functor $\downarrow \otimes_{C^*_\text{r}G} L^2\hat{G}$, we get the Hilbert space of square-integrable sections of $(\mathcal{H}_x)_{x \in \hat{G}}$ with the representation of $G$ by pointwise multiplication. Hence the functor $\downarrow \otimes_{C^*_\text{r}G} L^2\hat{G}$ forgets everything about the field $(\mathcal{H}_x)$ except the a.e.-equality class of its dimension function.

The dimension function of a continuous field of Hilbert spaces is automatically lower semi-continuous. Therefore, if $d: \hat{G} \to \mathbb{N}$ is not equal a.e. to a lower semi-continuous function, then the corresponding square-integrable representation cannot come from a Hilbert module over $C^*_\text{r}G$. To construct examples of such measurable functions, let $d$ be the characteristic function of a compact subset $K \subseteq \hat{G}$. Suppose that $d' \colon \hat{G} \to \mathbb{N}$ is lower semi-continuous and that $d' \leq d$ a.e.. It follows that $d' \leq 1$ and that $d' = 0$ on the open set $\hat{G} \setminus K$. Thus $d'$ is the characteristic function of an open subset $U \subseteq K$. If $K$ is a compact set with non-zero Haar measure and empty interior, then $d' = 0$ is the only lower semi-continuous dimension function with $d' \leq d$ a.e.. Nevertheless, $d' \neq d$ a.e..

The square-integrable $G$-Hilbert space associated to $d$ is $L^2(K, dx)$, on which $G$ acts by pointwise multiplication. Suppose that $\mathcal{H} \subseteq L^2(K, dx)$ is relatively continuous and complete. Let $\mathcal{H} \subseteq L^2(K)$ be the closure of $\mathcal{H}$. Then $(\mathcal{H}, \mathcal{H})$ is continuously square-integrable. Therefore, the dimension function of $\mathcal{H}$ is lower semi-continuous. By construction of $d$ this implies that $\mathcal{H} = \{0\}$. Consequently, $\{0\}$ is the only relatively continuous subset of the square-integrable $G$-Hilbert space $L^2(K, dx)$.

Conversely, if the dimension function of a separable $G$-Hilbert space is lower semi-continuous, there is a dense, relatively continuous subspace. The proof is left to the reader. However, this subspace is never unique. Even more, there are many Hilbert modules $\mathcal{F}$ over $C^*_\text{r}G$ for which $\mathcal{F} \otimes_{C^*_\text{r}G} L^2\hat{G} \cong L^2\hat{G}$. The most obvious source of non-uniqueness is modification on a set of measure zero. Let $S \subseteq \hat{G}$ be a closed subset of measure zero (for instance, a finite subset). The ideal

$$I_S = \{ f \in C_0(\hat{G}) \mid f|_S = 0 \} \subseteq C_0(\hat{G})$$

may be viewed as a Hilbert module over $C_0(\hat{G})$ and thus as a continuous field of Hilbert spaces over $\hat{G}$. Its dimension function is the characteristic function of $\hat{G} \setminus S$ and hence equal to 1 a.e.. Thus $I_S \otimes_{C^*_\text{r}G} L^2\hat{G} \cong L^2\hat{G}$. The generalized fixed point algebra in this example is $I_S$. Hence a generalized fixed point algebra for $L^2\hat{G}$ need not be isomorphic to $C_0(\hat{G})$, not even Morita–Rieffel equivalent to $C_0(\hat{G})$. 
The lack of uniqueness observed above can be overcome by restricting attention to \textit{maximal} relatively continuous subsets, that is, relatively continuous subsets that are not contained in any larger relatively continuous subset. Since \(I_x \subseteq C_0(\hat{G})\), the subspace \(\mathcal{M}_x\) cannot be maximal. However, even if we insist on maximality, we do not obtain uniqueness of the generalized fixed point algebra, because a continuous field is not yet determined by its dimension function.

If \((\mathcal{H}_x)_{x \in \hat{G}}\) is a continuous field of Hilbert spaces with \(\dim \mathcal{H}_x = n\) for all \(x \in \hat{G}\), then \((\mathcal{H}_x)\) “is” an \(n\)-dimensional complex vector bundle over \(\hat{G}\). That is, there is an \(n\)-dimensional complex vector bundle \(E\) over \(\hat{G}\) such that the space of continuous sections of \((\mathcal{H}_x)\) is isomorphic to \(C_0(\hat{G}, E)\) as a module over \(C_0(\hat{G})\). The corresponding square-integrable representation of \(G\) is \(L^2(\hat{G})\) because the dimension function is constant. Hence it only depends on the dimension \(n\). However, non-isomorphic vector bundles yield non-isomorphic Hilbert modules \(\mathcal{F}(\mathcal{E}, \mathcal{A})\) over \(C^*_r G\).

The generalized fixed point algebra associated to a vector bundle \(E \rightarrow \hat{G}\) is \(C_0(\hat{G}, \operatorname{End}(E))\). Especially, for the \(n\)-dimensional trivial vector bundle \(C^n\) we get

\[
C_0(\hat{G}, \operatorname{End}(\mathbb{C}^n)) \cong C_0(\hat{G}, \mathbb{M}_n).
\]

If \(E\) is a line bundle, then \(\operatorname{End}(E)\) is trivial. Hence the generalized fixed point algebras associated to vector bundles \(E\) and \(E'\) are isomorphic if \(E' \cong E \otimes L\) for a complex line bundle \(L\). The converse also holds: If the generalized fixed point algebras are isomorphic, then \(E\) and \(E'\) differ by tensoring with a line bundle. We leave the proof as an exercise in vector bundle theory for the interested reader. In particular, the generalized fixed point algebra is isomorphic to \(C_0(\hat{G}, \mathbb{M}_n)\) if and only if \(E \cong \mathbb{C}^n \otimes L = L \oplus L \oplus \cdots \oplus L\) is a direct sum of \(n\) copies of the same line bundle. This can be shown easily by studying the operation on \(\mathcal{E}\) of the matrix units in \(\mathbb{M}_n\).

Using the criterion above, it is not hard to find vector bundles for which the generalized fixed point algebra is not isomorphic to \(C_0(\hat{G}, \mathbb{M}_n)\). While this cannot happen for \(G = \mathbb{Z}\) because there are no non-trivial complex vector bundles on the circle, examples exist for \(G = \mathbb{Z}^2\). The corresponding dual group is the 2-torus \(\hat{G} \cong T^2\). The first Chern class yields an isomorphism from the set of isomorphism classes of line bundles over \(T^2\) to \(H^2(T^2, \mathbb{Z}) \cong \mathbb{Z}\). The multiplicativity of the total Chern character implies that \(c_1(L \oplus L) = c_1(L) + c_1(L)\) for any line bundle \(L\), so that \(c_1(L \oplus L)\) is divisible by 2. If we let \(L_0\) be the line bundle corresponding to a generator of \(H^2(T^2, \mathbb{Z})\) and let \(E := \mathbb{C} \oplus L_0\), then \(c_1(E) = c_1(\mathbb{C}) + c_1(L_0) = c_1(L_0)\) is not divisible by 2. We conclude that the generalized fixed point algebra associated to \(E\) is not isomorphic to \(C_0(\hat{G}, \mathbb{M}_2)\).
We claim that the relatively continuous subset \( R \) of \( L^2(G)^n \) associated to a vector bundle \((\mathcal{H}, G)\) over \( G \) is always maximal. Hence we cannot rule out the lack of uniqueness of \( \text{Fix}(L^2(G)^n, R) \) by requiring maximality. If \( R \subseteq R’ \) and \( R’ \) is relatively continuous, then \( \mathcal{F}(L^2(G)^n, R’) \) corresponds to a continuous field of Hilbert spaces \((\mathcal{H}, G)\) over \( G \). We have \( \mathcal{H} \subseteq \mathcal{H}’ \), for all \( x \) and \( \dim \mathcal{H}_x = n \) outside a set of measure zero. Lower semi-continuity implies that \( \dim \mathcal{H}_x = \dim \mathcal{H}_y = n \) for all \( x \). Therefore, \( (\mathcal{H}_x) = (\mathcal{H}_y) \) and hence \( R’ = R \).

Finally, we claim that no non-zero continuously square-integrable \( G \)-Hilbert space is ideal. Let \((\mathcal{H}, G)\) be a non-zero continuous field of Hilbert spaces over \( G \). Choose a non-zero continuous section \( f \in \mathcal{C}_0(G, (\mathcal{H})) \). Let \( U \subseteq G \) be an open set with \( f(s) \neq 0 \) for all \( s \in U \). There is a bounded, positive, measurable function \( \phi : G \to C \) whose restriction to \( U \) is not equal a.e. to a continuous function. Hence \( \phi^2 \|f\|^2 \) is not equal a.e. to a continuous function, so that the square-integrable section \( \phi \cdot f \) of \((\mathcal{H})\) is not continuous. Therefore, the operator of pointwise multiplication by \( \phi \) is not \( R \)-continuous, although it is equivariant and adjointable.

9. PROPER COEFFICIENTS

The last section shows that there are significant differences between continuously square-integrable, square-integrable, and arbitrary equivariant Hilbert modules for \( B = \mathbb{C} \). Nevertheless, if the group action on \( B \) is “sufficiently proper”, these differences disappear. That is, any Hilbert \( B \)-module is square-integrable and contains a unique dense, complete, relatively continuous subspace. This happens if \( B \) is proper in Kasparov’s sense and, more generally, if the induced group action on the (not necessarily separated) spectrum of \( B \) is proper.

**Definition 9.1.** Let \( X \) be a not necessarily separated topological space. Let \( G \) be a locally compact group and let \( G \times X \to X \) be a continuous action of \( G \) on \( X \). We call \( X \) a proper \( G \)-space iff for all \( x, y \in X \) there are neighborhoods \( U_x \) and \( U_y \) of \( x \) and \( y \) in \( X \) such that the set

\[
\{ g \in G \mid g(U_x) \cap U_y \neq \emptyset \} \subseteq G
\]

is relatively compact.

This definition is equivalent to Bourbaki’s definition [2, III.4.4]. We call \( K \subseteq X \) quasi-compact iff any open covering of \( K \) has a finite subcovering and relatively quasi-compact iff \( K \) is contained in a quasi-compact subset of \( X \).

The following lemma shows that Definition 9.1 contains the usual definition of proper actions on separated, locally compact spaces.
**Lemma 9.1.** Let $X$ be a not necessarily separated, proper $G$-space. Let $K, L \subseteq X$ be relatively quasi-compact. Then there are open neighborhoods $U_K$ and $U_L$ of $K$ and $L$, respectively, such that the set
\[
\{ g \in G \mid g(U_K) \cap U_L \neq \emptyset \} \subseteq G
\]
is relatively compact.

**Proof.** The proof is a straightforward exercise in topology. We may assume without loss of generality that $K$ and $L$ are quasi-compact, not just relatively quasi-compact. For $x \in K$, $y \in L$, there are open neighborhoods $U_{x,y}$ of $x$ and $y$ such that $g(U_{x,y}) \cap U_{x,y} = \emptyset$ for all $g$ outside a compact subset of $G$, because the action on $X$ is proper. By quasi-compactness, for fixed $x$ finitely many of the open sets $U_{x,y}$ cover $L$. Let $U_L^*$ be their union and let $U_K^*$ be the intersection of the corresponding $U_{x,y}^*$. Then $(U_L^*)_{x \in K}$ is an open covering of $K$. Finitely many of these sets suffice to cover $K$. Let $U_K^*$ be their union and let $U_L^*$ be the intersection of the corresponding open neighborhoods $U_L^*$ of $L$. These sets have the desired properties.

Let $B$ be a $G$-$C^*$-algebra and let $P$ be its primitive ideal space, equipped with the Jacobson topology and the continuous action of $G$ defined by $g \cdot p := \{ b \in B \mid b \in p \}$ for $g \in G$ and $p \in P$. It makes no difference to use the space of irreducible representations of $B$ instead because we only use the lattice of open subsets of $P$.

**Definition 9.2.** A $G$-$C^*$-algebra is called **spectrally proper** iff its primitive ideal space is a proper $G$-space.

We claim that proper $G$-$C^*$-algebras are spectrally proper. Let $X$ be a proper, locally compact $G$-space. By the Dauns-Hoffmann Theorem, the center of $\mathcal{M}(B)$ is isomorphic to $C_c(P)$. It follows that essential homomorphisms from $C_0(X)$ to the center of $\mathcal{M}(B)$ correspond to continuous maps $P \to X$ [8]. As a result, $B$ is a proper $G$-$C^*$-algebra iff there is an equivariant, continuous map $P \to X$ for a separated, locally compact, proper $G$-space $X$. This implies that $P$ is proper, as desired.

We remark that for proper $G$-$C^*$-algebras, generalized fixed point algebras are also defined by Kasparov [5, Section 3.2]. Using cut off functions, it is not hard to check that his construction is a special case of ours, that is, $B^G = \text{Fix}(B)$.

We recall some well-known facts about the primitive ideal space to fix our notation. If $b \in B$, $p \in P$, let $b_p$ be the image of $b$ in the quotient $B/p$. Open subsets $U \subseteq P$ correspond to closed ideals in $B$ via
\[
U \mapsto B_U := \bigcap_{p \in P \setminus U} p = \{ b \in B \mid b_p = 0 \text{ for all } p \in P \setminus U \}.
\]
We have $U_1 \subseteq U_2$ if and only if $B_{U_1} \subseteq B_{U_2}$. If $U_1, U_2 \subseteq P$ are relatively quasi-compact and open, the same holds for $U_1 \cup U_2$. Hence the family $\mathcal{C}$ of all relatively quasi-compact, open subsets of $P$ is directed. Therefore,

$$B_c := \bigcup_{U \in \mathcal{C}} B_U$$

is a $\ast$-ideal in $B$. This ideal is dense in $B$ because the sets

$$U_{b, t} := \{ p \in P \mid \| b_p \| > t \}$$

are open and relatively quasi-compact for all $b \in B$, $t > 0$. Functional calculus allows us to approximate $b$ by elements of $U_{b, t}$ with $t > 0$.

**Theorem 9.1.** Let $B$ be a spectrally proper $G$-C*-algebra and let $\mathcal{E}$ be a $G$-equivariant Hilbert module over $B$. Let $B_c$ be as above and let $\mathcal{E}_c := \mathcal{E} \cdot B_c$. Then $\langle \mathcal{E}_c \mid \mathcal{E}_c \rangle \subseteq C_c(G, B)$ and $\mathcal{E}_c$ is a dense, relatively continuous subspace of $\mathcal{E}$. In particular, $\mathcal{E}$ is square-integrable.

The completion $\mathcal{R}_0$ of $\mathcal{E}_c$ is the only dense, complete, relatively continuous subspace of $\mathcal{E}$. We have $\xi \in \mathcal{R}_0$ if and only if $\xi \in \mathcal{E}_c$ and $\langle \xi \mid \xi \rangle \in C^*_c(G, B)$. Any relatively continuous subset of $\mathcal{E}$ is contained in $\mathcal{R}_0$.

**Proof.** Let $U \subseteq P$ be open and relatively compact and let $\mathcal{E}_U := \mathcal{E} \cdot B_U$. By Cohen's Factorization Theorem, $\mathcal{E}_U$ is a closed linear subspace of $\mathcal{E}$. The subset

$$V := \{ g \in G \mid gU \cap U \neq \emptyset \} \subseteq G$$

is open and relatively compact by Lemma 9.1. We may write elements of $\mathcal{E}_U$ in the form $\xi \cdot b, \eta \cdot c$ with $\xi, \eta \in \mathcal{E}$, $b, c \in B_U$. Equation (20) yields

$$\langle \xi \cdot b \mid \eta \cdot c \rangle(g) = b^* \cdot \langle \xi \mid \eta \rangle(g) \cdot \beta_e(c) \in B_U \cdot B \cdot \beta_e(B_U) \subseteq B_{U \cap V}.$$ 

Hence $\langle \mathcal{E}_U \mid \mathcal{E}_U \rangle \subseteq C_c(V, B) \subseteq C_c(G, B)$. It follows that $\langle \mathcal{E}_c \mid \mathcal{E}_c \rangle \subseteq C_c(G, B)$. Since $B_c$ is dense in $B$, $\mathcal{E}_c$ is dense in $\mathcal{E}$. Proposition 6.5 yields $\mathcal{E}_c \subseteq \mathcal{E}_0$, so that $\mathcal{E}_c \subseteq \mathcal{E}$ is a dense, relatively continuous subspace and $\mathcal{E}$ is square-integrable.

Since $\| \langle \xi \rangle \| \leq \| \xi \|^2$, it follows that $\| \langle \xi \rangle \| \leq C_U \cdot \| \xi \|$ for all $\xi \in \mathcal{E}_U$ with some $C_U > 0$. Hence the norms $\| \cdot \|$ and $\| \cdot \|_u$ are equivalent on $\mathcal{E}_c$.

Let $\mathcal{R} \subseteq \mathcal{E}$ be a dense, complete, relatively continuous subspace. We claim that $\mathcal{R}$ contains $\mathcal{E}_c$, so that $\mathcal{R}_0 \subseteq \mathcal{R}$. Proposition 6.4 implies that $\mathcal{R} \cdot B_c \subseteq \mathcal{R}$. Since $\mathcal{R}$ is dense in $\mathcal{E}$, $\mathcal{R} \cdot B_c$ is dense in $\mathcal{E}_c$ with respect to the norm $\| \cdot \|$ and hence also with respect to the norm $\| \cdot \|_u$ because these two norms are equivalent on $\mathcal{E}_c$. Since $\mathcal{R}$ is complete, it follows that $\mathcal{E}_c \subseteq \mathcal{R}$. Hence $\mathcal{E}_c \subseteq \mathcal{R}$.
If \( \xi \in \mathcal{R}_0 \), then \( \xi \in \mathcal{R}_0 \) and \( \langle \xi | \zeta \rangle \in C^*_r(G, B) \) by Proposition 6.3. Assume conversely that \( \xi \in \mathcal{R}_0 \) and \( \langle \xi | \zeta \rangle \in C^*_r(G, B) \). We claim that \( \xi \in \mathcal{R}_0 \). Since \( B \subseteq B \) is a dense \(*\)-ideal, there is an approximate identity \((u_i)_{i \in I}\) for \( B \) with \( u_i \in B \) for all \( i \in I \). Thus \( \xi \cdot u_i \in \mathcal{R}_0 \subseteq \mathcal{R}_0 \) for all \( i \in I \). Let \( \mathcal{R} \subseteq \mathcal{R}_0 \) be the completion of \( \{ \xi \} \). Proposition 6.4 applied to \( \mathcal{R} \) yields that \( \xi \cdot u_i \to \xi \) in the norm \( \| \cdot \| \). Hence \( \xi \in \mathcal{R}_0 \) as asserted.

Therefore, any relatively continuous subset \( \mathcal{R} \subseteq \mathcal{E} \) is contained in \( \mathcal{R}_0 \).

**Corollary 9.1.** Let \( B \) be a spectrally proper \( G\)-C*-algebra. The functor

\[
\mathcal{F} \mapsto \mathcal{F} \otimes_{C^*_r(G, B)} L^2(G, B)
\]

is an equivalence between the C*-categories of Hilbert \( C^*_r(G, B) \)-modules and \( G \)-equivariant Hilbert \( B \)-modules. That is, any \( G \)-equivariant Hilbert module \( \mathcal{E} \) over \( B \) arises in this way for a unique Hilbert module \( \mathcal{F} \) over \( C^*_r(G, B) \), and the map \( \mathcal{B}(\mathcal{F}) \to \mathcal{B}(\mathcal{E}) \) is an isomorphism.

**Proof.** Let \( \mathcal{E} \) be a Hilbert \( B, G \)-module. By Theorem 9.1, there is a unique dense, complete, relatively continuous subset \( \mathcal{R} \subseteq \mathcal{E} \). Hence there is no difference between continuously square-integrable Hilbert \( B, G \)-modules and Hilbert \( B, G \)-modules. Theorem 6.1 shows that isomorphism classes of Hilbert modules over \( C^*_r(G, B) \) and Hilbert \( B, G \)-modules correspond to each other bijectively. Since \( \mathcal{R} \subseteq \mathcal{E} \) is unique, we have \( u(\mathcal{R}) = \mathcal{R} \) for all \( u \in \mathcal{B}(\mathcal{E}) \). Hence \( \mathcal{B}(\mathcal{F}) \cong \mathcal{B}(\mathcal{E}) \) by Corollary 5.1.

For proper \( G\)-C*-algebras, results similar to Corollary 9.1 have been known for some time. The statement closest to ours is due to Tu [12, Proposition 6.24]. Implicitly, Corollary 9.1 is used in the (so far unpublished) proof by Kasparov and Skandalis that the Baum–Connes assembly map with proper coefficients is surjective [6]. The first related result I could find is due to Kasparov [5, Section 3.2]. Besides defining a fixed point algebra \( B^\mathcal{E} \) for a proper \( G\)-C*-algebra \( B \), he also associates to a Hilbert \( B, G \)-module \( \mathcal{E} \) a Hilbert module \( \mathcal{E}^\mathcal{E} \) over \( B^\mathcal{E} \) and shows that \( \mathcal{B}(\mathcal{E}^\mathcal{E}) \cong \mathcal{B}(\mathcal{E})^\mathcal{E} \).

**REFERENCES**