Betti Numbers of Modules of Exponent Two over Regular Local Rings

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Let \((R, m, K)\) be a regular local ring of dimension \(n\) and let \(M\) be a finite length module over \(R\). This paper gives an affirmative answer to Horrocks’ questions when \(m^2M = 0\), that is, in this case the rank of the \(i\)th syzygy of \(M\) is at least \(\binom{n}{i-1}\) and the \(i\)th Betti number of \(M\) is at least \(\binom{n}{i}\).

\section*{INTRODUCTION}

Let \((R, m, K)\) be a regular local ring of dimension \(n\) and let \(M\) be a finite length module over \(R\). A well-known question asks: Must \(\beta_i(M)\), the \(i\)th Betty number of \(M\), be at least \(\binom{n}{i}\), where \(i\) is an integer between 0 and \(n\) (these numbers are achieved when \(M = K = R/m\))\? (The Betti numbers of \(M\) may be defined as the ranks of the free modules in a minimal free resolution \(F_i = 0 \to F_n \to \cdots \to F_0 \to 0\) for \(M\); here, \(H_i(F_i) = 0\) for \(i \geq 1\), \(H_0(F_i) \equiv M\), and the minimality means that \(\text{Im } F_{i+1} \subseteq mF_i\) for \(0 \leq i \leq n - 1\). From this it follows easily that the \(i\)th Betty number is simply \(\dim_K \text{Tor}_i^R(M, K)\)). In fact, in \([BE1]\) Buchsbaum and Eisenbud conjectured even more strongly than with \(M, R,\) and \(F\) as above, for \(1 \leq i \leq n\) the rank of the \(i\)th map \(F_i \to F_{i-1}\) in a minimal free resolution \(F\) of \(M\) (which is the same as the rank of the \(i\)th syzygy of \(M\)) is at least \(\binom{n}{i-1}\). An affirmative answer to this question will give an affirmative answer to the question above since the rank of \(F_i\) is the sum of the ranks of the maps \(F_{i+1} \to F_i\) and \(F_i \to F_{i-1}\). Although Buchsbaum and Eisenbud made the

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conjectures prior to Hartshorne’s problem list [Ha] in which Horrocks submitted the two questions, these questions are usually referred to as Horrocks’ questions or Horrocks’ conjectures. For more background and related problems, see [ChE]. There has been considerable study of these questions. See, for example, [A, BE1, BE2, Ca1, Ca2, Ch, ChEM, D, EG2, HeK, Ho, HuU, Sa].

By the exponent of $M$ we mean the smallest non-negative integer $t$ such that $m^tM = 0$. In this paper we will confirm Horrocks’ questions for the exponent two case.

**Theorem 0.1.** Let $M$ be a module of exponent two over a regular local ring. Then the rank of the $i$th syzygy of $M$ is at least $\binom{n+1}{i}$. In particular, the $i$th Betti number of $M$ is at least $\binom{n}{i}$. We give a brief description of Koszul complexes: the reader is referred to [Se] for further details. Because the Koszul complex on the $x_i$ gives a free resolution of $K$, it can be used to calculate the Betti numbers of $M$. If $x = x_1, \ldots, x_n$ is any sequence of elements of a ring $R$ the Koszul complex $K(x; R)$ may be defined as the tensor product, over $R$, of the $n$ length one Koszul complexes $K_i(x_i; R)$, each of which has the form $0 \to R \to R \to 0$, where the map from $K_i(x_i; R) = R$ to $K_j(x_j; R) = R$ is given by multiplication by $x_i$. Alternatively, one may identify $K(x; R)$ with a free module $G$ on $n$ generators $u_i$, where the differential $d$ maps $u_i$ to $x_i$ in $K(x; R) = R$, and then the entire complex may be identified with the exterior algebra $\wedge G$, where the map $d$ is extended to $\wedge G$ in the unique way that makes it a derivation of degree $-1$ (so that if $v \in \wedge^i G$ and $w \in \wedge^j G$ are homogeneous elements of $\wedge G$ of respective degrees $i$ and $j$, then $d(v \wedge w) = (dv) \wedge w + (-1)^i v \wedge (dw)$).

To simplify notation, if $\langle t \rangle$ denotes a sequence of integers $\langle t_1, \ldots, t_i \rangle$ of length $i$ in $\{1, \ldots, n\}$ then we write $u_{\langle t \rangle}$ for $u_{t_1} \wedge \cdots \wedge u_{t_i}$, and if $1 \leq j \leq i$ we write $\langle t \rangle - j$ for this sequence with its $j$th term omitted, a sequence of length $i - 1$. With these conventions we have the explicit formula

$$d_i(u_{\langle t \rangle}) = \sum_{j=1}^{i} (-1)^{i-j} x_j u_{\langle t \rangle - j}.$$  

We may then define $K(x; M)$ as $K(x; R) \otimes_R M$, and write $H_i(x; M)$ to denote $H_i(K(x; M))$. Since $K(x; R)$ is a free resolution of $K$ over $R$, we have $H_i(x; M) \cong \text{Tor}_i^R(M, K)$, and so $\beta_i(M) = \dim_K H_i(x; M)$.

Unless otherwise specified, tensor products, $\text{Hom}$, and direct sums are taken over $K$ for the rest of this introduction.

Let $x_1, \ldots, x_n$ be minimal generators of the maximal ideal $m$. Let $V$ be the $K$-vector space spanned by a set of pre-image of a basis of $M/mM$ ($V$
is essentially $M/mM$ and $W = mM$. Each $x_i$ induces a map $f_i$ from $V$ to $W'$ and kills $W$. We can make $V \oplus W$ into a module over $K[X_1, \ldots, X_n]$ by letting $X_i$ act by taking $V$ to $W$ by the map $f_i$ and by killing $W$. It is easy to see that $V \oplus W$ is actually gr $M$. We can now compare the original Koszul complex and the Koszul complex of gr $M$ (over gr $R$) with respect to the $X$'s. Let $d_i$ and $d'_i$ denote the differential maps for the Koszul complexes for $M$ and for gr $M$, respectively. The $k$-modules $W = mM$ and $M/mM$ can be seen both as $R$-modules and as gr $R$-modules. Since both the images of $d_{i+1}$ and of $d'_{i+1}$ are contained in the direct sum of $(r_i)$ copies of $mM$ and these are in turn contained in the kernels of $d_i$ and $d'_i$, it is easy to see that the two Koszul complexes have homology of the same length. Hence the Betti numbers of $M$ over $R$ are the same to the Betti numbers of gr $M$ over gr $R$. Thus, we may assume that $R = K[x_1, \ldots, x_n]$ is the polynomial ring of $n$ variables and $M = V \otimes_K W$ with its module structure complete determined by $f_1, \ldots, f_n \in \text{Hom}_K(V, W).

Let $U$ denote the $n$-dimensional $K$-vector space spanned by the generators $u_i$. Then $G \equiv U \otimes R$ and we may identify $\wedge G \equiv \wedge U \otimes R$. Hence $K(\langle x \rangle M)$ is isomorphic, as a $K$-vector space, with $(\wedge U \otimes R) \otimes_K M \equiv \wedge U \otimes M$. Thus the complex $K(\langle x \rangle M)$ may be identified with the vector space $\wedge^i U \otimes M = \wedge^i U \otimes (V \oplus W) = (\wedge^i U \otimes V) \oplus (\wedge^i U \otimes W)$. The differential $d_i$ kills $\wedge^i U \otimes W$ and has image contained in $\wedge^{i-1} U \otimes W$. Let $\phi_i$ denote the restriction of $d_i$ to $\wedge^i U \otimes V$, with the target restricted to $\wedge^{i-1} U \otimes W$. Then the map $\phi_i$ may be described directly in terms of $f_1, \ldots, f_n \in \text{Hom}_K(V, W)$ by the formula

$$\phi_i(u_{(i)} \otimes v) = \sum_{j=1}^{i} (-1)^{i-j} u_{(i)-j} \otimes f_j(v).$$

In this situation it is clear that we may identify the image of $d_{i+1}$ with $\text{Im} \phi_{i+1}$ and the kernel of $d_i$ with $(\text{Ker} \phi_i) \oplus (\wedge^i U \otimes W)$, from which we see that we may identify $H_i(x; M)$ with $(\text{Ker} \phi_i) \oplus (\wedge^i U \otimes W/\text{Im} \phi_{i+1})$. Let $\dim V = r$ and $\dim W = s$. Thus, the $i$th Betti number, $\beta_i(M) = \beta_i$, is simply $(\dim_K(\wedge^i U \otimes V) - \text{rank} \phi_i) + (\dim_K(\wedge^i U \times W) - \text{rank} \phi_{i+1}) = (\binom{n}{i} r + s) - (\text{rank} \phi_i + \text{rank} \phi_{i+1})$. For $1 \leq i \leq n$, let $\rho_i(M) = \rho_i$ denote the rank of the $i$th map in a minimal free resolution of $M$. Then $\rho_i = \sum_{j=1}^{n} (-1)^{i-j} \beta_j$. Applying the result above, we find that $\rho_i = (\sum_{j=1}^{n} (-1)^{i-j}) (r + s) - \text{rank} \phi_i = (\binom{n}{i-1} (r + s) - \text{rank} \phi_i$.

Hence, for fixed choices of $n, r, s$ we minimize both the Betti number of $M$ and the ranks of the maps in its minimal free resolution by choosing the maps $f_i$ in “general position,” i.e., the minimum values of the $\beta_i$ and of the $\rho_i$ will be achieved on a suitably small open subset of the affine space
of dimension \( nrs \) that parameterizes the choices of the \( f_j \). We may use as the parameters the entries of \( n \) matrices of size \( r \times s \) that give the \( f_j \). Since we may enlarge \( K \) without affecting the Betti numbers, the minimum Betti numbers are realized for example, by choosing matrices for all the \( f_j \) whose entries are \( nrs \) algebraically independent indeterminates over the prime field of \( K \). Thus, we are led to investigate the ranks of the linear maps \( \phi_i \) when the \( f_i \) are in sufficiently general position.

The conjectures that \( \beta_i(M) \geq \binom{r}{i} \) and \( \rho_i(M) \geq \binom{s-1}{i} \) are therefore equivalent to the conjectures that for all choices of the \( f_i \) (or when the \( f_i \) are in general position), \( \text{rank} \ \phi_i + \text{rank} \ \phi_{i+1} \leq \binom{r}{i} (r + s - 1) \) and \( \text{rank} \ \phi_i \leq \binom{r-1}{i} (r + s - 1) \), respectively. If we write \( a(n, i; r, s), 1 \leq i \leq n \), for the highest possible rank of \( \phi_i \) we have the following results.

**Proposition 0.2.** Let \( M \) be a module of exponent two over a regular local ring. If \( n \leq r + s - 1 \) then \( a(n, i; r, s) \leq \min(\binom{r}{i}, \binom{s-1}{i}) \leq \binom{s-1}{i}(r + s - 1) \), and in particular \( a(r + s - 1, i; r, s) = \min(\binom{r}{i}, \binom{s-1}{i}) \). If \( n > r + s - 1 \) then \( a(n, i; r, s) = a(n - 1, i; r, s) + a(n - 1, i - 1; r, s) \).

Hence Proposition 0.2 implies Theorem 0.1. It also enables us to find a formula for \( a(n, i; r, s) \) easily. Proposition 0.2 says that when \( r \) and \( s \) are fixed, \( a(n, i; r, s) \) is a completely increasing triangle which is stable after row \( r + s - 1 \) (see Definition 2.1). For example, we can write the triangle for \( \min(2\binom{r}{i}, 3\binom{s-1}{i}) \) up to row 5

\[
\begin{array}{cccccc}
& & & 2 & & \\
& & 2 & 3 & & \\
& 2 & 6 & 12 & 3 & \\
2 & 10 & 20 & 15 & 3 \\
\end{array}
\]

and this triangle coincides with the triangle for \( a(n, i; 2, 3) \) at row 4 (and row 5). We can easily fill in the rest of this triangle for \( a(n, i; 2, 3) \) since Proposition 0.2 tells us this triangle is stable after row 4. (It is suspected that this is the triangle for \( a(n, i; 2, 3) \), but this issue is not pursued in this paper since it is inconsequential.) We can also use Lemma 2.2 to conclude that \( a(n, i; 2, 3) \leq 2(\binom{r-1}{i}) + (\binom{s-1}{i}) + (\binom{r}{i}) + 3(\binom{s-1}{i}) \) and equality holds at least for \( n \geq 4 \). Thus when \( \dim M/MM = 2 \) and \( \dim mM = 3 \), the rank of the \( i \)th syzygy of \( M \) is at least \( \binom{r-1}{i} + 2(\binom{s-1}{i}) + 4(\binom{r}{i}) + (\binom{s-1}{i}) \) and the \( i \)th Betti number of \( M \) is at least \( \binom{r}{i} + 2(\binom{s-1}{i}) + 4(\binom{r}{i}) + (\binom{s-1}{i}) \).
The material in Section 1 is straightforward, but we devote a lot of space to set the notation. We will prove the first part of Proposition 0.2 there as well. In Section 2 we prove the second part of Proposition 0.2 but we leave the most tricky step to Section 3.

1

For each $i$, let $T_i$ be the set of the $i$-element subsets $t$ of $\{1, \ldots, n\}$. Let $\langle t \rangle$ be the increasing sequence of elements of $t$. Order $T_i$ according to the lexicographical order of $\langle t \rangle$. Define $u_t$ to be $u_{\langle t \rangle}$ (see the Introduction for definition). We will always use the ordered basis $(u_t)_{t \in T_i}$ for $K_i(x; R)$. Denote by $A_i^n(x_1, \ldots, x_n)$ the associated matrix of the differential map from $K_i(x; R)$ to $K_{i-1}(x; R)$ with respect to these bases. Our matrices act on the right, so that the co-kernel of the map represented by a matrix is obtained by killing the row space. For instance, we have

$$A_2^3(x_1, x_2, x_3) = \begin{pmatrix} -x_2 & x_1 & 0 \\ -x_3 & x_1 & \end{pmatrix}.$$}

In this fashion, it is easy to check that

$$A_i^n(x_1, \ldots, x_n) = \begin{pmatrix} -A_{i-1}^{n-1}(x_2, \ldots, x_n) & x_1I \\ 0 & A_i^{n-1}(x_2, \ldots, x_n) \end{pmatrix}, \quad (1.1)$$

where $I$ is the $(n-1) \times (n-1)$ identity matrix.

Suppose given two linear maps $B : S \to T$ and $C : S' \to T'$, where $B$ and $C$ are matrices relative to the ordered bases $(v_j)_j$, $(w'_j)_j$, and $(v'_j)_j$, $(w'_j)_j$, respectively. There is an induced map $B \otimes C : S \otimes S' \to T \otimes T'$. Relative to the ordered bases $(v_j \otimes v'_j, v_2 \otimes v'_2, \ldots, v_j \otimes v'_j)$ and $(w_j \otimes w'_j, w_2 \otimes w'_2, \ldots, w_j \otimes w'_j)$ the matrix $B \otimes C$ can be thought of as the block matrix obtained by replacing each entry $c$ of $C$ by the matrix $cB$. On the other hand, relative to the ordered bases $(v_1 \otimes v'_1, v_2 \otimes v'_2, \ldots, v_1 \otimes v'_1)$ and $(w_1 \otimes w'_1, w_2 \otimes w'_2, \ldots, w_2 \otimes w'_2)$ one gets instead the block matrix obtained by replacing each element $b$ of $B$ by $bC$.

Fix a basis for $V$ and $W$, respectively. Let the matrix of $f_i$ with respect to these bases be $\langle d_i \rangle_{i \times i}$. Note that the elements $d_i$ are in $K$. Here, we describe the matrices of the Koszul complex in question slightly differently. Write the matrix

$$A_i^n(x_1, \ldots, x_n) = x_1B(n, i, 1) + \cdots + x_nB(n, i, n).$$
where the \( B(n, i, j) \) are suitable matrices with entries in \( K \) (they are in fact 0, 1, or \(-1\)). Then the matrix of \( \phi_i \) which one obtains by tensoring the Koszul complex with \( M = V \otimes W \) is 
\[
\sum D \otimes B(n, i, 1) + \cdots + B(n, i, n).
\]
Hence, with one ordering of the bases the matrix of \( \phi_i \) can be easily checked to be \( A^n_i(D_1, \ldots, D_s) \), the block matrix obtained by replacing \( x_i \) by \( D \) and the 0’s by a suitable size zero matrix. With the other ordering of the bases, the matrix of \( \phi_i \) can be checked to be
\[
\begin{pmatrix}
A^n_i(D_{11}) & \cdots & A^n_i(D_{1s}) \\
\vdots & \ddots & \vdots \\
A^n_i(D_{r1}) & \cdots & A^n_i(D_{rs})
\end{pmatrix},
\]
where \( D_{jk} = (d_{jk}, \ldots, n d_{jk}) \) for \( j = 1, \ldots, r \) and \( k = 1, \ldots, s \).

**Definition 1.1.** If we stack the matrices \( D_1, \ldots, D_s \) together, they form an \( n \times r \times s \) cubical array \( D \). We can write \( D^j = (D_{jk})_{ik} \) and \( D^{+k} = (D_{jk})_{ij} \) for the slices of \( D \) from each respective direction. However, we will usually describe \( D \) by specifying \( D \) or \( D^{+k} \). We will write \( A^n_j(D) \) for the matrix in (1.2). We will write \( a(n, i; r, s) \) for the rank of the matrix \( A^n_j(D) \) or the rank of \( A^n_j(D_1, \ldots, D_s) \) (they are equal by the previous paragraph) when \( D \) is in general position, that is, when the \( d_{jk} \) in (1.2) can be considered distinct indeterminates. This will be the sharp upper bound for the ranks of \( \phi_i \) for all relevant \( M \).

Conversely, suppose given any \( n \times r \times s \) cubical array \( D \) with \( D \) as in the previous paragraph. Let \( V = K^r \) and \( W = K^s \). We can make \( V \oplus W \) into a \( K[x_1, \ldots, x_n] \)-module by letting \( x_i \) act by taking \( V \) to \( W \) by the matrix \( D \) (with respect to the standard basis) and by killing \( W \). We call this module \( M(D) \). The matrix of \( \phi_i \) for \( M(D) \) is still \( A^n_j(D) \).

When we choose a new set of minimal generators for \( m \), \( D \) is affected and we say we do an \( n \)-operation to \( D \). When we change the bases for \( V \) and \( W \), we say we do an \( r \)- and an \( s \)-operation, respectively, to \( D \). None of these operations will change the rank of the corresponding \( \phi_i \).

First, let’s consider the case \( s = 1 \). Assume \( D \) is in general position. In this situation, the cubical array \( D \) is basically just an \( n \times r \) matrix \( D_{+1} \). We can do \( n \)- and \( r \)-operations so that \( D_{+1} \) has 1 on the diagonal and 0 elsewhere. By substituting in this special set of values, it’s easy to compute \( a(n, i; r, 1) \). We can similarly compute \( a(n, i; 1, s) \).

**Definition 1.2.** If \( j > n \), let \( e^n_j \) be the zero \( n \)-vector. Otherwise, write \( e^n_j \) for the \( n \)-vector which has 1 as the \( j \)th entry and 0 elsewhere. The dimension \( n \) is often omitted when understood. Write \( e^n_{i+k} \) for the \( n \times r \times s \) cubical array \( D \) with \( D_{jk} = e^n_{i+k-1} \).
With this notation we have the following lemma.

**Lemma 1.3.** For any \( n, i, r, \) and \( s \), the following are true:

(a) \( a(n, i; r, 1) = \text{rank } A^r_i(\delta^s_{i-1}) = \binom{n-1}{r-1} + \binom{n-2}{r-2} + \cdots + \binom{n-s-1}{r-s-1} = \binom{n}{r-1} - \binom{n-s-1}{r-s-1} \)

(b) \( a(n, i; 1, s) = \text{rank } A^n_i(\delta^s_{i-1}) = \binom{n-1}{r-1} + \binom{n-2}{r-2} + \cdots + \binom{n-s-1}{r-s-1} = \binom{n}{r-1} - \binom{n-s-1}{r-s-1} \).

**Proof.** First note that \( a(n, i; 1, 1) = \binom{n-1}{r-1} = \binom{n}{r-1} - \binom{n-s-1}{r-s-1} \). By definition \( a(n, i; 1, s) \) is the rank of

\[
\begin{pmatrix}
0 & I_{(n-1)_1} & -A^{n-1}_{i-1}(e_1) & 0 & \cdots & -A^{n-1}_{i-1}(e_{s-1}) & 0 \\
0 & 0 & 0 & A^{n-1}_{i-1}(e_1) & \cdots & 0 & A^{n-1}_{i-1}(e_{s-1})
\end{pmatrix}
\]

Hence \( a(n, i; 1, s) = \binom{n-1}{r-1} + \text{rank } A^{n-1}_{i-1}(\delta^s_{i-1}) = \binom{n-1}{r-1} + a(n - 1, i; 1, s - 1) \) and (b) follows by induction on \( s \).

Similarly, we have \( a(n, i; r, 1) = \binom{n-1}{r-1} + a(n - 1, i - 1; r - 1, 1) \) and (a) follows from induction on \( r \).

From Lemma 1.3 we see that \( a(n, i; r, 1) = \binom{n}{r-1} \) (the full column rank) if and only if \( r \geq i \) and \( a(n, i; 1, 2) = \binom{n}{r} \) (the full row rank) if and only if \( s \geq n - i + 1 \). It turns out these relations between \( i, r, \) and \( s \) are very important.

**Lemma 1.4.** For any \( n, i, r, \) and \( s \) one has \( a(n, i; r, s) = a(n, n + 1 - i; s, r) \).

**Proof.** Let \( D \) be an \( n \times r \times s \) cubical array in general position and let \( K = K(\mathbf{x}; M(D)) \). Then it is easy to see \( \text{Hom}_R(K, M(D)) \) and \( K_{n-\mathbf{x}}(\mathbf{M}(D')) \) are isomorphic where \( D' \) is the cubical array such that \( (D')^k_{j\ell} = (D^k_{j\ell}) \). Since \( D' \) is an \( n \times s \times r \) cubical array in general position, the lemma follows.

**Lemma 1.5.** If \( i \leq r \) then \( a(r + s - 1; i, r, s) = \text{rank } A^{r+s-1}_{i-1}(\delta^s_{i-1}) = \binom{r+s-1}{i-1} \). If \( i \geq r \) then \( a(r + s - 1; i, r, s) = \text{rank } A^{r+s-1}_{i-1}(\delta^s_{i-1}) = \binom{r+s-1}{i-1} \). In both cases they have full ranks.

**Proof.** We prove the first half of the lemma by induction on \( r + s \). By Lemma 1.3 we will assume \( r, s \geq 2 \).
Let $D$ be an $n \times r \times s$ cubical array in general position. Notice that $A_i^{r+s-1}(D)$ is an $(r+s-1) \times (r+s-1) \times s$ matrix. Notice as well $(r+s-1)r - (r+s-1)s$ is a positive multiple of $r/i - s/(s + (r - i))$. If we assume $r \geq i$ then this is non-negative. This shows that $A_i^{r+s-1}(D)$ has at most as many columns as rows. It suffices to show that $A_i^{r+s-1}(\delta_{s,r}^{r+s-1})$ has full column rank.

Assume the induction hypothesis for $r+s-1$. By exchanging rows and columns, it is easy to see that $A_i^{r+s-1}(\delta_{s,r}^{r+s-1})$ has the same rank as the matrix

$$
\begin{pmatrix}
I_1 & \ast & 0 \\
0 & A_i^{r+s-2}(\delta_{r+1,s}^{r+s-2}) & 0 \\
\ast & 0 & A_i^{r+s-2}(\delta_{r+1,s}^{r+s-2})
\end{pmatrix},
$$

where $I_1$ is the $(r+s-2) \times (r+s-2)$ identity matrix. By induction, we can do row and column operations so that this matrix becomes

$$
\begin{pmatrix}
I_1 & \ast & 0 \\
0 & I_2 & 0 \\
0 & 0 & I_3 \\
\ast & 0 & I_3 \\
\ast & 0 & 0
\end{pmatrix},
$$

where $I_2$ and $I_3$ are the identity matrices of size $(r+s-2)\times(s-1)$ and $(r+s-2)(s-1)$, respectively. The third and the last row-blocks might not be there in case $A_i^{r+s-2}(\delta_{r+1,s}^{r+s-2})$ or $A_i^{r+s-2}(\delta_{r+1,s}^{r+s-2})$ is a square matrix. It is obvious that this matrix has full column rank.

We can prove the second half of the lemma similarly, but it also follows easily from Lemma 1.4 and the first half of the lemma.

**Corollary 1.6.** If $i \leq r$ or $n - i + 1 \leq s$ then $a(n, i; r, s) \leq \min\binom{n}{r}, \binom{n}{i-1}s$ and equality holds for $n \geq r + s - 1$.

**Proof.** It suffices to check for $n > r + s - 1$ that the equality holds. When $n \geq r + s - 1$ and $i \leq r$, we know by Lemma 1.5 that $a(n, i; n - r - 1) = \binom{n}{i-1}(n - r - 1)$, which is the full column rank. Since by assumption $n - r + 1 \geq s$, we must have $a(n, i; r, s) = \binom{n}{i-1}s$. The other case follows similarly.
Lemma 1.7. If $i \leq r$ or $n - i + 1 \leq s$ then $a(n, i; r, s) \leq \binom{n-1}{i-1}(r + s - 1)$.

Proof. We will demonstrate the case for $i \leq r$. The other case follows similarly or from Lemma 1.4.

Let $c = \binom{n-1}{i-1}(r + s - 1) - \min\binom{i}{r}, \binom{n}{i+1}s$. We need to check that $c$ is nonnegative. By straightforward computation, we can see that $c$ is nonnegative. If $\binom{i}{r} \geq \binom{n}{i+1}s$ if and only if $r/i \geq s/(n - i + 1)$. If $\binom{i}{r} \geq \binom{n}{i+1}s$ then $c$ is a positive multiple of $(r - 1)/(i - 1) - s/(n - i + 1)$, which is no less than $r/i - s/(n - i + 1)$ since $r/i \geq 1$. Hence $c$ is non-negative. If $\binom{i}{r} \leq \binom{n}{i+1}s$ then $c$ is a positive multiple of $(s - 1)/(n - i) - r/i \geq s/(n - i + 1) - r/i \geq 0$. We see that $c$ is again non-negative.

To see the first half of Proposition 0.2 is true, note that when $n \leq r + s - 1$ either $i \leq r$ or $n - i + 1 \leq s$. We can then apply Corollary 1.6 and Lemma 1.7. It remains to compute $a(n, i; r, s)$ for $n > r + s - 1$. We will develop the general formula in the next section.

2

Definition 2.1. A triangle $T$ is a collection of real numbers $T(n, i)$ where $n, i \in \mathbb{Z}$ and there exists $n_0 \in \mathbb{Z}$ such that $T(n, i) = 0$ unless $n \geq n_0$ and $n_0 \leq i \leq n$. The sequence $(T(n, i))$, is the $n$th row of $T$. The first row whose elements are not all zero is called the initial row of $T$. We say that the triangle $T$ is increasing after row $N$ when one has $T(n, i) \geq T(n, i) + T(n, i - 1)$ for $n \geq N$. We will also say that the triangle $T$ is stable after row $N$ when one has $T(n + 1, i) = T(n, i) + T(n, i - 1)$ for $n \geq N$. The triangle $T$ is called completely increasing (or completely stable) if $T$ is increasing (or stable respectively) after the initial row.

For instance, the Pascal triangle $P(n, i) = \binom{n}{i}$ is a completely increasing and completely stable triangle whose initial row is the zeroth row. A more important example is the triangle $a(n, i; r, s)$ if we fix $r$ and $s$. This triangle has the first row as its initial row.
Lemma 2.2. Suppose given a triangle $T$ and a fixed integer $N$. Then

$$T(n, i) = \sum_{k=0}^{n-N} T(N, i - k) \binom{n - N}{k}$$

$$+ \sum_{m=N+1}^{n} \sum_{k=0}^{n-m} [T(m, i - k) - T(m - 1, i - k)$$

$$- T(m - 1, i - k - 1)] \binom{n - m}{k} \tag{2.1}$$

for all $n \geq N$.

Proof. We will prove this lemma by induction on $n$. The statement holds obviously when $n = N$. Now we assume the induction hypothesis for $n - 1$. We will write $T'(m, j) = T(m, j) - T(m - 1, j) - T(m - 1, j - 1)$ to shorten terminology. Thus

$$T(n, i) = T(n - 1, i) + T(n - 1, i - 1) + T'(n, i) \binom{n - n}{0}$$

$$= \sum_{k=0}^{n-1} T(N, i - k) \binom{n - 1 - N}{k}$$

$$+ \sum_{m=N+1}^{n-1} \sum_{k=0}^{n-1-m} T'(m, i - k) \binom{n - 1 - m}{k}$$

$$+ \sum_{k=0}^{n-1-N} T(N, i - 1 - k) \binom{n - N - 1}{k}$$

$$+ \sum_{m=N+1}^{n-1} \sum_{k=0}^{n-1-m} T'(m, i - 1 - k) \binom{n - m - 1}{k}$$

$$+ T'(n, i) \binom{n - n}{0}$$

$$= \sum_{k=0}^{n-N} T(N, i - k) \binom{n - N}{k}$$

$$+ \sum_{m=N+1}^{n-1} \sum_{k=0}^{n-m} T'(m, i - k) \binom{n - m}{k} + T'(n, i) \binom{n - n}{0}$$

$$= \sum_{k=0}^{n-N} T(N, i - k) \binom{n - N}{k}$$

$$+ \sum_{m=N+1}^{n} \sum_{k=0}^{n-m} T'(m, i - k) \binom{n - m}{k}.$$
Corollary 2.3. If $T$ is stable after row $N$ then $T(n, i) = \sum_{k=0}^{n-N} T(N, i-k)(n-k)^N$ holds for all $n \geq N$. Conversely, if $T$ is a triangle which is increasing after row $N$ and if there exists $M \geq N$ such that $T(n, i) = \sum_{k=0}^{n-M} T(N, i-k)(n-k)^N$ holds for all $n \geq M$ then $T$ is also stable after row $N$.

In particular, if two completely increasing triangles agree at row $N$ and after row $M$ for some $M \geq N$ and if one of the triangles is stable after row $N$ then so is the other.

Proof. The first statement is obvious. For the second statement, notice that since $T$ is increasing after row $N$, each summand in the second term of (2.1) is nonnegative. For the hypothesis to be true, we should have $T(m, i-k) - T(m-1, i-k) - T(m-1, i-k-1) = 0$ for all $m \geq N + 1$ and for all $i$ and $k$. Hence $T$ is stable after row $N$.

Lemma 2.4. Fix $r$ and $s$. The triangle $a(n, i; r, s)$ is completely increasing.

Proof. It is obvious that the first row of $a(n, i; r, s)$ is its initial row. Let $n \geq 1$ and let $D$ be the $(n+1) \times r \times s$ cubical array in general position. Then

$$a(n+1, i; r, s)$$

$$= \text{rank } A_{i+1}^{n+1}(1_{D}, 2_{D}, \ldots, n_{D})$$

$$\geq \text{rank } A_{i+1}^{n+1}(0_{D}, 2_{D}, \ldots, n_{D})$$

$$= \text{rank } \left( \begin{array}{cc}
-A_{i-1}^{n}(1_{D}, \ldots, n_{D}) & 0 \\
0 & A_{i}^{n}(1_{D}, \ldots, n_{D})
\end{array} \right)$$

$$= a(n, i-1; r, s) + a(n, i; r, s).$$

Definition 2.5. Let $N \geq rs$. Define $\Delta_{rs}^n$ to be the cubical array $D$ with $D_{jk} = e_{j+(k-1)r}$.

We have the following important lemma.

Lemma 2.6. If $n \geq rs$ then $a(n, i; r, s) = \text{rank } A_{i}^{n}(\Delta_{rs}^{n})$. In particular, this implies that $a(n+1, i; r, s) = a(n, i; r, s) + a(n, i-1; r, s)$ for $n \geq rs$. In other words, the triangle $a(n, i; r, s)$ is stable after row $rs$.

Proof. Let $D$ be the $n \times r \times s$ cubical array in general position and assume $n \geq rs$. Let $D'$ be the $n \times r \times s$ cubical array obtained by doing $n$-operations to $D$ so that the $n \times r$ matrix $D'_{s \times 1}$ has the identity matrix on the rows from row 1 to row $r$ and zero elsewhere. We know that the rank of $A_{i}^{n}(D')$ stays maximal. Let $D''$ be the cubical array with $D''_{s \times 1} = D'_{s \times 1}$ and $D''_{s \times k}$, $2 \leq k \leq s$, in the general position. The cubical array is more specialized than $D$, but still more generalized than $D'$. Hence the rank of
$A^n_r(D')$ is still maximal. We may assume $D = D''$. Continue doing $n$-operations so that the $n \times r$ matrix $D_{s2}$ has the identity matrix on the rows from row $r + 1$ to row $2r$ and zero elsewhere while the first layer still retains its special form. Again, we may assume that $D_{s1}$ and $D_{s2}$ have the special form, while $D_{s k}$, $3 \leq k \leq s$, are in general position. Continue doing so; it can be achieved that $D_{s k}$, $1 \leq k \leq s$, has the identity matrix on the rows from row $(k - 1)r + 1$ to row $kr$ and zero elsewhere. Hence the first half of the lemma is true.

In particular, if $n \geq rs$ we should have $x_{n+1}$ kills $M(\Delta_{rs}^{n+1})$, and hence $n+1D$ is the zero matrix. Exchanging $x_1$ and $x_{n+1}$, we may assume $D = 0$. The result now follows using the same argument as in Lemma 2.4.

**Remark.** The reader may notice that the triangle $a(n, i; r, s)$ is stable after row $rs$ for a simpler reason. The dimension of $\text{Hom}_k(V, W)$ over $k$ is $rs$. Thus if $n > rs$ after a change of bases we may assume one of the linear forms acts like 0 in which case the minimal free resolution comes from the lower dimension via a mapping cone.

**Definition 2.7.** An index diagram $\mathbf{M}$ is a matrix with non-negative integer entries. We often need to consider a cubical array $D$ with exactly one non-zero entry, say the $l_{jk}$th entry, in $D_{jk}$ for each pair $(j, k)$. In this situation we shall call the matrix $(l_{jk})_{r \times s}$ the index diagram of $D$.

Conversely, given $n$ and given an $r \times s$ index diagram $\mathbf{M} = (l_{jk})$ we call the $n \times r \times s$ cubical array $D$ with $D_{jk} = e_{l_{jk}}^n$ the $n$th associated cubical array of the index diagram $\mathbf{M}$, which we denote by $\Delta_{\mathbf{M}}^n$.

Looking closely at the proof of Lemma 1.5, we can see the main point is the equality

$$a(r + s - 1, i; r, s) = \binom{r + s - 2}{i - 1} + a(r + s - 2, i; r, s - 1)$$

$$+ a(r + s - 2, i - 1; r - 1, s). \quad (2.2)$$

This equality suggests a way to find a basis for the column space of $A^n_r(\Delta_{rs})$. Assume $n \geq rs$ and write $a_t = a(n, i; r, s)$. We know that by Lemma 2.6 the rank of the matrix associated with the cubical array $X^n_{rs} = X$ with $X_{jk} = x_{j+(k-1)r}e_{l_{jk}}^{n+(k-1)r}$ is $a_r$. (This seems to be a terrible choice of notation since the $x$ are also used for the minimal generators for the maximal ideal $m$. I am still using these because I am accustomed to mentally picturing the matrix in the form of (1.1). This is probably not a very good reason but fortunately the $x$ are algebraically independent over $K$ in the polynomial ring $K[x_1, \ldots, x_r]$.) This says that one of the $a_r$-minors
of $A^n(X)$ is not zero. Hence we can pick $a_i$ entries from $A^n(X)$ such that no two of them lie on the same row or on the same column to form a non-vanishing term to the corresponding $a_i$-minor. For instance, when $s = 1$ we have

$$A^n_1(X) = \begin{pmatrix}
A^n_1(X_{11}) \\
A^n_1(X_{12}) \\
\vdots \\
A^n_1(X_{1r})
\end{pmatrix} = \begin{pmatrix}
A^n_1(x_1, 0, 0, \ldots, 0) \\
A^n_1(0, x_2, 0, \ldots, 0) \\
\vdots \\
A^n_1(0, \ldots, 0, x_r, \ldots, 0)
\end{pmatrix}.
$$

We can choose all the $x_1$ in $A^n_1(X_{11})$, then we choose all the $x_2$ that are choosable (that is, not already on the same row or column as the ones already picked) in $A^n_1(X_{12})$, then all the choosable $x_3$ in $A^n_1(X_{13})$, and so on. This gives the term

$$x_1^{i_1 - 1} \cdots x_r^{i_r - 1},$$

which is a non-vanishing term of the corresponding $a_i$-minor (cf. Lemma 1.3).

**Definition 2.8.** For any $n, r, and s$, choose all the $x_1$ in $A^n_1(X_{11})$ (the $(1,1)$-block of $A^n(X)$), then choose all the choosable $x_2$ in $A^n_1(X_{21})$ (the $(2,1)$-block of $A^n(X)$), and continue doing so until we have chosen all the choosable $x_{rs}$ in $A^n_1(X_{rs})$ (the $(r,s)$-block of $A^n(X)$). Let the number of entries thus chosen be $b(n, i; r, s)$. We will call the $b(n, i; r, s)$ columns where the chosen elements belong to the standard columns of the matrix $A^n_1(X^n)$ (and of $A^n_1(\Delta^n)$ after they are specialized).

More generally we can define the standard columns with respect to an arbitrary index diagram. Suppose given an $r \times s$ index diagram $M = (l_{jk})$. We choose in the $(1,1)$-block the entries at the position of $x_{i,1}$ in $A^n_1(x_1, \ldots, x_n)$, then we choose in the $(2,1)$-block all the choosable entries at the position of $x_{i,2}$, and so on. The columns these chosen entries belong to are called standard with respect to $M$.

**Remark.** By permuting the generators for the maximal ideal $m$ it is easy to see that as long as all the entries of an $r \times s$ index diagram $M$ are distinct, the number of the standard columns with respect to $M$ is still $b(n, i; r, s)$.

Let $n \geq rs$. Note that $b(n, i; r, s) = a(n, i; r, s)$ when $r = 1$ or $s = 1$. Note as well that in the $b(n, i; r, s)$-minor of $A^n_1(X)$ containing these chosen entries no other terms can cancel the term given by these entries. This says that $a(n, i; r, s)$ is at least $b(n, i; r, s)$ and that the standard columns of $A^n_1(X)$ are linearly independent.
To compute $b(n, i; r, s)$, note that after choosing all the $x_i$ in $A^n_i(x_1, \ldots, x_n)$ (there are $\binom{n}{i}$ of them) the rest of the chosen entries come from $A^{n-1}_i(x_1, \ldots, x_n)$ with respect to $(j + (k - 1)r)_{1 \leq j \leq r, 2 \leq k \leq s}$ and from $A^{n-1}_{i-1}(x_1, \ldots, x_n)$ with respect to $(j + (k - 1)r)_{2 \leq j \leq r, 1 \leq k \leq s}$. Hence

$$b(n, i; r, s) = \binom{n-1}{i-1} + b(n - 1, i; r, s - 1) + b(n - 1, i - 1; r - 1, s)$$  \hspace{1cm} (2.3)

holds for $n \geq rs$. We now venture to give a formula for $b(n, i; r, s)$

**Lemma 2.9.** Let $n \geq rs$. Then

$$b(n, i; r, s) = \sum_{k=0}^{n-r-s+1} a(r + s - 1, i - k; r, s) \binom{n-r-s+1}{k}.$$ \hspace{1cm} (2.4)

**Proof.** When $r = 1$ or $s = 1$ we know that $b(n, i; r, s) = a(n, i; r, s)$. Since in this case $a(n, i; r, s)$ is stable after row $r + s - 1$ by Lemma 1.3, the lemma follows from Corollary 2.3 with $N = r + s - 1$. We may assume $r, s \geq 2$. First by applying Corollary 2.3 to the Pascal triangle we have

$$\binom{n-1}{i-1} = \sum_{k=0}^{n-r-s+1} \binom{r+s-2}{i-k-1} \binom{n-r-s+1}{k}.$$ \hspace{1cm} (2.5)

Now by induction on $r + s$ and by using (2.3), (2.2), and (2.5) we have

$$b(n, i; r, s) = \binom{n-1}{i-1} + \sum_{k=0}^{n-r-s+1} a(r + s - 2, i - k; r, s - 1)$$

$$\times \binom{n-r-s+1}{k}$$

$$+ \sum_{k=0}^{n-r-s+1} a(r + s - 2, i - k - 1; r - 1, s)$$

$$\times \binom{n-r-s+1}{k}$$

$$= \sum_{k=0}^{n-r-s+1} a(r + s - 1, i - k; r, s) \binom{n-r-s+1}{k}.$$ \hspace{1cm} \[\blacksquare\]

Of course, at this point $b(n, i; r, s)$ is defined only for $n \geq rs$. But if we use (2.4) to define $b(n, i; r, s)$ for $n \geq r + s - 1$ and let $b(n, i; r, s) = 0$
otherwise, Lemma 2.9 says that the triangle \( a(n, i; r, s) \) and the triangle \( b(n, i; r, s) \) agree at row \( r + s - 1 \), and that the triangle \( b(n, i; r, s) \) is stable after row \( r + s - 1 \). For the rest of the paper we will show that \( a(n, i; r, s) = b(n, i; r, s) \) for \( n \geq rs \). Corollary 2.3 tells us the stability property of \( b(n, i; r, s) \) would force \( a(n, i; r, s) \) to be stable after row \( r + s - 1 \), which is the second half of Proposition 0.2 (see Corollary 2.14).

To achieve this, we will show that for \( n \geq rs \) the standard columns of \( \mathcal{A}_{n}^{\mu}(\Delta_{rs}^{\mu}) \) span the column space of \( \mathcal{A}_{n}^{\mu}(\Delta_{rs}^{\mu}) \), but first we need to know how to distinguish between standard and non-standard columns.

**Definition 2.10.** The rows and columns of \( \mathcal{A}_{n}(x_1, \ldots, x_n) \) can be labeled by \( i \)-element and \( (i-1) \)-element subsets \( s_i \) and \( t_i \) of \( \{1, \ldots, n\} \), respectively. We will simply write \( s \) and \( t \) when \( n \) is understood. The \((s, t)\)-entry is non-zero if and only if \( t \) is a subset of \( s \). In the row of \( s \) only those \( x_i \) with \( l \in s \) would appear, and in the column of \( t \) only those \( x_i \) with \( l \notin t \) would appear. Denote by \( v_t \) the column of \( t \) in \( \mathcal{A}_{n}(x_1, \ldots, x_n) \). We call an element in \( t \) a label index (of \( v_t \)).

Denote by \( v_t(l_1, \ldots, l_r) \) the corresponding column in \( \mathcal{A}_{n}^{\mu}(\Delta_{rs}^{\mu}) \) where \( M_0 \) is the transpose of \((l_1, \ldots, l_r)\). When \( M = (l_{jk}) \) is given, we use \( v_t[k] \) (or simply \( v_t[k] \)) to denote the column \( v_t(l_{1k}, \ldots, l_{rk}) \) in \( \mathcal{A}_{n}^{\mu}(\Delta_{rs}^{\mu}) \).

We will see that the positions of the label indices in the index diagram will indicate if a column is standard or not.

Let \( M \) be the index diagram of \( \Delta_{rs}^{\mu} \). Suppose \( r = 1 \). If \( k \) is a label index, then \( v_t[k] = 0 \) is non-standard. We may assume \( k \) is not a label index. If \( v_t[k] \) is non-standard, then a \( x_{k'} \) is already chosen for some \( k' < k \). Since \( x_{k} \) appears in the row of \( s \) where \( s = t \cup \{k\} \), this implies \( k' \) is a label index. On the other hand, if \( k' \) is the smallest label index and \( k' < k \), then \( v_{t \cup \{k\} \setminus \{k'\} \setminus \{k\}} \) is standard since no \( x_{k''} \) with \( k'' < k' \) lie in the same row as \( x_{k'} \). The fact that \( x_{k' \setminus \{k\}} \) is standard forces \( v_t[k] \) to be non-standard. Hence \( v_t[k] \) is non-standard if and only if the smallest label index of \( v_t[k] \) is at most \( k \). The situation is even more transparent when \( s = 1 \). With a similar argument it is easy to see that \( v_t[1] \) is non-standard if and only if \( 0 \) if and only if \( 1, \ldots, r \) are all label indices. In both situations we can see the conditions in the following definition are satisfied.

**Definition 2.11.** We say the \( k_0 \)-position of \( t \) is singular in a given index diagram if and only if

(i) there is at least one label index in each row of the index diagram,

(ii) the farthest left label index in each row (they are called the principal label indices) can only lie right below or to the right of the one in the previous row, and cannot lie to the right of the \( k_0 \) column.
We will see that the principal label indices indicate what preempt the column in question from becoming a standard column.

**Lemma 2.12.** Let $\mathbf{M}$ be an $r \times s$ index diagram with distinct non-zero entries. The column $\nu_{\mathbf{t}}(k_0)$ is non-standard in $A^n_i(\Delta^n_M)$ if and only if the $k_0$-position of $\mathbf{t}$ is singular in the index diagram $\mathbf{M}$.

**Proof.** We will write $\mathbf{t} - 1 = (t - 1) : t \in \mathbf{t}$.

Let $\mathbf{M} = (l_{jk})_{r \times s}$. We will prove this lemma by induction on $r + s$. With some minor modification the argument above works for the case $r = 1$. Hence we may assume $r, s \geq 2$. For $k_0 \leq s$, the column $\nu_{\mathbf{t}}(k_0)$ is non-standard in $A^n_i(\Delta^n_M)$ if and only if $\nu_{\mathbf{t}}(k_0)$ is non-standard in $A^n_i(\Delta^n_M)$ where $\mathbf{M}' = (l_{jk})_{r \times k_0}$. Thus we may assume $k_0 = s > 1$. Since we may permute the minimal generators in the maximal ideal, we may assume $l_{1s} = 1$.

Suppose $1$ is not a label index. Since $l_{js} \neq 1$, by (1.1) we have

$$\nu_{\mathbf{t}}(l_{js}) = \begin{pmatrix} 0^{s-1} \\ \nu_{(t-1)s-1}(l_{js} - 1) \end{pmatrix}$$

for each $j$. One can see that $\nu_{\mathbf{t}}(l_{js})$ is non-standard if and only if $\nu_{(t-1)s-1}(l_{js} - 1)$ is non-standard in $A^n_{i-1}(\Delta^{n-1}_{M_1})$ with $M_1 = (l_{i,k+1 - 1})_{r \times (s-1)}$. This happens if and only if the $(s-1)$-position of $\mathbf{t} - 1$ is singular in $M_1$ by induction if and only if the $s$-position of $\mathbf{t}$ is singular in $\mathbf{M}$.

If $1$ is a label index, then by (1.1) again we have

$$\nu_{\mathbf{t}}(l_{js}) = \begin{pmatrix} \nu_{(t \setminus 1)s-1}(l_{js} - 1) \\ 0^{s-1} \end{pmatrix}$$

for each $j$. Again, one can see that $\nu_{\mathbf{t}}(l_{js})$ is non-standard if and only if $\nu_{(t \setminus 1)s-1}(l_{js} - 1)$ is non-standard in $A^n_{i-1}(\Delta^{n-1}_{M_2})$ with $M_2 = (l_{i,j+1 - 1})_{(r-1) \times s}$. By induction this happens if and only if the $s$-position of $\mathbf{t} \setminus 1$ is singular in $M_2$ if and only if the $s$-position of $\mathbf{t}$ is singular in $\mathbf{M}$.

We devote Section 3 to prove the following proposition.

**Proposition 2.13.** Let $n \geq rs$. The standard columns of $A^n_i(\Delta^n_M)$ span the column space of $A^n_i(\Delta^n_M)$. Hence $b(n, i; r, s) = a(n, i; r, s)$ for $n \geq rs$.

The following is the second half of Proposition 0.2.

**Corollary 2.14.** For $n \geq r + s - 1$ we have $a(n + 1, i; r, s) = a(n, i; r, s) + a(n, i - 1; r, s)$. In other words, the triangle $a(n, i; r, s)$ is stable after row $r + s - 1$.
Proof. By Proposition 2.13, the triangle \( a(n, i; r, s) \) and the triangle \( b(n, i; r, s) \) agree after row \( rs \), and by Lemma 2.9 agree at row \( r + s - 1 \). The lemma now follows Corollary 2.3.

3

We will prove Proposition 2.13 by induction on \( r + s \). We know this result is true when \( r \) or \( s = 1 \), so we may assume \( r, s \geq 2 \). By induction \( v_t[k] \) is in the space spanned by the standard columns of \( A^n_t(\Delta^n_{rs}) \) for all \( t \) and \( k < s \). It suffices to check only the non-standard columns \( v_q[s] \).

If \( i \leq r \) then by Definition 2.10 and Lemma 2.12, all the columns of \( A^n_t(\Delta^n_{rs}) \) are standard. We may assume \( i \geq r + 1 \). We first concentrate on the case \( i = r + 1 \). In this situation there are exactly \( r \) label indices, so for a nonstandard column there is exactly one label index in each row of the index diagram.

**Definition 3.1.** Suppose given an \( r \times s \) index diagram \( M = (l_{jk}) \) with distinct non-zero entries and given \( t = (t_1, \ldots, t_s) \) such that in each row of \( M \) there is exactly one element of \( t \). Define \( [t] \) to be the sequence \([k_1, \ldots, k_s] \) such that \((l_{1k_1}, \ldots, l_{sk_s}) = t\).

By consulting the index diagram one can easily recover \( t \) from \([t]\) and vice versa. The set \( t \) gives the actual label indices, but the sequence \([t]\) gives the “addresses” of the label indices in the index diagram. With this notation we can see that the \( k \)-position of \( t \) is singular if and only if \( k_1 \leq k_2 \cdots \leq k_s \leq k \).

We describe the sign of \( x_t \) in the \( v_t \) of \( A^n_t(x_1, \ldots, x_n) \). If \( l \in t \) then \( x_l \) does not appear in this column at all. But if \( l \notin t \) then by our construction in Section 1, the sign is positive if and only if the number of the label indices less than \( t \) is even:

**Definition 3.2.** Suppose given the index diagram \( M = (l_{jk}) \). Define \( v_t[j_0, k_0] \) to be \( v_t(l_{jk_0}) \), the corresponding column in the \((j_0, k_0)\)-block of \( A^n_t(\Delta^n_{rs}) \).

With this definition, we have

\[
v_t[k_0] = \begin{pmatrix}
v_t[1, k_0] \\
\vdots \\
v_t[r, k_0]
\end{pmatrix}
\]

Let \( M \) be an index diagram with distinct non-zero entries. Suppose \( t \) is such that \([t] = [k_1, \ldots, k_s] \). Then \( v_t[j_0, k_0] = 0 \) if and only if \( k_{j_0} = k_0 \).
When $k_{j_0} \neq k_0$ the column vector $v_t[j_0, k_0]$ has exactly one non-zero entry 1 or $-1$. When $M = \Delta^\circ_{s^n}$, the sign is positive if and only if the number of $j$ such that $k_j < k_0$ or $k_j = k_0$ with $j < j_0$ is even.

**Definition 3.3.** Suppose $t$ is such that $[t] = [k_1, \ldots, k_r]$. Define $\pi_t = \pi \in S_r$ to be such that:

(i) $k_{\pi(1)}$ are non-decreasing, and
(ii) if $k_{j_1} = k_{j_2}$ and $j_1 < j_2$ then $\pi(j_1) < \pi(j_2)$.

**Definition 3.4.** Let $(k_i)$ be a sequence. Define the *absolute set* of $(k_i)$, denoted $(k_i)$, by disregarding the order in the sequence $(k_i)$ while maintaining the frequency.

**Lemma 3.5.** Suppose given a non-standard column $v_{k_i}[s]$ in $A_{r+1}^n(\Delta^s_{r^n})$ with $[t_0] = [k_1, \ldots, k_r]$. Let $T_k = \{ t: \lvert [t], k \rvert = \lvert [t_0], s \rvert \}$. Then

$$
\sum_{k=1}^s \sum_{t \in T_k} \text{sg } \pi_t v_t[k] = 0.
$$

*Proof.* It suffice to check for each $j_0$ with $1 \leq j_0 \leq r$ one has

$$
\sum_{k=1}^s \sum_{t \in T_k} \text{sg } \pi_t v_t[j_0, k] = 0.
$$

Let $[t] = [k_1, \ldots, k_r] \in T_k$ and let $k_j = k'$. If $k = k'$ then $v_t[j_0, k] = 0$. We may assume $k \neq k'$. Let $t'$ be such that $[t'] = [k_1, \ldots, k_{j_0-1}, k, k_{j_0+1}, \ldots, k_r]$. Notice that $t' \in T_{k'}$. Write $\pi$ and $\pi'$ for $\pi_t$ and $\pi_{t'}$, respectively.

Both $v_t[j_0, k]$ and $v_{t'}[j_0, k']$ have their respective unique non-zero entries at the row of $t \cup t'$ (an $(r + 1)$-element set). Let $m$ be the number of $j$ such that $k_{\pi(j)} < k$ or $k_{\pi(j)} = k$ with $j < j_0$ and let $m'$ be the number of $j$ such that $k_{\pi'(j)} < k'$ or $k_{\pi(j)} = k'$ with $j < j_0$. Then

$$
\text{sg } \pi v_t[j_0, k] + \text{sg } \pi' v_{t'}[j_0, k']
$$

$$
= (0, \ldots, \text{sg } \pi(-1)^m + \text{sg } \pi'(-1)^{m'}, \ldots, 0)^\triangledown.
$$

We might as well assume $k < k'$. We should have $m < m'$ and

$$
\begin{cases}
  k_{\pi(1)} \leq \cdots \leq k_{\pi(m+1)} = k \leq \cdots \leq k_{\pi'(r)} \\
  k_{\pi(1)} \leq \cdots \leq k_{\pi'(m')} = k' \leq \cdots \leq k_{\pi'(r)},
\end{cases}
$$

where $\pi(j) = \pi'(j)$ for $1 \leq j \leq m$ and $m' + 1 \leq j \leq r$, $\pi(j) = \pi'(j + 1)$ for $m + 1 \leq j \leq m' - 1$ and $\pi(m + 1) = \pi(m')$. One can check that
$sg\pi = (-1)^{m-m-1}sg\pi'$. Hence $sg\pi \nu_k[j_0, k] + sg\pi' \nu_k[j_0, k'] = 0$. The terms of the left-hand side of (3.2) are either 0 or can be paired up this way. Hence the Eq. (3.2) holds.

In the left-hand side of (3.1) one can see that $\nu_k[s]$ is the only non-standard column in the $s$th column-block since the position of any other $t \in T_s$ is non-singular. Hence by induction we have that $\nu_k[s]$ is the space spanned by the standard columns of $A_{r+1}^{n}(\Delta_{rs})$. Next we look at the general case $i \geq r + 1$.

Take a non-standard column $\nu_k[s]$ in $A_{r}^{n}(\Delta_{rs})$. Let $t_0 = t_1 - t_0$. By Lemma 2.12 we have $\nu_k[s]$ is a non-standard column in $A_{r}^{n}(\Delta_{rs})$. Hence (3.1) is true using the notation in Lemma 3.5.

Let $\bar{t}$ denote retaining only the rows in (3.1) that are labeled by an $(r + 1)$-element set not intersecting $u$. Then we still have

$$\sum_{k=1}^{n} \sum_{t \in T_s} sg\pi t \nu_k[u][k] = \sum_{k=1}^{n} \sum_{t \in T_s, t \cap u = \emptyset} sg\pi t \nu_k[u][k] = 0. \quad (3.3)$$

It follows that in $A_{r}^{n}(\Delta_{rs})$

$$\sum_{k=1}^{n} \sum_{t \in T_s, t \cap u = \emptyset} sg\pi t \nu_k[u][k] = 0 \quad (3.4)$$

since in the left-hand side of (3.4) the row of $s$ not containing $u$ is zero and the row of $s$ containing $u$ is the same as the row of $s - u$ in (3.3).

In the left-hand side of (3.4) one of the terms is $\nu_k[s]$ and it is the only non-standard column in the $s$th column-block of $A_{r}^{n}(\Delta_{rs})$. Hence by induction we have that $\nu_k[s]$ is in the space spanned by the standard columns of $A_{r}^{n}(\Delta_{rs})$. The proof of Proposition 2.13 is now complete.

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