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Characterization of multivariate heavy-tailed distribution families via copula

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ABSTRACT

The multivariate regular variation (MRV) is one of the most important tools in modeling multivariate heavy-tailed phenomena. This paper characterizes the MRV distributions through the tail dependence function of the copula associated with them. Along with some existing results, our studies indicate that the existence of the lower tail dependence function of the survival copula is necessary and sufficient for a random vector with regularly varying univariate marginals to have a MRV tail. Moreover, the limit measure of the MRV tail is explicitly characterized. Our analysis is also extended to some more general multivariate heavy-tailed distributions, including the subexponential and the long-tailed distribution families.

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1. Introduction

Heavy-tail analysis has played an increasingly important role in insurance, finance and risk management in recent years. It is well known that financial returns are usually heavy-tailed, and thus the corresponding risk management often relies on heavy-tail analysis. In the field of reinsurance, one often needs to assess some extreme values making heavy-tail analysis techniques indispensable. For a comprehensive review on heavy-tailed distributions and their applications, we refer to [1,5,8,17–19].

In the heavy-tail analysis, one of the most important classes of distributions is the regular variation class, which includes all of the distributions (or equivalently random variables) with a regularly varying tail. The regular variation class has been widely used in modeling heavy-tailed phenomena. In order to efficiently characterize the multivariate heavy-tailed phenomena, the regularly varying distributions are extended to the multivariate case by de Haan and Resnick [6], leading to the concept of multivariate regular variation (MRV). Since then, MRV has been widely used in a variety of fields including queuing theory, stochastic networks, telecommunications, insurance and finance; see, for example, [10,14,9].

In the literature, one can find various equivalent formulations for the MRV distributions. In the present paper, we shall establish another characterization of MRV via the concept of copula, which is defined as a multivariate distribution function with uniform marginal distributions over the unit interval [0, 1]; see [7,16]. The copula enables us to construct a multivariate distribution function from the marginal (possibly different) distribution functions in a way that takes their dependence structure into account.

Our research is motivated by the interplay of the copula and the MRV distributions in their concrete applications. One typical situation arises in the assessment of aggregate risks, as discussed in [20]. Its mathematical setup is as follows.

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Let $\mathbf{X} := (X_1, \ldots, X_d)$ be a non-negative MRV random vector, representing various losses in a multivariate portfolio, and consider the corresponding aggregate risk of a form $\|\mathbf{X}\|$, where $\|\mathbf{X}\|$ denotes an aggregation model such as $\sum_{i=1}^{n} X_i$ and $\left(\sum_{i=1}^{n} X_i^2\right)^{1/2}$. As we know, the MRV distributions are defined using some limiting properties; therefore, it is technically challenging if we handle such assessment problems without imposing any additional assumptions. To establish meaningful results, it is often necessary to conduct the assessment by specifying a copula for the dependence structure of $\mathbf{X}$ in the existing literature.

In such cases, a question arises naturally: what conditions are required on a copula C and a set of marginal distributions so that the resulting joint distribution of $\mathbf{X}$ has a MRV tail? Of course, we should assume that the univariate marginal distributions of $\mathbf{X}$ are all regularly varying, since a MRV distribution must have regularly varying univariate marginals. To the best of our knowledge, this question has remained open until today and the present paper aims to fill this gap. For presentation convenience, we shall discuss the required conditions in terms of the survival copula of $\mathbf{X}$ instead of its copula. One result relevant to the above question is the Theorem 2.3 by Li and Sun [13]. While their results are expressed in terms of copulas, we can equivalently express those results in terms of the survival copulas as stated in Theorem 3.1 below. According to Li and Sun [13], if $\mathbf{X}$ has a MRV tail, the lower tail dependence functions of the marginals of any dimension of the survival copula exist, and can be explicitly expressed through the limit measure of the MRV distribution.

In this paper, we have established the converse of the results from Li and Sun [13]. By Theorem 3.2 below, we formally show that the existence of the lower tail dependence functions is also sufficient for $\mathbf{X}$ to have a MRV tail; moreover, the limit measure can be explicitly expressed in terms of the lower tail dependence function. To illustrate our results, we present Examples 3.1 and 3.2 to demonstrate that the tail dependence function of a general copula does not necessarily exist, and that the joint of univariate regularly varying marginals without satisfying the tail dependence function existence condition does not necessarily have a MRV tail. Moreover, we analyzed many important copula families, and found that the tail dependence function existence condition is satisfied for all of them. Finally, our analysis has also been extended to some other multivariate heavy-tailed distributions, which are more general than the MRV class and include both the subexponential distribution family and the long-tailed distribution family.

The outline of the paper is as follows. In Section 2, we present some preliminaries on the MRV distribution and copula. Section 3 contains our main results, where a characterization of MRV class is established, and some examples are presented. Section 4 investigates the subexponential and long-tailed distribution families. Finally, the Appendix presents the proof of our main result Theorem 3.2.

2. Preliminaries

A measurable function $U: \mathbb{R} \mapsto \mathbb{R}$ is regularly varying at ∞ with index ρ, if it holds $\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\rho}$ for any real number $x > 0$. Here, ρ is called the exponent of variation, and if $\rho = 0$, U is called a slowly varying function. For convenience, throughout the paper we use $\mathcal{R}_\rho$ to denote the class of all survival functions that are regularly varying with index ρ. We may also denote $X \in \mathcal{R}_\rho$ if the random variable X has a survival distribution from the class $\mathcal{R}_\rho$. Denote the punctured space $\mathbb{E}_d = [0, \infty]^d \setminus \{\mathbf{0}\}$, where $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d$, and let $\{\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \ldots\}$ denote a sequence of independent and identically distributed d-dimensional random vectors valued in $[0, \infty)^d$. In what follows, a lowercase bold letter, such as $\mathbf{x}$, denotes a real number vector $(x_1, \ldots, x_d)$ of an appropriate dimension d.

Definition 2.1. The distribution function of $\mathbf{X}$ in $[0, \infty]^d$ has a multivariate regularly varying tail, if there exists a non-null Radon measure $\mu_{\mathbf{X}}$ on $\mathcal{B}_d$, the Borel σ-field of $\mathbb{E}_d$, and a normalizing function $b(u): \mathbb{R} \mapsto \mathbb{R}$ with $b(u) \to \infty$ as $u \to \infty$, such that

$$u \Pr\left(\frac{\mathbf{X}}{b(u)} \in \cdot\right) \xrightarrow{v} \mu_{\mathbf{X}}(\cdot), \quad \text{as } u \to \infty,$$

in $M_+(\mathbb{E}_d)$, where $M_+(\mathbb{E}_d)$ denotes the space of non-negative Radon measures on $\mathbb{E}_d$, and the symbol $\xrightarrow{v}$ stands for vague convergence on $M_+(\mathbb{E}_d)$.

Remark 2.1. In the above definition for the MRV distribution, we concentrate on the nonnegative random vectors. Beyond the nonnegative orthant, readers may refer to [19].

Let C be a symmetric copula function, $C: [0, 1]^d \to [0, 1]$ and $\widehat{C}$ be the corresponding survival copula. We use C_k and $\widehat{C}_k$ to denote the k-dimensional marginals of C and $\widehat{C}$ respectively. For more details regarding copulas and survival copulas, we refer to [16].

Remark 2.2. Technically, the symmetric condition on C is not necessary for our results. Imposing this condition is for presentational convenience. For a general copula, its k-dimensional marginals may not have a uniform form, and this will significantly complicate our analysis and the presentation of our results as well.

Definition 2.2. The lower and upper tail dependence functions of a copula C_k, denoted by $\lambda_k^{(l)}$ and $\lambda_k^{(u)}$ respectively, are defined as

$$\lambda_k^{(l)}(u_1, \ldots, u_k) = \lim_{t \to 0^+} \frac{C_k(tu_1, \ldots, tu_k)}{t},$$

and

$$\lambda_k^{(u)}(u_1, \ldots, u_k) = \lim_{t \to 0^+} \frac{\widehat{C}_k(tu_1, \ldots, tu_k)}{t},$$

provided that the corresponding limit exists, where $u_k \neq 0$ and $1 \leq k \leq d$.

Remark 2.3. (1) Note that C is the copula of a random vector $(-X_1, \ldots, -X_n)$, if and only if C is the survival copula of the random vector $(X_1, \ldots, X_n)$.

(2) If a copula C has a lower tail dependence function $\lambda^{(l)}$ and an upper tail dependence function $\lambda^{(u)}$, then its survival copula has a lower tail dependence function $\lambda^{(u)}$ and an upper tail dependence index $\lambda^{(l)}$; vice versa.

3. Multivariate regular variation

3.1. Conditions and main results

Without any loss of generality, we assume $\mathbf{X}$ has a continuous joint distribution; thus, it has a unique copula and a unique survival copula, according to the Sklar's Theorem; see [16]. Hereafter, we let $\widehat{C}$ be the survival copula associated with $\mathbf{X}$, or equivalently, the copula associated with $-\mathbf{X} \equiv (-X_1, \ldots, -X_d)$. While all the theorems established in the present paper can be extended to the case with a general survival copula, we assume, for presentational convenience, that $\widehat{C}$ is a symmetric survival copula in the sense that its marginals of the same dimension are identical. We write its k-dimensional marginal by $\widehat{C}_k$, and denote its lower tail dependence function by

$$\lambda_k(u_1, \ldots, u_k) := \lim_{t \to 0^+} \frac{\widehat{C}_k(tu_1, \ldots, tu_k)}{t},$$

for $k = 1, \ldots, d$, provided that the limit exists. To proceed, we let F_i denote the univariate distribution of the ith component X_i in the vector $\mathbf{X}$ for $i = 1, \ldots, d$, and impose the following two conditions:

C1. The tail distribution $\overline{F}_1$ is regularly varying at infinity with index $-\alpha < 0$, denoted $\overline{F}_1 \in \mathcal{R}_{-\alpha}$, i.e. $\overline{F}_1(x) = x^{-\alpha} L_1(x)$ for some function L_1 slowly varying at infinity.

C2. $\overline{F}_1, \ldots, \overline{F}_d$ have equivalent tails:

$$\lim_{x \to \infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = c_i,$$

for some constant $c_i > 0$, $i = 2, 3, \ldots$.

Remark 3.1. The equivalent tail condition C2 is usually not satisfied by the original data in specific applications. In heavy-tail analysis, we often adopt certain monotone transformations to transform the original data to satisfy the equivalent tail condition; for details, we refer to [19].

Let us temporarily assume that $\mathbf{X}$ has a MRV distribution and proceed to introduce a necessary condition on its survival copula $\widehat{C}$ in terms of its tail dependence function. Such a condition is developed by Li and Sun [13] (also see [20]). Their results are expressed in terms of the copula of X. For completeness, in the present paper, we rephrase their results in terms of the survival copula and present them in the following Theorem 3.1. In the theorem, the functions τ_k are defined by

$$\tau_k(x_1, \ldots, x_k) := \lim_{u \to \infty} u \Pr\left(X_1 > b(u)x_1, \ldots, X_k > b(u)x_k\right), \quad k = 1, \ldots, d,$$

where the existence of the limit follows trivially from the definition of the MRV distributions.

Theorem 3.1. *Let* $\mathbf{X} = (X_1, \ldots, X_d)$ *be a random vector, which possesses a survival copula* $\widehat{C}$ *and univariate marginal distribution functions* F_i, $i = 1, \ldots, d$, *satisfying conditions* C1 *and* C2. *If, furthermore,* $\mathbf{X}$ *has a MRV distribution, then the lower tail dependence functions* $\lambda_k(x_1, \ldots, x_k)$ *of* $\widehat{C}$ *exist and are given by*

$$\lambda_k(x_1, \ldots, x_k) = \tau_k\left((c_1^{-1}x_1)^{-1/\alpha}, \ldots, (c_k^{-1}x_k)^{-1/\alpha}\right), \tag{3.1}$$

for $k = 1, \ldots, d$.

Proof. See Theorem 2.3 by Li and Sun [13]. □

Remark 3.2. The expression of the lower tail dependence functions $\lambda_k(x_1, \ldots, x_k)$ in (3.1) corresponds to the upper tail dependence functions of the copula of $\mathbf{X}$ presented in their Theorem 2.3 by Li and Sun [13]. Moreover, while the results in [13] are expressed in terms of the limit measure $\mu_{\mathbf{X}}$, the expression for $\lambda_k(x_1, \ldots, x_k)$ given in (3.1) can be easily recovered from their results.

As mentioned before, our main contribution in the present paper is to investigate the converse of the above Theorem 3.1. We have the following Theorem 3.2, which implies that the existence of the lower tail dependence functions of $\widehat{C}$ is also sufficient for $\mathbf{X}$ to have a MRV distribution; moreover, the limit measure of the resulting MRV distribution can be explicitly expressed via these lower tail dependence functions.

Theorem 3.2. *Let* $\mathbf{X} = (X_1, \ldots, X_d)$ *be a random vector, which possesses a survival copula* $\widehat{C}$ *and univariate marginal distributions* F_i, $i = 1, \ldots, d$, *satisfying conditions* C1 *and* C2. *If, additionally, the joint distribution of* $\mathbf{X}$ *is continuous, and the lower tail dependence functions of* $\widehat{C}$

$$\lambda_k(u_1, \ldots, u_k) \quad \text{exists for } k = 1, \ldots, d, \tag{3.2}$$

then $\mathbf{X}$ *has a MRV distribution; more specifically, we have*

$$u \Pr\left(\frac{\mathbf{X}}{b(u)} \in \cdot\right) \xrightarrow{v} \mu_{\mathbf{X}}(\cdot), \quad u \to \infty, \tag{3.3}$$

where $b(u) = \left(\frac{1}{\overline{F}_1}\right)^{\leftarrow}(u) \equiv \inf\left\{t \geq 0 \colon \frac{1}{\overline{F}_1(t)} \leq u\right\}$, *and the limit measure* $\mu_{\mathbf{X}}(\cdot)$ *is given by*

$$\mu_{\mathbf{X}}([\mathbf{0}, \mathbf{x}]^c) = \sum_{j=1}^{k} c_j x_j^{-\alpha} - \sum_{1 \leq i < j \leq k} \lambda_2(c_i x_i^{-\alpha}, c_j x_j^{-\alpha}) + \cdots + (-1)^{k+1} \lambda_d(c_1 x_1^{-\alpha}, \ldots, c_k x_k^{-\alpha}). \tag{3.4}$$

Proof. See the Appendix. □

3.2. Remarks and examples

Remark 3.3. According to Theorems 3.1 and 3.2, the tail dependence function existence condition (3.2) is critical for conclusion on whether the resulting joint distribution has a MRV tail or not. To demonstrate that condition (3.2) is not generally satisfied by all copulas, we present Example 3.1 below; and to illustrate that conditions C1 and C2 without the additional assumption of (3.2) may not be sufficient for $\mathbf{X}$ to have a MRV tail, we present a counterexample in Example 3.2 below.

Example 3.1. Consider the following diagonal copula:

$$C_\delta(u_1, u_2) = \min\left\{u_1, u_2, \frac{1}{2}(\delta(u_1) + \delta(u_2))\right\},$$

where

$$\delta(u) = \begin{cases} 1, & u = 1, \\ u, & u \leq \dfrac{1}{2}, \\ x_{2i-1}, & x_{2i-1} \leq u < x_{2i}, \\ x_{2i}^2 + 2(u - x_{2i}), & x_{2i} \leq u < x_{2i+1}, \end{cases}$$

$x_1 = \frac{1}{2}$, and $x_{2i} = \sqrt{x_{2i-1}}$ and $x_{2i+1} = 2x_{2i} - x_{2i}^2$ for $i = 1, 2, \ldots$. Kortschak and Albrecher [11] have formally showed that C_δ satisfies all the conditions required in defining a copula. Nevertheless, the lower tail dependence function of its survival copula does not exist, as explained below.

Clearly, the survival copula of C_δ is given by

$$\widehat{C}_\delta(u_1, u_2) = u_1 + u_2 - 1 + C_\delta(1 - u_1, 1 - u_2),$$

and simple manipulation leads to

$$\frac{\widehat{C}_\delta(t, t)}{t} = 1 + \min\left(0, \frac{t - 1 + \delta(1 - t)}{t}\right).$$

We show the non-existence of $\lim_{t\to 0^+} \frac{\widehat{C}_\delta(t,t)}{t}$ by deriving two different limits for two distinct sequences of t tending to 0. In fact, for $t = 1 - x_{2i}$,

$$\lim_{i\to\infty} \frac{\widehat{C}_d(1-x_{2i}, 1-x_{2i})}{1-x_{2i}} = 1 + \lim_{i\to\infty} \min\left(0, \frac{-x_{2i}+x_{2i}^2}{1-x_{2i}}\right) = 1 + \lim_{i\to\infty} \min(0, -x_{2i}) = 0,$$

and for $x = 1 - x_{2i-1}$,

$$\lim_{i\to\infty} \frac{\widehat{C}_d(1-x_{2i-1}, 1-x_{2i-1})}{1-x_{2i-1}} = 1 + \lim_{i\to\infty} \min\left(0, \frac{1}{1-x_{2i-1}}\right) = 1.$$

Example 3.2. To show that conditions C1 and C2 are not sufficient for the resulting joint distribution to have a MRV tail, we introduce an example from [3], where the example is used to show that a long-tailed distribution may not have a MRV tail.

For any $x_1, x_2 \geq 0$, let

$$\overline{\mathbf{F}}(x_1, x_2) = \Pr(X_1 > x_1, X_2 > x_2) = \frac{1 + \gamma \sin\{\log r_1(\mathbf{x})\} \cos\left\{\frac{1}{2}\pi r_2(\mathbf{x})\right\}}{r_1(\mathbf{x})},$$

where $r_1(\mathbf{x}) = 1 + x_1 + x_2$, $r_2(\mathbf{x}) = (x_1 - x_2)/r_1(\mathbf{x})$ and $0 < |\gamma| \leq \frac{1}{12}$. Clearly, X_1 and X_2 have the same marginal distribution F which is asymptotically Pareto:

$$\overline{F}(x) = \Pr(X_1 > x) = \Pr(X_2 > x) \sim x^{-1}, \quad \text{as } x \to \infty.$$

Thus, conditions C1 and C2 are satisfied by (X_1, X_2). However, according to [3], its joint distribution does not have a MRV tail.

Remark 3.4. Although a general copula does not necessarily satisfy condition (3.2) as we have seen in Example 3.1, we find that many important copula families actually satisfy this condition. Next, we shall analyze some of those copula families.

Example 3.3. Suppose that $\widehat{C}$ is a copula from the Farlie–Gumbel–Morgenstern family (abbreviated FGM), i.e.,

$$\widehat{C}(u_1, \ldots, u_d) = \left(1 + \theta \prod_{i=1}^{d} (1-u_i)\right) \prod_{i=1}^{d} u_i,$$

with some parameter θ satisfying $|\theta| \leq 1$, or from the Cuadras–Augé family (abbreviated CA), i.e.,

$$\widehat{C}(u_1, \ldots, u_d) = (\min(u_1, \ldots, u_d))^{\theta} \left(\prod_{i=1}^{d} u_i\right)^{1-\theta},$$

with some parameter $\theta \in (0, 1)$. Then, it is easy to check that the tail dependence functions λ_k exist, and $\lambda_k = 0$ for $k = 2, \ldots, d$. Therefore, according to Theorem 3.2, if the random vector $\mathbf{X}$ satisfies conditions C1 and C2, and has a survival copula $\widehat{C}$ from either the FGM family or the CA family, then $\mathbf{X}$ has a MRV tail with a limit measure $\mu_{\mathbf{X}}$ defined by

$$\mu_{\mathbf{X}}([\mathbf{0}, \mathbf{x}]^c) = \sum_{i=1}^{d} c_i x_i^{-\alpha}, \quad \text{for } \mathbf{x} \in \mathbb{R}_+^d.$$

Example 3.4. Suppose that $\widehat{C}$ is the Fréchet–Hoeffding copula (abbreviated FH), i.e.

$$\widehat{C}(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d).$$

Then, it is easy to verify

$$\lambda_k(x_1, \ldots, x_k) = \min(x_1, \ldots, x_k),$$

for $\mathbf{x} \in \mathbb{R}_+^k$ and $k = 2, \ldots, d$. Thus, according to Theorem 3.2, if the random vector $\mathbf{X}$ satisfies conditions C1 and C2, and has a FH survival copula $\widehat{C}$, then $\mathbf{X}$ has a MRV distribution with a limit measure $\mu_{\mathbf{X}}$ defined by

$$\mu_{\mathbf{X}}([\mathbf{0}, \mathbf{x}]^c) = \max(c_1 x_1^{-\alpha}, \ldots, c_d x_d^{-\alpha}), \quad \text{for } \mathbf{x} \in \mathbb{R}_+^d.$$

Example 3.5. In this example, we consider the well-known Archimedean copula family. If we let $\widehat{C}$ be an Archimedean copula, then it admits the following representation:

$$\widehat{C}(u_1, \ldots, u_d) = \psi\left(\psi^{\leftarrow}(u_1) + \cdots + \psi^{\leftarrow}(u_d)\right),$$

where $\psi: [0, \infty] \to [0, 1]$ is a nonincreasing and continuous function, strictly decreasing on $[0, \inf\{\mathbf{x}: \psi(x) = 0\})$ and satisfying the conditions $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$, and $\psi^{\leftarrow}(u) = \inf\{x \geq 0: \psi(x) \leq u\}$ with $\psi^{\leftarrow}(\infty) = 0$ and $\psi^{\leftarrow}(0) = \inf\{u: \psi(u) = 0\}$ by convention. As demonstrated by McNeil and Nešlehová [15] in their Example 2.1, a function $\widehat{C}$ possessing the above representation is not necessarily an appropriate copula. According to [15], a necessary and sufficient condition for $\widehat{C}$ to be an appropriate copula is that the generator ψ is d-monotone on $[0, \infty)$.

The extremal behaviours of Archimedean copulas are usually characterized by the limiting properties of either the generator ψ or its inverse $\psi^{\leftarrow}$. In particular, as shown by Charpentier and Segers [2] in their Theorem 3.1, the lower tail dependence function of an Archimedean copula always exists when the inverse of its generator is regularly varying at 0, or equivalently the generator is regularly varying at 1. It is worth noting that the monograph [16] by Nelsen lists as many as twenty two bivariate Archimedean copulas in its Table 4.1, and almost all of their generators can be easily verified as regularly varying. Nevertheless, the conjecture that all the Archimedean copulas have a lower tail dependence function is wrong. Counterexamples include Examples 3 and 4 from [12].

Next, we assume that $\widehat{C}$ is the survival copula of $\mathbf{X}$, and that $\mathbf{X}$ satisfies conditions C1 and C2. Moreover, we suppose that the generator ψ of $\widehat{C}$ is regularly varying at 1. Then, by Theorem 3.1 from [2], we can obtain the explicit forms of the lower tail dependence functions λ_k of $\widehat{C}$ as shown in the following three cases. We shall then further apply Theorem 3.1 to conclude that $\mathbf{X}$ has a MRV tail, and obtain its limit measure $\mu_{\mathbf{X}}$ as shown below for each case.

(a) Suppose $\psi \in \mathcal{R}_{-\rho}$ with $\rho > 0$, or equivalently $\psi^{\leftarrow} \in \mathcal{R}_{-\rho^{-1}}$. Then, according to [2], $\lambda_k(x_1, \ldots, x_k) = \left(\sum_{i=1}^{k} x_i^{-\rho^{-1}}\right)^{-\rho}$ for $k \geq 2$, and consequently our Theorem 3.2 implies that $\mathbf{X}$ has a MRV distribution with a limit measure $\mu_{\mathbf{X}}$ given by

$$\mu_{\mathbf{X}}([\mathbf{0}, \mathbf{x}]^c) = \sum_{j=1}^{k} c_j x_j^{-\alpha} - \sum_{1\leq i<j\leq k} \left((c_1 x_1^{-\alpha})^{-\frac{1}{\rho}} + (c_2 x_2^{-\alpha})^{-\frac{1}{\rho}}\right)^{-\rho} + \cdots + (-1)^{d+1} \left(\sum_{i=1}^{d} (c_i x_i^{-\alpha})^{-\frac{1}{\rho}}\right)^{-\rho}.$$

(b) Suppose $\psi \in \mathcal{R}_0$, or equivalently $\psi^{\leftarrow} \in \mathcal{R}_{-\infty}$, i.e.,

$$\lim_{t\to 0} \frac{\psi^{\leftarrow}(tx)}{\psi^{\leftarrow}(t)} = \begin{cases} 0, & \text{if } x > 1, \\ \infty, & \text{if } 0 < x < 1. \end{cases}$$

Then, according to [2], $\lambda_k(x_1, \ldots, x_k) = \min\{x_1, \ldots, x_k\}$ for $k \geq 2$, and consequently our Theorem 3.2 implies that $\mathbf{X}$ has a MRV distribution with a limit measure $\mu_{\mathbf{X}}$ given by $\mu_{\mathbf{X}}([\mathbf{0}, \mathbf{x}]^c) = \max\{x_1, \ldots, x_d\}$.

(c) Suppose $\psi \in \mathcal{R}_{-\infty}$, or equivalently $\psi^{\leftarrow} \in \mathcal{R}_0$. Then, according to [2], $\lambda_k(x_1, \ldots, x_k) \equiv 0$ for $k \geq 0$, and consequently our Theorem 3.2 implies that $\mathbf{X}$ has a MRV distribution with a limit measure $\mu_{\mathbf{X}}$ given by $\mu_{\mathbf{X}}([\mathbf{0}, \mathbf{x}]^c) = \sum_{i=1}^{d} c_i x_i^{-\alpha}$.

4. Other multivariate heavy-tailed families

In this section, we shall extend our results established in Section 3 to two other important multivariate heavy-tailed families of distributions: the long-tailed family and the subexponential family. The MRV class of distributions is a subset of either of these two families. In the univariate case, they are defined as follows.

Definition 4.1. (1) The survival function $\overline{F}$ of a random variable is in class $\mathcal{L}(\alpha)$ (called long-tailed class) for some $\alpha \geq 0$, denoted $\overline{F} \in \mathcal{L}(\alpha)$, if and only if

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = e^{\alpha y}, \quad \text{for } y > 0.$$

(2) The survival function $\overline{F}$ of a random variable is in class $\mathcal{S}(\alpha)$ (called the subexponential class) for some $\alpha \geq 0$, denoted by $\overline{F} \in \mathcal{S}(\alpha)$, if and only if $\overline{F} \in \mathcal{L}(\alpha)$ and

$$\lim_{x\to\infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = D,$$

where $\overline{F * F}$ denotes $1 - F * F$, $F * F$ is the twofold convolution of F, and D is a constant.

When $\alpha = 0$ in the above definition, the resulting classes $\mathcal{L}(0)$ and $\mathcal{S}(0)$ are actually large enough to contain all the regularly varying distributions. They satisfy the following inclusion relations:

$$\mathcal{R}_{-\alpha} \subset \mathcal{S}(0) \subset \mathcal{L}(0).$$

For more details, we refer to [1,4,8]. In the multivariate case, the long-tailed class and subexponential class of distributions are defined as follows.

Definition 4.2. (1) A random vector $\mathbf{X}$ in $[0, \infty)^d$ belongs to the multivariate long-tailed class $\mathcal{L}(\nu, \mathbf{b})$, denoted $\mathbf{X} \in \mathcal{L}(\nu, \mathbf{b})$, if and only if there exist a non-null Radon measure ν on $\mathcal{B}_d$, the Borel σ-field of $\mathbb{E}_d$, and a function vector $\mathbf{b}(u) :=$

$(b_1(u), \ldots, b_d(u))$ satisfying $b_i(u) \to \infty$ as $u \to \infty$ for $i = 1, \ldots, d$, such that

$$u \Pr[\mathbf{X} - \mathbf{b}(u) \in \cdot] \xrightarrow{v} \nu(\cdot). \tag{4.1}$$

in $M_+(\mathbb{E}_d)$.

(2) A random vector $\mathbf{X}$ in $[0, \infty)^d$ belongs to the multivariate subexponential class $\mathcal{S}(\nu, \mathbf{b})$, denoted $\mathbf{X} \in \mathcal{S}(\nu, \mathbf{b})$, if and only if the following two conditions are satisfied:

(i) $\mathbf{X} \in \mathcal{L}(\nu, \mathbf{b})$;
(ii) there exists a function vector $\mathbf{b}(u) := (b_1(u), \ldots, b_d(u))$ satisfying $b_i(u) \to \infty$ as $u \to \infty$ for $i = 1, \ldots, d$, such that

$$u \Pr[\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{b}(u) \in \cdot] \xrightarrow{v} \nu^{(2)}(\cdot), \tag{4.2}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are two independent copies of the random vector $\mathbf{X}$, and $\frac{1}{2}\nu^{(2)} = \nu * \mathbf{F}$ denoting the convolution of measure ν and the joint distribution $\mathbf{F}$ of $\mathbf{X}$.

According to [3], if a random vector $\mathbf{X}$ belongs to the multivariate long-tailed class (or the multivariate subexponential class), then the univariate marginal distributions of $\mathbf{X}$ are still long-tailed (or subexponential). In other words, if (4.1) (or (4.2)) holds, then there are some constants $\alpha_i \geq 0$ such that $X_i \in \mathcal{L}(\alpha_i)$ (or $X_i \in \mathcal{S}(\alpha_i)$) for $i = 1, \ldots, d$. In the present paper, we are interested in the converse result as stated in the following Theorems 4.1 and 4.2. Theorem 4.1 can be proved in a way very similar to that used for Theorem 3.2, and hence we omit its proof.

Theorem 4.1. *Assume that all the conditions in Theorem 3.2 are satisfied except replacing condition* C1 *by the following one:*

C1'. *The univariate survival distribution $\overline{F}_i \in \mathcal{L}(\alpha_i)$ for some $\alpha_i \geq 0$, $i = 1, \ldots, d$.*

Then, $\mathbf{X}$ belongs to the multivariate long-tailed class; more specifically,

$$u \Pr(\mathbf{X} - \mathbf{b}(u) \in \cdot) \xrightarrow{v} \nu_{\mathbf{X}}(\cdot), \quad u \to \infty,$$

where $\mathbf{b}(u) = (b(u), \ldots, b(u))$, $b(u) = \left(\frac{1}{\overline{F}_1}\right)^{\leftarrow}(u)$, and the limit measure $\nu_{\mathbf{X}}(\cdot)$ is defined by

$$\nu_{\mathbf{X}}([\mathbf{0}, \mathbf{x}]^c) = \sum_{j=1}^{k} c_j e^{-\alpha_j x_j} - \sum_{1 \leq i < j \leq k} \lambda_2(c_i e^{-\alpha_i x_i}, c_j e^{-\alpha_j x_j}) + \cdots + (-1)^{k+1} \lambda_d(c_1 e^{-\alpha_1 x_1}, \ldots, c_k e^{-\alpha_k x_k}).$$

Remark 4.1. Let $\alpha_i = 0$ in Theorem 4.1, i.e., $X_i \in \mathcal{L}(0)$, for $i = 1, \ldots, d$. Then, for any $\mathbf{x} \in \mathbb{R}_+^d$,

$$\nu_{\mathbf{X}}([\mathbf{0}, \mathbf{x}]^c) = \sum_{j=1}^{k} c_j - \sum_{1 \leq i < j \leq k} \lambda_2(c_i, c_j) + \cdots + (-1)^{k+1} \lambda_d(c_1, \ldots, c_k), \tag{4.3}$$

which implies that the limit measure $\nu_{\mathbf{X}}$ is degenerate when the marginal distributions are all from class $\mathcal{L}(0)$.

Theorem 4.2. *Assume that all the conditions in Theorem 3.2 are satisfied except replacing condition* C1 *by the following one:*

C1''. *The univariate survival distribution $\overline{F}_i$ is from class $\mathcal{S}(\alpha_i)$ for some $\alpha_i \geq 0$, $i = 1, \ldots, d$.*

Then, $\mathbf{X}$ belongs to the multivariate subexponential class $\mathcal{S}(\nu_{\mathbf{X}}, \mathbf{b})$, where the limit measure $\nu_{\mathbf{X}}$ and the normalizing function $\mathbf{b}$ are as defined in Theorem 4.1.

Proof. Clearly, $\overline{F}_i \in \mathcal{S}(\alpha_i)$ implies $\overline{F}_i \in \mathcal{L}(\alpha_i)$, and hence, by Theorem 4.1, $\mathbf{X} \in \mathcal{L}(\nu_{\mathbf{X}}, \mathbf{b})$. Consequently, using Corollary 2.4 in [3], we immediately obtain $\mathbf{X} \in \mathcal{S}(\nu_{\mathbf{X}}, \mathbf{b})$. □

Remark 4.2. Let $\alpha_i = 0$ in Theorem 4.2, i.e., $X_i \in \mathcal{S}(0)$ for $i = 1, \ldots, d$. Then, by Remark 4.1, the limit measure $\nu_{\mathbf{X}}$ is degenerate as shown in (4.3), and hence its convolution with the joint distribution $\mathbf{F}$ of $\mathbf{X}$ is itself. Therefore, in this case, we have

$$u \Pr[\mathbf{X}_1 + \mathbf{X}_2 - \mathbf{b}(u) \in \cdot] \xrightarrow{v} 2\nu^{(0)},$$

where $\nu^{(0)}$ denotes the degenerate measure given by the right hand side of Eq. (4.3).

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Appendix

Proof of Theorem 3.2. We first note that conditions C1 and C2 imply

$$(x^{-\alpha}-\varepsilon_1)\overline{F}_i(t)\le \overline{F}_i(tx)\le (x^{-\alpha}+\varepsilon_1)\overline{F}_i(t), \tag{A.1}$$

and

$$(c_i-\varepsilon_2)\overline{F}_1(t)\le \overline{F}_i(t)\le (c_i+\varepsilon_2)\overline{F}_1(t) \tag{A.2}$$

for any $x>0$, $\varepsilon_1>0$, $\varepsilon_2>0$, large enough t, and $i=1,2,\dots,d$. Recall that $\widehat{C}_k$ is the k-dimensional marginals of $\widehat{C}$. Thus, $\widehat{C}_k(u_1,\dots,u_k)$ is nondecreasing in each of its arguments u_i, and this fact, combined with the second inequalities in (A.1) and (A.2), further implies that

$$\begin{aligned}\widehat{C}_k\left(\overline{F}_{i_1}(tx_{i_1}),\dots,\overline{F}_{i_k}(tx_{i_k})\right) &\le \lim_{\varepsilon_1\to0^+,\varepsilon_2\to0^+}\widehat{C}_k\left((x_{i_1}^{-\alpha}+\varepsilon_1)(c_{i_1}+\varepsilon_2)\overline{F}_1(t),\dots,(x_{i_k}^{-\alpha}+\varepsilon_1)(c_{i_k}+\varepsilon_2)\overline{F}_1(t)\right)\\ &=\widehat{C}_k\left(c_{i_1}x_{i_1}^{-\alpha}\overline{F}_1(t),\dots,c_{i_k}x_{i_k}^{-\alpha}\overline{F}_1(t)\right),\end{aligned}$$

where $\{i_1,\dots,i_k\}$ denotes a subset of $\{1,\dots,d\}$ for $k\le d$, and the last equality is due to the assumption that $\mathbf{X}$ has a continuous distribution. The last display implies

$$\begin{aligned}\limsup_{t\to\infty}\frac{\widehat{C}_k\left(\overline{F}_{i_1}(tx_{i_1}),\dots,\overline{F}_{i_k}(tx_{i_k})\right)}{\overline{F}_1(t)} &\le \limsup_{t\to\infty}\frac{\widehat{C}_k\left(c_{i_1}x_{i_1}^{-\alpha}\overline{F}_1(t),\dots,c_{i_k}x_{i_k}^{-\alpha}\overline{F}_1(t)\right)}{\overline{F}_1(t)}\\ &=\lambda_k\left(c_{i_1}x_{i_1}^{-\alpha},\dots,c_{i_k}x_{i_k}^{-\alpha}\right),\end{aligned} \tag{A.3}$$

in a similar way, by using the first inequalities in (A.1) and (A.2), we can obtain

$$\liminf_{t\to\infty}\frac{\widehat{C}_k\left(\overline{F}_{i_1}(tx_{i_1}),\dots,\overline{F}_{i_k}(tx_{i_k})\right)}{\overline{F}_1(t)}\ge\lambda_k\left(c_{i_1}x_{i_1}^{-\alpha},\dots,c_{i_k}x_{i_k}^{-\alpha}\right). \tag{A.4}$$

Combining (A.3) and (A.4) yields

$$\lim_{t\to\infty}\frac{\widehat{C}_k\left(\overline{F}_{i_1}(tx_{i_1}),\dots,\overline{F}_{i_k}(tx_{i_k})\right)}{\overline{F}_1(t)}=\lambda_k\left(c_{i_1}x_{i_1}^{-\alpha},\dots,c_{i_k}x_{i_k}^{-\alpha}\right),$$

and hence

$$\lim_{u\to\infty}u\Pr\left(X_{i_1}>b(u)x_{i_1},\dots,X_{i_k}>b(u)x_{i_k}\right)=\lambda_k\left(c_{i_1}x_{i_1}^{-\alpha},\dots,c_{i_k}x_{i_k}^{-\alpha}\right).$$

So, for any $\mathbf{x}\in\mathbb{R}_+^d$,

$$\begin{aligned}u\Pr\left(\frac{\mathbf{X}}{b(u)}\in[\mathbf{0},\mathbf{x}]^c\right) &= u\Pr\left(\bigcup_{k=1}^{d}\left(\frac{X_k}{b(u)}>x_k\right)\right)\\ &=\sum_{k=1}^{d}u\Pr\left(\frac{X_k}{b(u)}>x_k\right)-\sum_{1\le k<j\le d}u\Pr\left(\frac{X_k}{b(u)}>x_k,\frac{X_j}{b(u)}>x_j\right)\\ &\quad+\cdots+(-1)^{d+1}u\Pr\left[\bigcap_{k=1}^{d}\left(\frac{X_k}{b(u)}>x_k\right)\right]\\ &\to\sum_{k=1}^{d}c_kx_k^{-\alpha}-\sum_{1\le k<j\le d}u\lambda_2(c_kx_k^{-\alpha},c_jx_j^{-\alpha})+\cdots+(-1)^{d+1}\lambda_d(c_1x_1^{-\alpha},\dots,c_dx_d^{-\alpha}),\end{aligned}$$

as $u\to\infty$. By this, the proof is complete. □

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