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On integrable deformations of superstring sigma models related to $AdS_n \times S^n$ supercosets

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Abstract

We consider two integrable deformations of 2d sigma models on supercosets associated with $AdS_n \times S^n$. The first, the "η-deformation" (based on the Yang–Baxter sigma model), is a one-parameter generalization of the standard superstring action on $AdS_n \times S^n$, while the second, the "λ-deformation" (based on the deformed gauged WZW model), is a generalization of the non-abelian T-dual of the $AdS_n \times S^n$ superstring. We show that the η-deformed model may be obtained from the λ-deformed one by a special scaling limit and analytic continuation in coordinates combined with a particular identification of the parameters of the two models. The relation between the couplings and deformation parameters is consistent with the interpretation of the first model as a real quantum deformation and the second as a root of unity quantum deformation. For the $AdS_2 \times S^2$ case we then explore the effect of this limit on the supergravity background associated with the λ-deformed model. We also suggest that the two models may form a dual Poisson–Lie pair and provide direct evidence for this in the case of the integrable deformations of the coset associated with S^2.

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1. Introduction

Recently there has been significant interest in two special integrable models that are closely associated with the superstring sigma model on $AdS_n \times S^n$. First, in [1] a particular integrable deformation of the $AdS_5 \times S^5$ supercoset model was considered, generalizing the bosonic Yang–Baxter sigma model of [2–4]. Second, in [5,6] (generalizing the bosonic model of [7]) an integrable model based on the $\hat{F}/\hat{F}$ gauged WZW model was constructed, which is also closely associated with the $AdS_5 \times S^5$ supercoset. The latter model may be interpreted as an integrable deformation of the non-abelian T-dual of the $AdS_5 \times S^5$ supercoset action.

We shall simply refer to the first model as the "η-model" and to the second as the "λ-model". As they contain, as special points, the original F/G coset model and its non-abelian T-dual model respectively, one may suspect that they are related by some sort of duality provided one properly identifies their parameters. Indeed, we shall provide evidence (in the simplest 2d target space case) that they are such a pair of Poisson–Lie dual models [8–12] hence representing the two "faces" of a single interpolating or "double" theory.

At the same time, it turns out there is also another, more surprising, relation: the η-model can be obtained directly from the λ-model as a special limit (combined with an analytic continuation), which in some sense cuts off the asymptotically flat region.[2] The special point $\varkappa = i$ of the η-model is a pp-wave background [13] that for low-dimensional examples is equivalent in the light-cone gauge to the Pohlmeyer-reduced (PR) model for the coset theory. This provides therefore a direct link between the special limit of the λ-model and the PR model (conjectured in [5,6] and recently made explicit in [14]).

This special limit is of particular interest for understanding the relation of the λ-model to the q-deformation of the light-cone gauge S-matrix [15] for q being a phase. For q real the S-matrix is unitary and has been shown to be in perturbative agreement [16,17] with the η-model of [1]. For q equal a phase, unitarity can be restored [18], and the resulting S-matrix has been conjectured to be related to the λ-model [5,6]. However, as the λ-model has no isometries one cannot fix the associated light-cone gauge and hence there is no apparent connection to the S-matrix of [15]. An important feature of the special limit is that it generates isometries. It is therefore natural to conjecture that taking an appropriate limit in the λ-model associated with the $AdS_5 \times S^5$ supercoset will give the deformed model whose light-cone gauge S-matrix is that of [18].

We shall start in Section 2 with a review of the actions of the η-deformed and λ-deformed models, considering in detail the relation between the parameters and also the truncations to the bosonic models.

Then in Section 3 we shall describe the scaling limit and analytic continuation that allows one to obtain the metric of the η-model from that of the λ-model. We shall discuss the action of this limit on the corresponding supergravity solution of [19,20] in Section 4 for the models related to the $AdS_2 \times S^2$ supercoset.

Finally, in Section 5 we will conjecture that the two models form a dual Poisson–Lie pair [8,9] and directly verify this in the case of the integrable deformations of the coset associated with S^2.

In Appendix A we shall give different simple forms of the conformally-flat metrics of the deformed models associated with S^2, while in Appendix B we will discuss an alternative proposal for the dilaton of the models related to the $AdS_2 \times S^2$ supercoset.

[2] This is somewhat similar to how the $AdS_5 \times S^5$ background is related to the D3-brane geometry when one decouples the asymptotic region.

2. Deformed models

2.1. Supercoset based actions

We shall consider two integrable 2d models based on the supercosets

$$\frac{\widehat{F}}{G_1 \times G_2} \supset \frac{F_1}{G_1} \times \frac{F_2}{G_2} \ , \tag{2.1}$$

where $\widehat{F}$ is a supergroup (e.g. $PSU(2,2|4)$ in $AdS_5 \times S^5$ case) and F_i and G_i are bosonic subgroups. The superalgebra $\hat{\mathfrak{f}}$ of $\widehat{F}$ admits the usual $\mathbb{Z}_4$ grading, with the zero-graded part corresponding to the algebra of $G_1 \times G_2$, and the bilinear form $\mathrm{STr} = \mathrm{Tr}_{F_1} - \mathrm{Tr}_{F_2}$.

The first "η-model" is defined by the deformed supercoset action of [1] (generalizing the bosonic model of [2])[3]

$$\hat{I}_{h,\eta}(g) = \frac{h}{2}\int d^2x \ \mathrm{STr}\big[g^{-1}\partial_+ g \ P_\eta \frac{1}{1-\frac{2\eta}{1-\eta^2}R_g P_\eta} \ g^{-1}\partial_- g\big] \ , \tag{2.2}$$

where $g \in \widehat{F}$ and

$$P_\eta = P_2 + \frac{1-\eta^2}{2}(P_1 - P_3) \ , \qquad R_g = \mathrm{Ad}_g^{-1} \, R \, \mathrm{Ad}_g \ . \tag{2.3}$$

Here $\mathrm{Ad}_g(M) = gMg^{-1}$, P_r are projectors onto the $\mathbb{Z}_4$-graded spaces of $\hat{\mathfrak{f}}$ and the constant matrix R is an antisymmetric solution of the non-split modified classical Yang–Baxter equation for $\hat{\mathfrak{f}}$. The overall coupling h is the analog of string tension and η is the deformation parameter.[4] This action possesses the following $\mathbb{Z}_2$ symmetry:

$$\text{parity} \ , \qquad h \to h \ , \qquad \eta \to -\eta \ . \tag{2.4}$$

In the undeformed limit, the action (2.2) reduces to the standard supercoset action [21,22]

$$\hat{I}_{h,0}(g) = \frac{h}{2}\int d^2x \ \mathrm{STr}\big[g^{-1}\partial_+ g \ P \ g^{-1}\partial_- g\big] \ ,$$
$$P = P_\eta\Big|_{\eta=0} = P_2 + \frac{1}{2}(P_1 - P_3) \ . \tag{2.5}$$

The global $\widehat{F}$ symmetry of this undeformed action is broken by the η-deformation to its abelian Cartan subgroup.

The second "λ-model" [6] (generalizing the bosonic model of [7,23]) is defined by the action

[3] We choose Minkowski signature in 2d with $d^2x = dx^0 dx^1$ and $\partial_\pm = \partial_0 \pm \partial_1$.

[4] Here the bilinear form Tr (STr) is related to the usual matrix trace tr (supertrace str) by $\mathrm{Tr} = \nu^{-1}\mathrm{tr}$ for some representation-dependent normalization ν. We fix this normalization ν such that in the undeformed limit h plays the role of the usual string tension in $AdS_n \times S^n$ backgrounds. In particular, this means that in the $AdS_2 \times S^2$ case with $\eta = 0$ the bosonic part of the action is given by

$$I_{h,0}(g) = \frac{h}{2}\int d^2x \big[-(1+\rho^2)\partial_+ t\partial_- t + \frac{1}{1+\rho^2}\partial_+\rho\partial_-\rho + (1-r^2)\partial_+\varphi\partial_-\varphi + \frac{1}{1-r^2}\partial_+ r\partial_- r\big] \ .$$

$$\hat{I}_{k,\lambda}(f,A) = \frac{k}{4\pi}\Big(\int d^2x \ \mathrm{STr}\big[\frac{1}{2}f^{-1}\partial_+ f f^{-1}\partial_- f + A_+\partial_- f f^{-1} - A_- f^{-1}\partial_+ f - f^{-1}A_+ f A_- + A_+A_-\big] - \frac{1}{3}\int d^3x\, \epsilon^{abc}\,\mathrm{STr}\big[f^{-1}\partial_a f f^{-1}\partial_b f f^{-1}\partial_c f\big] + (\lambda^{-2}-1)\int d^2x \ \mathrm{STr}\big[A_+P_\lambda A_-\big]\Big)\,, \tag{2.6}$$

where $f \in \widehat{F}$, $A_\pm \in \hat{\mathfrak{f}}$ and

$$P_\lambda = P_2 + \frac{1}{\lambda^{-1}+1}(P_1 - \lambda P_3)\,. \tag{2.7}$$

The first three lines of (2.6) correspond to the $\widehat{F}/\widehat{F}$ gauged WZW model with coupling (level) k and λ is a deformation parameter. This action possesses the following $\mathbb{Z}_2$ symmetry

$$\begin{aligned} &\text{parity}\,, \qquad k \to -k\,, \qquad \lambda \to \lambda^{-1}\,, \\ &A_+ \to \Lambda A_+\,, \qquad A_- \to \mathrm{Ad}_f(A_- - f^{-1}\partial_- f)\,, \end{aligned} \tag{2.8}$$

where $\Lambda = I + (\lambda^{-2}-1)P_\lambda = P_0 + \lambda^{-2}P_2 + \lambda^{-1}P_1 + \lambda P_3$.

In contrast to (2.2) this action has no global symmetry (there is a $G_1 \times G_2$ gauge symmetry, which in the end we will always fix). The interpretation of this action can be understood by considering the special limit $k \to \infty$, $\lambda \to 1$ combined with scaling $f \to 1$ as [7]

$$\begin{aligned} &f = \exp(-\frac{4\pi}{k}v) = 1 - \frac{4\pi}{k}v + \mathcal{O}(k^{-2})\,, \\ &\lambda = 1 - \frac{\pi}{k}h + \mathcal{O}(k^{-2})\,, \qquad k \to \infty\,, \end{aligned} \tag{2.9}$$

where the $\hat{\mathfrak{f}}$ valued field v and the constant h are kept fixed in the limit. This leads to the following action[5]

$$\hat{I}_{k\to\infty,\lambda\to 1}(f\to 1, A) = \int d^2x \ \mathrm{STr}\big[v\,(\partial_- A_+ - \partial_+ A_- + [A_-, A_+])\big] + \frac{h}{2}\int d^2x \ \mathrm{STr}\big[A_+ P A_-\big]\,, \tag{2.10}$$

where $P = P_\lambda\Big|_{\lambda=1}$ is given in (2.5). This may be interpreted as a first-order action interpolating between the supercoset action (2.5) (if one first integrates out v giving $A_\pm = g^{-1}\partial_\pm g$) and its non-abelian T-dual model (if one first integrates out $A_\pm$).

Thus the meaning of (2.6) is a deformation of the first-order interpolating action (2.10). If one first integrates out $A_\pm$ in (2.6) and gauge-fixes the supergroup element f the resulting sigma model may be viewed as a deformation of the non-abelian T-dual of the original supercoset model (2.5). At the same time, explicitly integrating out f in (2.6) is not possible in general, so (2.6) does not apparently have a direct relation to a deformation of the supercoset model (2.5).

While there is a close on-shell connection between the models (2.2) and (2.6) at the level of classical Hamiltonian (Poisson-bracket) structures [1,5,6], establishing their correspondence at

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5 Note that $\frac{h}{2} = \frac{\kappa^2}{4\pi}$, where κ^2 is the string tension parameter used in [7,5,6] (the definition of $\partial_\pm$ used therein had an extra factor of $1/2$ compared to that used here).

the level of the actions (and thus eventually at the quantum level) remains an open problem that we will attempt to address below.[6]

2.2. Relations between parameters

Let us now comment on relations between the deformation parameters of the two models (2.2) and (2.6). The deformation parameters in the two actions of [1] and [6] may be defined in terms of the parameter $\epsilon^2 \in \mathbb{R}$ that appears in the deformed classical Poisson algebra relations.[7]

The relation to the parameter η of [1] (or $\varkappa$ introduced in [16]) is given by

$$\epsilon^2 = \frac{4\eta^2}{(1+\eta^2)^2} = \frac{\varkappa^2}{1+\varkappa^2} \,, \qquad \epsilon^2 \in [0,1] \,, \quad \eta^2 \in [0,1] \,, \quad \varkappa^2 \in [0,\infty] \,, \tag{2.11}$$

where

$$\varkappa = \frac{2\eta}{1-\eta^2} \tag{2.12}$$

is a natural deformation parameter appearing in the bosonic part of the model (2.2). Here the ranges describe the deformation considered in [1,16]. Note that we could also take

$$\eta^2 \in [1,\infty] \,, \tag{2.13}$$

to cover the ranges $\epsilon^2 \in [0,1]$ and $\varkappa^2 \in [0,\infty]$. This is a consequence of the fact that the complex η^2 plane covers the complex ϵ^2 and $\varkappa^2$ planes twice. This can be seen explicitly from the relation

$$\epsilon^2(\eta^2) = \epsilon^2(\frac{1}{\eta^2}) \,. \tag{2.14}$$

The deformation parameter λ in the action (2.6) of [6] is related to ϵ^2 by

$$\begin{gathered}\epsilon^2 = -\frac{(1-\lambda^2)^2}{4\lambda^2} = -\frac{1}{4b^2(1+b^2)} \,, \\ \epsilon^2 \in [-\infty,0] \,, \quad \lambda^2 \in [0,1] \,, \quad b^2 \in [0,\infty] \,,\end{gathered} \tag{2.15}$$

where we have introduced

$$b^2 = \frac{\lambda^2}{1-\lambda^2} \,, \tag{2.16}$$

[6] Note that integrability, together with expected quantum UV finiteness, suggest that classical relations may in some way extend to the quantum level.

[7] For both deformed models, there was a paper focussing on the bosonic case, [4] and [5], written before the papers discussing the deformation of the superstring, [1] and [6] respectively. The parameter η_b of [4] is related to the parameter η of [1] by

$$\eta_b = \frac{2\eta}{1-\eta^2} \,,$$

while the parameter λ_b of [5] is related to the parameter λ of [6] by

$$\lambda_b = \lambda^2 \,.$$

To avoid confusion, we will always use the definitions of parameters as given in the papers discussing the superstring [1,6].

which is again a natural deformation parameter in the bosonic part of (2.6). Here the ranges describe the deformation considered in [6], but we could also take

$$\lambda^2 \in [1, \infty] , \qquad b^2 \in [-\infty, -1] , \tag{2.17}$$

to cover the range $\epsilon^2 \in [-\infty, 0]$. This is again a consequence of the fact that the complex λ^2 or b^2 planes cover the complex ϵ^2 plane twice, which can be seen explicitly from the relations

$$\epsilon^2(\lambda^2) = \epsilon^2(\frac{1}{\lambda^2}) , \qquad \epsilon^2(b^2) = \epsilon^2(-1 - b^2) . \tag{2.18}$$

For a particular value of ϵ^2 there are four equivalent values of η, b and λ and two equivalent values of $\varkappa$ as described in the table:

η	$-\eta$	$-\eta^{-1}$	η^{-1}
$\varkappa$	$-\varkappa$	$\varkappa$	$-\varkappa$
λ	λ^{-1}	$-\lambda$	$-\lambda^{-1}$
b	$\pm\sqrt{-1-b^2}$	$-b$	$\mp\sqrt{-1-b^2}$

The first and second columns and the third and fourth columns give rise to equivalent theories in both the two deformations as they are related by the $\mathbb{Z}_2$ symmetries (2.4) and (2.8). Furthermore, restricting to the bosonic models, the first and third columns and the second and fourth columns give rise to identical deformed theories. This is a consequence of the fact that the bosonic truncation of (2.2) depends only on $\varkappa$, while the bosonic truncation of (2.6) depends only on λ^2.

Comparing (2.11) and (2.15) suggests that the parameters of the two deformed models may be related by an analytic continuation (choosing signs so that $\lambda = 0, 1$ corresponds to $\eta = i, 0$)

$$\eta = i\frac{1-\lambda}{1+\lambda} , \qquad \lambda = \frac{i-\eta}{i+\eta} , \tag{2.19}$$

or, equivalently,

$$b^2 = -\frac{1}{2} + \frac{i}{2\varkappa} , \qquad \varkappa = \frac{i}{1+2b^2} = i\frac{1-\lambda^2}{1+\lambda^2} . \tag{2.20}$$

Below we will see that (2.20) is indeed the relation that allows one to obtain the η-model (2.2) as a special limit (combined with an analytic continuation) of the λ-model (2.6).

In addition, this will require us to relate the overall couplings of the two models by the following analytic continuation (assuming the plus sign in (2.19))

$$\frac{k}{\pi} = i\frac{h}{\varkappa} , \qquad \text{i.e.} \qquad h = \frac{k}{\pi(1+2b^2)} . \tag{2.21}$$

Indeed, (2.21) is implied by (2.20) and the expression for λ in (2.9), which was required to obtain the interpolating model (2.10) for large k: with $\lambda \to 1 - \frac{\pi h}{k}$ we find from (2.16) that $b^2 \to \frac{k}{2\pi h}$ and thus, from (2.20), that $\varkappa \to \frac{i\pi h}{k}$, in agreement with (2.21).

The relation (2.21) is also consistent with the Pohlmeyer reduction limit, which in the context of the η-deformation [1] corresponds to taking $\varkappa \to \pm i$, as discussed in [13], with h being proportional to the level of the underlying G/H gauged WZW model. This then ties in with the Pohlmeyer reduction limit of the deformation of [6] for which k plays the role of the level [14].

Remarkably, (2.21) corresponds to the expected relation between the quantum deformation parameters q for the two models (cf. [1,16,5,6]):

$$q = e^{-\frac{\varkappa}{h}} \qquad \leftrightarrow \qquad q = e^{-\frac{i\pi}{k}} \ , \tag{2.22}$$

with the real q corresponding to the η-model (2.2) and the root of unity q to the λ-model (2.6). Indeed, $q = \exp(-\frac{i\pi}{k})$ is the standard expectation for the q-deformation parameter of a WZW type model.

2.3. Bosonic actions

It is useful to consider explicitly the bosonic parts of the two models (2.2) and (2.6). We shall concentrate on the part corresponding to one (compact) F/G factor in (2.1). The bosonic counterpart of the η-model action (2.2) is

$$I_{h,\eta}(g) = -\frac{h}{2}\int d^2x \, \mathrm{Tr}\big[J_+ P \frac{1}{1-\varkappa R_g P} J_-\big] \ , \qquad J_a = g^{-1}\partial_a g \ , \tag{2.23}$$

$$\varkappa \equiv \frac{2\eta}{1-\eta^2} \ , \qquad\qquad R_g = \mathrm{Ad}_g^{-1} R \, \mathrm{Ad}_g \ , \tag{2.24}$$

where $g \in F$, $P = P_2$ is the projector onto the F/G coset part of the algebra $\mathfrak{f}$ of F and R is a solution of the modified classical YBE for $\mathfrak{f}$. For $\varkappa = 0$ this becomes the standard F/G coset sigma model.

To make the structure of this action more transparent let us rewrite it in a first-order form. Since $\frac{1}{1-\varkappa R_g P} = \sum_{n=0}^{\infty}(\varkappa R_g P)^n$ and $P^2 = P$, introducing an auxiliary field B_a in the coset part of $\mathfrak{f}$ (i.e. $PB_a = B_a$) we get

$$-\mathrm{Tr}\big[J_+ P \frac{1}{1-\varkappa R_g P} J_-\big] \ \to \ -\mathrm{Tr}\big[-B_+(1-\varkappa R_g)B_- + B_+ J_- + B_- J_+\big] \ . \tag{2.25}$$

Replacing B_a by the field A_a in $\mathfrak{f}$, adding a term $A_a C_a$ where $C_a \in \mathfrak{g}$ is in the algebra of G and then redefining $\mathrm{Ad}_g(A_a) = g A_a g^{-1} \to A_a$ we find the following first-order form of (2.23), which has a right-action G-gauge symmetry

$$I_{h,\eta}(g, A, C) = -\frac{h}{2}\int d^2x \, \mathrm{Tr}\big[-A_+(1-\varkappa R)A_- + A_+ D_- g\, g^{-1} + A_- D_+ g\, g^{-1}\big] \ , \tag{2.26}$$

$$D_a g \equiv \partial_a g - g C_a \ , \qquad g' = gu \ , \quad C'_a = u^{-1}C_a u + u^{-1}\partial_a u \ , \quad u \in G \ . \tag{2.27}$$

This model has parameters $(h, \varkappa)$ and for $\varkappa \neq 0$ the global F symmetry is broken to its Cartan torus directions.[8] In the first-order action (2.26) the deformation corresponds simply to adding the quadratic $\varkappa A_+ R A_-$ term. Indeed, we can rewrite (2.26) as

$$\begin{aligned} I_{h,\eta}(g, A, C) = -\frac{h}{2}\int d^2x \, \mathrm{Tr}\big[&D_+ g\, g^{-1} D_- g\, g^{-1} \\ &- (A_+ - D_+ g\, g^{-1})(A_- - D_- g\, g^{-1}) + \varkappa A_+ R A_-\big] \ . \end{aligned} \tag{2.28}$$

For $\varkappa = 0$ one can integrate out A_a giving the standard coset sigma model action.[9]

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[8] The canonical choice of R annihilates Cartan generators and preserves (up to factors) the positive and negative root generators: $R(T_i) = 0, \ R(E_+) = -iE_+, \ R(E_-) = iE_-$.

[9] The simplicity of the first-order action (2.26) is related to the simplicity of the corresponding classical Hamiltonian description [4]. At the same time, its superstring generalization is not straightforward as P_η in (2.2) is not a projector and hence $P_\eta^2 \neq P_\eta$.

The bosonic part of the λ-model action (2.6) has parameters (k, λ) and a local G symmetry

$$I_{k,\lambda}(f, A, C) = k\big[I_{\rm gWZW}(f, A) - \frac{b^{-2}}{4\pi}\int d^2x\ {\rm Tr}(A_a - C_a)^2\big] , \qquad b^{-2} \equiv \lambda^{-2} - 1 . \tag{2.29}$$

Here $f \in F$, $A_a \in \mathfrak{f}$ is the gauge field of the F/F gauged WZW model and $C_a \in \mathfrak{g}$ (the term $(A_a - C_a)^2$ is equivalent to $(PA_a)^2$). b is a natural deformation parameter (like $\varkappa$ in (2.24)). The case of $b \to 0$ corresponds to $A_a = C_a$ or the F/G gauged WZW model. Another limit is as in (2.9), i.e. $k \to \infty$ and $b \to \infty$: $\lambda = 1 - \frac{\pi}{k}h + \ldots$ implies $b^{-2} = \frac{2\pi}{k}h + \ldots$. Then setting $f = 1 - \frac{4\pi}{k}v + \ldots$ where $v \in \mathfrak{f}$ we find from (2.29) the bosonic truncation of (2.10) [7]

$$I_{k\to\infty,\lambda\to 1} = -\int d^2x\ {\rm Tr}\big[\, v\, F_{+-}(A) + \frac{h}{2}(A_a - C_a)^2\big] , \tag{2.30}$$

where F_{ab} is the field strength of A_a. This is the interpolating action for the F/G coset sigma model and its non-abelian T-dual: if we first integrate over v we get $A_a = g^{-1}\partial_a g$, $g \in F$, and thus the original F/G coset model with tension h; if we first integrate over A_a and C_a we get a sigma model for v which is the non-abelian dual of the F/G coset model.

This suggests that (2.29) may be viewed as an interpolating model between the λ-deformation of the non-abelian T-dual model (a model for the field f found by first integrating out A_a and C_a) and a deformation of the F/G coset sigma model found by parameterizing A_a in terms of the fields g and $\tilde{g}$ (e.g., as $A_a = g^{-1}\partial_a g + \epsilon_{ab}\tilde{g}^{-1}\partial_b\tilde{g}$) and integrating out all fields $(f, \tilde{g}, C)$ other than g. The latter procedure need not, however, give a local action for g away from the $k \to \infty,\ b^{-2} \to \frac{2\pi}{k}h$ point.[10]

While the actions (2.26) and (2.29) look very different, having, in particular, different symmetries, one possibility is that they may be viewed as two dual faces of a “doubled” model related by Poisson–Lie type duality [8–10]. The η-model may then be the analog of the “solvable” member of the dual pair. We shall provide explicit evidence for this in Section 5 below.

Another possibility to relate the λ-model to the η-model is by a limit that will break the F/G symmetric structure of (2.29) to reflect the presence of the R-matrix in (2.23), (2.26). This limit will involve a certain scaling (and analytic continuation) of the group element f plus the map between the parameters (2.20), (2.21). We shall demonstrate the existence of such limit on various relevant F/G coset examples in the next section. We shall then study the effect of this limit on the corresponding supergravity backgrounds in Section 4.

3. Relating the λ-model to the η-model by a limit

The target space backgrounds that correspond to the η-model (2.2), (2.23) have abelian isometries associated with the Cartan directions of the algebra of F that are preserved by R-matrix. At the same time, the backgrounds that correspond to the λ-model (2.6), (2.29) (found by integrating out A_a and fixing a G-gauge on f) do not have isometries at all.[11] To be able to relate the cor-

[10] At the same time, since the deformed η-model action (2.26) depends not only on the current but also explicitly on g it does not allow a dualization in an obvious way, i.e. an analog of a dual model should be non-local.

[11] This is also a common feature of backgrounds corresponding to F/G gauged WZW models with a non-abelian G, but for a non-trivial λ deformation it applies also to the abelian G case [7].

responding metrics we thus need to take a certain scaling limit of the λ-model in the coordinates corresponding to the Cartan directions of F.[12]

Below we shall first explicitly demonstrate the existence of such limits on particular low-dimensional cases, $AdS_2 \times S^2$ and $AdS_3 \times S^3$, and then explain the general construction for S^n and similar spaces related by analytic continuation. We shall also explain the relation to the Pohlmeyer reduced model.

3.1. $AdS_2 \times S^2$

In the case of $AdS_2 \times S^2$ the relevant bosonic coset space is

$$\frac{SO(1,2)}{SO(1,1)} \times \frac{SO(3)}{SO(2)} . \tag{3.1}$$

Starting with the λ-model action (2.29), integrating out the gauge field and gauge-fixing the $SO(1,2) \times SO(3)$ field f as[13]

$$f = \big[\exp(it\sigma_3)\exp(\xi\sigma_1)\big]\oplus\big[\exp(i\varphi\sigma_3)\exp(i\zeta\sigma_1)\big] , \tag{3.2}$$

we find the following metric[14]

$$\begin{aligned}2\pi k^{-1}ds^2 = \frac{1}{1+2b^2}\big[&-dt^2+\cot^2 t\, d\xi^2 - 4b^2(1+b^2)(\cosh\xi\, dt - \cot t \sinh\xi\, d\xi)^2 \\ &+ d\varphi^2 + \cot^2\varphi\, d\zeta^2 + 4b^2(1+b^2)(\cos\zeta\, d\varphi + \cot\varphi \sin\zeta\, d\zeta)^2\big] .\end{aligned} \tag{3.3}$$

Note that here, for the AdS_2 part, we are considering a different patch of the deformed space than used in [19] which corresponds to

$$\tilde f = \big[\exp(\tilde\xi\sigma_2)\exp(\tilde t\sigma_1)\big]\oplus\big[\exp(i\varphi\sigma_1)\exp(i\zeta\sigma_1)\big] , \tag{3.4}$$

leading instead to

$$\begin{aligned}2\pi k^{-1}\widetilde{ds}^2 = \frac{1}{1+2b^2}\big[&d\tilde\xi^2 - \coth^2\tilde\xi\, d\tilde t^2 + 4b^2(1+b^2)(\cosh\tilde t\, d\tilde\xi + \coth\tilde\xi \sinh\tilde t\, d\tilde t)^2 \\ &+ d\varphi^2 + \cot^2\varphi\, d\zeta^2 + 4b^2(1+b^2)(\cos\zeta\, d\varphi + \cot\varphi \sin\zeta\, d\zeta)^2\big] ,\end{aligned} \tag{3.5}$$

i.e. related to (3.3) via the analytic continuation

$$\tilde\xi = it , \qquad \tilde t = \xi . \tag{3.6}$$

The reason we consider the patch (3.3) is that it admits a special (singular) field redefinition with which we can recover the metric corresponding to the η-deformed $AdS_2 \times S^2$ model [4,1].

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[12] The special role of these coordinates may be anticipated from the fact that the λ-model (2.29) can be viewed as a deformation of the F/F gauged WZW model, which is a topological theory [24]. In the F/F gauged WZW model the gauge symmetry ($f' = w^{-1} f w$, $w \in F$) allows one to gauge away all but the Cartan directions, i.e. to choose $f = e^{\varphi_i T_i}$, $[T_i, T_j] = 0$, so that the Lagrangian becomes $L = \partial_+\varphi_i\partial_-\varphi_i + A_{+i}\partial_-\varphi_i - A_{-i}\partial_+\varphi_i$ with $\varphi_i = a_i =$ const as the only solutions. One may then use these moduli parameters a_i to define certain limits of the deformed background.

[13] Here σ_i are Pauli matrices and $\{(\sigma_1\oplus 0), (0\oplus i\sigma_1)\}$ generates the gauge group. We also take $\mathrm{Tr} = 2\mathrm{tr}$, where tr is the usual matrix trace, i.e. $\nu = \frac{1}{2}$ in footnote 4.

[14] We shall use the following notation to relate the bosonic part of the action to the metric: $I = \int d^2x\, G_{mn}(X)\partial_+X^m\partial_-X^n$ with $ds^2 = G_{mn}(X)dX^m dX^n$, i.e. we will absorb all overall constants in the action into the metric. All the bosonic backgrounds we will consider below will not have a non-trivial B field [27,20].

Let us now consider the following (complex) coordinate redefinition $(t, \xi; \varphi, \zeta) \to (t, \rho; \varphi, r)$ combined with infinite imaginary shifts of the (t, φ) directions (turning them into isometries):

$$t \to t + \frac{i}{2}\log\big[\frac{1-\varkappa^2\rho^2}{1+\rho^2}\big] + i\log\gamma\ , \qquad \xi \to \frac{1}{2}\log\big[-\frac{1-\varkappa\rho}{1+\varkappa\rho}\big]\ ,$$
$$\varphi \to \varphi + \frac{i}{2}\log\big[\frac{1+\varkappa^2 r^2}{1-r^2}\big] + i\log\gamma\ , \qquad \zeta \to \frac{i}{2}\log\big[-\frac{1+i\varkappa r}{1-i\varkappa r}\big]\ , \qquad \gamma\to\infty\ . \quad (3.7)$$

Here we have introduced the parameter $\varkappa$, which is assumed to be related to b by (2.20). We shall also assume that k is related to h by (2.21), i.e.

$$b^2 = -\frac{1}{2} + \frac{i}{2\varkappa}\ , \qquad h = \frac{k}{\pi(1+2b^2)}\ . \quad (3.8)$$

Then the metric (3.3) transforms into

$$2h^{-1}ds^2 = \frac{1}{1-\varkappa^2\rho^2}\big[-(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2}\big] + \frac{1}{1+\varkappa^2 r^2}\big[(1-r^2)d\varphi^2 + \frac{dr^2}{1-r^2}\big]\ , \quad (3.9)$$

i.e. becomes exactly the η-deformed $AdS_2 \times S^2$ metric [1,16,13,25] with h as a tension. Indeed, this metric corresponds to (2.23) with g parameterized as

$$g = \big[\exp(\frac{it}{2}\sigma_3)\exp(\frac{1}{2}\operatorname{arcsinh}\rho\,\sigma_2)\big]\oplus\big[\exp(\frac{i\varphi}{2}\sigma_3)\exp(\frac{i}{2}\arcsin r\,\sigma_2)\big]\ , \quad (3.10)$$

and the R-matrix chosen to annihilate the Cartan directions $\{i\sigma_3\oplus 0, 0\oplus i\sigma_3\}$.

This relation between (3.3) and (3.9) involving complex coordinate redefinitions (3.7) and a complex map between parameters (3.8) suggests that the λ-model and η-model may correspond to different real "slices" of some larger complexified model.

To shed more light on the meaning of the infinite imaginary shift of t and φ in (3.7) that plays a central role in the above relation between (3.3) and (3.9) it is useful to repeat the discussion using a simpler (algebraic) choice of coordinates in which the metric becomes conformally flat. Starting with (3.3) and doing the coordinate redefinition $(t, \xi; \varphi, \zeta) \to (x, y; p, q)$

$$t = \arccos\sqrt{x^2-y^2}\ , \qquad \xi = \operatorname{arccosh}\frac{x}{\sqrt{x^2-y^2}}\ , \qquad x^2-y^2<1\ ,$$
$$\varphi = \arccos\sqrt{p^2+q^2}\ , \qquad \zeta = \arccos\frac{p}{\sqrt{p^2+q^2}}\ , \qquad p^2+q^2<1\ , \quad (3.11)$$

we find

$$2\pi k^{-1}ds^2 = \frac{1}{1-x^2+y^2}\big[-(1+2b^2)dx^2 + \frac{dy^2}{1+2b^2}\big] + \frac{1}{1-p^2-q^2}\big[(1+2b^2)dp^2 + \frac{dq^2}{1+2b^2}\big]\ . \quad (3.12)$$

Formally continuing to the region for which $x^2-y^2>1$ represents (3.5), i.e. the original metric of [19]. Furthermore, one can check that $x^2-y^2=1$ is a curvature singularity and hence the two patches covered by (3.3) and (3.5) are separated by this singularity.

Using again the relation between (k, b) and $(h, \varkappa)$ in (3.8) and making an infinite rescaling of the coordinates

$$x \to \gamma \varkappa x\,, \qquad y \to \gamma y\,, \qquad p \to \gamma \varkappa p\,, \qquad q \to \gamma q\,, \qquad \gamma \to \infty\,, \tag{3.13}$$

we get

$$2h^{-1} ds^2 = \frac{1}{y^2 - \varkappa^2 x^2}\big(dy^2 + dx^2\big) + \frac{1}{q^2 + \varkappa^2 p^2}\big(-dq^2 + dp^2\big)\,. \tag{3.14}$$

This may be interpreted as the metric of η-deformed $H^2 \times dS_2$ (Euclidean AdS_2 times 2d de Sitter space)[15] background which is related to $AdS_2 \times S^2$ by an analytic continuation.[16] We will elaborate on this limit (giving its alternative form) focussing on the S^2 part of (3.3) in Appendix A.

The infinite scaling limit (3.13) relating the λ-model to the η-model amounts to dropping the constants 1 in the denominators in (3.12). It thus corresponds to decoupling the asymptotically flat region of the λ-model metric (3.12) so that the η-model metric may be interpreted as emerging in a "near-horizon" limit (combined with an analytic continuation of the parameters according to (2.20), (2.21)).

3.2. $AdS_3 \times S^3$

Let us now consider the λ-deformed action (2.6), (2.29) for the coset corresponding to $AdS_3 \times S^3$:

$$\frac{SO(2,2)}{SO(2,1)} \times \frac{SO(4)}{SO(3)}\,. \tag{3.15}$$

Parameterizing the gauge-fixed group-valued field f (for the parts associated with AdS_3 and S^3 respectively) as

$$\begin{aligned} f = &\big[\exp(it(\sigma_3 \oplus -\sigma_3))\exp(\xi(\sigma_1 \oplus \sigma_1))\exp(i\psi(\sigma_3 \oplus \sigma_3))\big] \\ &\oplus \big[\exp(i\varphi(\sigma_3 \oplus -\sigma_3))\exp(i\zeta(\sigma_1 \oplus \sigma_1))\exp(i\phi(\sigma_3 \oplus \sigma_3))\big]\,, \end{aligned} \tag{3.16}$$

and integrating out the gauge field, we find the following metric (cf. (3.3))[17]

$$\begin{aligned} 2\pi k^{-1} ds^2 = \frac{1}{1+2b^2}\big[&-dt^2 + J^2 + \coth^2\xi\, K^2 - 4b^2(1+b^2)(\cosh^2\xi\,(dt-K)^2 - J^2) \\ &+ d\varphi^2 + \tilde{J}^2 + \cot^2\zeta\, \tilde{K}^2 + 4b^2(1+b^2)(\cos^2\zeta\,(d\varphi+\tilde{K})^2 + \tilde{J}^2)\big]\,, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} J &= \csc(2t)\big(\sin(2\psi)d\xi - \coth\xi(\cos(2t) - \cos(2\psi))d\psi\big)\,, \\ K &= \csc(2t)\big(\tanh\xi(\cos(2t) + \cos(2\psi))d\xi - \sin(2\psi)d\psi\big)\,, \\ \tilde{J} &= \csc(2\varphi)\big(\sin(2\phi)d\zeta + \cot\zeta(\cos(2\varphi) - \cos(2\phi))d\phi\big)\,, \\ \tilde{K} &= \csc(2\varphi)\big(\tan\zeta(\cos(2\varphi) + \cos(2\phi))d\zeta + \sin(2\phi)d\phi\big)\,. \end{aligned} \tag{3.18}$$

[15] The $\varkappa \to \infty$ limit of (3.14) gives the same metric as $\varkappa = 0$ but with reversed overall sign and the roles of coordinates interchanged.

[16] Note that the "flat-slicing" or Poincaré-patch like real coordinates do not exist for S^2 but exist for its analytic continuation dS_2.

[17] Here we take $\mathrm{Tr} = \mathrm{tr}$, where tr is the usual matrix trace, i.e. $\nu = 1$ in footnote 4.

Taking the same limit as in the $AdS_2 \times S^2$ case, i.e. using the redefinitions (3.7) and (3.8), we find that (3.17) becomes

$$2h^{-1}ds^2 = \frac{1}{1-\varkappa^2\rho^2}\big[-(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2}\big] + \rho^2 d\psi^2 + \frac{1}{1+\varkappa^2 r^2}\big[(1-r^2)d\varphi^2 + \frac{dr^2}{1-r^2}\big] + r^2 d\phi^2 \ . \tag{3.19}$$

This is precisely the metric [16,13,26] that corresponds to the deformed $AdS_3 \times S^3$ η-model action (2.2), (2.23) with $g \in F$ parameterized as

$$g = \big[\exp(\frac{it}{2}(\sigma_3 \oplus -\sigma_3) + \frac{i\psi}{2}(\sigma_3 \oplus \sigma_3))\exp(\frac{1}{2}\operatorname{arcsinh}\rho\,(\sigma_2 \oplus -\sigma_2))\big] \oplus \big[\exp(\frac{i\varphi}{2}(\sigma_3 \oplus -\sigma_3) + \frac{i\phi}{2}(\sigma_3 \oplus \sigma_3))\exp(\frac{i}{2}\arcsin r\,(\sigma_2 \oplus -\sigma_2))\big] \ , \tag{3.20}$$

and the R-matrix chosen to annihilate the Cartan directions $\{(i\sigma_3 \oplus 0 \oplus 0 \oplus 0),\ (0 \oplus i\sigma_3 \oplus 0 \oplus 0),\ (0 \oplus 0 \oplus i\sigma_3 \oplus 0),\ (0 \oplus 0 \oplus 0 \oplus i\sigma_3)\}$.[18]

3.3. S^n and analytic continuations to AdS_n, dS_n and H^n

Let us now describe a systematic procedure for taking the above limit, relating the actions of the λ-model and η-model in the general $AdS_n \times S^n$ case by considering for simplicity the F/G coset corresponding to the S^n factor, i.e.

$$\frac{SO(n+1)}{SO(n)} \ . \tag{3.21}$$

We shall use the antisymmetric real matrices as the familiar basis of the algebra $\mathfrak{so}(n+1)$[19]

$$(T_{ab})_{ij} = \delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi} \ , \qquad a, b, i, j = 1, \ldots, n+1 \ , \tag{3.22}$$

with the projector onto the coset being given by

$$P_2(M) = -\sum_{a=2}^{n+1} \mathrm{Tr}(M\,T_{1a})\,T_{1a} \ . \tag{3.23}$$

In general, we will choose to parameterize the gauge-fixed field $f \in F = SO(n+1)$ in the action (2.29) as

$$f = \exp(2\varphi T_{12})\exp(2\zeta T_{23})\exp(2\phi_1 T_{34})\exp(2\chi T_{45})\exp(2\phi_2 T_{56})\ldots \tag{3.24}$$

and then take a sequence of limits of the following type

$$\Psi \to \Psi + i\log\gamma \ , \qquad \gamma \to \infty \ , \tag{3.25}$$

first on $\Psi = \varphi$ and then on every other field in (3.24), i.e. on ϕ_1, then on ϕ_2, etc. This effectively picks out a Cartan subalgebra of $\mathfrak{so}(n+1)$

[18] Note that there are actually two choices of solution to the corresponding modified classical YBE, one of which gives the required deformation (3.19) – see [28].

[19] Here we will also take $\mathrm{Tr} = \frac{1}{2}\mathrm{tr}$, where tr is the usual matrix trace, i.e. $\nu = 2$ in footnote 4.

$$\{T_{12},\ T_{34},\ T_{56},\ \ldots\}\ , \tag{3.26}$$

and the angles $\varphi, \phi_1, \phi_2, \ldots$ will become isometries of the resulting metric.

A couple of comments are in order. First, it is worth noting that for n odd the last exponential factor in (3.24) is in the sequence and hence the prescription tells us that we should take the limit in the corresponding field. In the S^3 and S^5 examples below this final limit is not necessary: the previous limits already lead to this direction being an isometry and hence the limit (3.25) would be trivial (the same should also be true for all odd n). A related observation is that it always appears to be possible to truncate easily from $n = 2N + 1$ to $n = 2N$ by just setting this final angle to zero. It transpires that to go from $n = 2N$ to $n = 2N - 1$ is not so trivial. This is not so much to do with taking the limit, rather with the field redefinitions and analytic continuations that we need to perform to recover the metrics of [16,13,29].

In the following we will consider the two non-trivial cases $n = 3$ (already discussed in Section 3.2 above) and $n = 5$, with the $n = 2$ and $n = 4$ examples following as simple truncations. It will be useful to define the following functions

$$\mathrm{f}\,(r) = \frac{1}{1+\varkappa^2 r^2}\ , \qquad \mathrm{g}\,(r) = \frac{1}{1-r^2}\ , \qquad \mathrm{v}\,(r,\theta) = \frac{1}{1+\varkappa^2 r^4 \sin^2\theta}\ . \tag{3.27}$$

n = 3 and **n = 2**: Starting with (2.29) and taking the limits as described above we end up with a metric with two isometric directions φ and ϕ_1. There are then two analytic continuations/coordinate redefinitions that are of particular interest. The first is given by

$$\varphi \to \varphi + \frac{i}{2}\log\big[\frac{1+\varkappa^2 r^2}{1-r^2}\big]\ , \qquad \zeta \to \frac{i}{2}\log\big[-\frac{1+i\varkappa r}{1-i\varkappa r}\big]\ , \qquad \phi_1 \to \phi_1\ , \tag{3.28}$$

and the resulting metric is as in (3.19) (with $\phi = \phi_1$)

$$2h^{-1}ds^2 = \mathrm{f}\,(\mathrm{g}^{-1}d\varphi^2 + \mathrm{g}\,dr^2) + r^2 d\phi_1\ . \tag{3.29}$$

This metric is precisely the deformation of S^3 arising from the corresponding η-model [16,13, 28,26]: it follows from the η-model action (2.2), (2.23) with $g \in F$ parameterized as

$$g = \exp(\phi_1 T_{34})\exp(\varphi T_{12})\exp(\arcsin r\, T_{13})\ , \tag{3.30}$$

and the R-matrix chosen to annihilate the Cartan directions $\{T_{12}, T_{34}\}$. The second change of variables is given by

$$\varphi \to i\varkappa\varphi + \frac{i}{2}\log\big[\frac{1-r^2}{1+\varkappa^2 r^2}\big]\ , \qquad \zeta \to \frac{i}{2}\log\big[\frac{1-r}{1+r}\big]\ , \qquad \phi_1 \to i\varkappa\phi_1\ , \tag{3.31}$$

with the resulting metric being

$$2h^{-1}ds^2 = \mathrm{g}\,(\mathrm{f}^{-1}d\varphi^2 + \mathrm{f}\,dr^2) + r^{-2}d\phi_1^2\ . \tag{3.32}$$

This metric is related to (3.29) by two T-dualities – in each of the isometric directions φ and ϕ_1. Furthermore, there is a formal map between the two metrics (3.29) and (3.32) given by

$$\varphi \to i\varkappa\varphi\ , \qquad r \to \frac{i}{\varkappa r}\ , \qquad \phi_1 \to i\varkappa\phi_1\ . \tag{3.33}$$

To recover the corresponding expressions for $n = 2$ one can consistently truncate by setting $\phi_1 = 0$.

n = 5 and **n = 4**: Taking the limits as described above, from (2.29) we find a metric with three isometric directions φ, ϕ_1 and ϕ_2. There are again two analytic continuations/coordinate redefinitions that are of particular interest. The first is given by

$$\begin{aligned} &\varphi \to \varphi + \frac{i}{2}\log\big[\frac{1+\varkappa^2 r^2}{1-r^2}\big] , \qquad \phi_1 \to i\varkappa\phi_1 + i\log\cos\theta , \qquad \phi_2 \to \phi_2 , \\ &\zeta \to \frac{i}{2}\log\big[-\frac{1+i\varkappa r}{1-i\varkappa r}\big] , \qquad \chi \to \frac{i}{2}\log\big[-\frac{1-\sin\theta}{1+\sin\theta}\big] , \end{aligned} \tag{3.34}$$

and the resulting metric is (with f, g, v defined in (3.27))

$$\begin{aligned} 2h^{-1}ds^2 = \mathrm{f}\,(\mathrm{g}^{-1}d\varphi^2 + \mathrm{g}\,dr^2) + \frac{(d\phi_1 + \varkappa r^4 \mathrm{v}\,\sin\theta\cos\theta\, d\theta)^2}{r^2 \mathrm{v}\,\cos^2\theta} \\ + r^2 \mathrm{v}\, d\theta^2 + r^2\sin^2\theta\, d\phi_2^2 . \end{aligned} \tag{3.35}$$

As shown in [29], this metric is T-dual to the metric constructed in [16], which follows from the η-model (2.2), (2.23) of [4,1] with $g \in F$ parameterized as

$$g = \exp(\phi_2 T_{56})\exp(\phi_1 T_{34})\exp(\theta T_{35})\exp(\varphi T_{12})\exp(\arcsin r\, T_{13}) , \tag{3.36}$$

and the R-matrix chosen to annihilate the Cartan directions $\{T_{12}, T_{34}, T_{56}\}$. Here the T-duality should be done in just the ϕ_1 isometry, making the metric diagonal but generating a non-zero B-field, in agreement with the background found in [16].[20]

The second change of variables is given by

$$\begin{aligned} &\varphi \to i\varkappa\varphi + \frac{i}{2}\log\big[\frac{1-r^2}{1+\varkappa^2 r^2}\big] , \qquad \phi_1 \to i\varkappa\phi_1 + i\log\cos\theta , \qquad \phi_2 \to i\varkappa\phi_2 , \\ &\zeta \to \frac{i}{2}\log\big[\frac{1-r}{1+r}\big] , \qquad \chi \to \frac{i}{2}\log\big[\frac{1-\sin\theta}{1+\sin\theta}\big] , \end{aligned} \tag{3.37}$$

leading to

$$\begin{aligned} 2h^{-1}ds^2 = \mathrm{g}\,(\mathrm{f}^{-1}d\varphi^2 + \mathrm{f}\,dr^2) + \frac{(d\phi_1 + \varkappa r^4 \mathrm{v}\,\sin\theta\cos\theta d\theta)^2}{r^2 \mathrm{v}\,\cos^2\theta} \\ + r^2 \mathrm{v}\, d\theta^2 + r^{-2}\csc^2\theta\, d\phi_2^2 . \end{aligned} \tag{3.38}$$

This metric (related to (3.35) by two T-dualities) is also T-dual to the metric found in [16]: here one needs three T-dualities – in each of the isometric directions φ, ϕ_1 and ϕ_2. There is again a formal map between the two metrics (3.35) and (3.38) given by

$$\begin{aligned} &\varphi \to i\varkappa\varphi , \qquad r \to \frac{i}{\varkappa r} , \qquad \phi_1 \to \phi_1 + i\log\sin\theta , \\ &\theta \to i\log\big[-i\tan\frac{\theta}{2}\big] , \qquad \phi_2 \to i\varkappa\phi_2 . \end{aligned} \tag{3.39}$$

To obtain similar expressions for the $n = 4$ case one can consistently truncate by setting $\phi_2 = 0$ in the $n = 5$ expressions.

Let us now briefly outline the analytic continuations to AdS_n, dS_n and H^n. These geometries are all based on different real forms of the complexified coset space $\frac{SO(n+1,\mathbb{C})}{SO(n,\mathbb{C})}$, i.e.

[20] Note that all the λ-model backgrounds corresponding to the choice of f in (3.24) have no B field [27,20].

$$S^n = \frac{SO(n+1)}{SO(n)} \ , \qquad AdS_n = \frac{SO(2,n-1)}{SO(1,n-1)} \ ,$$
$$dS_n = \frac{SO(1,n)}{SO(1,n-1)} \ , \qquad H^n = \frac{SO(1,n)}{SO(n)} \ . \tag{3.40}$$

After a brief study of the group elements of interest (3.24), (3.30), (3.36) one can see that for H^n there is essentially one analytic continuation of the basis (3.22), while for AdS_n and dS_n there are many potentially inequivalent ones, which in turn may lead to metrics covering different coordinate patches of the η-model and λ-model metrics.

For AdS_n one choice of analytic continuation is given by

$$T_{1\hat{a}} \to iT_{1\hat{a}} \ , \qquad T_{2\hat{a}} \to iT_{2\hat{a}} \ , \qquad \hat{a} = 3, \dots, n+1 \ , \tag{3.41}$$

for which the subalgebra commuting with T_{12}, spanned by $T_{\hat{a}\hat{b}}$, remains $\mathfrak{so}(n-1)$. This corresponds to analytically continuing the fields as follows

$$\varphi \to t \ , \qquad \phi_i \to \psi_i \ , \qquad \zeta \to i\xi \ , \qquad \chi \to \hat{\chi} \ , \qquad r \to i\rho \ , \qquad \theta \to \hat{\theta} \ . \tag{3.42}$$

Here we also need to flip the overall sign of the metrics. Other possible analytic continuations involve $T_{12} \to iT_{12}$, so that the subalgebra commuting with this generator is then $\mathfrak{so}(1, n-2)$. It is an analytic continuation of this form that is required to obtain the first line of (3.5) from the second line and was considered in the supergravity constructions of [19,20].

For dS_n one choice of the analytic continuation is given by

$$T_{12} \to iT_{12} \ , \qquad T_{2\hat{a}} \to iT_{2\hat{a}} \ , \qquad \hat{a} = 3, \dots, n+1 \ , \tag{3.43}$$

for which the subalgebra commuting with T_{12}, spanned by $T_{\hat{a}\hat{b}}$, remains $\mathfrak{so}(n-1)$. This corresponds to analytically continuing the fields as follows

$$\varphi \to it \ , \qquad \phi_i \to \psi_i \ , \qquad \zeta \to i\xi \ , \qquad \chi \to \hat{\chi} \ , \qquad r \to \rho \ , \qquad \theta \to \hat{\theta} \ . \tag{3.44}$$

The remaining analytic continuations, which we will not explore in detail here, involve leaving T_{12} as is, so that the subalgebra commuting with this generator is again $\mathfrak{so}(1, n-2)$.

To recover the coset and deformed models associated with H^n we analytically continue

$$T_{1\bar{a}} \to iT_{1\bar{a}} \ , \qquad \varphi \to i\varphi \ , \qquad r \to ir \ , \qquad \bar{a} = 2, \dots, n+1 \ , \tag{3.45}$$

and, as for AdS_n, flip the overall sign of the metrics. It will also be useful to give the direct analytic continuation of the fields from AdS_n to H^n, i.e. combining the inverse of (3.42) and (3.45)

$$t \to i\varphi \ , \qquad \phi_i \to \psi_i \ , \qquad \xi \to -i\zeta \ , \qquad \chi \to \hat{\chi} \ , \qquad \rho \to r \ , \qquad \theta \to \hat{\theta} \ . \tag{3.46}$$

3.4. Relation to the Pohlmeyer-reduced model for $AdS_n \times S^n$ and the $\eta \to i/\lambda \to 0$ limit

The Pohlmeyer-reduced model is conjectured to be related to the λ-model at the special point in the parameter space $\lambda = 0$ or $b = 0$ [5,6], or, equivalently, according to (2.20), $\eta = i$ or $\varkappa = i$. For this point the relation between the overall couplings (2.21) becomes $h = \frac{k}{\pi}$. As discussed beneath (2.29) the $b \to 0$ limit of the λ-model gives the F/G gauged WZW model. On the other hand, it was shown in [13] that for the η-models arising as deformations of $AdS_2 \times S^2$ and $AdS_3 \times S^3$ models the $\varkappa \to i$ limit of (3.29) can be taken in such a way (combining it with a coordinate redefinition) that it gives a string action in a pp-wave type background, whose light-cone gauge-fixing is the Pohlmeyer reduction (PR) [27,30,31] of these $AdS_n \times S^n$ models.[21]

In Section 3.3 we considered a sequence of special coordinate redefinitions that led from the λ-model to (T-duals) of the η-model. In the cases of S^2 and S^3 there was only one limit in this sequence (3.28). One can thus see the emergence of the PR model from the λ-model in a special limit (cf. also [5,14]).

In the $AdS_5 \times S^5$ case the $\varkappa \to i$ limit of the η-model did not lead directly to the PR model, but rather to a closely related theory with an imaginary B field [13]. It is now clear that there is a natural "intermediate" candidate model for recovering the PR model found by making only the first coordinate redefinition in the sequence (3.25), (3.34) along with the corresponding one for AdS_5

$$\begin{aligned} &t \to t + \frac{i}{2}\log\big[\frac{1-\varkappa^2\rho^2}{1+\rho^2}\big] + i\log\gamma\ , \qquad \xi \to \frac{1}{2}\log\big[-\frac{1-\varkappa\rho}{1+\varkappa\rho}\big]\ , \\ &\varphi \to \varphi + \frac{i}{2}\log\big[\frac{1+\varkappa^2 r^2}{1-r^2}\big] + i\log\gamma\ , \qquad \zeta \to \frac{i}{2}\log\big[-\frac{1+i\varkappa r}{1-i\varkappa r}\big]\ , \qquad \gamma\to\infty\ , \end{aligned} \tag{3.47}$$

and using the relation of the parameters in (3.8). It is interesting to note that considering the analytic continuation to $H^5 \times dS_5$ given in (3.44), (3.46) this becomes

$$\begin{aligned} &\varphi \to \varphi + \frac{1}{2}\log\big[\frac{1-\varkappa^2 r^2}{1+r^2}\big] + \log\gamma\ , \qquad \zeta \to \frac{i}{2}\log\big[-\frac{1-\varkappa r}{1+\varkappa r}\big]\ , \\ &t \to t + \frac{1}{2}\log\big[\frac{1+\varkappa^2\rho^2}{1-\rho^2}\big] + \log\gamma\ , \qquad \xi \to \frac{1}{2}\log\big[-\frac{1+i\varkappa\rho}{1-i\varkappa\rho}\big]\ , \qquad \gamma\to\infty\ , \end{aligned} \tag{3.48}$$

which for $\varkappa^2 \in (0,-1]$ is a real field redefinition and real limit. Furthermore, for $\varkappa$ in this range the map between the parameters (3.8) also becomes real. Therefore, this limit of the $AdS_5 \times S^5$ λ-model can be thought of as first an analytic continuation to $H^5 \times dS_5$, then a real limit and field redefinition and finally analytically continuing back.

Following this procedure we find a somewhat involved metric, which has isometric directions t and φ and importantly is real for $\varkappa^2 \in (0,-1]$.[22] Therefore, it is natural to conjecture that the light-cone gauge-fixing of this model is related to the kink S-matrix of [18].[23]

The limit of [13]

$$t = \epsilon x^- + \frac{x^+}{\epsilon}\ , \qquad \varphi = \epsilon x^- - \frac{x^+}{\epsilon}\ ,$$

[21] If one takes the $\varkappa \to i$ limit of the η-model without rescaling the coordinates the resulting action gives the same model without the potential term, i.e. one time and one space dimension decouple. The metric in the "transverse" directions is that of the $SO(2)\times SO(1,1)$ and $\frac{SO(3)}{SO(2)} \times \frac{SO(1,2)}{SO(2)}$ gauged WZW models for $n=2$ and $n=3$ respectively.

[22] Recall that if we take the second special limit for ϕ_1 in (3.34) the off-diagonal terms in the resulting metric (3.35) are imaginary for this range of $\varkappa$.

[23] This discussion is also true if we only consider the first coordinate redefinition in the sequence (3.25), (3.37), however, the resulting metrics are diffeomorphic as they are related by the map

$$t \to i\varkappa t\ , \qquad \rho \to -\frac{i}{\varkappa\rho}\ , \qquad \varphi \to i\varkappa\varphi\ , \qquad r \to \frac{i}{\varkappa r}$$

which is real for $\varkappa^2 \in (0,-1]$.

$$\rho = \tan\alpha\ , \quad r = \tanh\beta\ , \quad \varkappa = \sqrt{-1+\epsilon^2}\ , \quad \epsilon \to 0\ , \tag{3.49}$$

for the $AdS_3 \times S^3$ η-model gives a pp-wave type model whose light-cone gauge fixing is the Pohlmeyer reduction of strings on $AdS_3 \times S^3$ [31] with axial gauging of the associated gauged WZW model. In higher dimensions the gauge group of the PR theory is no longer abelian and hence axial gauging is not possible. Therefore, the limit (3.49) needs a mild modification to extract the vector gauged model

$$\begin{aligned} &t = \epsilon x^- + \frac{x^+}{\epsilon}\ , \qquad \varphi = \epsilon x^- - \frac{x^+}{\epsilon}\ , \\ &\rho = \cot\alpha\ , \quad r = \coth\beta\ , \quad \varkappa = \sqrt{-1-\epsilon^2}\ , \quad \epsilon \to 0\ . \end{aligned} \tag{3.50}$$

Taking this limit in the model obtained by the special limit (3.47) of the λ-model associated with $AdS_5 \times S^5$ we find a pp-wave type metric (recall that in this limit we get from (2.21) that $h = \frac{k}{\pi}$)

$$\begin{aligned} 2h^{-1}ds^2 = &-4dx^-dx^+ + \frac{1}{2}(\cos\alpha - \cosh\beta)\,(dx^+)^2 \\ &+ ds^2_{A\perp}(\alpha, \psi_1, \hat\chi, \psi_2) + ds^2_{S\perp}(\beta, \phi_1, \chi, \phi_2)\ , \end{aligned} \tag{3.51}$$

where the "transverse" metrics $ds^2_{A\perp}$ and $ds^2_{S\perp}$ are those of the gauged WZW model for $\frac{SO(5)}{SO(4)}$ and $\frac{SO(1,4)}{SO(4)}$ respectively.[24] The light-cone gauge-fixing of this model ($x^+ = \mu\tau$) corresponds therefore to the Pohlmeyer-reduced theory for strings on $AdS_5 \times S^5$ [27]. Note that as for the $AdS_2 \times S^2$ and $AdS_3 \times S^3$ cases, the roles of the AdS_n and S^n are effectively interchanged, i.e. the $\varkappa \to i$ limit of the deformed AdS_5 metric leads to the PR model for the string on $\mathbb{R} \times S^5$ and vice versa.

4. Supergravity backgrounds for deformed models: $AdS_2 \times S^2$

Having discussed the form of the metrics corresponding to the η-model and λ-model let us now consider their extension to the full type IIB supergravity backgrounds expected to be associated with the superstring actions (2.2) and (2.6). The direct construction of such backgrounds supporting the metrics of η-model turns out to be quite non-trivial [16,29]. At the same time, the RR backgrounds supporting the λ-model metrics appear to be much simpler and they were found explicitly in the $AdS_n \times S^n$ cases in [19] ($n = 2, 3$) and [20] ($n = 5$).

Given that the metrics of η-model can be obtained, as explained above, from the metrics of the λ-model by a special scaling limit and analytic continuation, one may reconstruct the full supergravity backgrounds that emerge when this limit is applied to the solutions of [19,20]. This will be explored below on the simplest $AdS_2 \times S^2$ example. Surprisingly, the resulting limiting background will be different from the one constructed in [29], even though the two share the same metric (3.9). Understanding the proper meaning of this solution (that takes a very simple form in the algebraic coordinates introduced in (3.11), (3.12)) will be left for the future.

To discuss the deformed backgrounds associated with the $AdS_2 \times S^2$ supercoset it is useful to follow [29] and consider the compactification of 10d type IIB supergravity to four dimensions

[24] We parameterize the gauge-fixed field $f_{\rm PR} \in SO(5) \times SO(1,4)$ of the PR model as

$$f_{\rm PR} = \big[\exp(2\alpha T_{23})\exp(\psi_1 T_{34})\exp(\hat\chi T_{45})\exp(\psi_2 T_{56})\big]\oplus\big[\exp(2i\beta T_{23})\exp(\phi_1 T_{34})\exp(\chi T_{45})\exp(\phi_2 T_{56})\big]\ ,$$

and integrate out the gauge field.

on T^6 retaining only the metric, dilaton and a single RR 1-form potential $A = A_m dx^m$.[25] The resulting bosonic 4d action is then given by

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\Big[e^{-2\Phi}\big[R + 4(\nabla\Phi)^2\big] - \frac{1}{4}F_{mn}F^{mn}\Big]\,. \tag{4.1}$$

The corresponding equations of motion are

$$R + 4\nabla^2\Phi - 4(\nabla\Phi)^2 = 0\,, \qquad R_{mn} + 2\nabla_m\nabla_n\Phi = \frac{e^{2\Phi}}{2}(F_{mp}F_n{}^p - \frac{1}{4}g_{mn}F^2)\,,$$
$$\partial_n(\sqrt{-g}F^{mn}) = 0\,. \tag{4.2}$$

The first two equations imply that the dilaton should satisfy $\nabla^2 e^{-2\Phi} = 0$.

4.1. Angular coordinates

Our starting point will be the supergravity solution of [19] supporting the λ-model metric (3.5)[26]

$$\begin{aligned}
2\pi k^{-1}\widetilde{ds}^2 = \frac{1}{1+2b^2}\big[&d\tilde\xi^2 - \coth^2\tilde\xi\, d\tilde t^2 + 4b^2(1+b^2)(\cosh\tilde t\, d\tilde\xi + \coth\tilde\xi\sinh\tilde t\, d\tilde t)^2\\
&+ d\varphi^2 + \cot^2\varphi\, d\zeta^2 + 4b^2(1+b^2)(\cos\zeta\, d\varphi + \cot\varphi\sin\zeta\, d\zeta)^2\big]\,,\\
e^{\tilde\Phi} = \frac{e^{\tilde\Phi_0}}{\sinh\tilde\xi\,\sin\varphi}&\,,\\
\sqrt{2\pi k^{-1}}\tilde A = -4\sqrt{\frac{b^2(b^2+1)}{1+2b^2}}&\,e^{-\tilde\Phi_0}\\
&\times\big[c_1\cos\varphi\cos\zeta\, d(\cosh\tilde\xi\sinh\tilde t) + c_2\cosh\tilde\xi\cosh\tilde t\, d(\cos\varphi\sin\zeta)\big]\,.
\end{aligned} \tag{4.3}$$

Here the free constants c_1 and c_2 satisfy

$$c_1^2 + c_2^2 = 1\,, \tag{4.4}$$

and encode the usual freedom of $U(1)$ electromagnetic duality rotations in 4d. The choice $c_1 = c_2 = \frac{1}{\sqrt{2}}$ ensures symmetry between the two coset factors.

Analytically continuing the AdS_2 coset part to the patch of interest (3.2)

$$\tilde\xi = it\,, \qquad \tilde t = \xi\,, \qquad e^{\tilde\Phi_0} = ie^{\Phi_0}\,, \tag{4.5}$$

gives the following solution of the equations of motion (4.2) supporting the metric (3.3)

$$\begin{aligned}
2\pi k^{-1}ds^2 = \frac{1}{1+2b^2}\big[&-dt^2 + \cot^2 t\, d\xi^2 - 4b^2(1+b^2)(\cosh\xi\, dt - \cot t\sinh\xi\, d\xi)^2\\
&+ d\varphi^2 + \cot^2\varphi\, d\zeta^2 + 4b^2(1+b^2)(\cos\zeta\, d\varphi + \cot\varphi\sin\zeta\, d\zeta)^2\big]\,,\\
e^{\Phi} = \frac{e^{\Phi_0}}{\sin t\,\sin\varphi}&\,,
\end{aligned}$$

[25] The corresponding 10d 5-form strength will be expressed in terms of the product of the 2-form F and holomorphic 3-form on T^6 as in (A.19) of [29].

[26] In Appendix B we discuss an alternative choice of the dilaton based on the proposal of [6].

$$\sqrt{2\pi k^{-1}}A = 4i\sqrt{\frac{b^2(b^2+1)}{1+2b^2}}e^{-\Phi_0}$$
$$\times\left[c_1\cos\varphi\cos\zeta\, d(\cos t\sinh\xi) + c_2\cos t\cosh\xi\, d(\cos\varphi\sin\zeta)\right]. \qquad (4.6)$$

The 1-form of the supergravity solution in (4.3) is real for real b. The analytic continuation to this new patch leads to an imaginary 1-form if b is real.

This raises an interesting question. If this background does correspond to the λ-deformation (2.6) [6] of the superstring sigma model, then for some (perfectly legitimate) choices of the $SO(1,2)$ gauge-fixed group field (3.2) we should end up with an action that is not manifestly real. However, the reality of the action (2.6) seems to follow in the usual way from considering the real form of the superalgebra. The non-reality should only manifest itself in the fermionic sector (as i appears in the RR flux) and could arise from an obstruction in the procedure of gauge-fixing the supergroup field of (2.6) and integrating out the superalgebra-valued gauge field, but it is not immediately clear why this should happen. At the same time, the imaginary RR flux may be expected, given that (2.6) can be interpreted as a deformation of the non-abelian T-dual of the $AdS_n \times S^n$ string model with the duality applied to all space–time dimensions including time (cf. [13,20,32]). Note, however, that the gauge field in the action (2.6) of the λ-model belongs to the superalgebra, and thus the non-abelian T-duality in (2.10) is performed also in the fermionic directions (cf. [35]), which may also have an effect on the issue of the reality of the corresponding RR flux.

As here we are interested in the special limit (and analytic continuation) (3.7) of the above background combined with the analytic continuation of the parameters (i.e. with b and k taken complex as in (2.20), (2.21)) we may formally consider the solutions of the complexified theory, discussing the reality issue only at the end. It is worth recalling however, as discussed in Section 3.4, that if we analytically continue to $H^2 \times dS_2$ using (3.44), (3.46), while the background (3.3) still has an imaginary 1-form, the special limits we consider below become real for real b (as in (3.48) compared to (3.47)).

The first limit we will take is as in (3.7) combined with infinite shift of the dilaton

$$\begin{aligned}
&t \to t + \frac{i}{2}\log\big[\frac{1-\varkappa^2\rho^2}{1+\rho^2}\big] + i\log\gamma\,, \qquad \xi \to \frac{1}{2}\log\big[-\frac{1-\varkappa\rho}{1+\varkappa\rho}\big]\,,\\
&\Phi_0 \to \Phi_0 + \log\big[-\frac{\gamma^2}{4}\big]\,,\\
&\varphi \to \varphi + \frac{i}{2}\log\big[\frac{1+\varkappa^2 r^2}{1-r^2}\big] + i\log\gamma\,, \qquad \zeta \to \frac{i}{2}\log\big[-\frac{1+i\varkappa r}{1-i\varkappa r}\big]\,,\\
&\gamma \to \infty\,.
\end{aligned} \qquad (4.7)$$

Starting from (4.6) we then get the following solution of the 4d supergravity equations (4.2) supporting the metric (3.9) of the η-model

$$\begin{aligned}
2h^{-1}ds^2 &= -\frac{1+\rho^2}{1-\varkappa^2\rho^2}dt^2 + \frac{d\rho^2}{(1-\varkappa^2\rho^2)(1+\rho^2)} + \frac{1-r^2}{1+\varkappa^2 r^2}d\varphi^2\\
&\quad + \frac{dr^2}{(1+\varkappa^2 r^2)(1-r^2)}\,,\\
e^{\Phi} &= e^{\Phi_0 + i(t+\varphi)}\frac{\sqrt{1+\rho^2}\sqrt{1-r^2}}{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2 r^2}}\,,
\end{aligned}$$

$$
\begin{aligned}
\sqrt{2h^{-1}}A &= \frac{2\sqrt{1+\varkappa^2}e^{-\Phi_0-i(t+\varphi)}}{\sqrt{1+\rho^2}\sqrt{1-r^2}} \\
&\quad\times\big[c_1 r\, d\big(t-\frac{i}{2}\log(1+\rho^2)\big)-c_2\rho\, d\big(\varphi-\frac{i}{2}\log(1-r^2)\big)\big]\,, \\
\sqrt{2h^{-1}}e^{\Phi}F &= -\frac{2\sqrt{1+\varkappa^2}}{\sqrt{1+\rho^2}\sqrt{1-r^2}}\big[c_1(e^0\wedge e^3-\rho r\, e^1\wedge e^2-ir\, e^0\wedge e^2-i\rho\, e^1\wedge e^3) \\
&\quad+c_2(\rho r\, e^0\wedge e^3+e^1\wedge e^2-i\rho\, e^0\wedge e^2+ir\, e^1\wedge e^3)\big]\,,
\end{aligned}
\tag{4.8}
$$

where we have defined the frame fields

$$
\begin{aligned}
e^0 &= \frac{\sqrt{1+\rho^2}}{\sqrt{1-\varkappa^2\rho^2}}dt\,, \qquad e^1=\frac{d\rho}{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\rho^2}}\,, \\
e^2 &= \frac{\sqrt{1-r^2}}{\sqrt{1+\varkappa^2r^2}}d\varphi\,, \qquad e^3=\frac{dr}{\sqrt{1+\varkappa^2r^2}\sqrt{1-r^2}}\,.
\end{aligned}
$$

This background looks strange: the $\varkappa\to 0$ limit of (4.8) gives the undeformed $AdS_2\times S^2$ metric supported by a non-trivial complex dilaton and RR flux that explicitly depend on t and φ. While t and φ are still isometries of the metric and $e^{\Phi}F$, which enter the classical GS superstring action, the dilaton and RR 1-form are only invariant under the combined transformation[27]

$$
t\to t+c\,, \qquad \varphi\to\varphi-c\,. \tag{4.9}
$$

This is different from the expected Bertotti–Robinson type flux supporting $AdS_2\times S^2$.

If we instead consider the $\varkappa\to\infty$ limit of (4.8), as taken in [33], i.e. first rescaling

$$
t\to\varkappa^{-1}t\,, \qquad \rho\to\varkappa^{-1}\rho\,, \qquad \varphi\to\varkappa^{-1}\varphi\,, \qquad r\to\varkappa^{-1}r\,, \qquad h\to h\varkappa^2\,, \tag{4.10}
$$

we find the following real supergravity solution

$$
\begin{aligned}
2h^{-1}ds^2 &= -\frac{dt^2}{1-\rho^2}+\frac{d\rho^2}{1-\rho^2}+\frac{d\varphi^2}{1+r^2}+\frac{dr^2}{1+r^2}\,, \qquad e^{\Phi}=\frac{e^{\Phi_0}}{\sqrt{1-\rho^2}\sqrt{1+r^2}}\,, \\
\sqrt{2h^{-1}}A &= 2e^{-\Phi_0}\big[c_1 r\, dt-c_2\rho\, d\varphi\big]\,, \\
\sqrt{2h^{-1}}e^{\Phi}F &= -\frac{2}{\sqrt{1-\rho^2}\sqrt{1+r^2}}\big[c_1 dt\wedge dr-c_2 d\varphi\wedge d\rho\big]\,.
\end{aligned}
\tag{4.11}
$$

This is precisely the solution of the "mirror" model constructed in [33] and is related to a $dS_2\times H^2$ background by T-dualities in t and φ, giving an imaginary RR flux as might be expected (cf. [32]).

The second limit we will consider is

$$
\begin{aligned}
t &\to i\varkappa t+\frac{i}{2}\log\big[\frac{1+\rho^2}{1-\varkappa^2\rho^2}\big]+i\log\gamma\,, \qquad \xi\to\frac{1}{2}\log\big[\frac{1-i\rho}{1+i\rho}\big]\,, \\
\Phi_0 &\to \Phi_0+\log\big[-\frac{\gamma^2}{4}\big]\,,
\end{aligned}
$$

[27] Formally the dilaton and RR 1-form are invariant under separate shifts in t and φ if one is also allowed to shift Φ_0. Note also that the linear terms in the dilaton have their origin in the large distance asymptotics of the background corresponding to the gWZW model when the metric becomes flat while the dilaton becomes linear, cf. (4.6), (4.7).

$$\varphi \to i\varkappa\varphi + \frac{i}{2}\log\big[\frac{1-r^2}{1+\varkappa^2 r^2}\big] + i\log\gamma\ , \qquad \zeta \to \frac{i}{2}\log\big[\frac{1-r}{1+r}\big]\ ,$$
$$\gamma \to \infty\ . \tag{4.12}$$

The resulting solution of (4.2) is given by

$$2h^{-1}ds^2 = -\frac{1-\varkappa^2\rho^2}{1+\rho^2}dt^2 + \frac{d\rho^2}{(1-\varkappa^2\rho^2)(1+\rho^2)} + \frac{1+\varkappa^2 r^2}{1-r^2}d\varphi^2$$
$$+ \frac{dr^2}{(1+\varkappa^2 r^2)(1-r^2)}\ ,$$
$$e^{\Phi} = e^{\Phi_0 - \varkappa(t+\varphi)}\frac{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2 r^2}}{\sqrt{1+\rho^2}\sqrt{1-r^2}}\ ,$$
$$\sqrt{2h^{-1}}A = -\frac{2i\sqrt{1+\varkappa^2}e^{-\Phi_0+\varkappa(t+\varphi)}}{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2 r^2}}$$
$$\times \big[c_1\rho\, d\big(t - \frac{1}{2\varkappa}\log(\frac{1-\varkappa^2\rho^2}{\varkappa^2\rho^2})\big) + c_2 r\, d\big(\varphi - \frac{1}{2\varkappa}\log(\frac{1+\varkappa^2 r^2}{\varkappa^2 r^2})\big)\big]\ ,$$
$$\sqrt{2h^{-1}}e^{\Phi}F = -\frac{2i\sqrt{1+\varkappa^2}}{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2 r^2}}$$
$$\times \big[c_1(\varkappa^2\rho r\, e^0\wedge e^3 - e^1\wedge e^2 - \varkappa\rho\, e^0\wedge e^2 + \varkappa r\, e^1\wedge e^3)$$
$$+ c_2(e^0\wedge e^3 + \varkappa^2\rho r\, e^1\wedge e^2 + \varkappa r\, e^0\wedge e^2 + \varkappa\rho\, e^1\wedge e^3)\big]\ , \tag{4.13}$$

where the frame fields are given by

$$e^0 = \frac{\sqrt{1-\varkappa^2\rho^2}}{\sqrt{1+\rho^2}}dt\ , \qquad e^1 = \frac{d\rho}{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\rho^2}}\ ,$$
$$e^2 = \frac{\sqrt{1+\varkappa^2 r^2}}{\sqrt{1-r^2}}d\varphi\ , \qquad e^3 = \frac{dr}{\sqrt{1+\varkappa^2 r^2}\sqrt{1-r^2}}\ .$$

There is a formal map between the two solutions (4.8) and (4.13) given by

$$t \to i\varkappa t\ , \qquad \rho \to -\frac{i}{\varkappa\rho}\ , \qquad \varphi \to i\varkappa\varphi\ , \qquad r \to \frac{i}{\varkappa r}\ . \tag{4.14}$$

The metric of (4.13) is the double T-dual (in t and φ) of the metric of (4.8). However, this T-duality relation does not obviously extend to the full backgrounds as shifts in t and φ are not isometries of the dilaton and the RR 1-form.[28] Again they are only invariant under the combined transformation (4.9).

[28] It may still be possible to define a generalization of the T-duality rules that will apply in the present situation. The dilaton coupling in the string action is given by $\sqrt{-h}R^{(2)}\Phi = -2\,\partial^2\omega\,\Phi$ (in conformally flat coordinates $h_{\alpha\beta} = e^{2\omega}\eta_{\alpha\beta}$). Therefore, if Φ has a term linear in a target-space direction (which is otherwise isometric, i.e. enters the string action only through its derivatives), we can integrate by parts and then perform the T-duality transformation in the usual manner. The resulting action will have a term proportional to $(\partial\omega)^2$ whose role is to cancel the conformal anomaly. As the dilaton coupling term is subleading in α' the T-dual classical superstring action can be found by the usual rules. One can then formally read off the corresponding metric, B field and e^{Φ} times the RR fluxes from the resulting action. They need not by themselves satisfy the Type IIB supergravity equations of motion as these follow from the vanishing of the one-loop Weyl anomaly beta-functions and thus are sensitive to the full dilaton coupling and, in particular, the central charge shift mentioned above. The resulting dilaton of the T-dual background may then be determined by solving these equations.

The $\varkappa \to 0$ limit of (4.13) is much simpler than that of (4.8)[29]

$$2h^{-1}ds^2 = -\frac{dt^2}{1+\rho^2} + \frac{d\rho^2}{1+\rho^2} + \frac{d\varphi^2}{1-r^2} + \frac{dr^2}{1-r^2} , \qquad e^{\Phi} = \frac{e^{\Phi_0}}{\sqrt{1+\rho^2}\sqrt{1-r^2}} ,$$
$$\sqrt{2h^{-1}}A = 2ie^{-\Phi_0}\big[c_1\rho\, d\varphi + c_2 r\, dt\big] ,$$
$$\sqrt{2h^{-1}}e^{\Phi}F = -\frac{2i}{\sqrt{1+\rho^2}\sqrt{1-r^2}}\big[c_1 d\varphi \wedge d\rho + c_2 dt \wedge dr\big] . \tag{4.15}$$

Performing T-dualities in both t and φ we recover the standard Bertotti–Robinson solution with constant dilaton and homogeneous RR flux:

$$2h^{-1}ds^2 = -(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2} + (1-r^2)d\varphi^2 + \frac{dr^2}{1-r^2} , \qquad e^{\Phi} = e^{\Phi_0} ,$$
$$\sqrt{2h^{-1}}A = 2e^{-\Phi_0}\big[c_1\rho\, dt - c_2 r\, d\varphi\big] ,$$
$$\sqrt{2h^{-1}}e^{\Phi}F = -2\big[c_1 dt \wedge d\rho - c_2 d\varphi \wedge dr\big] . \tag{4.16}$$

This suggests that if the metric and $e^{\Phi}F$ of the solution (4.13) can be formally T-dualized for $\varkappa \neq 0$ (e.g. by applying the standard T-duality rules to just these combinations of the background fields, see footnote 28) it will give a real "background" for the metric (3.9) (the T-duality in t will remove the factor of i in F). It would be interesting to see if this bears any relation to the η-deformation (2.2) of the $AdS_2 \times S^2$ supercoset model. Having a factorized (but not isometric) dilaton, this background will be obviously different from the solution constructed in [29][30] and its meaning remains to be understood. Finally, given that the standard Bertotti–Robinson solution appears (after T-dualities) in the $\varkappa \to 0$ limit of (4.13), while the "mirror" model (4.11) of [33] appears in the $\varkappa \to \infty$ limit of (4.8), it would be interesting to see if the map (4.14) between the two backgrounds (4.8), (4.13) is related to the "mirror duality" of [33,34].

Finally, let us note that the $\varkappa \to i$ limit of (4.8) or (4.13) can be taken as in (3.49)[31]

$$t = \epsilon x^- + \frac{x^+}{\epsilon} , \qquad \varphi = \epsilon x^- - \frac{x^+}{\epsilon} , \qquad \rho = \tan\alpha ,$$
$$r = \tanh\beta , \qquad \varkappa = \sqrt{-1+s\,\epsilon^2} . \tag{4.17}$$

[29] The apparent divergence of the RR potential turns out to be a total derivative and can therefore be removed by an appropriate gauge transformation

$$\sqrt{2h^{-1}}A \to \sqrt{2h^{-1}}A + d\Big(\frac{2i\sqrt{1+\varkappa^2}e^{-\Phi_0+\varkappa(t+\varphi)}}{\varkappa\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2 r^2}}(c_1\rho + c_2 r)\Big)$$
$$= \frac{2i\sqrt{1+\varkappa^2}e^{-\Phi_0+\varkappa(t+\varphi)}}{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2 r^2}}\big[c_1\rho\, d\big(\varphi - \frac{1}{2\varkappa}\log(1+\varkappa^2 r^2)\big) + c_2 r\, d\big(t - \frac{1}{2\varkappa}\log(1-\varkappa^2\rho^2)\big)\big] .$$

[30] In [29] the independence of the dilaton and RR fields from the isometric directions of the metric was assumed from the start.

[31] One can also use (3.50)

$$t = \epsilon x^- + \frac{x^+}{\epsilon} , \qquad \varphi = \epsilon x^- - \frac{x^+}{\epsilon} , \qquad \rho = \cot\alpha , \qquad r = \coth\beta , \qquad \varkappa = \sqrt{-1-s\,\epsilon^2}$$

leading to the same pp-wave type background. This is a consequence of the formal map (4.14) between (4.8) and (4.13).

Choosing $s = 1$ for the solution (4.8) and $s = -1$ for (4.13) and then sending $\epsilon \to 0$, in both cases we find the following pp-wave background

$$2h^{-1}ds^2 = -4dx^-dx^+ + \frac{1}{2}(\cos 2\alpha - \cosh 2\beta)(dx^+)^2 + d\alpha^2 + d\beta^2 ,$$
$$e^\Phi = e^{\Phi_0} , \qquad \sqrt{2h^{-1}}A = 2e^{-\Phi_0}\big[\tilde{c}_1 \cos\alpha \sinh\beta + \tilde{c}_2 \sin\alpha \cosh\beta\big]dx^+ , \tag{4.18}$$

where $\tilde{c}_{1,2} = \pm c_{1,2}$. This is the pp-wave background of [13], whose light-cone gauge-fixing ($x^+ = \mu\tau$) yields the Pohlmeyer-reduced theory for $AdS_2 \times S^2$, equivalent [27] to the $\mathcal{N} = 2$ supersymmetric sine-Gordon model. If we had taken the opposite signs for s in (4.17) we would have ended up with the same solution with $x^+ \to ix^+$. The light-cone gauge-fixing of this model gives the Pohlmeyer-reduced theory for $H^2 \times dS^2$.

Let us also note that if we set $\varkappa = i$ in the solutions (4.8) and (4.13) without the rescaling of $x^\pm$ in (4.17) we find a simple string background given by a flat metric with vanishing RR 1-form and a dilaton linear in the null direction $t + \varphi$ (the factor of $\pm i$ in the dilaton can be removed by a simple analytic continuation of t and φ).

4.2. Algebraic coordinates

The λ-model solutions (4.3) and (4.6) take remarkably simple forms in the algebraic coordinates introduced in (3.11), (3.12). The solution (4.6) becomes[32]

$$2\pi k^{-1}ds^2 = \frac{1}{1-x^2+y^2}\big[-(1+2b^2)dx^2 + \frac{dy^2}{1+2b^2}\big]$$
$$\qquad + \frac{1}{1-p^2-q^2}\big[(1+2b^2)dp^2 + \frac{dq^2}{1+2b^2}\big] ,$$
$$e^\Phi = \frac{e^{\Phi_0}}{\sqrt{1-x^2+y^2}\sqrt{1-p^2-q^2}} ,$$
$$\sqrt{2\pi k^{-1}}A = 4i\sqrt{\frac{b^2(b^2+1)}{1+2b^2}}e^{-\Phi_0}\big[c_1 p\, dy + c_2 x\, dq\big] . \tag{4.19}$$

Note that a formal analytic continuation of this background by setting $x = iy'$, $y = ix'$ gives a real solution

$$2\pi k^{-1}ds^2 = \frac{1}{1-x'^2+y'^2}\big[-\frac{dx'^2}{1+2b^2} + (1+2b^2)dy'^2\big]$$
$$\qquad + \frac{1}{1-p^2-q^2}\big[(1+2b^2)dp^2 + \frac{dq^2}{1+2b^2}\big] ,$$
$$e^\Phi = \frac{e^{\Phi_0}}{\sqrt{1-x'^2+y'^2}\sqrt{1-p^2-q^2}} ,$$

32 This form of the solution manifestly realizes the observation of [20] that the λ-deformation amounts to rescaling the tangent space directions of the gauged WZW model for F/G (here $\frac{SO(1,2)}{SO(1,1)} \times \frac{SO(3)}{SO(2)}$, given by the point $b = 0$) while leaving the dilaton invariant and with the RR flux depending on the deformation parameter only through an overall constant factor.

$$\sqrt{2\pi k^{-1}}A = -4\sqrt{\frac{b^2(b^2+1)}{1+2b^2}}e^{-\Phi_0}\big[c_1 p\, dx' + c_2 y'\, dq\big]\,. \tag{4.20}$$

If instead we formally continue (4.19) to the region for which $x^2 - y^2 > 1$, we find (after setting $e^{\widetilde{\Phi}_0} = ie^{\Phi_0}$) a different real background, which represents the solution (4.3), i.e. the original solution of [19] corresponding to the metric in the coordinate patch in (3.5).

Using the relations (2.20), (2.21) between the parameters and taking the scaling limit (3.13) combined with a redefinition of the dilaton $e^{\Phi_0} \to i\gamma^2 e^{\Phi_0}$ the solution (4.19) becomes simply

$$\begin{aligned}
[ll]2h^{-1}ds^2 &= \frac{1}{y^2-\varkappa^2x^2}\big(dx^2+dy^2\big) + \frac{1}{q^2+\varkappa^2p^2}\big(-dq^2+dp^2\big)\,,\\
e^{\Phi} &= \frac{e^{\Phi_0}}{\sqrt{y^2-\varkappa^2x^2}\sqrt{q^2+\varkappa^2p^2}}\,,\\
\sqrt{2h^{-1}}A &= -2i\sqrt{1+\varkappa^2}\,e^{-\Phi_0}\big(c_1 p\, dy + c_2 x\, dq\big)\,.
\end{aligned} \tag{4.21}$$

One can check directly that the supergravity equations of motion (4.2) are indeed satisfied.[33] This solution may be interpreted as a deformation of an $H^2 \times dS_2$ background (for which an imaginary RR flux could be expected, cf. [32]). For $\varkappa = 0$ the dilaton is non-constant but it can be eliminated by T-dualities in the x and p directions, which along with sending $y \to y^{-1}$ and $q \to q^{-1}$ leaves the metric invariant.

The metric and $e^{\Phi}F$ of (4.21) are invariant under separate rescalings of (x, y) and (p, q), however, as discussed above the dilaton and RR 1-form are only invariant when these rescalings are correlated as $(x, y) \to e^{\tilde{c}}(x, y)$, $(p, q) \to e^{-\tilde{c}}(p, q)$, which corresponds to the symmetry (4.9) of the backgrounds (4.8), (4.13).

A similar background representing the deformation of $AdS_2 \times S^2$ may be found using a different real slice of the diagonal coordinates as in (A.8). Setting

$$\begin{aligned}
&y = e^{i\varphi}\cosh v\,, \quad x = ie^{i\varphi}\sinh v\,, \quad r = \tanh v\,,\\
&q = e^{it}\cos\alpha\,, \quad p = ie^{it}\sin\alpha\,, \quad \rho = \tan\alpha\,,
\end{aligned} \tag{4.22}$$

we find that (4.21) then transforms into the background (4.8) found earlier.

5. Poisson–Lie duality interpretation

Apart from the relation between the λ-model and η-model through a scaling limit and analytic continuation described in Section 3, which is somewhat unexpected (though partly prompted by the natural map between the parameters (2.20)–(2.22)), one may anticipate that the two models may be in some sense dual to each other. Indeed, the undeformed limit of the η-model is the standard supercoset model, while the undeformed limit of the λ-model is the non-abelian T-dual of the latter (cf. (2.10), (2.30)). A natural suggestion is then that the two models may be related by the Poisson–Lie (PL) duality of [8,9].

Below we will directly verify this conjecture on the simplest example of the bosonic S^2 coset. The corresponding metric of the λ-model is in the second line of (3.3) (or, in diagonal form, the second line of (3.12)), and its η-model counterpart is in the second line of (3.9). We are going to

[33] To recall, c_1 and c_2 are arbitrary constants satisfying $c_1^2 + c_2^2 = 1$, so a symmetric choice is $c_1 = c_2 = \frac{1}{\sqrt{2}}$.

compare them with the PL dual pair of models associated with the $SL(2,\mathbb{C})$ double [9,11]: the first corresponds to the $SU(2)$ subgroup and the second to the Borel subgroup B_2 (upper triangular matrices with reals on diagonal). The corresponding metrics are given, e.g., in equations (3.18) and (3.19) of [11] with two free parameters a, b and with an overall coefficient T.[34]

The first metric is

$$ds_1^2 = \frac{\mathrm{T\,a}}{\mathrm{a}^2 + (\mathrm{b} - \cos\theta)^2}(d\theta^2 + \sin^2\theta d\varphi^2) \,. \tag{5.1}$$

Setting $\mathrm{b} = 0$ (which is required to get the integrable model we are interested in here) and

$$\mathrm{T} = \frac{h}{2\varkappa} \,, \qquad \mathrm{a} = \varkappa^{-1} \,, \tag{5.2}$$

we find that (5.1) becomes precisely the corresponding η-model metric in (3.9) (where $r = \cos\theta$).

The second metric of the PL dual pair is [11]

$$ds_2^2 = \frac{\mathrm{T\,a_1}}{2(1+\mathrm{a_1}z)}\Big(\frac{dz^2}{\rho^2} + \big[d\rho + \big(\frac{\mathrm{b}-1}{\mathrm{a}} + \frac{z - \frac{\mathrm{a_1}}{4}\rho^2}{1+\mathrm{a_1}z}\big)\frac{dz}{\rho}\big]^2\Big) \,,$$
$$\mathrm{a_1} \equiv \frac{2\mathrm{a}}{\mathrm{a}^2 + (\mathrm{b}-1)^2} \,. \tag{5.3}$$

Setting $\mathrm{b} = 0$ and doing a field redefinition to put this metric into a diagonal form

$$z = \frac{1}{2}(\mathrm{a} + \mathrm{a}^{-1})\big[(p+q)^2 - 1\big] \,, \qquad \rho = (\mathrm{a} + \mathrm{a}^{-1})\sqrt{p^2 - q^2 - 1} \,, \tag{5.4}$$

we find

$$ds_2^2 = \frac{\mathrm{T}}{p^2 - q^2 - 1}\big(\mathrm{a}\,dp^2 + \mathrm{a}^{-1}dq^2\big) \,. \tag{5.5}$$

Making further redefinitions

$$\mathrm{T} = \frac{k}{2i\pi} \,, \qquad \mathrm{a} = -i(1+2b^2) \,, \qquad q \to iq \,, \tag{5.6}$$

we obtain the metric of the λ-deformation of the non-abelian T-dual of S^2 in the algebraic coordinates used in (3.11), (3.12), (4.19)

$$ds^2 = \frac{k}{2\pi}\frac{1}{1 - p^2 - q^2}\big[(1+2b^2)dp^2 + \frac{dq^2}{1+2b^2}\big] \,. \tag{5.7}$$

Note that the definitions in (5.2) and (5.6) are related by the map (2.20), (2.21) precisely as required by our general discussion in Sections 2.2 and 3.1.

This implies that in the S^2 coset case, the η-deformation of [4] is Poisson–Lie dual to an analytic continuation of the λ-deformation of [7,5]. A similar relation should then be expected in general.

[34] We denote the parameters a, b of [9,11] by roman letters.

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While this paper was in preparation there appeared Ref. [37] giving a general Hamiltonian construction of the relation of the two deformed models via the Poisson–Lie duality complementing our discussion in Section 5.

Appendix A. Different forms of deformed metrics in $SO(3)/SO(2)$ case

The λ-deformed metric [7] corresponding to the S^2 coset (given in the second line of (3.3)) can be written, after a simple change of coordinates, $z = \cos\zeta,\ w = \cos\varphi\, \sin\zeta$, in the following form (ignoring overall factors)

$$ds^2 = \frac{1}{1-z^2}\big[dz^2 + \frac{z^2}{1-w^2}dw^2 + m^2(wdz+zdw)^2\big] ,$$
$$m^2 \equiv 4b^2(1+b^2) = -\varkappa^{-2} - 1 . \tag{A.1}$$

The non-abelian T-dual of S^2 is found in the limit $m \to \infty$ with $z = 1 - \frac{1}{2m^2}Z^2,\ w = 1 - \frac{1}{2m^2}W^2$ giving [9,36] $ds^2 = Z^{-2}(dW^2 + \frac{1}{4}[d(W^2+Z^2)]^2)$.

Introducing the new coordinates X, Y and P, Q as

$$e^Y = z\sqrt{1+m^2w^2} , \qquad \cos X = \sqrt{\frac{1-w^2}{1+m^2z^2}} , \tag{A.2}$$

$$P = e^Y \cos X = z\sqrt{1-w^2} , \quad Q = e^Y \sin X = \sqrt{1+m^2}\, zw , \quad P + iQ = e^{Y+iX} , \tag{A.3}$$

we can put (A.1) into the conformally-flat form (cf. (3.12))

$$ds^2 = \frac{1}{e^{-2Y} - \frac{1}{1+m^2}(1+m^2\cos^2 X)}(dX^2 + dY^2) \tag{A.4}$$

$$= \frac{1}{1 - P^2 - \frac{1}{1+m^2}Q^2}(dP^2 + dQ^2) . \tag{A.5}$$

Here the $m = 0$ limit corresponds to the $SO(3)/SO(2)$ gauged WZW metric.[35]

One option to take a limit of this metric is to do an infinite rescaling of P and Q (combined with the replacement of m by $\varkappa$ as in (A.1)), i.e. to drop the constant 1 in (A.5) (and reverse

[35] The curvature of (A.5) is (setting $1 + m^2 = -\varkappa^{-2}$): $R = -2\frac{1-\varkappa^2+(1+\varkappa^2)(P^2+\varkappa^2Q^2)}{1-P^2+\varkappa^2Q^2}$.

overall sign of the metric). This leads to a scale-invariant (i.e. it has an isometry) metric as in (3.13), (3.14) that is a deformation of H^2

$$ds^2 = \frac{1}{P^2 - \varkappa^2 Q^2}(dP^2 + dQ^2) . \tag{A.6}$$

Alternatively, we may consider the first form of the metric (A.4) and set $P + iQ = \exp(Y + iX) = \gamma \exp(iU - V)$

$$\begin{aligned} &Y = \ln\gamma + iU , \quad X = iV , \quad \text{i.e.} \\ &P = \gamma\, e^{iU} \cosh V , \quad Q = i\gamma\, e^{iU} \sinh V , \qquad \gamma \to \infty , \end{aligned} \tag{A.7}$$

i.e. use a different real slice where U, V are real while P, Q are not. Then the e^{-2Y} term in (A.4) drops out and we find

$$ds^2 = \frac{1}{\cosh^2 V + \varkappa^2 \sinh^2 V}(dU^2 + dV^2) . \tag{A.8}$$

This is, indeed, the metric of the η-deformed S^2 space,[36] i.e. it is equivalent to the second line of (3.9) ($\varphi = U,\ r = \tanh V$) [25,4,13].

A similar discussion can be repeated for the AdS_2 coset part of (3.3), obtaining the first line of (3.9) in the limit.

Appendix B. An alternative dilaton for the deformed models: $AdS_2 \times S^2$

The dilaton discussed in Section 4 (see (4.3), (4.6)) is the one assumed as a starting point for constructing supergravity solutions for the λ-model in [19,20] and originates from integrating out the gauge field $A_\pm$ (see [23] and references there) in the bosonic truncation (2.29) of (2.6), i.e.

$$e^{2\Phi} = \frac{1}{\det\big[(\mathrm{Ad}_f - 1 - (\lambda^{-2} - 1)P_\lambda)|_{\hat{\mathfrak{f}}_0 \oplus \hat{\mathfrak{f}}_2}\big]} , \tag{B.1}$$

where the operator under the determinant is restricted to act on the bosonic subalgebra of the superalgebra $\hat{\mathfrak{f}}$ and f is taken to be a bosonic coset representative.

For the λ-model associated with $AdS_2 \times S^2$ this gives the dilaton in (4.3) for the coset representative (3.4). For the coset representative (3.2) we find the dilaton in (4.6), i.e.

$$e^{\Phi} = \frac{e^{\Phi_0}}{\sin t \sin\varphi} . \tag{B.2}$$

In [6] an alternative expression for the dilaton was proposed, which is given by the superdeterminant arising from integrating out the complete gauge field in (2.6)

$$e^{2\Phi} = \frac{1}{\mathrm{sdet}\big[(\mathrm{Ad}_f - 1 - (\lambda^{-2} - 1)P_\lambda)|_{\hat{\mathfrak{f}}}\big]} , \tag{B.3}$$

where now the operator under the superdeterminant acts on the full superalgebra $\hat{\mathfrak{f}}$. As we are interested in the bosonic supergravity background, the group field f may still be taken to be a

36 The sphere metric may be written as $\frac{1}{\cosh^2 V}(dV^2 + dU^2) = d\alpha^2 + \cos^2\alpha\, dU^2$, $\tan\frac{\alpha}{2} = \tanh\frac{V}{2}$.

bosonic coset representative. Then the operator under the superdeterminant factorizes and (B.3) can be written as

$$e^{2\Phi} = \frac{\det\big[(\mathrm{Ad}_f - 1 - (\lambda^{-2} - 1) P_\lambda)\big|_{\hat{\mathfrak{f}}_1 \oplus \hat{\mathfrak{f}}_3}\big]}{\det\big[(\mathrm{Ad}_f - 1 - (\lambda^{-2} - 1) P_\lambda)\big|_{\hat{\mathfrak{f}}_0 \oplus \hat{\mathfrak{f}}_2}\big]} \,. \tag{B.4}$$

The denominator factor of (B.4) is identical to (B.1) and therefore for the λ-model associated with $AdS_2 \times S^2$ its contribution to the dilaton (for the coset representative (3.2)) is again given by (B.2).

To compute the contribution of the fermionic numerator factor we need to consider the full superalgebra in (2.1), (2.6) and not just its bosonic truncation. Starting with the superalgebra $\mathfrak{psu}(1,1|2)$,[37] which has the bosonic subalgebra $\mathfrak{so}(1,2) \oplus \mathfrak{so}(3)$ required for the $AdS_2 \times S^2$ case (the bosonic gauge group in (2.1) remains unchanged), we find the contribution of the numerator of (B.4) to e^Φ to be

$$\begin{aligned}&(1+\lambda^4+2\lambda^2\cosh 2\xi)\cos^2 t + (1+\lambda^4+2\lambda^2\cos 2\zeta)\cos^2\varphi - (1-\lambda^2)^2 \\ &\quad - 4\lambda(1+\lambda^2)\cos t\cos\varphi\cosh\xi\cos\zeta \,.\end{aligned} \tag{B.5}$$

Combining this expression with (B.2) we arrive at the following (alternative to (B.2)) proposal for the dilaton

$$\begin{aligned}e^\Phi = \frac{e^{\Phi_0}}{\sin t\sin\varphi}\Big[&(1+\lambda^4+2\lambda^2\cosh 2\xi)\cos^2 t + (1+\lambda^4+2\lambda^2\cos 2\zeta)\cos^2\varphi \\ &- (1-\lambda^2)^2 - 4\lambda(1+\lambda^2)\cos t\cos\varphi\cosh\xi\cos\zeta\Big] \,.\end{aligned} \tag{B.6}$$

One can indeed check that together with the metric of (4.6) this solves the dilaton equation, i.e. the first equation of (4.2) as well as the trace of the Einstein equation (the second equation of (4.2)).

The remaining equations involving RR flux are no longer satisfied, i.e. the RR background needs to be modified. How this should be done is not clear, but it is worth noting that as the trace of the Einstein equation in (4.2) is still satisfied, the simplest consistent ansatz is for only a single RR 1-form potential to be non-zero.[38]

Let us note that in the algebraic coordinates (3.11), (3.12) the dilaton (B.6) is given by

$$e^\Phi = 2e^{\Phi_0}\frac{(1+\lambda^2)^2(x^2+p^2) - 4\lambda(1+\lambda^2)xp - (1-\lambda^2)^2(1-y^2+q^2)}{\sqrt{1-p^2-q^2}\sqrt{1-x^2+y^2}} \,. \tag{B.7}$$

Here the denominator is the contribution from the bosonic sector (B.1), i.e. the dilaton considered earlier in (4.19). Again one can check that together with the metric of (4.19) this expression (B.7) solves the dilaton equation and the trace of the Einstein equation.

Now let us take the two special limits (4.7) and (4.12) of the new dilaton (B.6) (note that here we will no longer need the infinite shift of the constant part of the dilaton). This leads to

$$e^\Phi = e^{\Phi_0}\frac{\sqrt{1+\rho^2}\sqrt{1-r^2}\cos(t-\varphi) + i\sqrt{1+\varkappa^2}\rho r}{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2 r^2}} \tag{B.8}$$

[37] We use the matrix representation of $\mathfrak{psu}(1,1|2)$ given in Appendix C of [13].

[38] One can try some simple ansatzes, such as using the same RR 1-form as in (4.6), or, alternatively, demanding that $e^\Phi F$ is unchanged, but neither of these proposals work.

for the limit (4.7), relating to the metric in (4.8), and to

$$e^{\Phi} = e^{\Phi_0} \frac{\sqrt{1-\varkappa^2\rho^2}\sqrt{1+\varkappa^2 r^2}\cosh[\varkappa(t-\varphi)] + i\varkappa\sqrt{1+\varkappa^2}\rho r}{\sqrt{1+\rho^2}\sqrt{1-r^2}} \tag{B.9}$$

for the limit (4.12), relating to the metric in (4.13). In the $\varkappa \to \infty$ limit of (B.8) (using (4.10)) we recover the dilaton of the "mirror" model (4.11), while taking the $\varkappa \to 0$ limit of (B.9) we recover the T-dual of the dilaton of the background (4.15). Furthermore, we can recover the dilatons of (4.8) and (4.13) from (B.8) and (B.9) respectively (up to trivial signs) via an additional infinite constant shift of $t-\varphi$ (along with compensating shifts of the constant part of the dilaton). Equivalently, the expressions in (4.8) and (4.13) can be found directly from (B.6) by decorrelating the limits in the AdS_2 and S^2 λ-models, i.e. using two separate parameters γ for t and φ in (4.7) or (4.12).

For $\varkappa = i$, when the metrics of (4.8) and (4.13) become flat, any "null" dilaton $e^{\Phi} = F(t \pm \varphi)$ solves the dilaton equation and the trace of the Einstein equation in (4.2). Indeed, for $\varkappa = i$ the dilatons (B.8), (B.9) take this form. Further, if we take $\varkappa = i$ without rescaling the coordinates, so that the metric is Ricci flat, then asking that the RR fluxes vanish implies that e^{Φ} is also a linear function of $t \pm \varphi$.

Let us note that the dilatons (B.8), (B.9) are complex, so their interpretation as part of supergravity solutions is unclear. Also, with the dilatons (B.8), (B.9) having non-trivial (non-linear) dependence on t and φ the resulting background would be truly non-isometric (with no chance of simplifying T-duality transform). This suggests that to recover the η-model from the λ-model we should indeed consider the decorrelated limit of (B.6) (with two separate infinite γ parameters), leading again to the solutions (4.8) and (4.13), for which the dilatons *are* linear in $t \pm \varphi$.

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