Variational formulation of an age-physiology dependent population dynamics

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Abstract

We transform a deterministic age-physiological factor population dynamics problem into its variational form. The internal/external heterogeneity of a population profoundly affects its dynamics, therefore, apart from age $a$, a second independent variable, $g$, say, referred to as the physiological parameter of individuals will also be a basis for classification. Using the well-known Ostrogradski or Gauss formula, we prove the existence and uniqueness theorems for the classical weak solution of the model.

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1. Introduction

Most practical problems in science and engineering are described by non-linear equations that may not be solved explicitly using analytical techniques. When faced with such complex problems, it is often natural to resort to methods that yield approximate solutions [15]. The calculus of variation seeks to combine trial functions into satisfactory approximate solutions [17]. This paper analyzes the variational form of a quasi-linear age-physiology dependent population model which may be used to describe the population dynamics of some genetic defects such as Sickle-Cell Anaemia (SCA) [10], and provides existence and uniqueness theorems. Also, this

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model can suggestively describe the cellular density of blood cell under the abnormal condition of SCA.

Sickle-Cell Anaemia is an inherited disease often known as the ‘great masquerader’ [3]. In tropical regions where malaria is prevalent, the disease is common, and any human population living in such environment is at risk [10]. Such a population is generally comprised of three genotypic groups, namely, Normal (AA), Carriers (AS), and Sicklers (SS). Also, age (being an important and general property of biological systems) is often used as a means of classifying individuals into different cohorts. SCA is used here as a basis of elucidation, and also for the model framework to have some real biological meaning. More details on the model can be found in [10,13,14].

In a realistic mathematical model of the dynamics of 2-sex population, one would want to include key factors such as age, physiological factors, density dependence, gestation period and mating pattern. However, the need for mathematical tractability and convenience imposes constraints on the number of parameters that a single model can accommodate [12]. Despite many years of interest in the study of demographic models, the variational formulation of population models had remained almost unattended, except in [5], to the best of the author's knowledge. Also, mating patterns, gestation period, and physiological factors appear to have been the least favoured among these factors. Skakauskas and Sowunmi include gestation period into their models as reported in [12]. But, in the sequel, we shall consider the latter (i.e., the physiological variable).

There is an urgent need to search for more realistic models which will mimic the actual dynamics of infectious as well as inherited diseases. The context for this paper is motivated by the desire to extend the methods in [10,13] to its variational form where the proof of the existence and uniqueness theorem was based on the Contraction Mapping Principle [16]. Thus, it is in order to generalize the former method of solution (obtained via the method of characteristics) that we decided to discuss the variational formulation of the model equations. Despite our wealth of knowledge about sickling process, there is no general treatment for SCA (an important medical problem which has not of recent received much attention mathematically), and this problem is especially acute because of the relatively large number of people afflicted. Our focus therefore is basically from the mathematical point of view.

The rest of the paper is organized as follows: In the next section, the model equation is formulated, and for a complete derivation, see [9,10]. The additional factor \( g \), say, could represent mass, size, caloric content, ... but in the case of Sickle Cell-Anaemia, it is worth taking \( g \) to be the level (quantity) of haemoglobin F (HbF) because it can be used as a clinical index for the severity of the disease and, it is widely accepted that persistent high levels of HbF in SCA is associated with mild clinical and haematological features [8]. The idea of test function is introduced in Section 3 where an \textit{a priori} estimate of the weak solution is given. The existence of this weak solution is proved in Section 4 via the Ostrogradski formula, while in Section 5, we use a suitable test function due to Oleinik (but extended into its variables to suit the purpose of this study) to prove uniqueness of a weak solution. Some of the proofs are long, but robust and rigorous enough, where obvious details are omitted to avoid triviality. The main results are Theorems 4.1 and 5.1, which generalize those obtained in [5].

2. The model equations

Let \( u(t,a,g) \) be the population density of individuals at time \( t \), aged \( a \), with physiological structure \( g \). By applying a conservation law [9], the system of equations describing the dynamics of an age-physiology structured population is given by the set of first-order quasi-linear partial
differential equations
\[ \frac{\partial u_i}{\partial t}(t, a, g_i) + \frac{\partial u_i}{\partial a}(t, a, g_i) + G_i(a) \frac{\partial u_i}{\partial g_i}(t, a, g_i) = -R_i(a)u_i, \]
\[ u_i(0, a, g_i) = u_i(a, g_i) \quad \text{initial condition}, \]
\[ u_i(t, 0, g_i) = B_i(t, g_i) \quad \text{boundary condition}, \]
where \( G_i(a) \) is the velocity of \( g_i \), \( R_i(a) \) the death modulus of individuals of the \( i \)-class aged \( a \), \( i = 1, 2, 3 \), and 1 := AA, 2 := AS, 3 := SS.

\( t, a \in \mathbb{R}^+ \), \( g \in \Omega \subset \mathbb{R}^+ \); in particular \( t \in [0, T] \), \( a \in [0, A] \) and \( \Omega \) is a measurable set, \( [0, T] \times \Omega := \varphi \). \( A \) is the life span of the species and in the sequel, we will drop the suffix ‘\( i \)’, and shall refer to it as appropriate. It is important to point out here that the model equations are related to SCA in the sense that the parameter \( g_i \) appearing in the renewal function \( B_i(t, g_i) \) describes how such a factor is distributed among newborns. The explicit forms of the birth function are not necessary in our analysis, but can be found in [10] for the reader so disposed.

3. Variational formulation

We define the weak generalized solution of system (2.1) by multiplying it by a suitable test function \( w \in H_0^1(\varphi_0) \), where \( \varphi_0 := [0, A] \times \Omega \), with the assumption that \( R(a) := R \), is a constant where necessary.

\[ \langle u_t + u_a + G(a)u_g, w \rangle = -R\langle u, w \rangle, \quad w := w(a, g). \]  

\( u_t \) simply denotes the partial derivatives with respect to time. For all \( u \) in \( C^2([0, A] \times \Omega) \), \( u \) is a distribution in \( L^2([0, A] \times \Omega) \) and \( w \in H_0^1(\varphi_0) \), that is,

\[ \int_0^\infty \int_0^\infty (u_t \cdot w + u_a \cdot w + G(a)u_g \cdot w) \, da \, dt + R \int_0^\infty \int_0^\infty u \cdot w \, da \, dt = 0. \]  

(3.2)

Let

\[ I := \int_0^\infty \int_0^\infty u_t \cdot w \, da \, dt = \int_0^\infty \left( \int_0^\infty u_t \cdot w \, dt \right) \, da, \]

integration by parts yields

\[ I = \int_0^\infty \left[ u_\infty(a, g)w(a, g) - u_0(a, g)w(a, g) - \int_0^\infty u \cdot w_t \, dt \right] \, da \]

\[ = \int_0^\infty w(a, g)(u_\infty - u_0) \, da - \int_0^\infty \int_0^\infty u \cdot w_t \, dt \, da, \]

\[ J := \int_0^\infty \left( \int_0^\infty w \cdot u_a \, da \right) \, dt = \int_0^\infty \left( uw_\infty - u_0 - \int_0^\infty u \cdot w_a \, da \right) \, dt \]

\[ = 0 - \int_0^\infty B(t, g)w(0, g) \, dt - \int_0^\infty \int_0^\infty u \cdot w_a \, da \, dt, \]
\[ K := \int_0^\infty \int_0^\infty G(a) \cdot w \cdot u_g \, da \, dt. \]

Hence,

\[ I + J + K = \int_0^\infty \int_0^\infty \left[ G(a)w_u - u(w_a + w_t) \right] \, da \, dt \]
\[ + \int_0^\infty w(u_\infty - u_0) \, da - \int B(t, g)w(0, g) \, dt, \]

where \( w_a \) and \( w_t \) are taken in the sense of distributions [2], and the integrals are understood as functionals on

\[ W^{2,2}([0, A] \times \Omega) \cap W^{1,1}([0, A] \times \Omega) \cap C([0, A] \times \Omega). \]

Denote \( W^{2,2}(\varphi_0) \cap W^{1,1}(\varphi_0) \cap C(\varphi_0) := \Theta \), then by Sobolev’s theorem, \( W^{2,2}(\varphi_0) \) is compactly embedded in \( L^\infty(\varphi_0) \), see [1].

Let \( L(u, w) \) defined by the left-hand side of Eq. (3.2) above be a bilinear form on \( L^\infty(\varphi_0 \times [0, T]) \). The analysis below considers the case in which the death modulus \( R(a) \) is not constant. Thus, we have the following:

**Lemma 3.1.** A weakly differentiable function \( u \in C^2(\varphi_0) \) is a mild solution of Eq. (3.1) in \( \varphi_0 \) if by Riesz representation theorem [1], there exists a linear functional \( w \in \Theta \) satisfying Eq. (3.2) such that \( \forall u \in C^1(\varphi_0) \) and any fixed \( w \in W^{1,2}, w \mapsto \langle Lu, w \rangle := L(u, w) \) is a bounded linear functional on \( \Theta \).

**Proof.** This problem, more often than not is related to the *a priori* estimates, and without loss of generality, we normalize \( B(t, g) \) by assuming that

\[ \int_0^\infty B(t, g) \, dt = B(g) = 1. \]

Now, let

\[ \| \phi \psi \|_{W^{2,2}}^2(\cdot, \cdot, g) := \int_0^\infty \int_0^\infty \phi \psi \, da \, dt, \]

and

\[ \| \phi \psi \|_{W^{1,2}}^2(\cdot, \cdot, g) := \int_0^\infty \phi \psi \, da. \]

Then, define

\[ L(u, w) := \int_0^\infty \int_0^\infty \left[ G(a)w \cdot u_g - u(w_t + w_a) \right] \, da \, dt \]
\[ + \int_{0}^{\infty} w(u_{\infty} - u_0) \, da - w(0, g) = - \int_{0}^{\infty} \int_{0}^{\infty} R(a) u \cdot w \, da \, dt. \]  
(3.3)

The right-hand side (RHS) of Eq. (3.3) becomes

\[
\text{RHS} := \int_{0}^{\infty} \int_{0}^{\infty} R(a) u \cdot w \, da \, dt \leq |R| \|u \cdot w\|_{L^{\infty}(\varphi_0)} \leq C_* \|u\|_{W^{2,2}(\varphi_0)}^2 \|w\|_{W^{2,2}(\varphi_0)}^2,
\]

where \(C_*\) depends on the population parameter \(R(a)\) [7]. Also, the left-hand side (LHS) of (3.3) satisfies the inequality

\[
\left| L(u, w) \right|_{L^{\infty}(\varphi_0 \times [0,T])} \leq \left\| G(a) w u_g - u(w_t - w_a) \right\|_{W^{2,2}(\varphi_0)}^2 + \left\| w(u_{\infty} - u_0) \right\|_{W^{1,2}(\varphi_0)}^2 + \sup_{0 \leq a \leq A} |w(a, g)| \leq \tilde{C}_* \|u\|_{H^2} \|w\|_{H^2},
\]

(3.4)

where \(\tilde{C}_*\) is a positive constant depending on \(G(a)\). Hence, by approximating arbitrary functions \(u, w\) and their derivatives in \(W^{1,2}(\varphi_0)\), and \(C^1(\varphi_0)\) by functions in \(\Theta\) leads to the above inequality in (3.4). \(L(u, w)\) is thus a bilinear form on \(L^{\infty}(\varphi_0 \times [0, T])\), which is bounded. \(\Box\)

In fact, the regularity of \(u\) implies the boundedness of \(L(u, w)\) and hence, the above estimates. Let \(u = u(t, a, g) \in [0, T] \times [0, A] \times \Omega := Q\).

Then, find \(u\) such that

\[
\begin{align*}
  u_t + u_a + G(a) u_g &= - R(a) u \in Q, \\
  u(0, a, g) &= u_0(a, g) \geq 0 \quad \text{is in } \varphi_0, \\
  u(t, 0, g) &= B(t, g) \quad \text{on } \partial Q.
\end{align*}
\]

(3.5)

Now, suppose \(u_0 \geq 0\), and \(u_0 \in L^{1}_{loc}(\varphi_0)\), then there exists a suitable smooth test function \(w(a, g)\) in \(D(Q)\) such that \(w(A, g) = 0, w \in C^1(\bar{\varphi}_0)\), and the following proposition holds:

**Proposition 3.2.** Let \(u \geq 0\) on \(Q\) such that

(i) \(u \in L^2(\varphi_0; H^1_0(\Omega))\).

(ii) \(\forall w \in D(Q), \int_{0}^{T} \int_{0}^{A} (u_t \cdot w + u_a \cdot w + G(a) u_g \cdot w) \, da \, dt = 0\).

(iii) \(\forall t, a \geq 0, u(t, a) \in L^1(\Omega)\) and \(u(t, a) \rightarrow u_0\) as \(t \rightarrow \infty\) in \(L^1(\Omega)\).

Then, \(u\) is a weak solution of (3.5).

**Proof.** Let \(w \in C^1(Q)\) and vanishing on \(\partial Q\), by regularization process [4], we write

\[
w_n = w * \rho_n, \quad w_n \in D(Q),
\]

and by approximation, (ii) remains true for \(w \in C^1(Q)\) and \(w \equiv 0\) on \(\partial Q\).
Letting \( w \in C^1(Q), \rho \in C^\infty(Q), 0 \leq \rho \leq s, \rho(s) = 0, s < 0, \) and \( \rho(s) = 1, \) for \( s \geq 1, \) \( \rho' \geq 0 \) and setting \( \rho_n(t) = \rho(nt), \) \( w_n(t, a) := w(t, a)\rho_n(t), \) \( w_n \in C^1(\bar{Q}), \) \( w = 0 \) on \( \partial Q, \) we have the following:
\[
\int \int (u_t w_n + u_a w_n + G(a) u_g w_n) \, da \, dt = - \int \int R(a) u \cdot w_n \, da \, dt.
\]
Integration by parts yields after some little manipulations and rearrangements
\[
\int \int G(a) w_n u_g - u(w_{n,a} + w_{n,t}) \rho_{n,t} \, da \, dt
= \int B(t, g) w(0, g) \rho_{n,t} \, dt - \int w(u_\infty - u_0) \rho_n(t) \, da,
\]
where \( \rho_{n,t} := \frac{\partial}{\partial t} \rho_n, \) see [11], and since
\[
u(t, \cdot, \cdot) \to u_0(\cdot, \cdot) \quad \text{as} \ t \to \infty,
\]
the last expression vanishes. By setting \( w(0, g) \equiv 0, \)
\[
\int B(t, g) w(0, g) \rho_n(t)
\]
vanishes identically. It remains to show that the LHS given below equals zero.
\[
\int \int G(a) w u_g - u(w_{n,a} + w_{n,t}) \rho_{n,t} = \frac{1}{n} \int \int G(a) w u_g - \frac{1}{n} \int \int u(w_a + w_t) \rho_{n,t}.
\]
Choose \( \epsilon > 0, \) so that for \( n \) large enough,
\[
\|u(t, \cdot, \cdot) - u_0(\cdot, \cdot)\|_{L^1(\Omega)} \leq \epsilon, \quad 0 \leq t \leq \frac{1}{n},
\]
then,
\[
\left| \int \int G(a) w \cdot u_g - u(w_a + w_t) \right| \leq \int \int G(a) \gamma(\cdot) \|u - u_0\|_{L^1(\Omega)} \cdot \|\rho_n\|
\leq \epsilon \|\rho_n\|
\]
and
\[
\lim_{n \to \infty} \int \frac{1}{n} \gamma(\cdot)(u - u_0) = 0,
\]
where \( \gamma(\cdot) \) is a constant depending on the parameters of the equation. Hence \( u(t, \cdot, \cdot) \to u_0(\cdot, \cdot) \)
as \( t \to \infty, \) and this terminates the proof. \( \square \)

The above conclusion certainly characterizes the dynamics of a stable population (which is rare indeed!). Populations with seasonal life cycles could have such dynamics.
4. Existence of a weak solution

Let $T$ be the final time, then the following holds:

**Theorem 4.1.** Let $u_0(a, g) \in L^1(\varphi_0)$, then (2.1) has a weak solution for $T = +\infty$.

**Proof.** Suppose $u_0 \in C_c(\varphi_0)$; the space of continuous and weakly sequentially compact maps. We may use barrier method by approximating the initial data as $u_{0,n} = u_0 + \frac{1}{n} \Rightarrow u_{0,n} \geq \frac{1}{n}$; $n \in \mathbb{N}$. We therefore re-write (3.5) as follow: Find $u$ such that

$$
\begin{align*}
&u_t + u_a + G(a)u_g + Ru = 0, \\
u(0, a, g) = u_{0,n} \quad \text{in } \varphi_0, \\
u(t, 0, g) = B(t, g) \quad \text{in } \varphi.
\end{align*}
$$

By the classical maximum principle [6], $u_n$ is bounded, i.e.,

$$
\frac{1}{n} \leq u_n(t, a, g) \leq N + \frac{1}{n},
$$

where $N = \sup_{a, g \in \varphi_0} u_0(a, g)$.

The above problem (4.1) has a solution $u_n$ in $C^1(\bar{Q})$ owing to classical result. By regularity, $u \in C^\infty(Q)$, and since $\frac{1}{n+1} \leq \frac{1}{n}$; $u_{0,n+1} \leq u_{0,n}$, then

$$
u_{n+1}(t, a, g) \leq u_n(t, a, g) \quad \text{in } \bar{Q},
$$

so that

$$
0 \leq \frac{1}{n+1} \leq u_{n+1}(t, a, g) \leq u_n(t, a, g) \leq N + \frac{1}{n}.
$$

The sequence $\{u_n\}$ is decreasing and bounded. Also

$$
\lim_{n \to \infty} u_n(t, a, g) = u(t, a, g)
$$

exists for any $t, a, g$ in $\bar{Q}$, and by Lebesgue Monotone Convergence Theorem, $u_n \rightharpoonup u$ in $L^1(\varphi_0)$ since $\varphi_0$ is bounded. Next, to show that $u$ is a weak solution, choose $w_n = u_n - \frac{1}{n}$ as a suitable test function and multiplying (2.1) by $w_n$, we have

$$
P := \int_0^T \int_0^A (u_{n,t} + u_{n,a} + G(a)u_{n,g} + R(a)u_n)(u_n - \frac{1}{n}) \, da \, dt,
$$

where

$$
u_{n,t} = \frac{\partial}{\partial t} u_n.
$$

Integrating by parts the first two terms and using Ostrogradski formula (also known as divergence theorem or Gauss formula) [5] we obtain

$$
\begin{align*}
\int_0^T \int_0^A (u_{n,t} + u_{n,a})u_n \, da \, dt &= \frac{1}{2} \left\{ \int_0^A u_n^2(T, a, g) \, da + \int_0^T u_n^2(t, A, g) \, dt \right\} \\
&\quad - \frac{1}{2} \left\{ \int_0^A u_{n,0}(a, g) \, da + \int_0^T u_n^2(t, 0, g) \, dt \right\}.
\end{align*}
$$
and Eq. (4.2) becomes

\[
P = \frac{1}{2} \left\{ \int_0^A u_n^2(T, a, g) \, da + \int_0^T u_n^2(t, A, g) \, dt \right\} \quad \text{and} \quad u_n^2(t, 0, g) \leq (N + \frac{1}{n})^2 \text{mes}(0, T),
\]

where \( \text{mes}(\alpha, \beta) \) denotes the measure of the interval \([\alpha, \beta]\), then

\[
P \leq \frac{1}{2} \int_0^A u_n^2(T, a, g) \, da - \frac{1}{2} \left\{ \left( N + \frac{1}{n} \right)^2 \text{mes}(0, A) + \left( N + \frac{1}{n} \right)^2 \text{mes}(0, T) \right\}
\]

\[
+ \left(1 - \frac{1}{n}\right) \left[ \frac{\partial}{\partial g} \int_0^A G(a) \left( \int_0^T u_n(t, a, g) \, dt \right) \, da + \int_0^T A \int_0^T R(a) u_n^2(t, a, g) \, da \, dt \right]
\]

\[
- \frac{1}{n} \int_0^T A \int_0^T (u_{n,t} + u_{n,a}) \, da \, dt.
\]

\( T \) being arbitrary, we assume without loss of reality that \( u_n(T, a, g) \) is also bounded, hence

\[
P \leq -\frac{1}{2} \left( N + \frac{1}{n} \right)^2 \text{mes}(0, T) + \left(1 - \frac{1}{n}\right) \left[ \frac{\partial}{\partial g} \int_0^A G(a) \left( \int_0^T u_n(t, a, g) \, dt \right) \, da
\]

\[
+ \int_0^T A \int_0^T R(a) u_n^2(t, a, g) \, da \, dt \right] - \frac{1}{n} \int_0^T A \int_0^T (u_{n,t} + u_{n,a}) \, da \, dt.
\]

We now assume for the sake of brevity that \( |u_{n,-}| \leq N + \frac{1}{n} \). Since \( R(a) \) and \( G(a) \) are both vital rates, without loss of reality, let \( \gamma(a) = \max(R(a), G(a)) \), then

\[
P \leq -\frac{1}{2} \left( N + \frac{1}{n} \right)^2 \text{mes}(0, T) + \left(1 - \frac{1}{n}\right) \left[ \left( N + \frac{1}{n} \right) \text{mes}(0, T) \int_0^A G(a) \, da
\]

\[
+ \left( N + \frac{1}{n} \right)^2 \text{mes}(0, T) \int_0^A R(a) \, da \right] - \frac{2}{n} \left( N + \frac{1}{n} \right) \text{mes}(0, A) \cdot \text{mes}(0, T),
\]
[0, A] ≠ φ; [0, T] ≠ φ, and without loss of generality, let \( \operatorname{mes}(0, A) = \operatorname{mes}(0, T) = 1 \). Then,

\[
P \leq -\frac{1}{2} \left( N + \frac{1}{n} \right)^2 + \left( \frac{n-1}{n} \right) \left[ \left( N + \frac{1}{n} \right) A \int_0^A G(a) da + \left( N + \frac{1}{n} \right)^2 A \int_0^A R(a) da \right] \]

\[
- \frac{2}{n} \left( N + \frac{1}{n} \right) \leq - \left( N + \frac{1}{n} \right) \left( N + \frac{3}{n} \right) + \left( \frac{n-1}{n} \right) \left[ \left( N + \frac{1}{n} \right) A \int_0^A \gamma(a) \left( 1 + N + \frac{1}{n} \right) da \right].
\]

\[
|P| \leq \left( \frac{3}{n} + N \right) \left( \frac{1}{n} + N \right) + \left( N + \frac{1}{n} \right) \left( \frac{n-1}{n} \right) \left( 1 + N + \frac{1}{n} \right) A \int_0^A \gamma(a) da
\]

\[
\leq \left( \frac{1}{n} + N \right) \left[ \frac{3}{n} + N + \left( 1 + N + \frac{1}{n} \right) \left( \frac{n-1}{n} \right) A \int_0^A \gamma(a) da \right]
\]

\[
\leq C_1(N, n, \gamma(a)),
\]

where \( C_1 \) is a constant depending on \( N, n \) and \( \gamma(a) \). It is therefore clear that \( u_n \to u \), and \( u_{a,t} \to u_t \) in \( D(Q) \) a.e., and by density, passing to the limit as \( n \to \infty \), \( u_n \to u \) in \( L^1(Q) \) a.e. Hence \( u \) is a weak solution of (3.5).

5. Uniqueness of weak solution

This is established using a suitable choice of the test function.

**Theorem 5.1.** Problem (3.5) has at most one weak solution.

**Proof.** Let \( u \) and \( \tilde{u} \) be two weak solutions of (3.5), then for any test function \( w \), we have

\[
\int \int \left\{ (u_t - \tilde{u}_t) \cdot w + (u_a - \tilde{u}_a) \cdot w + G(a)(u_g - \tilde{u}_g) \cdot w \right\} da dt = 0.
\]

(5.1)

Define

\[
w(t, a) := \begin{cases} 
\int_t^T \{ u(s, a, g) - \tilde{u}(s, a, g) \} ds, & 0 < t < T, \\
0, & \text{if } t \geq T.
\end{cases}
\]

(5.2)

This particular test function is due to Oleinik as stated in [11].

For \( T > 1 \),

\[
\frac{\partial w}{\partial t} = -\{ u(t, a, g) - \tilde{u}(t, a, g) \} \in L^2(Q),
\]

since \( u \in L^2(\varphi_0, H^1_0(\Omega)) \), \( w(t) \in H^1_0[0, A] \) with \( w(T) \equiv 0 \).

Now,

\[
\int \int \left\{ (u_t - \tilde{u}_t) + (u_a - \tilde{u}_a) + G(a)(u_g - \tilde{u}_g) \right\} \left( \int_t^T (u - \tilde{u}) ds \right) da dt = 0.
\]

(5.3)
Integrating by parts using the regularity of $u$ and $\tilde{u}$ yields
\begin{align*}
&\int \frac{1}{2} \left[ \int_t^T (u - \tilde{u}) \, ds \right]^2 \, da + \int \frac{1}{2} \left[ \int_t^T (u - \tilde{u}) \, ds \right]^2 \, dt \\
&\quad + \int \int \left\{ G(a)(u_g - \tilde{u}_g) \int_t^T (u - \tilde{u}) \, ds \right\} \, da \, dt = 0.
\end{align*}
(5.4)

The last expression can be approximated as follows: $u$ and $\tilde{u}$ being continuous, we can write
\begin{align*}
\left\{ \left( \frac{\partial}{\partial g} \int \int G(a)(u - \tilde{u}) \right) \int_t^T (u - \tilde{u}) \, ds \right\} \, da \, dt,
\end{align*}
(5.5)

and
\begin{align*}
\int_t^T (u - \tilde{u}) \, ds = \int_0^T \left\{ u(s-t, a, g) - \tilde{u}(s-t, a, g) \right\} \, ds
\leq C \| u(\cdot, a, g) - \tilde{u}(\cdot, a, g) \|_{L^1(\varphi_0)}.
\end{align*}

Hence, Eq. (5.5) becomes
\begin{align*}
\left\{ \left( \frac{\partial}{\partial g} \int \int G(a)(u - \tilde{u}) \right) \int_t^T \left[ u(s-t, a, g) - \tilde{u}(s-t, a, g) \right] ds \right\} \, da \, dt
\leq \tilde{c} \| u - \tilde{u} \| \text{ by Lemma (3.1)},
\end{align*}
(5.6)

where $\tilde{c}$ is a positive constant depending on the parameters $G(a)$, $t$, $\frac{\partial}{\partial g} (\cdot)$.

Therefore, (5.6) vanishes if and only if $u = \tilde{u}$ a.e. on $Q$, hence the proof. \qed

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**References**


