Existence for a Class of Semilinear Problems at Resonance

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1. INTRODUCTION

The purpose of this paper is to establish sufficient conditions for the existence of solutions for semilinear equations at resonance. Our results extend recent results of the author [15] concerning periodic solutions of ordinary differential equations. As an illustrative example, consider the equation

$$Au + e^u = f,$$  (1.1)

where $A$ is a linear operator on a suitable function space and $f$ is a suitable function. Under assumptions that allow for $N(A)$, the null space of $A$, to have dimension $v$, $1 \leq v < \infty$, we obtain sufficient conditions for (1.1) to have a solution. Our results are independent of sign conditions in the sense that they apply equally well to (1.1) with $A$ replaced by $-A$. Our basic assumptions are (i) $A$ is a Fredholm operator of index zero with domain $D(A) \subseteq L^\infty(G)$ ($G \subseteq \mathbb{R}^m$, $m \geq 1$, is a bounded open set) and codomain $L^1(G)$; (ii) $A$ has a partial inverse $K$ which is compact as a mapping from the range of $A$ in $L^1(G)$ into $L^\infty(G)$, and (iii) $N(A)$ contains the constants. These hypotheses eliminate a number of interesting cases of (1.1) from our consideration. For example, our results do not apply to the question of existence for the Neumann problem

$$\Delta u + e^u = f(x), \quad x \in G \subseteq \mathbb{R}^m, \quad m \geq 2,$$  (1.2)

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial G$$  (1.3)

for a bounded set $G$. The reader is referred to the paper [6] of Kazdan and Warner for some results related to (1.2), (1.3). To the best of our knowledge the general existence question for (1.2), (1.3) remains open. For results on (1.2), (1.3) with $\Delta$ replaced by $-\Delta$ and related problems, much more is known, and the reader is referred to [1, 2, 3, 7, 12, 16].

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On the other hand, our results do apply to the problem

$$\Delta^2 u - e^u = f(x), \quad x \in G,$$  \hspace{1cm} (1.4)

$$\partial u / \partial n = \partial \Delta u / \partial n = 0 \quad \text{on } \partial G$$  \hspace{1cm} (1.5)

for $G \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$. We are also able to remove the assumption made in [15] concerning the periodic problem that $N(A)$ contains only the constants; this has some useful practical consequences since it means that we need not, in many cases, know $N(A)$ in order to decide the solvability of (1.1).

Our main results make use of a well known continuation theorem of Mawhin; for the convenience of the reader we describe Mawhin's theorem in Section 2. In Section 3 we prove our main result, in Section 4 we present applications to periodic solutions for ordinary differential equations and extensions to delay equations. In Section 5 we apply our main result to a class of elliptic boundary value problems which includes (1.4), (1.5).

2. A CONTINUATION THEOREM OF MAWHIN

The proof of our main result depends upon a continuation theorem of Mawhin; his theorem leads to simpler proofs than other arguments might allow for. Let $X$ and $Z$ be normed vector spaces, $L: D(L) \subseteq X \rightarrow Z$ a linear Fredholm mapping of index zero (index$(L) = \dim \ker(L) - \text{codim Im}(L)$) and $N: X \rightarrow Z$ a continuous mapping. It follows that there exist continuous projections $P: X \rightarrow X$ and $Q: Z \rightarrow Z$ such that $\ker(Q) = \ker(L) = \text{codim Im}(Z - Q)$. Moreover the mapping $L: D(L) \cap \ker(P) \rightarrow \text{Im}(L)$ is invertible; denote its inverse by $K$. Let $\Omega$ be an open bounded subset of $X$. The mapping $N$ is said to be $L$-compact on $\Omega$ if $QN(\Omega)$ is bounded and $K(I - Q)N: \Omega \rightarrow X$ is compact. Let $J$ be an isomorphism from $\text{Im}(Q)$ onto $\ker(L)$; such a $J$ exists since these subspaces have the same finite dimension.

**Theorem A** (Mawhin [9, 11]). Let $L$ be a Fredholm mapping of index zero and let $N$ be $L$-compact on $\Omega$. Suppose

(i) For each $\lambda \in [0, 1[$ and each $x \in \text{Dom } L \cap \partial \Omega$,

$$Lx + \lambda Nx \neq 0.$$  

(ii) $QNx \neq 0$ for each $x \in \ker L \cap \partial \Omega$ and $d(JQN, \Omega \cap \ker L, 0) \neq 0$. Then the equation $Lx + Nx = 0$ has at least one solution in $D(L) \cap \Omega$. Here $d(\cdot, \cdot, \cdot)$ refers to the Brouwer degree.
3. A General Result

In the following $G$ will denote a bounded open connected set in $\mathbb{R}^v$, $v \geq 1$. We will denote the real Lebesgue spaces $L^1(G)$ and $L^\infty(G)$ simply by $L^1$ and $L^\infty$, respectively, and their respective norms by $| \cdot |_1$ and $| \cdot |_\infty$. If $A$ is a linear operator $D(A)$, $N(A)$, and $R(A)$ will respectively denote the domain, kernel, and range of $A$.

We will use the following hypotheses.

(h1) $A: D(A) \subseteq L^\infty \to L^1$ is a Fredholm map of index zero (i.e., $R(A)$ is closed in $L^1$, $N(A)$ and $L^1/R(A)$ are finite dimensional with equal dimension).

(h2) $R(A) \perp N(A)$, i.e., $\int_G fg = 0$ for all $f \in R(A)$, $g \in N(A)$. By (h1) we may write $L^1 = R(A) \oplus N(A)$.

(h3) Let $K = (A | D(A) \cap R(A))^{-1}$ and $Q: L^1 \to N(A)$ be the projection parallel to $R(A)$. Then $K(I - Q): L^1 \to L^\infty$ is a compact mapping.

(h4) The constant functions are contained in $N(A)$.

(h5) The functions in $N(A)$ have the unique continuation property. That is, if $w \in N(A)$ and $w(t) = 0$ on a set of positive measure, then $w(t) = 0$ for almost all $t \in G$.

Let $g: G \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We will make the following assumptions on $g$.

(C1) There is a number $\beta \geq 0$ such that $g(t, u) \geq -\beta$ for all $(t, u) \in G \times \mathbb{R}$ and $|g(t, u)| \leq \beta$ for all $t \in G$ and $u \leq 0$.

(C2) $\lim_{u \to -\infty} g(t, u) = \infty$, uniformly for $t \in G$.

We define a measurable function $g^- : G \to \mathbb{R}$ by

$$g^-(t) = \lim_{u \to -\infty} \sup_{s < u} g(t, u)$$

for all $t \in G$. By (C1) we have $g^- \in L^\infty$.

Let $f \in L^1$. We consider the equation

$$Au(t) + g(t, u(t)) = f(t). \quad (3.1)$$

**Theorem 3.1.** Assume (h1)–(h5) and (C1), (C2). Then there will exist a solution to Eq. (3.1) provided

$$\int_G [g^-(t) - f(t)] |w(t)| \, dt < 0 \quad (3.2)$$

for all $0 \neq w \in N(A)$ satisfying $w \leq 0$. 
Remark 3.1. By a solution to 3.1 we mean a function $u \in D(A)$ satisfying (3.1) a.e. on $G$.

Remark 3.2. Condition (3.2) is a Landesman–Lazer condition of the type first appearing in [8]; see also Williams [17].

Remark 3.3. Theorem 1 is exactly what one would predict from the main result of Williams [17] with our hypotheses. Unfortunately, the methods used in [8, 17] are Hilbert space methods and neither the direct application of those methods nor approximation schemes as in [1, 3, 12] will yield Theorem 1 or the applications we obtain. Incidentally, Kazdan and Warner [7] show that sub- and super-solution methods cannot be effective in establishing existence for (1.2), (1.3), even if $e^{\mu}$ is replaced by any (even slowly) increasing function.

For applications, the following corollary may be useful, since one does not need full knowledge of $N(A)$ in order to apply it.

**Corollary 3.1.** Assume (hl)-(h5) and (C1), (C2). Then there is a solution to (3.1) provided $g^{-}(t) \leq f(t)$ a.e. on $G$ and $g^{-}(t) < f(t)$ on a set of positive measure in $G$.

**Proof of Theorem 1.** Define $N: L^{\infty} \rightarrow L^{1}$ by $N u(t) = g(t, u(t)) - f(t)$ for $u \in L^{\infty}$ and $t \in G$; then $N$ is continuous and maps bounded sets in $L^{\infty}$ into bounded sets in $L^{1}$. Equation (3.1) is equivalent to the equation

$$Au + Nu = 0. \quad (3.3)$$

The underlying hypotheses of Mawhin's theorem (Theorem A, Section 2) hold and it remains to verify (i) and (ii) of that theorem. We thus consider

$$Au + \lambda Nu = 0, \quad \lambda \in [0, 1[, \quad (3.4)$$

or, equivalently,

$$Au + \lambda g(t, u) = \lambda f, \quad \lambda \in [0, 1[. \quad (3.5)$$

Since the constants are contained in $N(A)$, if $(u, \lambda)$ is a solution of (3.5) we must have

$$\int_{G} g(t, u) \, dt = \int_{G} f \, dt. \quad (3.6)$$

From (3.5) and (C1),

$$|Au(t)| \leq g(t, u(t)) + 2\beta + |f(t)|. \quad (3.7)$$
Integrating (3.7) over \( G \) and using (3.6) yields
\[
\iiint_G |Au| \leq \iiint_G f + \iiint_G |f'| + 2\beta |G| = c_1,
\] (3.8)
valid for any \( \lambda \in [0, 1] \). We now write \( u = x_0 + x_1 \), where \( x_0 \in N(A) \) and \( x_1 \in R(A) \cap D(A) \). By (3.8) and (h1) there is a number \( k > 0 \) such that for \( \lambda \in [0, 1] \)
\[
|x_1|_\infty \leq k |Ax_1|_1 \leq kc_1.
\] (3.9)

We will now obtain an a priori bound on \( x_0 \) independent of \( 0 < \lambda < 1 \). Suppose there is a sequence \( \{(\lambda_n, u_n)\} \) of solutions to (3.5) with \( u_n = x_{0n} + x_{1n} \), \( x_{0n} \in N(A) \), \( x_{1n} \in R(A) \), and \( |x_{0n}|_\infty \to \infty \). By the unique continuation property we have \( |x_{0n}(t)| \to \infty \) a.e. on \( G \). To see this, let \( y_{0n} = x_{0n}/|x_{0n}|_\infty \). The kernel of \( A, N(A) \), is finite dimensional and we may therefore assume that \( y_{0n} \to y_0 \in N(A) \), and \( |y_0|_\infty = 1 \). If there were a set of positive measure on which \( |x_{0n}(t)| \) remained bounded, on that set we would have \( y_0(t) = 0 \) which by unique continuation implies \( y_0(t) = 0 \) a.e. in \( G \), contradicting \( |y_0|_\infty = 1 \).

Let \( G_+ = \{t: x_{0n}(t) \to \infty \} \), and suppose \( |G_+| > 0 \). Pick a number \( \tilde{c} > 0 \) such that
\[
(\tilde{c}/2)|G_+| - 2\beta |G| > 2 \iiint_G f.
\] (3.10)

For any \( M > 0 \) there exists a number \( n_1 > 0 \) and a set \( Q \subseteq G_+ \) with \( |Q| \geq |G_+|/2 \) such that for \( t \in Q \) and \( n \geq n_1 \) we have \( x_{0n}(t) \geq M + kc_1 \); thus for \( t \in Q \)
\[
u_n(t) \geq x_{0n}(t) - |x_{1n}|_\infty
\geq x_{0n}(t) - kc_1 \geq M.
\]

By (C2) we may choose \( M \) so large that \( g(t, u_n(t)) \geq \tilde{c} \) for \( n \geq n_1 \) and \( t \in Q \). We have
\[
n \int_G f = \int_G g(t, u_n) \geq \int_Q g(t, u_n) - 2\beta |G| - \int_G f
\geq \int_Q \tilde{c} - 2\beta |G| - \int_G f
\geq (\tilde{c}/2)|G_+| - 2\beta |G| - \int_G f > \int_G f
\]}
by (3.10). We must conclude that \(|G_+| = 0\). We therefore have \(x_{0n}(t) \to -\infty\) a.e. on \(G\). Now using (C1) and (3.9) we have

\[
\lim_{n \to \infty} \sup_{G} g(t, u_n(t)) = g^-(t) \text{ a.e.}
\]

Let \(y_{0n} = x_{0n}/|x_{0n}|_\infty\). Without loss of generality we may assume \(y_{0n} \to \tilde{y}_0 \in N(A)\), and now \(\tilde{y}_0(t) \leq 0\) a.e. in \(G\), \(\tilde{y}_0 \neq 0\). Multiplying each side of

\[
Au_n + \lambda_n g(t, u_n) = \lambda_n f
\]

by \(\tilde{y}_0\) and taking limits yields, after an application of Fatou's lemma,

\[
\int_G f|\tilde{y}_0| = -\int_G f\tilde{y}_0 = \lim_{n \to \infty} \sup_{G} \int_G g(t, u_n) \tilde{y}_0 = \lim_{n \to \infty} \sup_{G} \int_G g(t, u_n) \tilde{y}_0 \leq \int_G g^-(t) |\tilde{y}_0| < \int_G f|\tilde{y}_0|
\]

by the hypotheses in the statement of this theorem. This contradiction shows that \(|x_{0n}|_\infty\) remains bounded; thus by (3.9) there exists an a priori bound on all solutions \(u = x_0 + x_1\) to (3.5). This verifies (i) of Theorem A. It remains to verify (ii).

Because \(N(A)\) is contained algebraically in each of \(L^1\) and \(L^\infty\) the mapping \(J\) in Theorem A may be taken to be the natural identity imbedding from \(N(A)\) regarded as a subspace of \(L^1\) into \(N(A)\) regarded as a subspace of \(L^\infty\). Let \(w_1, \ldots, w_m\) be an orthogonal basis for \(N(A)\) normalized so that \(\int_G w_i^2 = 1\), \(1 \leq i \leq m\). We may represent the projector \(Q: L^1 \to N(A)\) by

\[
Q u = \sum_{i=1}^{m} \int_G u w_i \, dt \, w_i.
\]

The mapping \(T = JQN|N(A)\) may be identified with the mapping \(\tilde{T}: \mathbb{R}^m \to \mathbb{R}^m\) defined for \(\hat{\alpha} = (a_1, \ldots, a_m) \in \mathbb{R}^m\) by \(\tilde{T}(\hat{\alpha}) = \hat{c} = (c_1, \ldots, c_m)\) with

\[
c_i = \int_G [g(t, \hat{\alpha} \cdot \hat{w}) - f(t)] \, w_i \, dt, \quad \hat{w} = (w_1, \ldots, w_m)
\]

for \(1 \leq i \leq m\). We will show that \(d(JQN, N(A) \cap \Omega_r, 0) \neq 0\) for large \(r\) \((\Omega_r = \{u \in L^\infty: |u|_\infty < r\})\) by showing \(d(\tilde{T}, S_r, 0) \neq 0\) for all large \(r\), where \(S_r = \{\hat{\alpha} \in \mathbb{R}^m: |\hat{\alpha}| < r\}\).

We claim that if \(|\hat{\alpha}|\) is large then \(\tilde{T}(\hat{\alpha}) \cdot \hat{\alpha} > 0\), which, as is well known, implies \(\tilde{T}\) is homotopic to the identity on large \(S_r\), so that \(d(\tilde{T}, S_r, 0) = 1\) for
large $r$. Let $\bar{a} = \hat{a}/|\hat{a}|$. Suppose for all integers $n > 0$ there exist $\hat{a}_n$ with $|\hat{a}_n| > n$ and $\bar{T}(\hat{a}_n) \cdot \hat{a}_n \leq 0$. Then $T(\hat{a}_n) \cdot \hat{a}_n \leq 0$ so that

$$\int_G \left[ g(t, \hat{a}_n \cdot \hat{w}) - f(t) \right] \hat{a}_n \cdot \hat{w} \, dt \leq 0.$$ 

Without loss of generality we may assume $\hat{a}_n \to \bar{a}$ in $R^n$ with $|\bar{a}| = 1$. Moreover, $|\hat{a}_n \cdot \hat{w}|_\infty \to \infty$ so that $|\hat{a}_n \cdot \hat{w}(t)| \to \infty$ a.e. by unique continuation, and $\bar{a} \cdot \hat{w}(t) = 0$ on a set of at most measure zero. Let $G^\pm = \{t: \hat{a}_n \cdot \hat{w}(t) \to \pm \infty\}$, and suppose $|G^+| > 0$. Clearly $\bar{a} \cdot \hat{w}(t) > 0$ a.e. on $G^+$, and if we let $F_n(t) = [g(t, \hat{a}_n \cdot \hat{w}) - f(t)]$ then

$$0 \geq \liminf_{n \to \infty} \bar{T}(\hat{a}_n) \cdot \hat{a}_n$$

$$= \liminf_{n \to \infty} \int_G F_n(t) \hat{a}_n \cdot \hat{w} \, dt$$

$$= \liminf_{n \to \infty} \left[ \int_{G^+} F_n(t) \hat{a}_n \cdot \hat{w} \, dt + \int_{G^-} F_n(t) \hat{a}_n \cdot \hat{w} \, dt \right]$$

$$\geq \int_{G^+} \liminf F_n(t) \hat{a}_n \cdot \hat{w} \, dt + \int_{G^-} \liminf F_n(t) \hat{a}_n \cdot \hat{w} \, dt$$

$$= \int_{G^+} \bar{a} \cdot \hat{w} \, dt + \int_{G^-} [g^-(t) - f(t)] \bar{a} \cdot \hat{w} \, dt$$

$$= \infty.$$ 

Thus we must have $|G^+| = 0$. Thus $|G^-| = |G|$ and $\hat{a}_n \cdot \hat{w}(t) \to -\infty$ a.e. on $G$ so that, using the notation above, $\bar{a} \cdot \hat{w} \leq 0$ a.e. on $G$ and $\bar{a} \cdot \hat{w} \not\equiv 0$. Taking limits again as above,

$$0 \geq \liminf_{n \to \infty} \bar{T}(\hat{a}_n) \cdot \hat{a}_n$$

$$\geq \int_G [g^-(t) - f(t)] \bar{a} \cdot \hat{w} \, dt$$

$$= -\int_G [g^-(t) - f(t)] |\bar{a} \cdot \hat{w}| \, dt$$

$$> 0$$

by the hypothesis of the theorem. Thus we must have $\bar{T}(\hat{a}) \cdot \hat{a} > 0$ if $|\hat{a}| \geq r$ for some large $r$, and our argument is completed as indicated above.
4. Periodic Solutions

Consider the ordinary differential equation
\[ x^{(m)} + a_{m-1}x^{(m-1)} + \cdots + a_1x' + g(t, x) = f(t), \]  
(4.1)
where \( a_1, \ldots, a_{m-1} \) are real constants, \( g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( T \)-periodic in its first variable, \( g(t + T, x) = g(t, x) \) for all \( t, x \in \mathbb{R} \), and \( f: \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( T \)-periodic, \( T > 0 \).

It is classical that if we define \( D(A) \subseteq L^\infty(0, T) \) by
\[ D(A) = \{ u \in C^{m-1}[0, T]: u^{(i)}(0) = u^{(i)}(T), 0 \leq i \leq m - 1 \} \]
and \( Au \) for \( u \in D(A) \) by
\[ Au = u^{(m)} + a_{m-1}u^{(m-1)} + \cdots + a_1u' \]
then (h1)–(h5) are satisfied. From Theorem 3.1 we obtain

**Theorem 4.1.** Suppose \( g \) satisfies (C1), (C2) with \( G = [0, T] \). If
\[ \int_0^T |g(t) - f| |w(t)| dt < 0 \]
for all \( w \in N(A) \) with \( w \leq 0, w \neq 0 \), there is a \( T \)-periodic solution of (4.1).

**Proof:** Immediate, together with the observation that if \( u \) is a solution to \( Au + g(t, u) = f \) then \( u \) extends \( T \)-periodically to be a solution of (4.1) on \((-\infty, \infty)\). Moreover, it is easy to show that \( u^{(m)} \) is also continuous, and thus \( u \) is a classical \( T \)-periodic solution of (4.1).

Corollary 3.1 also applies in this case; the following example illustrates that corollary.

**Example 4.1.** Let \( a_1, \ldots, a_{m-1} \) be any \( m - 1 \) real numbers and let \( p, f: \mathbb{R} \rightarrow \mathbb{R} \) be continuous and \( T \)-periodic with \( p(t) > 0 \) for all \( t, f(t) \geq 0 \) for all \( t \), and \( f \neq 0 \). Let \( g: \mathbb{R} \rightarrow \mathbb{R} \) be continuous with \( g(x) > 0 \) for all \( x \in \mathbb{R} \) and \( g(-\infty) = 0, g(\infty) = \infty \). Then the following ordinary differential equation has a \( T \)-periodic solution:
\[ x^{(m)} + a_{m-1}x^{(m-1)} + \cdots + a_1x' + p(t)g(x) = f(t). \]  
(4.2)

It follows from the results of [15] that if the homogeneous equation
\[ x^{(m)} + a_{m-1}x^{(m-1)} + \cdots + a_1x' = 0 \]
has only the constants for its $T$-periodic solutions then the condition on $p$ may be weakened to $p(t) \geq 0$ for all $t$ and $p \not= 0$. The following example shows this to be false in the more general situation.

**Example 4.2.** Consider the third order equation

$$x'' + x' + p(t) g(x) = 2 - \cos t,$$

where $g$ is as in Example 4.1 and $p$ is continuous and $2\pi$-periodic with $p(t) > 0$ for $0 < t < \pi/2$ and $p(t) = 0$ for $\pi/2 \leq t \leq 2\pi$. Suppose (4.3) has a $2\pi$-periodic solution $x(t)$; then multiplying each side of (4.3) by $\cos t$ and integrating from 0 to $2\pi$ leads to the contradiction

$$0 < \int_{0}^{\pi/2} p(t) g(x(t)) \cos t \, dt = \int_{0}^{2\pi} p(t) g(x(t)) \cos t \, dt = -\int_{0}^{2\pi} \cos^2 t \, dt.$$

Thus (4.3) has no $2\pi$-periodic solutions, although it satisfies all the requirements of Example 4.1 except $p(t) > 0$ for all $t$.

Now consider the delay (or advance) differential equation

$$x^{(m)} + a_{m-1} x^{(m-1)} + \cdots + a_1 x' + g(t, x(t - r)) = f(t),$$

where $a_1, \ldots, a_{m-1}$ are again real numbers, $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $T$-periodic in its first variable, $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $T$-periodic, and $r$ is any real number. Define two measurable functions $g_+, g_-: \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ by

$$g_+(t) = \lim \inf_{x \rightarrow -\infty} g(t, x),$$

$$g_-(t) = \lim \sup_{x \rightarrow \infty} g(t, x).$$

Suppose there are numbers $\delta_+, \delta_-$ such that $g(t, x) \geq \delta_+$ for all $x \geq 0$, and $t \in \mathbb{R}$ and $g(t, x) \leq \delta_-$ for $x \leq 0$, $t \in \mathbb{R}$, and $\alpha, \beta > 0$ such that $|g(t, x)| \leq g(t, x) + \alpha |x| + \beta$ for all $t, x \in \mathbb{R}$. Then we have

**Theorem 4.2.** Suppose

(i) The only $T$-periodic solutions to

$$x^{(m)} + a_{m-1} x^{(m-1)} + \cdots + a_1 x' = 0$$

are the constants.

(ii) $\int_{0}^{T} g_- < \int_{0}^{T} f < \int_{0}^{T} g_+.$

Then there exists a $T$-periodic solution of (4.4) provided $\alpha < \alpha_o$ with $\alpha_o$ sufficiently small.
Proof. The proof is very similar to that of Theorem 1 in [15] for ordinary differential equations and we refer the reader to that.

Remark 4.1. Mawhin [11, p. 71] has earlier shown that in the case of first order scalar equations Theorem 4.2 with \( r = 0 \) is valid without our growth restriction on \( g(t, x) \) for \( x \leq 0 \). Other recent results on Eq. (4.1) or vector analogues use the assumption that \( N(A) \) consists of the constants only and that the nonlinear terms are bounded or of slow growth; e.g., [5, 10, 13, 14]. Our results here do not include these. In the case of bounded nonlinearity for (4.1) it is clear (because (h1)–(h5) hold) that William’s result in [17] may be formulated for (4.1). We expect to consider systems of equations elsewhere.

5. A Neumann Problem

Let \( G \subseteq \mathbb{R}^2 \) or \( \mathbb{R}^3 \) be a smooth, bounded, connected open set. We consider the Neumann problem

\[
\begin{align*}
-\Delta^2 u + g(x, u) &= f(x), & x &\in G, \\
\frac{\partial u}{\partial n} &= \frac{\partial (\Delta u)}{\partial n} = 0 & \text{on } \partial G,
\end{align*}
\]

where \( g: G \times \mathbb{R} \to \mathbb{R} \) is continuous and \( f \in L^1(G) \), and \( \partial/\partial n \) denotes the outward normal derivative.

The fundamental solution and thus the Neumann function for

\[
\begin{align*}
-\Delta^2 u &= 0, \\
\frac{\partial u}{\partial n} &= \frac{\partial (\Delta u)}{\partial n} = 0 & \text{on } \partial G
\end{align*}
\]

is continuous on \( \overline{G} \); see [4, p. 4f].

Let \( N(x, y) \) denote Neumann’s function. Then any solution of

\[
\begin{align*}
-\Delta^2 u &= f, \\
\frac{\partial u}{\partial n} &= \frac{\partial (\Delta u)}{\partial n} = 0 & \text{on } \partial G
\end{align*}
\]

satisfies

\[
u(x) = \int_G N(x, y) f(y) \, dy = Kf.
\]

Moreover, because \( N \) is continuous \( K \) maps \( L^1(G) \) into \( L^\infty(G) \) compactly.

Let \( Q: L^1(G) \to L^1(G) \) be defined by

\[
Qu = |G|^{-1} \int_G f.
\]
Let $X_1 = K(I - Q)L^1(G)$; we have $X_1 \perp \mathbb{R}$. Let $D(A) = X_1 \oplus \mathbb{R}$ and define $A: D(A) \subseteq L^\infty \to L^1$ for $x = x_1 + x_0$, $x_1 \in X_1$ and $x_0 \in \mathbb{R}$ by $Ax = y$, where $Qy = 0$ and $Ky = x_1$. Because $K$ is continuous and $(I - Q)L^1$ is closed in $L^1$, we see $A$ is a Fredholm operator and clearly it has index zero. Any solution of

$$Au + g(x, u) = f(x), \quad x \in G, \quad (5.4)$$

is a formal solution of (5.1), (5.2). We will call any solution of (5.4) a mild solution of (5.1), (5.2). It is not hard to show that if $f \in L^\infty$ and $u$ is a mild solution of (5.1), (5.2) then $u$ is a weak solution of (5.1), (5.2). For all $v \in C^4(\bar{G})$ satisfying $\partial v / \partial n = \partial (\Delta v) / \partial n = 0$ on $\partial G$ we have

$$-\langle u, A^2 v \rangle + \langle g(x, u), v \rangle = \langle f, v \rangle, \quad (5.5)$$

where $\langle f, g \rangle = \int_G fg$.

To see this, recall that $N(x, y) = N(y, x)$ so that $\langle Ku, v \rangle = \langle u, Kv \rangle$ for all $u, v \in L^\infty$ with $Qu = Qv = 0$. Thus, if $u = u_1 + \alpha$, $v = v_1 + \beta$ are in $D(A)$ with $u_1, v_1 \in X_1$, $\alpha, \beta \in \mathbb{R}$ and $Au, Av \in L^\infty$ then

$$\langle Au, v \rangle = \langle Au_1, KAv + \beta \rangle - \langle Au_1, KAv_1 \rangle = \langle KAu_1, Av_1 \rangle = \langle u_1, Av_1 \rangle = \langle u, Av \rangle.$$ 

Thus, if $v \in C^4 \cap D(A)$ then in this case $Av = -A^2 v$ and $\langle Au, v \rangle = \langle u, -A^2 v \rangle$. Equation (5.5) now follows since if $f \in L^\infty$, $Au \in L^\infty$ also.

**Theorem 5.1.** Let $g$ satisfy (C1), (C2) (see Section 3) and suppose $f \in L^1(G)$. Then (5.1), (5.2) has a mild solution provided

$$\int_G g^- < \int_G f.$$ 

If $f \in L^\infty$ then this mild solution is also a weak solution of (5.1), (5.2).

**Proof.** Immediate application of our comments above together with Theorem 3.1.

**Remark 5.1.** If we replace $-A^2 u$ in (5.1) with $\Delta^2 u$ then we obtain the same conclusion; however, in this case results have also been obtained by other methods (see [1, 3, 12]).

**Remark 5.2.** Similar results could be obtained for general elliptic operators of order $2m$ with constant coefficients provided $2m > n$, where $G \subseteq \mathbb{R}^n$. 

SEMILINEAR PROBLEMS AT RESONANCE

REFERENCES