Choquet Simplexes in Finite Dimension – A Survey

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Abstract: This survey covers various geometric results related to Choquet simplexes in the Euclidean space $\mathbb{E}^d$; it describes the known properties of Choquet simplexes and marks still open problems.

Keywords: convex set, choquet simplex, simplex, generalized simplex, simplicial cone.

1. Introduction

In 1956 Choquet [12, 13] defined a simplex (afterwards called a Choquet simplex) as a convex set $S$ in a linear space $E$ of any dimension such that the intersection of any two homothetic copies of $S$, if nonempty, is again a homothetic copy of $S$, possibly degenerated into a point:

$$(u + \lambda S) \cap (v + \mu S) = w + \nu S, \quad u, v, w \in E, \quad \lambda, \mu, \nu \geq 0.$$ 

By using the technique of measure representation (cf. [32]), it was shown later that finite-dimensional compact Choquet simplexes are precisely the simplexes in the usual sense, i.e., they are the convex hulls of finitely many affinely independent points. The properties of Choquet simplexes were essentially used in the Choquet representation theory (see, e.g., [1, 2, 32, 48] for references). In particular, Choquet has pointed out that the lattice structure of an ordered vector space $E$ is closely related with the simplicial character of the base of the positive cone of $E$ (see Section 2 below for details).

Independently of Choquet, Rogers and Shephard [36] proved a year later, while determining the lower bound for the volume of the difference body, that a compact convex body $K$ in the Euclidean space $\mathbb{E}^d$ is a simplex if and only if any nonempty intersection of $K$ and a translate of $K$ is a homothetic copy of $K$, possibly degenerated into a point:

$$K \cap (x + K) = z + \lambda K, \quad x, z \in \mathbb{E}^d, \quad \lambda \geq 0.$$ 

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These two approaches were further developed in various directions. One of them, mainly due to Gruber [24, 27], Simons [43], Bair and Fournier [3, 4, 5, 6, 19, 20, 21], follows the idea to describe all Choquet simplexes in $E^d$, especially non-closed or unbounded (see Section 4).

Another direction of investigation, mainly due to Schneider [39, 40] and Gruber [25, 26], is based on various generalizations of the Rogers-Shephard result and leads to characterizations of wider classes of convex bodies, including direct sums of simplexes, simplicial cones, and linear subspaces (see Section 5).

This paper is a survey of various geometric results related to Choquet simplexes in finite dimension; it summarizes the existing results and marks still open problems. The paper content is described by section headings as follows.

1. Introduction.
2. Positive cones and Choquet simplexes.
3. The Rogers-Shephard inequality.
4. Choquet simplexes in $E^d$.
5. Some generalizations.

In what follows, by a convex body we mean a closed convex set with nonempty interior in the Euclidean space $E^d$, $d \geq 2$. In particular, we distinguish bounded and unbounded convex bodies. As usual, $bdK$ and $intK$ stand for the boundary and the interior of $K$. A convex $d$-polyhedron is a convex body in $E^d$ that is a finite intersection of closed halfspaces. A convex $d$-polytope is a bounded $d$-polyhedron.

2. Positive Cones and Choquet Simplexes

The ideas of Choquet were preceded by the study of positive cones in (partially) ordered vector spaces. Let us recall that an ordered vector space is a real vector space $E$ equipped with a transitive, reflexive, and antisymmetric relation $\leq$ such that for any vectors $x, y, z \in E$ and a real number $\lambda \geq 0$ the following conditions hold:

- if $x \leq y$, then $x + z \leq y + z$,
- if $x \leq y$, then $\lambda x \leq \lambda y$.

The positive cone $C$ of $E$ is defined by

$$C = \{ x \in E \mid x \geq \theta \},$$

where $\theta$ denotes zero vector of $E$. As easily seen, $C$ a convex pointed cone, that is:

- $C + C \subset C$,
- $\lambda C \subset C$ for any real number $\lambda \geq 0$,
- $C \cap (-C) = \{ \theta \}$.

Clearly, we can define a partial order $\leq$ in $E$ by means of a convex pointed cone $C \subset E$ such that $x \leq y$ if and only if $y - x \in C$. For simplicity of consideration, we assume that $E = C - C$, i.e., that $C$ generates $E$. 
An ordered vector space $E$ is called a *vector lattice* provided any pair $x, y$ of vectors in $E$ has a least upper bound, denoted $x \vee y$ (or, equivalently, a greatest lower bound, denoted $x \wedge y$). The following simple proposition gives a geometric description of positive cones in vector lattices (see, e.g., Clarkson [14], Peressini [31], and Rosenthal [38]).

2.1. An ordered vector space $E$ is a vector lattice if and only if its positive cone $C = \{ x \in E \mid x \geq \theta \}$ satisfies the following condition: for any vectors $x, y \in E$ there is a vector $z \in E$ such that

\[(x + C) \cap (y + C) = z + C. \quad (1)\]

Proof. Let $E$ be a vector lattice and $x, y$ be vectors in $E$. Put $z = x \vee y$. If $v \in (x + C) \cap (y + C)$ then $x \leq v$ and $y \leq v$. Hence $z \leq v$ and $v \in z + C$. On the other hand, if $v \in z + C$ then $z \leq v$. As a result, $x \leq v$ and $y \leq v$. This implies $v \in (x + C) \cap (y + C)$.

Hence $C$ satisfies condition (1).

Conversely, let the positive cone $C$ of an ordered vector space $E$ satisfy condition (1). Choose a pair $x, y$ of vectors in $E$ and, for this pair, let $z$ be the respective vector from (1). Since $\theta \in C$, we have

\[z = z + \theta \in z + C = (x + C) \cap (y + C),\]

whence $x \leq z$ and $y \leq z$. Now suppose that a vector $v \in E$ satisfies the inequalities $x \leq v$ and $y \leq v$. Then $v \in (x + C) \cap (y + C)$, so $v \in z + C$ by (1). Hence $z \leq v$. Summing up, we have $z = x \vee y$. □

The geometric structure of the positive cone in an ordered vector space is not uniquely defined. For example, a two-dimensional ordered vector space $E^2$ is either a direct sum or a lexicographical union of two ordered lines (see, e.g., Birkhoff [7]); as a result, its positive cone is (up to a linear isomorphism) either the first quadrant $\{(x_1, x_2) \mid x_1 > 0, x_2 > 0\}$ or the cone $\{(x_1, x_2) \mid x_1 > 0\} \cup \{(0, x_2) \mid x_2 > 0\}$.

Positive cones with bases play an important role in the theory of ordered vector spaces. Suppose that $C(\neq \{\theta\})$ is a convex pointed cone in a vector space $E$. A convex subset $B$ of $C$ is a base for $C$ if each nonzero vector $x \in C$ has a unique representation $x = \lambda b$ for suitable $\lambda > 0$ and $b \in B$. Clearly, a convex set $B$ is a base for $C$ if and only if $C = \cup \{\lambda B \mid \lambda > 0\}$ and the affine hull of $B$ does not contain $\theta$.

Now assume that $B$ is a base for the positive cone $C$ of an ordered vector space $E$. Choquet [12, 13] has pointed out that the lattice structure of $E$ is associated with the simplicial character of $B$; he called a non-empty convex set $S$ in a real vector space $E$ a simplex (afterwards called a Choquet simplex) provided the intersection of any two homothetic copies of $S$, if nonempty, is again a homothetic copy of $S$ (possibly degenerated into a point), i.e., for any vectors $u, v \in E$ and real numbers $\lambda, \mu \geq 0$ there is a vector $w \in E$ and a real number $\nu \geq 0$ such that

\[(u + \lambda S) \cap (v + \mu S) = w + \nu S. \quad (2)\]

The following statement, known as the Choquet-Kendall theorem, was initially proved by Choquet [12, 13] for the case of ordered topological vector spaces and later generalized
by Kendall [28] for the case of ordered vector spaces without topology. See Bair and Fournier [3, 4], Goulet de Rugy [23], Peressini [31], and Rosenthal [38] for various proofs of the Choquet-Kendall theorem. Let us recall that a nonempty set \( X \) in a vector space \( E \) is called \textit{linearly compact} if the intersection of \( X \) with any line \( l \subset E \) is a compact subset of \( l \).

\[ \text{2.2. (The Choquet-Kendall Theorem)} \text{ Let } C \text{ be a convex pointed cone with a base } B \text{ in a vector space } E, \text{ and let } \leq \text{ be the partial order in } E \text{ generated by } C. \text{ Then the following conditions are equivalent:} \]

\begin{align*}
(a) & \quad E \text{ is a vector lattice with respect to } \leq, \\
(b) & \quad \text{for any vectors } x, y \in E, \text{ there is a vector } z \in E \text{ such that (1) holds,} \\
(c) & \quad B \text{ is a linearly compact Choquet simplex.} 
\end{align*}

The Choquet-Kendall theorem gives an implicit way to describe compact Choquet simplexes in the Euclidean space \( E^d \), observed by Choquet [13] and later by many others. Indeed, an ordered vector space \( E^d \) whose positive cone has a base, is the direct sum of \( d \) ordered lines (see, e.g., Birkhoff [7]). As a result, the positive cone \( C \) of such a space is a simplicial cone, i.e., \( C \) is the cone hull of \( d \) linearly independent vectors. Any base for \( C \), being a compact intersection of \( C \) and a hyperplane missing \( \theta \), is a usual \((d - 1)\)-dimensional simplex, that is, the convex hull of \( d \) affinely independent points. Since the notions of linear compactness and usual compactness in \( E^d \) coincide, the Choquet-Kendall theorem implies that compact Choquet simplexes of dimension \( d - 1 \) in \( E^d \) are precisely usual \((d-1)\)-simplexes. This conclusion obviously implies that compact Choquet simplexes in \( E^d \) are exactly usual simplexes.

\[ \text{3. The Rogers–Shephard Inequality} \]

For a compact convex body \( K \) in the Euclidean space \( E^d \), the set

\[ DK = K - K (= \{ x - y \mid x, y \in K \}) \]

is called the \textit{difference body} of \( K \). In 1920 Blaschke [8] asked for bounds for the volume \( V(DK) \) of the difference body in terms of the volume \( V(K) \) of the original body. The lower bound

\[ V(DK) \geq 2^d V(K), \]

with equality if and only if \( K \) is centrally symmetric, is a well-known consequence of the Brunn-Minkowski inequality (see, e.g., Bonnesen and Fenchel [9], Section 53).

The upper bound for \( V(DK) \) has a longer history. The inequality \( V(DK) \leq 6 V(K) \), proved by Rademacher [33] for \( d = 2 \), and the inequality \( V(DK) \leq 20 V(K) \), proved independently by Estermann [18] and Süss [46] for \( d = 3 \), give the best possible upper bounds in two and three dimensions. Moreover, the respective equalities occur only for
triangles and tetrahedra. Bonnesen and Fenchel [9], Section 53, proved that

\[ V(DK) \leq \sum_{k=0}^{d} \binom{d}{k} \mu(k) V(K), \quad d \geq 2, \quad (3) \]

where \( \mu(k) = \min\{k, d - k\} \). Since (3) is in compliance with the cases \( d = 2, 3 \) above, Bonnesen and Fenchel conjectured that (3) is sharp for every \( d \geq 2 \) and that equality holds if and only if \( K \) is a \( d \)-simplex. Finally, Rogers and Shephard [36] have sharpened (3) to the exact upper bound by proving the following statement.

3.1. (The Rogers-Shephard Theorem) For any compact convex body \( K \subset E^d \),

\[ V(DK) \leq \binom{2d}{d} V(K). \quad (4) \]

Equality holds in (4) if and only if \( K \) is a \( d \)-simplex.

Further extensions of the Rogers-Shephard inequality can be found, for example, in Böröczky, Jr. [10], Chakerian [11], Gardner and Zhang [22], Rogers and Shephard [37], Schneider [41], and Zhang [49].

The proof of (4), based on the calculation of the volume \( V(K \cap (x + K)) \), \( x \in DK \), uses the following two auxiliary statements.

(A) Equality holds in (4) if and only if the intersection \( K \cap (x + K) \) is homothetic to \( K \) for each \( x \in DK \):

\[ K \cap (x + K) = z + \lambda K, \quad z \in E^d, \quad \lambda \geq 0. \quad (5) \]

(B) \( K \) is a \( d \)-simplex if and only if for any \( x \in DK \) there is a vector \( z \in E^d \) and a real number \( \lambda \geq 0 \) such that property (5) holds.

Clearly, the restriction \( x \in DK \) is introduced merely to ensure that \( K \cap (x + K) \) is nonempty. For convenience, we will say that a set \( S \subset E^d \) satisfies the Rogers-Shephard condition provided every nonempty intersection \( S \cap (x + S) \), \( x \in E^d \), is a homothetic copy of \( S \), possibly degenerated into a point.

The original proof of statement (B) above, being rather long, was simplified in various ways by Eggleston, Grünbaum, and Klee [17] (for convex polytopes), Martini [29], and Schneider [42]. Theorem 3.2 below contains a proof of (B), which combines the arguments of Eggleston, Grünbaum, and Klee [17], Gruber [24], and Schneider [42].

3.2. (Choquet-Rogers-Shephard) For a compact convex body \( K \subset E^d \), the following conditions are equivalent:

(a) \( K \) is a \( d \)-simplex,

(b) \( K \) is a Choquet simplex,

(c) \( K \) satisfies the Rogers-Shephard condition.
Proof. (a) ⇒ (b). We can represent $K$ in a suitable coordinate system of $E^d$ as

$$K = \{(\xi_1, \ldots, \xi_d) \in E^d \mid \xi_1 \geq 0, \ldots, \xi_d \geq 0, \xi_1 + \cdots + \xi_d \leq 1\}.$$ 

Let $K_1$ and $K_2$ be homothetic copies of $K$: $K_1 = a + \lambda K$ and $K_2 = b + \mu K$ for suitable vectors $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$ in $E^d$ and real numbers $\lambda, \mu \geq 0$. Then

$$K_1 = \{(\xi_1, \ldots, \xi_d) \in E^d \mid \xi_1 \geq a_1, \ldots, \xi_d \geq a_d, \xi_1 + \cdots + \xi_d \leq \lambda + a_1 + \cdots + a_d\},$$

$$K_2 = \{(\xi_1, \ldots, \xi_d) \in E^d \mid \xi_1 \geq b_1, \ldots, \xi_d \geq b_d, \xi_1 + \cdots + \xi_d \leq \mu + b_1 + \cdots + b_d\},$$

and

$$K_1 \cap K_2 = \{(\xi_1, \ldots, \xi_d) \in E^d \mid \xi_1 \geq \max \{a_1, b_1\}, \ldots, \xi_d \geq \max \{a_d, b_d\}, \xi_1 + \cdots + \xi_d \leq \min \{\lambda + a_1 + \cdots + a_d, \mu + b_1 + \cdots + b_d\}\}.$$

Clearly, the intersection $K_1 \cap K_2$ is nonempty if and only if

$$\max \{a_1, b_1\} + \cdots + \max \{a_d, b_d\} \leq \min \{\lambda + a_1 + \cdots + a_d, \mu + b_1 + \cdots + b_d\}.$$ 

In this case $K_1 \cap K_2 = w + \nu K$ with $w = \max \{a_1, b_1\}, \ldots, \max \{a_d, b_d\}$ and

$$\nu = \min \{\lambda + a_1 + \cdots + a_d, \mu + b_1 + \cdots + b_d\} - \left(\max \{a_1, b_1\} + \cdots + \max \{a_d, b_d\}\right).$$

(b) ⇒ (c) is trivial.

(c) ⇒ (a). We proceed by induction on the dimension $d$. The statement is obvious for $d = 1$. Let (c) ⇒ (a) be true for all $d \leq n - 1$, $n \geq 2$, and $K$ be a compact convex body in $E^n$ satisfying condition (c). Choose an exposed point $p$ of $K$, and let $H$ be a hyperplane with the property $K \cap H = \{p\}$. Since the set of regular points is dense in the boundary of $K$, we can choose a regular point $q$ of $K$ such that the open line segment $[p, q]$ lies in the interior of $K$. Then every intersection

$$K_\lambda = K \cap (\lambda(p - q) + K), \quad 0 < \lambda < 1,$$

is a compact convex body. By (c), every $K_\lambda$ is homothetic to $K$. Then $K_\lambda = h_\lambda(K)$ for a suitable homothety $h_\lambda$. Since $K_\lambda \cap H = \{p\}$, we have $h_\lambda(p) = p$, whence $p$ is the center of the homothety $h_\lambda$.

Next we are going to prove that $K$ is a bounded convex cone with apex $p$. Let $L$ be the hyperplane supporting $K$ at $q$ and $M$ be the closed halfspace determined by $L$ and containing $K$. Let

$$C(K, p) = p + \cup \{\mu(K - p) \mid \mu \geq 0\}$$

be the convex cone with apex $p$ generated by $K$. We will show that $K = C(K, p) \cap M$. The inclusion $K \subseteq C(K, p) \cap M$ is trivial. Let $x$ be any point in $C(K, p) \cap M$. Then there are points $y, z \in K$ such that $y - p = \alpha(x - p)$ and $z - q = \beta(x - q)$ with $\alpha, \beta \geq 0$. If a real number $\lambda \in [0, 1]$ is sufficiently close to 1, then the segments $[p, y]$ and $[q, z] + \lambda(p - q)$ intersect in a point $x_\lambda$. Since $x_\lambda \in K_\lambda$ and $x_\lambda = h_\lambda(x)$, we deduce that $x \in K$. As a result,
$K$ is a bounded cone with apex $p$ and compact base $F = C(K, p) \cap L$. In particular, $F$ is a $(n - 1)$-dimensional face of $K$.

Let $v + K$ be a translate of $K$ such that its facet $v + F$ lies in $L$ and $F \cap (v + F) \neq \emptyset$. Obviously, $K \cap (v + K) \neq \emptyset$, and, by (c), $K \cap (v + K)$ is homothetic to $K$. Thus the facet $F \cap (v + F)$ of $K \cap (v + K)$ is homothetic to $F$, and, by the inductive assumption, $F$ is a $(n - 1)$-simplex. Hence $K = \text{conv}(p \cup F)$ is a $n$-simplex. 

\[ \square \]

4. Choquet Simplexes in $E^d$

The results on Choquet simplexes in $E^d$ can be roughly divided into two groups. One of them contains various generalizations of the Rogers-Shephard characterization of simplexes, while the other contains descriptions of various Choquet simplexes, especially nonclosed or unbounded, posed as a problem by Goullet de Rugy [23].

\textit{Bounded Choquet Simplexes}

Gruber [24] sharpened the Rogers-Shephard characterization of simplexes, by proving the following statement.

4.1. A compact convex body $K \subset E^d$ is a $d$-simplex if and only if it satisfies the Rogers-Shephard condition for any vector $x$ from an arbitrarily chosen neighborhood of the origin $O \subset E^d$.

If $V$ is a sufficiently small neighborhood of the origin $\theta$, then any intersection $K \cap (x + K)$, $x \in V$, has nonempty interior. This obviously implies the following statement mentioned by Gruber [24] (which also can be concluded from the proof of 3.2).

4.2. A compact convex body $K \subset E^d$ is a $d$-simplex if and only if any $d$-dimensional intersection $K \cap (x + K)$, $x \in E^d$, is a homothetic copy of $K$.

Furthermore, Gruber [24] extended the Rogers-Shephard criterion for the case of compact nonconvex sets as follows.

4.3. A compact (not necessarily convex) set $K \subset E^d$ with nonempty interior is a $d$-simplex if and only if it satisfies the Rogers-Shephard condition.

For a given real number $\alpha \in [0, 1]$, let $F_\alpha$ denote the family of compact convex bodies in $E^d$ such that each body $K \in F_\alpha$ contains a ball of radius $\alpha$ and is contained in the concentric ball of radius $1/\alpha$. Similarly, for a real number $\alpha \in [0, 1]$, let $F_\infty$ denote the family of compact convex bodies in $E^d$ such that each body $K \in F_\infty$ contains a ball of radius $\alpha$. The following results of Gruber [27] characterize $d$-simplexes within the classes $F_\alpha$ and $F_\infty$, respectively.

4.4. If $d \geq 3$ then for any given real number $0 < \alpha \leq 1$ there is a finite set $X_\alpha \subset E^d$ such that a convex body $K \in F_\alpha$ is a $d$-simplex if and only if $K$ satisfies the Rogers-Shephard condition for each $x \in X_\alpha$. 
4.5. If \( d \geq 3 \) then there exists an infinite sequence \( X \subset E^d \) of points convergent to \( \theta \) such that a compact convex body \( K \subset E^d \) is a \( d \)-simplex if and only if \( K \) satisfies the Rogers-Shephard condition for each \( x \in X \).

4.6. If \( d = 2 \) then for any given real number \( \alpha \geq 1 \), a planar convex body \( K \in F_{\infty} \) is a triangle if and only if \( K \) satisfies the Rogers-Shephard condition for each \( x \in \{(\alpha, 0), (0, \alpha), (\alpha, \alpha)\} \subset E^2 \).

Gruber [24] also conjectured (based on the case \( d = 1 \)) that if a bounded set \( K \subset E^d \) of positive measure satisfies the Rogers-Shephard condition, then \( K \) is necessarily the interior of a \( d \)-simplex together with the relative interiors of some of its \( r \)-dimensional faces, \( 0 \leq r \leq d - 1 \). This conjecture was confirmed by Eggleston [16]. In a weaker form, this result was independently discovered by Simons [43], who proved that a bounded convex set with nonempty interior in \( E^d \) is a Choquet simplex if and only if it is the intersection of \( d + 1 \) halfspaces each of them being closed or open.

**Unbounded Choquet Simplexes**

The following description of closed unbounded Choquet simplexes in \( E^d \) is due to Bair and Fourneau [3, 4, 19], although their results are particular cases of more general statements previously proved by Gruber [25].

4.7. The closed unbounded line-free Choquet simplexes in \( E^d \) are the convex cones whose bases are \( k \)-simplexes, \( 0 \leq k \leq d - 1 \). The closed unbounded Choquet simplexes in \( E^d \) containing lines are the direct sums of vector subspaces and line-free convex cones whose bases are \( k \)-simplexes, \( 0 \leq k \leq d - 1 \).

The study of nonclosed unbounded Choquet simplexes was initiated by Fourneau [19] (see also Bair and Fourneau [6]). For this purpose, Fourneau introduced the notion of *quasi-simplex* for the closure of a Choquet simplex, and showed that a convex body \( K \subset E^d \) is a quasi-simplex if and only if any \( d \)-dimensional intersection \( K \cap (x + K) \), \( x \in E^d \), is a homothetic copy of \( K \). Furthermore, it was proved in [19] that the line-free quasi-simplexes are precisely the generalized simplexes introduced by Rockafellar [35].

4.8. For a line-free convex body \( K \subset E^d \), the following conditions are equivalent:

(a) \( K \) is a quasi-simplex, i.e., any \( d \)-dimensional intersection \( K \cap (x + K) \), \( x \in E^d \), is a homothetic copy of \( K \),

(b) \( K \) is a generalized simplex, i.e.,

\[
K = \text{conv}\{x_1, \ldots, x_k\} + \sum_{i=k+1}^{d+1} [\theta, x_i],
\]

where \( x_1, \ldots, x_{d+1} \) are affinely independent points and \([\theta, x_i]\) denotes the halfline through \( x_i \) with origin \( \theta \).
As a consequence, open line-free Choquet simplexes in $E^d$ are exactly the interiors of generalized simplexes. The next auxiliary step towards a complete description of line-free Choquet simplexes consists in establishing a relation between line-free Choquet simplexes in $E^{d-1}$ and open Choquet simplexes in $E^d$.

4.9. If $S$ is a line-free Choquet simplex that lies in a hyperplane of $E^d$ missing the origin $\theta$, then the cone $P = \cup \{\lambda S | \lambda > 0\}$ is an open Choquet simplex in $E^d$.

Finally, Fourneau [19] obtained the following statement.

4.10. A line-free convex set $K \subset E^d$ of dimension $d$ is a Choquet simplex if and only if there are linearly independent linear functionals $f_1, \ldots, f_d$ on $E^d$ such that one of the following two cases holds:

(a) $K = \{x \in E^d | f_i(x) \geq 0, i = 1, \ldots, k, f_i(x) > 0, i = k + 1, \ldots, d\},$
(b) $K = \{x \in E^d | f_i(x) \geq 0, i = 1, \ldots, k, f_i(x) > 0, i = k + 1, \ldots, d, \text{ and either } \Sigma\{f_i(x) | i \in I\} \leq 1 \text{ or } \Sigma\{f_i(x) | i \in I\} < 1\},$ where $I$ is a nonempty subset of $\{1, \ldots, d\}.$

As mentioned by Fourneau [19], the description of Choquet simplexes that contain lines looks rather difficult, since they are not, in general, intersections of closed or open halfspaces. Some properties of Choquet simplexes, leading to the following classification in two and three dimensions, are studied by Fourneau [21].

4.11. The Choquet simplexes in two dimensions are:

(a) a closed halfplane, an open halfplane, or an open halfplane with a (pointed or not) ray of its boundary line,
(b) an open slab, an open slab with one of its boundary lines, or an open slab with a (pointed or not) ray of one of its boundary lines,
(c) the whole plane.

4.12. The Choquet simplexes in three dimensions are:

(a) an open halfspace or an open halfspace with any conical Choquet 2-simplex in its boundary plane,
(b) an open slab, an open slab with one of its boundary lines, or an open slab with a (pointed or not) ray of one of its boundary lines,
(c) an intersection of two open or closed nonparallel halfspaces,
(d) a set of type (c) with a conical Choquet 2-simplex, part of a two-dimensional face $F$ of the closure, which is bounded by another maximal face $F'$, the bounding ray being included if and only if $F'$ is a part of the simplex,
the intersection of a set of type (c) or (d) with an open halfspace, including the edge of the dihedron, the boundary plane of which is parallel to its edge,

(f) the whole space.

We finalize this section with one more approach to Choquet simplexes, developed in [44] and [45]. Since the relation "K is homothetic to M" is an equivalence relation on the family of convex (possibly unbounded) bodies in $E^d$, we can consider this family as the union of homothety classes. As above,

$$C(K, p) = p + \{ \mu (K - p) \mid \mu \geq 0 \}$$

stands for the convex cone with apex $p$ generated by $K$.

4.13. For a pair of compact convex bodies $K_1$ and $K_2$ in $E^d$, the following conditions are equivalent:

(a) the $d$-dimensional intersections $K_1 \cap (x + K_2)$, $x \in E^d$, belong to at most countably many homothety classes of convex bodies,

(b) the $d$-dimensional intersections $K_1 \cap (x + K_2)$, $x \in E^d$, belong to a unique homothety class of convex bodies,

(c) $K_1$ and $K_2$ are homothetic $d$-simplexes.

4.14. For a pair of line-free convex bodies $K_1$ and $K_2$ in $E^d$, the following conditions are equivalent:

(a) there is a line-free convex body $K \subset E^d$ such that every nonempty intersection $K_1 \cap (v + K_2)$, $v \in E^d$, is a homothetic copy of $K$ (possibly degenerated into a point),

(b) there is a closed line-free Choquet simplex $K$, i.e., $K$ is a $d$-simplex of the form $K = \text{conv} \{ x_0, x_1, \ldots, x_d \}$ or a simplicial $d$-cone of the form $K = \sum_{i=0}^{d} [x_0, x_i)$, where $x_0, x_1, \ldots, x_d$ are affinely independent points, such that one of the following conditions holds:

   (b1) both $K_1$ and $K_2$ are homothetic copies of $K$,

   (b2) one of $K_1, K_2$, say $K_1$, is a homothetic copy of $K$, and $K_2$ is a simplicial $d$-cone that is translate of a generated cone $C(K, x)$, where $x \in \{x_0, x_1, \ldots, x_d\}$,

   (b3) both $K_1$ and $K_2$ are simplicial $d$-cones that are translates of generated cones $C(K, x)$ and $C(K, z)$, where $x$ and $z$ are distinct points in $\{x_0, x_1, \ldots, x_d\}$ (this condition is possible only if $K$ is a $d$-simplex).

Probably, one can get a similar characterization of a pair of generalized simplexes by replacing condition (a) from 4.14 with a weaker condition (a') below.
(d') the d-dimensional intersections $K_1 \cap (v + K_2)$, $v \in E^d$, belong to a unique homothety class of convex bodies.

The problem to describe all pairs $K, K'$ of convex bodies in $E^d$ (not necessarily line-free) such that all nonempty intersections $K \cap (x + K')$, $x \in E^d$, are homothetic copies of a unique convex body in $E^d$ still remains open.

We also want to remark here that condition (a) from 4.13, being considered for the case of unbounded (even line-free) convex bodies in $E^d$, goes beyond the family of generalized simplexes. Indeed, for the convex polyhedral 3-cone
\[
K = \{(\xi_1, \xi_2, \xi_3) \in E^3 \mid \xi_3 \geq 0, |\xi_1| + |\xi_2| \leq \xi_3\},
\]
which is not a generalized simplex, the 3-dimensional intersections $K \cap (x + K)$, $x \in E^3$, belong to three homothety classes, generated by the convex bodies $K, K \cap ((1, 1, 0) + K), K \cap ((1, -1, 0) + K)$, respectively.

As easily seen, 4.14 implies the following result of Coquet and Dupin [15].

4.15. For a family of line-free convex bodies $K_1, \ldots, K_n$ in $E^d$, the following conditions are equivalent:

(a) for any points $x_1, \ldots, x_n \in E^d$ there is a point $z \in E^d$ such that
\[
(x_1 + K_1) \cap \cdots \cap (x_n + K_n) = z + K_1 \cap \cdots \cap K_n,
\]
(b) $K_1, \ldots, K_n$ are translates of the same simplicial cone.

5. Some Generalizations

A series of essential generalizations of the Rogers-Shephard result were initiated by Schneider [39, 40] and further developed by Gruber [25, 26].

Let $\mathcal{K}$ denote the family of compact convex bodies in $E^d$. For a body $K \in \mathcal{K}$ and a boundary point $p$ of $K$, let $N(K, p)$ denote the set of unit vectors $e \in E^d$ such that $e + p$ is an outward normal to $K$ at $p$. Given a pair of bodies $K$ and $M$ in $\mathcal{K}$, Schneider [39] introduced the following binary relation $K \sigma M$: for any boundary point $p$ of $K$, there is a boundary point $q$ of $M$ such that $N(K, p) = N(M, q)$. Obviously, $K \sigma M$ if $K$ and $M$ are homothetic bodies or when both of them are smooth. Schneider [39] posed the problem to describe the family $\mathcal{K}(\sigma)$ of bodies in $\mathcal{K}$ defined by
\[
\mathcal{K}(\sigma) = \{K \in \mathcal{K} \mid (x \in E^d, K \cap (x + K) \in \mathcal{K}) \Rightarrow K \cap (x + K) \sigma K\}.
\]

The main result of [39] is given in statement 5.1 below. (See also another proof of this fact by Gruber [25].)

5.1. $\mathcal{K}(\sigma)$ consists of the primitive convex $d$-polytopes in $E^d$ whose outward normals to the facets of $P$ form a strong positive basis.
Let us recall that a convex $d$-polytope $P \subset E^d$ is primitive provided the removal of any facet of $P$ leaves an unbounded polyhedron. Equivalently, a convex $d$-polytope is primitive if and only if the set $N(P)$ of outward normals to the facets of $P$ forms a positive basis for $E^d$.

A set $B$ of vectors is a strong positive basis for $E^d$ if it is a positive basis and for any nonempty disjoint subsets $B_1, B_2$ of $B$, the intersection $\text{pos} B_1 \cap \text{pos} B_2$ of their positive hulls is $\{\theta\}$. It is known (see, e.g., Reay [34]) that, for a given strong positive basis $B$ of $E^d$, the space $E^d$ is representable as the direct sum of subspaces $L_1, \ldots, L_k$ such that

$$B = (B \cap L_1) \cup \cdots \cup (B \cap L_k) \quad \text{and} \quad \text{card}(B \cap L_i) = \dim L_i + 1, \ i = 1, \ldots, k.$$ 

Since the relation $\text{card} N(P) = d + 1$ is characteristic for the $d$-simplexes, Gruber [25] concluded that $\mathcal{K}(\sigma)$ consists of the $d$-polytopes that are direct sums of simplexes. In a standard way, a convex $d$-polytope $P \subset E^d$ is the direct sum of simplexes $S_1, \ldots, S_k$ provided every point $x \in P$ is uniquely represented as the vector sum $x = x_1 + \cdots + x_k$, where $x_1 \in S_1, \ldots, x_k \in S_k$.

Furthermore, Gruber [25] extended Schneider’s result above to the case of unbounded convex bodies as follows.

5.2. If $\mathcal{K}_u$ denotes the family of unbounded convex bodies in $E^d$, then $K \in \mathcal{K}_u(\sigma)$ if and only if $K$ is the direct sum of simplexes, a line-free simplicial cone, and a subspace of $E^d$, or of only one or two of these types of sets.

Another result of Schneider [40] is formulated in statement 5.3 below.

5.3. If a convex $d$-polytope $P$ in $E^d$ is such that every $d$-dimensional intersection $P \cap (x + P)$, $x \in E^d$, has the same number of vertices as $P$, then $P$ is the direct sum of simplexes.

Based on this result, Schneider [40] introduces the following binary relation $\sigma_k$, $0 \leq k \leq d - 1$, on the family of convex $d$-polytopes in $E^d$: $P \sigma_k Q$ if and only if $P$ and $Q$ have the same number of $k$-faces. Let $\mathcal{P}(\sigma_k)$ denote the family of convex $d$-polytopes $P$ in $E^d$ such that every $d$-dimensional intersection $P \cap (x + P)$, $x \in E^d$, has the same number of $k$-faces as $P$. As mentioned in [40], the polytopes which are direct sums of simplexes form the class $\mathcal{P}(\sigma_0)$ and belong to every class $\mathcal{P}(\sigma_k)$, $1 \leq k \leq d - 1$. Furthermore, $\mathcal{P}(\sigma_{d-1})$ contains the primitive polytopes and the $d$-dimensional cross-polytope. Based on these observations, Schneider [40] asked whether, for any $k$ with $1 \leq k \leq d - 2$, the family $\mathcal{P}(\sigma_k)$ contains convex $d$-polytopes which are not direct sums of simplexes.

Following [40], McMullen, Schneider, and Shephard [30] proved that a convex $d$-polytope $P$ is primitive if and only if, for any $d$-polytope $Q$ with $N(Q) = N(P)$, every $d$-dimensional intersection $Q \cap (x + Q)$, $x \in E^d$, has the same number of facets as $Q$.

Gruber [26] extended Schneider’s considerations for the family $\mathcal{P}_u$ of unbounded convex $d$-polyhedra in $E^d$ as follows.

5.4. An unbounded line-free convex $d$-polyhedron $P$ belongs to $\mathcal{P}_u(\sigma_0)$ if and only if $P$ is the direct sum of simplexes and a simplicial cone.
5.5. An unbounded convex $d$-polyhedron $P$, with at least one edge, belongs to $P_u(\sigma_1)$ if and only if $P$ is the direct sum of simplexes, a simplicial cone, and a one-dimensional subspace, or of only one or two of these types of sets.

Any of these two statements implies one more result of Gruber [26].

5.6. An unbounded convex $d$-polyhedron $P$ is the direct sum of simplexes, a simplicial cone, and a subspace of dimension at least one, or of only one or two of these types of sets, if and only if any $d$-dimensional intersection $P \cap (x + P)$, $x \in \mathbb{R}^d$, is an affine image of $P$.

It is interesting to compare 5.6 above with the following deep result of Gruber [25].

5.7. A compact convex body $K \subset \mathbb{R}^d$ is the direct sum of simplexes if and only if any $d$-dimensional intersection $K \cap (x + K)$, $x \in \mathbb{R}^d$, is an affine image of $K$.

It is conjectured in [26] that 5.7 can be extended for the case of unbounded convex bodies in $\mathbb{R}^d$ by including simplicial cones and linear subspaces as additional factors (see also [44] for the respective conjecture on a pair of convex bodies).

Finally, we mention the paper of Wilker [47], containing a study of those sets $S \subset \mathbb{R}^d$ which satisfy the equality $S \cap (x + S) = z + \lambda S$ for given $x \in \mathbb{R}^d$ and $\lambda \geq 0$.

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References


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