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NOTES

A Note on the Cattle Problem of Archimedes

PETER SCHREIBER

Ernst-Moritz-Arndt-Universität, Fachrichtungen Mathematik/Informatik, F.-L.-Jahn-Strasse 15 a, D-17489 Greifswald, Federal Republic of Germany

The famous cattle problem, an ancient mathematical riddle ascribed to Archimedes, is unsolvable. The known "solutions" contradict one of the conditions given in the wording of the problem. © 1993 Academic Press, Inc.

Im berühmten Rinderproblem, einem dem Archimedes zugeschriebenen mathematischen Rätsel, ist die in der Literatur verbreitete "Lösung" im Widerspruch zu einer im Wortlaut der Aufgabe genannten Bedingung. © 1993 Academic Press, Inc.

Le problème fameux de boeufs, une énigme mathématique attribuée à Archimède, est insoluble. Les 'solutions' connues contredisent à une des conditions mentionnées dans le texte du problème. © 1993 Academic Press, Inc.

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The cattle problem (problema bovinum) is the most extensive and complicated of the ancient Greek mathematical riddles. It was rediscovered in a codex in the Herzog August library in Wolfenbüttel by the German poet G.E. Lessing, who published it for the first time in 1773. (Later on a second copy was found in Paris: Bibliothèque nationale, cod. Paris Graec. 2448 saec. XIV, fol. 57.) The Greek text begins with the following announcement:

> Problem found by Archimedes within the epigrams and sent by him to those in Alexandria which are concerned with such matters, in a letter to Eratosthenes of Cyrene.

Heath [1921, 97f] described the problem in these words:

> It is required to find the numbers of bulls and cows of each of four colours, or to find 8 unknown quantities. The first part of the problem connects the unknowns by seven simple equations; and the second part adds two more conditions to which the unknowns must be subject. If W, w be the numbers of white bulls and cows respectively and (X,x), (Y,y), (Z,z) represent the numbers of the other three colours, we have first the following equations

$$\begin{aligned} W &= (\tfrac{1}{2}+\tfrac{1}{3})X + Y, & w &= (\tfrac{1}{3}+\tfrac{1}{4})(X+x),\\ X &= (\tfrac{1}{4}+\tfrac{1}{5})Z + Y, & x &= (\tfrac{1}{4}+\tfrac{1}{5})(Z+z),\\ Z &= (\tfrac{1}{6}+\tfrac{1}{7})W + Y, & z &= (\tfrac{1}{5}+\tfrac{1}{6})(Y+y),\\ & & y &= (\tfrac{1}{6}+\tfrac{1}{7})(W+w). \end{aligned} \tag{1}$$

> Secondly it is required that $W + X$ is a square and $Y + Z$ is a triangular number. (2)

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(For another English description of the problem cf. [Archibald 1918]. There are several German adaptions of the Greek hexameters, e.g., [Nesselmann 1842, Krumbiegel 1880, Wertheim 1890]. The last has been republished several times, e.g., in [Wußing 1965] and recently, with a careful presentation of the solution of the equations (1), by Pieper [1991 88–89 and 162–164], but there is apparently no hexametric adaption in English.)

The general solution of the 7 homogenous linear equations (1) is

white bulls	$W = 829{,}318{,}560\, n$,	white cows	$w = 576{,}508{,}800\, n$,
black bulls	$X = 596{,}841{,}120\, n$,	black cows	$x = 391{,}459{,}680\, n$,
brown bulls	$Y = 331{,}950{,}960\, n$,	brown cows	$y = 435{,}137{,}040\, n$,
dappled bulls	$Z = 588{,}644{,}800\, n$,	dappled cows	$z = 281{,}265{,}600\, n$,

where n is an arbitrary natural number. The two additional conditions (2) merely imply that a very large value of n is required to satisfy these conditions but they do not alter the proportions between the eight sorts of cattle. (The full problem is treated, e.g., in [Krumbiegel & Amthor 1880] and [Bell 1895a, b].)

There have been doubts concerning the attribution of the problem to Archimedes, but Heiberg [1879, 26; 1881, 451–455] and other authorities did not hesitate to ascribe it to him, and most modern authors, e.g., Clagett [1970, 214], follow Heiberg without commentary. Furthermore Struve [1821], Nesselmann [1842], and some others were of the opinion that the two conditions (2) must have been added at some later period, although Heiberg did not even accept this hypothesis.

What no one has hitherto noticed however, is that the general solution of the equations (1) stands in contradiction to lines 7–8 of the Greek text of the riddle which says

In each sort of cattle there are many more bulls than cows.

This condition obviously fails for the brown cattle. In contrast to some other places in the text, there is no ambiguity as to the interpretation of these lines. The Greek text reads

ἐν δὲ ἑκάστῳ στίφει ἔσαν ταῦροι πλήθεσι βριθόμενοι

which Heiberg translated in Latin as "In singulis autem gregibus tauri erant numero praeualidi . . .". Only Nesselmann perhaps may have noticed the contradiction, because he translated the lines in question as "Es faßte Stiere jegliche Schaar mächtig in Haufen gedrängt . . ." [Nesselmann 1842 483].

What may we conclude from this contradiction? I think the system (1) of equations would have been a very difficult problem even without the conditions (2) for any Greek mathematician of the third century B.C., except, perhaps, Archimedes. Perhaps the inventor of the problem simply miscalculated. But there is also a legend that Archimedes posed the cattle problem simply to confound Eratosthenes and the other mathematicians in Alexandria. (Indeed a variant of this legend is told in a novel about Archimedes [Szava 1960 283–287].) Perhaps Archimedes had inserted the stipulation of many more bulls than cows in each sort of cattle to unmask those which were unable to solve even the equations (1). Modern mathematicians are able to solve the equations (and even under the additional conditions (2) as Amthor [1880] had first done) but their solutions do not take into account the exact wording of the problem.

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