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On $*$-paranormal contractions and properties for $*$-class A operators[☆]

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ABSTRACT

An operator $T \in B(\mathcal{H})$ is called a $*$-class $\mathcal{A}$ operator if $|T^2| \geqslant |T^*|^2$, and T is said to be $*$-paranormal if $\|T^*x\|^2 \leqslant \|T^2x\|$ for every unit vector x in $\mathcal{H}$. In this paper, we show that $*$-paranormal contractions are the direct sum of a unitary and a $C_{.0}$ completely non-unitary contraction. Also, we consider the tensor products of $*$-class $\mathcal{A}$ operators.

1. Introduction

Throughout this paper, $\mathcal{H}$ denotes an infinite dimensional complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $B(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For an operator $T \in B(\mathcal{H})$ set, as usual, $|T| = (T^*T)^{\frac{1}{2}}$ and $[T^*, T] = T^*T - TT^*$ (the self-commutator of T). An operator $T \in B(\mathcal{H})$ is said to be *hyponormal* if $[T^*, T]$ is nonnegative (equivalently, if $\|T^*x\| \leqslant \|Tx\|$ for every x in $\mathcal{H}$), and T is said to be $*$-paranormal if $\|T^*x\|^2 \leqslant \|T^2x\|$ for any unit vector x in $\mathcal{H}$. We introduce a new class of operators:

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Definition 1.1. An operator $T \in B(\mathcal{H})$ is said to be a $*$-class $\mathcal{A}$ operator if $|T^2| \geqslant |T^*|^2$.

A $*$-class $\mathcal{A}$ operator is a generalization of a hyponormal operator and $*$-class $\mathcal{A}$ operators form a subclass of the class of $*$-paranormal operators.

Theorem 1.2. *Each hyponormal operator is a $*$-class $\mathcal{A}$ operator.*

Proof. Suppose that T is a hyponormal. Then

$$|T^2|^2 = T^{*2}T^2 = T^*|T|^2T \geq T^*|T^*|^2T = |T|^4.$$

Therefore we have $|T^2| \geq |T|^2 \geq |T^*|^2$. $\square$

Theorem 1.3. *If $T \in B(\mathcal{H})$ is a $*$-class $\mathcal{A}$ operator, then T is a $*$-paranormal operator.*

Proof. Using the Hölder–McCarthy inequality, we have

$$\|T^2x\|^2 = \langle T^{*2}T^2x, x\rangle = \langle |T^2|^2x, x\rangle \geq \langle |T^2|x, x\rangle^2 \geq \langle |T^*|^2x, x\rangle^2 = \|T^*x\|^4.$$

for all $x \in \mathcal{H}$ such that $||x|| = 1$. Hence the proof is complete. $\square$

In this paper, we consider the properties of $*$-paranormal contractions and the tensor products for $*$-class $\mathcal{A}$ (and $*$-paranormal) operators.

2. On $*$-paranormal contractions

Recall that $T \in B(\mathcal{H})$ is a $C_{.0}$-contraction if $T^{*n}x \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in \mathcal{H}$. In the following, we write cnu part for the *completely non-unitary part* of a contraction. In this section, we consider the properties of $*$-paranormal contractions.

$*$-Class $\mathcal{A}$ operators are normaloid (indeed hereditarily normaloid, i.e., the restriction of a $*$-class $\mathcal{A}$ operator to an invariant subspace is again $*$-class $\mathcal{A}$, so normaloid).

Lemma 2.1. *If T is a $*$-class $\mathcal{A}$ operator and $\mathcal{M}$ is an invariant subspace of T, then $T\mid_{\mathcal{M}}$ is also a $*$-class $\mathcal{A}$ operator.*

Proof. Let

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp},$$

and P be the orthogonal projection onto $\mathcal{M}$. Since T is a $*$-class $\mathcal{A}$ operator, we have

$$P\left\{|T^2| - |T^*|^2\right\}P \geq 0.$$

Therefore, we see that

$$\begin{aligned} AA^* \leq AA^* + BB^* = PTT^*P \leq P(T^{*2}T^2)^{\frac{1}{2}}P \leq (PT^{*2}T^2P)^{\frac{1}{2}} \text{ by Hansen's inequality} \\ = |A^2|. \end{aligned}$$

This implies that A is a $*$-class $\mathcal{A}$ operator and the proof is complete. $\square$

Lemma 2.2. *The eigenvalues of a $*$-paranormal operator are normal (i.e., the corresponding eigenspaces are reducing).*

Proof. If $T \in B(\mathcal{H})$ is $*$-paranormal, $\lambda \in \sigma_p(T)$ and $Tx = \lambda x$ for some nontrivial $x \in \mathcal{H}$, $||x|| = 1$, then

$$\begin{aligned} ||(T^* - \overline{\lambda})x||^2 &= ||T^*x||^2 - \lambda\langle T^*x, x\rangle - \overline{\lambda}\langle x, T^*x\rangle + |\lambda|^2 \\ &\leqslant ||T^2x||||x|| - \lambda\langle x, Tx\rangle - \overline{\lambda}\langle Tx, x\rangle + |\lambda|^2 \\ &= 0. \quad \square \end{aligned}$$

Lemma 2.3. *If $T \in B(\mathcal{H})$ is $*$-paranormal, then $asc(T - \lambda) \leqslant 1$ for all complex numbers λ.*

Proof. Lemma 2.2 implies $(T-\lambda)^{-1}(0) \perp (T-\lambda)\mathcal{H}$; hence, if $x \in (T-\lambda)^{-2}(0)$ and $x \notin (T-\lambda)^{-1}(0)$, then $x = 0$ ($\Longrightarrow$ asc$(T - \lambda) \leqslant 1$). $\square$

Lemma 2.3 implies that $*$-paranormal operator have SVEP, the single-valued extension property, everywhere [1, Theorem 3.8]. Indeed, more is true: $*$-paranormal operators satisfy (Bishop's) property (β), where $A \in B(\mathcal{H})$ satisfies property (β) if, for an open subset $\mathcal{U}$ of the complex plane and a sequence $\{f_n\}$ of analytic functions $f_n : \mathcal{U} \longrightarrow \mathcal{H}$, $(A - \lambda)f_n(\lambda)$ converges uniformly to 0 on compact subsets of $\mathcal{U}$ implies f_n converges uniformly to 0 on compact subsets of $\mathcal{U}$. Recall, [20, Lemma 2.1 and Theorem 3.5], that a sufficient condition for $A \in B(\mathcal{H})$ to satisfy property (β) is that $\sigma_a(A) = \sigma_{na}(A)$, where σ_a denotes the approximate point spectrum and, for a sequence $\{x_n\} \subset \mathcal{H}$ of unit vectors,

$$\sigma_{na}(A) = \{\lambda \in \sigma_a(A) : ||(A - \lambda)x_n|| \longrightarrow 0 \Longrightarrow ||(A^* - \overline{\lambda})x_n|| \longrightarrow 0\}.$$

Proposition 2.4. *$*$-paranormal operators satisfy property (β).*

Proof. Let $T \in B(\mathcal{H})$ be $*$-paranormal. In view of the above, we have to prove that $\sigma_a(T) = \sigma_{na}(T)$. The Berberian extension theorem [4] says that there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and an isometric $*$-isomorphism $T \longrightarrow T^o \in B(\mathcal{K})$ preserving order such that $\sigma_a(T) = \sigma_a(T^o) = \sigma_p(T^o)$. It is immediate from the definition of $*$-paranormality that T^o is $*$-paranormal, hence (see Lemma 2.2) $\sigma_p(T^o)$ consists of normal eigenvalues. Let $\lambda \in \sigma_a(T)$ and let $\{x_n\} \subset \mathcal{H}$ be a sequence of unit vectors such that $||(T - \lambda)x_n|| \longrightarrow 0$ (as $n \longrightarrow \infty$). Denoting the equivalence class of $\{x_n\}$ (in $\mathcal{K}$) by $[x]$, it follows that $(T^o - \lambda)[x] = 0$; hence $(T^o - \lambda)^*[x] = 0$. Since $\{||(T - \lambda)x_n||\}$ is a convergent sequence, it follows [4, p. 112] that $\lim_{n\longrightarrow\infty} ||(T - \lambda)^*x_n|| = \lim_{n\longrightarrow\infty} ||(T - \lambda)x_n|| = 0$. Thus $\sigma_a(T) = \sigma_{na}(T)$. $\square$

The following theorem shows that (just as for paranormal contractions [5]) $*$-paranormal contractions in $B(\mathcal{H})$ are the direct sum of a unitary and a $C_{.0}$-contraction.

Theorem 2.5. *$*$-paranormal contractions are the direct sum of a unitary and a $C_{.0}$ cnu contraction.*

Proof. If $T \in B(\mathcal{H})$ is a contraction, then the sequence $\{T^nT^{*n}\}$ converges strongly to a contraction $A \in B(\mathcal{H})$ such that

$$0 \leqslant A \leqslant 1, \quad A^{-1}(0) = \{x \in \mathcal{H} : T^{*n}x \longrightarrow 0\}, \quad \text{and} \quad TAT^* = A;$$

furthermore, there exists an isometry $V : \overline{A(\mathcal{H})} \longrightarrow \overline{A(\mathcal{H})}$ such that

$$A^{\frac{1}{2}}T^* = VA^{\frac{1}{2}} \Longleftrightarrow TA^{\frac{1}{2}} = A^{\frac{1}{2}}V^*$$

on $\overline{A(\mathcal{H})}$, and

$$||A^{\frac{1}{2}}V^n x|| \longrightarrow ||x||$$

for every $x \in \overline{A(\mathcal{H})}$ [8]. For an $x \in \mathcal{H}$, let

$$x_n = A^{\frac{1}{2}}V^n x, \quad n \in \mathbb{N} \cup \{0\}.$$

Then, for all non-negative integers k,

$$T^k x_{n+k} = T^k A^{\frac{1}{2}} V^{k+n} x = A^{\frac{1}{2}} V^{*k} V^{k+n} x = A^{\frac{1}{2}} V^n x = x_n,$$

and for all $k \leqslant n$,

$$T^k x_n = x_{n-k}.$$

Evidently, the sequence $\{||x_n||\}$ is a bounded above increasing sequence: we prove that if T is $*$-paranormal, then A is a projection. We start by proving that $\{||x_n||\}$ is a constant sequence.

Let T be $*$-paranormal. Then, for all $n \geqslant 1$ and non-trivial $x \in \overline{A(\mathcal{H})}$,

$$\begin{aligned} ||x_n||^2 &= ||Tx_{n+1}||^2 \leqslant ||T^*(Tx_{n+1})||\,||x_{n+1}|| \leqslant ||T^3 x_{n+1}||^{\frac{1}{2}} ||Tx_{n+1}||^{\frac{1}{2}} ||x_{n+1}|| \\ &= ||x_{n-2}||^{\frac{1}{2}} ||x_n||^{\frac{1}{2}} ||x_{n+1}||; \end{aligned}$$

hence

$$||x_n|| \leqslant ||x_{n-2}||^{\frac{1}{3}} ||x_{n+1}||^{\frac{2}{3}} \leqslant \frac{1}{3}(||x_{n-2}|| + 2||x_{n+1}||).$$

Thus,

$$2(||x_{n+1}|| - ||x_n||) \geqslant ||x_n|| - ||x_{n-2}|| = (||x_n|| - ||x_{n-1}||) + (||x_{n-1}|| - ||x_{n-2}||).$$

Denoting $b_n := ||x_n|| - ||x_{n-1}||$, we have $2b_{n+1} \geqslant b_n + b_{n-1}$, where $b_n \geqslant 0$ and $b_n \longrightarrow 0$ as $n \longrightarrow \infty$. Suppose that there exists an integer $i \geqslant 1$ such that $b_i > 0$; then $b_{i+1} \geqslant b_i/2 > 0$, and it follows from an induction argument that $b_n \geqslant b_i/2 > 0$ for all $n > i$. Consequently we must have $b_n = 0$ for all n, which means $||x_{n-1}|| = ||x_n||$ for all $n \geqslant 1$. This implies that

$$||A^{\frac{1}{2}}V^n x|| = ||A^{\frac{1}{2}}x|| = ||x||$$

for every $x \in \overline{A(\mathcal{H})}$, i.e., the non-negative contraction $A^{\frac{1}{2}}$ is an isometry on $\overline{A(\mathcal{H})}$ $(= \overline{A^{\frac{1}{2}}(\mathcal{H})})$. Hence $\mathcal{H}$ admits the decomposition $\mathcal{H} = \overline{A(\mathcal{H})} \oplus A^{-1}(0)$ into A-invariant subspaces (observe that $(A^{\frac{1}{2}})^{-1}(0) = A^{-1}(0)$), $A^{\frac{1}{2}}|_{\overline{A(\mathcal{H})}}$ is the identity map (on $\overline{A(\mathcal{H})}$) and $A^{\frac{1}{2}}|_{A^{-1}(0)} = 0$. Hence A is a projection.

Recall from [13] that if A is a projection, then T admits a decomposition

$$T = T_u \oplus T_c, \quad T_c = S^* \oplus T_0,$$

where T_u is unitary and the cnu part T_c of T is the direct sum of a backward unilateral shift S^* and a $C_{.0}$-contraction T_0. We prove that S^* is missing from the direct sum. For this recall the (easily proved) fact that an operator $B = B_1 \oplus B_2$ has SVEP at a point if and only if its direct sum components have SVEP at the point. Since $*$-paranormal operators have SVEP (everywhere), it follows that if S^* is present in the direct sum for T, then it has SVEP everywhere. This, however, contradicts the well known fact that the backward unilateral shift does not have SVEP anywhere on its spectrum except for the boundary points of its spectrum. Hence $T = T_u \oplus T_0$. $\square$

It is immediate from Theorem 2.5 that a C_{11} $*$-paranormal contraction is unitary. More is true. Let $\mathbb{D}$ denote the unit disc (in the complex plane), and $\partial\mathbb{D}$ its boundary.

Proposition 2.6. *A $*$-paranormal operator T with spectrum $\sigma(T) \subseteq \partial\mathbb{D}$ is unitary.*

Proof. T being normaloid (see Lemma 2.1), T is a contraction. Suppose that T has a non-trivial cnu part T_c; then $T_c \in C_{.0}$ is normaloid (by Lemma 2.1) with $\sigma(T_c) = \{\lambda : |\lambda| = r(T_c) = 1\}$. Thus $\sigma(T_c)$ consists entirely of *the peripheral spectrum* of T_c [9, p. 225]. Choose a $\lambda \in \sigma(T_c)$; replacing T_c by $\frac{1}{\lambda}T_c$ if need be (evidently, $*$-paranormal operators are closed under multiplication by scalars), we may assume that $\lambda = 1$, and then it follows from [9, Proposition 54.2] that ($T_c - 1$ has ascent $\leqslant 1$ and) $\dim(\mathrm{T}_c^* - 1)^{-1}(0) > 0$. Since T_c and T_c^* have the same invariant vectors, $\dim(\mathrm{T_c} - 1)^{-1}(0) > 0$. This implies that 1 is an eigenvalue, hence a normal eigenvalue, of T_c. But then T_c has a unitary direct summand – a contradiction. Hence T is unitary. $\square$

Let $D_T = (1 - T^*T)^{\frac{1}{2}}$ denote the *defect operator* of $T \in B(\mathcal{H})$: we say that D_T is Hilbert-Schmidt, $D_T \in \mathcal{C}_2$, if D_T^2 is trace class. A C_{00}-contraction T is of the class C_0 of contractions if there exits an inner function u such that $u(T) = 0$: a C_{00}-contraction with Hilbert-Schmidt defect operator is a C_0-contraction [19]. The following corollary says that a $*$- paranormal contraction with Hilbert-Schmidt defect operator is the direct sum of a normal and a C_{10} contraction.

Corollary 2.7. *If $T \in B(\mathcal{H})$ is a $*$-paranormal contraction such that $D_T \in \mathcal{C}_2$, then the pure (i.e., the completely non-normal) part of T is a C_{10}-contraction.*

Proof. Decompose T into its normal and pure parts by $T = T_n \oplus T_p$. Then $T_p \in C_{.0}$ is cnu and $D_{T_p} \in \mathcal{C}_2$. Recall [18, p. 75] that the $C_{.0}$ contraction T_p has a triangulation

$$T_p = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix},$$

where $T_1 \in C_{00}$ and $T_2 \in C_{10}$. Since $D_{T_p} \in \mathcal{C}_2$ implies $D_{T_1} \in \mathcal{C}_2$, T_1 is a C_0-contraction and as such has a triangulation

$$T_1 = \begin{pmatrix} T_{11} & * \\ 0 & T_{12} \end{pmatrix},$$

where $\sigma(T_{11}) = \sigma_p(T_{11}) \subset \mathbb{D}$ and $\sigma(T_{12}) \subseteq \partial\mathbb{D}$ (the minimal function of T_{11} is a Blaschke product and the minimal function of T_{12} is a singular inner function [18]). Since T_1 is pure (and so does not have any eigenvalues, by Lemma 2.2), and since $\sigma(T_1) \subseteq \partial\mathbb{D}$ implies T_1 is unitary, we conclude that $T_p = T_2 \in C_{10}$. $\square$

An operator $T \in B(\mathcal{H})$ is said to be supercyclic if the homogeneous orbit $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbf{N} \cup \mathbf{0}\}$ is dense in $\mathcal{H}$ for some $x \in \mathcal{H}$. (Such an $\mathcal{H}$ is then necessarily separable.) It is known that paranormal operators in $B(\mathcal{X})$ are not supercyclic: $*$-paranormal operators satisfy a similar property.

Corollary 2.8. *$*$-paranormal operators are not supercyclic.*

Proof. Let $T \in B(\mathcal{H})$ be a $*$-paranormal operator such that T has a supercyclic vector. Since T satisfies property (β) and is normaloid, $\sigma(T) = \{\lambda : |\lambda| = r(T) = ||T||\}$ [14, Proposition 3.3.18]. Dividing by $||T||$ if need be, we may thus assume that T is a contraction with spectrum in $\partial\mathbb{D}$. But then T is unitary; since no unitary operator on an infinite dimensional Hilbert space can be supercyclic, we have a contradiction. $\square$

We prove next that $*$-paranormal operators are simply polaroid, where an operator $A \in B(\mathcal{H})$ is said to be *simply polaroid* if the isolated points of the spectrum of the operator are simple poles (i.e., order one poles) of the resolvent of the operator. The following notation and terminology will be required. The *quasinilpotent part* $H_0(A)$ and the *analytic core* $K(A)$ of $A \in B(\mathcal{H})$ are defined by

$$H_0(A) = \left\{x \in \mathcal{H} : \lim_{n \longrightarrow \infty} ||A^n x||^{\frac{1}{n}} = 0\right\}$$

and

$$\begin{aligned} K(A) = \{x \in \mathcal{H} : \text{there exists a sequence} \quad \{x_n\} \subset \mathcal{H} \quad \text{and} \quad \delta > 0 \\ \text{for which} \quad x = x_0, Ax_{n+1} = x_n \quad \text{and} \quad \|x_n\| \leqslant \delta^n \|x\| \quad \text{for all} \quad n = 1, 2, \ldots\}. \end{aligned}$$

$H_0(A)$ and $K(A)$ are (generally) non-closed hyperinvariant subspaces of A such that $(A)^{-q}(0) \subseteq H_0(A)$ for all $q = 0, 1, 2, \ldots$ and $AK(A) = K(A)$; also, if $\lambda \in \mathrm{iso}\sigma(A)$, then $\mathcal{H} = H_0(A-\lambda) \oplus K(A-\lambda)$, where $H_0(A-\lambda)$ and $K(A-\lambda)$ are closed [1].

Theorem 2.9. *$*$-paranormal operators are simply polaroid.*

Proof. Let $\lambda \in \mathrm{iso}\sigma(T)$, where $T \in B(\mathcal{H})$ is $*$-paranormal. Then

$$\mathcal{H} = H_0(T-\lambda) \oplus K(T-\lambda),$$

where $H_0(T-\lambda)$ and $K(T-\lambda)$ are closed, $\sigma(T_1) = \sigma(T|_{H_0(T-\lambda)}) = \{\lambda\}$ and $\sigma(T|_{K(T-\lambda)}) = \sigma(T)\setminus\{\lambda\}$. If $\lambda = 0$, then, T being normaloid, $T_1 = 0$ and $H_0(T) = T^{-1}(0)$. If instead $\lambda \neq 0$, then (recall, $*$-paranormal operators are closed under multiplication by scalars) we may assume that $\lambda = 1$. Applying Proposition 2.6 it follows that T_1 is unitary. Consequently [14, Theorem 1.5.14] $T_1 = I|_{H_0(T-1)}$, which implies that $H_0(T-1) = (T-1)^{-1}(0)$. Thus, in either case, we have that $H_0(T-\lambda) = (T-\lambda)^{-1}(0)$. The proof now follows from the implications

$$\begin{aligned} & \mathcal{H} = (T-\lambda)^{-1}(0) \oplus K(T-\lambda) \\ \Longrightarrow\; & (T-\lambda)\mathcal{H} = 0 \oplus (T-\lambda)K(T-\lambda) = K(T-\lambda) \\ \Longrightarrow\; & \mathcal{H} = (T-\lambda)^{-1}(0) \oplus (T-\lambda)\mathcal{H}. \quad \square \end{aligned}$$

Recall [7] that an operator $A \in B(\mathcal{H})$ is *hereditarily polaroid*, $A \in \mathcal{HP}$, if every part (i.e., restriction to a closed invariant subspace) of the operator is polaroid. Since every part of a $*$-paranormal operator is $*$-paranormal, $*$-paranormal operators are $\mathcal{HP}$ operators. For an operator $A \in B(\mathcal{H})$, let $H(\sigma(A))$ denote the set of functions f which are analytic on a neighborhood of $\sigma(A)$, and let $H_c(\sigma(A))$ denote those $f \in H(\sigma(A))$ which are non-constant on connected components of $\sigma(A)$. The following corollary is an immediate consequence of Theorem 2.9 and [7, Theorem 3.6].

Corollary 2.10. *If $T \in B(\mathcal{H})$ is a polynomially $*$-paranormal operator and $A \in B(\mathcal{H})$ is an algebraic operator which commutes with T, then $f(T+A)$ satisfies generalized Weyl's theorem for every $f \in H(\sigma(T+A))$ and $f(T^*+A^*)$ satisfies a-generalized Weyl's theorem for every $f \in H_c(\sigma(T+A))$.*

3. Tensor products for $*$-class $\mathcal{A}$ operators

For given non-zero operators $T, S \in B(\mathcal{H})$, let $T \otimes S$ denote the tensor product on the product space $\mathcal{H} \otimes \mathcal{H}$. The operation of taking tensor products $T \otimes S$ preserves many properties of $T, S \in B(\mathcal{H})$, but by no means all of them. Thus, the normaloid property is invariant under tensor products (see [16, pp. 623]), and $T \otimes S$ is normal if and only if T and S are [10,17]; however, there exist paranormal operators T and $S \in B(\mathcal{H})$ such that $T \otimes S$ is not paranormal [2]. In [6], Duggal showed that for non-zero $T, S \in B(\mathcal{H})$, $T \otimes S$ is p-hyponormal if and only if T, S are p-hyponormal, where an operator $T \in B(\mathcal{H})$ is said to be p-hyponormal if $|T|^{2p} - |T^*|^{2p} \geqslant 0$ for $0 < p \leqslant 1$. This result was extended to p-quasihyponormal operators and class $\mathcal{A}$ operators in [12,11], respectively.

In this section, we prove an analogous results for $*$-class $\mathcal{A}$ operators.

It is well known [3] that T is $*$-paranormal if and only if

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geqslant 0 \text{ for all } \lambda > 0.$$

Borrowing an argument from Ando's [2] show in the following that there exists a $*$-paranormal operator $T \in B(\mathcal{H})$ such that $T \otimes T$ is not $*$-paranormal:

If a bounded linear operator T is $*$-paranormal, then the tensor products $T \otimes 1$ and $1 \otimes T$ are $*$-paranormal. In fact, for each $\lambda > 0$

$$(T \otimes 1)^{*2}(T \otimes 1)^2 - 2\lambda(T \otimes 1)(T \otimes 1)^* + \lambda^2(1 \otimes 1) = (T^{*2}T^2 - 2\lambda TT^* + \lambda^2) \otimes 1 \geqslant 0,$$

because the tensor product of two positive operators is positive. Observe that $T \otimes T = (T \otimes I)(I \otimes T)$, and $T \otimes I$ double commutes with $I \otimes T$ (i.e., $(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)$ and $(T \otimes I)(I \otimes T^*) = (I \otimes T^*)(T \otimes I)$). We give below an example to show that the tensor product $T \otimes T$ is not necessarily $*$-paranormal.

Let $K = \bigoplus_{n=1}^{\infty} H_n$, where $H_n \cong H$. Given positive operators A and $B \in B(\mathcal{H})$, define the operator $T_{A,B}$ on K as follows:

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then a computation shows that the operator $T_{A,B}$ is $*$-paranormal if and only if

$$B^4 - 2\lambda A^2 + \lambda^2 \geqslant 0 \quad \textit{for all } \lambda > 0.$$

Now consider the operators

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}.$$

Then both C and D are positive and for every $\lambda > 0$

$$D - 2\lambda C + \lambda^2 = \begin{pmatrix} (1-\lambda)^2 & 2(1-\lambda) \\ 2(1-\lambda) & (2-\lambda)^2 + 4 \end{pmatrix}$$

is positive. Let $A = C^{\frac{1}{2}}$ and $B = D^{\frac{1}{4}}$. Then $T_{A,B}$ is $*$-paranormal because for $\lambda > 0$

$$T^{*2}T^2 - 2\lambda TT^* + \lambda^2 = B^4 - 2\lambda A^2 + \lambda^2 = D - 2\lambda C + \lambda^2$$

is positive. But the tensor product $T \otimes T$ is not $*$-paranormal. In fact, if $T \otimes T$ is $*$-paranormal, then

$$(T \otimes T)^{*2}(T \otimes T)^2 - 2(T \otimes T)(T \otimes T)^* + 1 \otimes 1$$

must be positive, and hence the compression of the left side to the canonical imbedding of $H \otimes H$ in $K \otimes K$ is also positive. However the compression coincides with

$$D \otimes D - 2C \otimes C + 1 \otimes 1 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 5 & 2 & 12 \\ 0 & 2 & 5 & 12 \\ 2 & 12 & 12 & 57 \end{pmatrix},$$

which is not positive.

Now we will consider the tensor products for $*$-class $\mathcal{A}$ operators. We need the following famous inequality (see, for example [17, Proposition 2.2]) as a useful tool.

Lemma 3.1. *Let S_i, T_i $(i = 1, 2)$ be nonzero positive operators. Then the following conditions are equivalent:*

- $S_1 \otimes T_1 \leq S_2 \otimes T_2$.
- *There exists $c > 0$ such that $S_1 \leq cS_2$ and $T_1 \leq c^{-1}T_2$.*

Theorem 3.2. *For nonzero operators S and $T \in B(\mathcal{H})$, $S \otimes T$ belongs to $*$-class $\mathcal{A}$ if and only if S and T belongs to $*$-class $\mathcal{A}$.*

Proof. Suppose that S and T are $*$-class $\mathcal{A}$ operators in $B(\mathcal{H})$. Then $|S^2| - |S^*|^2 \geq 0$ and $|T^2| - |T^*|^2 \geq 0$ implies

$$\begin{aligned} |(S \otimes T)^2| - |(S \otimes T)^*|^2 &= |S^2| \otimes |T^2| - |S^*|^2 \otimes |T^*|^2 \\ &= |S^2| \otimes |T^2| + |S^2| \otimes |T^*|^2 - |S^2| \otimes |T^*|^2 - |S^*|^2 \otimes |T^*|^2 \\ &= |S^2| \otimes (|T^2| - |T^*|^2) + (|S^2| - |S^*|^2) \otimes |T^*|^2 \geq 0, \end{aligned}$$

which implies $S \otimes T$ is a $*$-class $\mathcal{A}$ operator.

Conversely, suppose that $S \otimes T$ is a $*$-class $\mathcal{A}$ operator. Then

$$|S^*|^2 \otimes |T^*|^2 = |(S \otimes T)^*|^2 \leq |(S \otimes T)^2| = |S^2| \otimes |T^2|.$$

Now using Lemma 3.1, we have a positive real number c for which

$$|S^*|^2 \leq c|S^2| \quad \text{and} \quad |T^*|^2 \leq c^{-1}|T^2|.$$

This implies that

$$\|S\|^2 = \|S^*\|^2 = \sup_{\|x\|=1} \langle |S^*|^2 x, x\rangle \leq \sup_{\|x\|=1} \langle c|S^2|x, x\rangle \leq c\| \, |S^2| \, \| = c\|S^2\| \leq c\|S\|^2$$

and

$$\|T\|^2 = \|T^*\|^2 = \sup_{\|x\|=1} \langle |T^*|^2 x, x\rangle \leq \sup_{\|x\|=1} \langle c^{-1}|T^2|x, x\rangle \leq c^{-1}\| \, |T^2| \, \| = c\|S^2\| \leq c^{-1}\|T\|^2.$$

Clearly, we must have $c = 1$, and hence S and T are $*$-class $\mathcal{A}$ operators. □

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References

[1] P. Aiena, Fredholm and Local Spectral Theory with Applications to Multipliers, Kluwer, 2004.
[2] T. Ando, Operators with a norm condition, Acta Sci. Math. (Szeged) 33 (1972) 169–178.
[3] S.C. Arora, J.K. Thukral, On a class of operators, Glas. Math. 21 (1986) 381–386.
[4] S.K. Berberian, Approximate proper vectors, Proc. Amer. Math. Soc. 13 (1962) 111–114.
[5] B.P. Duggal, C.S. Kubrusly, Paranormal contractions have property PF, Far East J. Math. 14 (2004) 237–249.
[6] B.P. Duggal, Tensor products of operators – strong stability and p-hyponormality, Glasg. Math. J. 42 (2000) 371–381.
[7] B.P. Duggal, Hereditarily polaroid operators, SVEP and Weyl's theorem, J. Math. Anal. Appl. 340 (2008) 366–373.
[8] E. Durszt, Contractions as restricted shifts, Acta Sci. Math. (Szeged) 48 (1985) 129–134.
[9] H.G. Heuser, Functional Analysis, John Wiley and Sons, 1982.
[10] Jin-chuan Hou, On tensor products of operators, Acta Math. Sin. (N.S.) 9 (1993) 195–202.
[11] I.H. Jeon, B.P. Duggal, On operators with an absolute value condition, J. Korean Math. Soc. 41 (2004) 617–627.
[12] I.H. Kim, Tensor products of Log-hyponormal operators, Bull. Korean Math. Soc. 42 (2005) 269–277.
[13] C.S. Kubrusly, P.C.M. Vieira, D.O. Pinto, A decomposition for a class of operators, Adv. Math. Sci. Appl. 6 (1996) 523–530.
[14] K.B. Laursen, M.M. Neumann, Introduction to Local Spectral Theory, Clarendon Press, Oxford, 2000.
[15] V. Müller, in: Spectral Theory of Linear Operators, Operator Theory Advances and Applications, vol. 139, Birkhäuser Verlag, 2003.
[16] T. Saitô, in: Hyponormal Operators and Related Topics, Lecture Notes in Mathematics, vol. 247, Springer-Verlag, 1971.
[17] Jan Stochel, Seminormality of operators from their tensor products, Proc. Amer. Math. Soc. 124 (1996) 435–440.
[18] B. Sz-Nagy, C. Foiaş, Harmonic Analysis of Operators on Hilbert Space, North-Holland, Amsterdam, 1970.
[19] K. Takahashi, $C_{1.}$ contractions with Hilbert–Schmidt defect operator, J. Operator Theory 12 (1984) 331–337.
[20] A. Uchiyama, K. Tanahashi, Bishop's property (β) for paranormal operators, Oper. Matrices 3 (2009) 517–524.