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Compact maps and quasi-finite complexes

M. Cencelj [a,1], J. Dydak [b,2], J. Smrekar [c,*,3], A. Vavpetič [c], Ž. Virk [c]

[a] *IMFM, Univerza v Ljubljani, Jadranska ulica 19, SI-1111 Ljubljana, Slovenija*
[b] *University of Tennessee, Knoxville, TN 37996, USA*
[c] *Fakulteta za Matematiko in Fiziko, Univerza v Ljubljani, Jadranska ulica 19, SI-1111 Ljubljana, Slovenija*

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Abstract

The simplest condition characterizing quasi-finite CW complexes K is the implication $X\tau_h K \Rightarrow \beta(X)\tau K$ for all paracompact spaces X. Here are the main results of the paper:

Theorem 0.1. *If $\{K_s\}_{s\in S}$ is a family of pointed quasi-finite complexes, then their wedge $\bigvee_{s\in S} K_s$ is quasi-finite.*

Theorem 0.2. *If K_1 and K_2 are quasi-finite countable CW complexes, then their join $K_1 * K_2$ is quasi-finite.*

Theorem 0.3. *For every quasi-finite CW complex K there is a family $\{K_s\}_{s\in S}$ of countable CW complexes such that $\bigvee_{s\in S} K_s$ is quasi-finite and is equivalent, over the class of paracompact spaces, to K.*

Theorem 0.4. *Two quasi-finite CW complexes K and L are equivalent over the class of paracompact spaces if and only if they are equivalent over the class of compact metric spaces.*

Quasi-finite CW complexes lead naturally to the concept of $X\tau\mathcal{F}$, where $\mathcal{F}$ is a family of maps between CW complexes. We generalize some well-known results of extension theory using that concept.

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* Corresponding author.
E-mail addresses: matija.cencelj@guest.arnes.si (M. Cencelj), dydak@math.utk.edu (J. Dydak), jaka.smrekar@fmf.uni-lj.si (J. Smrekar), ales.vavpetic@fmf.uni-lj.si (A. Vavpetič), ziga.virk@fmf.uni-lj.si (Ž. Virk).
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1. Introduction

Definition 1.1. A *compact map* is a continuous function $f: X \to Y$ such that $f(X)$ is contained in a compact subset of Y. If for each $x \in X$ there is a neighborhood U_x such that $f|_{U_x}: U_x \to Y$ is compact then we say that f is *locally compact*.

The notation $K \in AE(X)$ or $X\tau K$ means that any continuous map $f: A \to K$, with A closed in X, extends over X. Also, $K \in AE_h(X)$ or $X\tau_h K$ indicates that any map $f: A \to K$, with A closed in X, extends over X up to homotopy. Employing the notation of [9] we use $K \in AE_{lc}(X)$ (or $X\tau_{lc}K$) to mean that any locally compact map defined on a closed subset of X extends over X to a locally compact map.

Definition 1.2. A CW complex K is an *absolute extensor of X with respect to compact maps* (notation $K \in AE_{cm}(X)$ or $X\tau_{cm}K$) if every compact map $f: A \to K$, A closed in X, extends to a compact map $g: X \to K$.

It follows from (4) of Proposition 5.1 that for X a paracompact space and K a CW complex $X\tau_h K$ and $X\tau_{lc}K$ are equivalent. Furthermore, also $X\tau_h K$ and $X\tau K$ are equivalent if either X is, in addition, metrizable or locally compact (apply (2) of the same proposition), or K is locally finite (apply (4)).

This paper is devoted to the question of extendability of compact maps and the main problem we are interested in is the following:

Problem 1.3. Given a class $\mathcal{C}$ of paracompact spaces characterize CW complexes K such that $X\tau_h K$ and $X\tau_{cm}K$ are equivalent for all $X \in \mathcal{C}$.

It turns out that $X\tau_{cm}K$ is equivalent to $\beta(X)\tau K$; see the more general Proposition 2.2 below.

Proposition 1.4. *If X is paracompact and K is a CW complex, then $X\tau_{cm}K$ if and only if $\beta(X)\tau K$.*

For separable metric spaces the following is true (see Section 5 for the proof).

Proposition 1.5. *If X is separable metric and K is a CW complex, then $X\tau_{cm}K$ (equivalently, $\beta(X)\tau K$) implies $X\tau K$.*

Problem 1.6. Let X be paracompact and let K be a CW complex. Does $X\tau_{cm}K$ (equivalently, $\beta(X)\tau K$) imply $X\tau_h K$?

Definition 1.7. Let $\mathcal{C}$ be a class of paracompact spaces. A CW complex K such that $X \in \mathcal{C}$ and $X\tau_h K$ imply $X\tau_{cm}K$ is called *weakly quasi-finite with respect to $\mathcal{C}$* or simply weakly *$\mathcal{C}$-quasi-finite*.

In view of 1.4 Problem 1.3 is related to the classical question in any dimension theory: Does Čech–Stone compactification preserve dimension? It is so in the theory of covering dimension (see [13] or [14]). In cohomological dimension theory it was a difficult problem (see [19]), finally solved in the negative due to a counterexample of Dranishnikov [4] which was followed by those of Dydak and Walsh [12], and Karinski [18]. Dydak and Mogilski [11] were the first to see a connection between existence of extension dimension preserving compactifications and K-invertible maps (see 2.18 for a more general definition of invertibility—for the class of all spaces X with $X\tau K$ Definition 2.18 corresponds to K-invertibility).

Chigogidze [1] established that not only there is a connection but, in fact, the two notions are equivalent.

Theorem 1.8 *(Chigogidze). For a countable simplicial complex K the following conditions are equivalent*:

1. *If X is any space then $X\tau K$ if and only if $\beta(X)\tau K$.*
2. *There exist a metrizable compactum E with $E\tau K$ and a K-invertible map $p: E \to I^\omega$ onto the Hilbert cube.*

Karasev [16] gave an intrinsic characterization of countable complexes P satisfying 1.8 and called them *quasi-finite complexes*. His definition was later generalized by Karasev–Valov [17] as follows:

Definition 1.9. A CW complex K is called *quasi-finite* if there is a function e from the family of all finite subcomplexes of K to itself satisfying the following properties:

(a) $M \subset e(M)$ for all M.
(b) If X is a Tychonoff space, K is an absolute extensor for X in the sense of [2], and A is any closed subset of X then every map $f: A \to M$ extends over X to a map $g: X \to e(M)$.

Obviously our Definition 1.7 of quasi-finiteness is weaker than 1.9. However, later on we show that the two are equivalent for certain classes of topological spaces.

In subsequent sections we discuss the relationship between properties $X\tau_h K$ and $\beta(X)\tau K$, under different hypotheses on X and K. The main focus is on quasi-finite CW complexes K.

In Section 2 we generalize the concept $X\tau_h K$ to the concept $X\tau_h\mathcal{F}$ where $\mathcal{F}$ is an arbitrary family of maps between CW complexes. It turns out that this concept is natural for extension theory. We introduce various notions of quasi-finite families and discuss relations among them. We also generalize the notion of a K-invertible map (see [17]) for a CW complex K to an $\mathcal{F}$-invertible map where $\mathcal{F}$ is a family of maps between CW complexes. We discuss relations between existence of invertible maps and quasi-finite families.

In Section 3 we apply our results to quasi-finite CW complexes, i.e. the case when the family $\mathcal{F}$ solely consists of the identity map on a CW complex, and deduce several geometric properties of quasi-finite CW complexes.

In Section 4 we prove the existence of "quasi-finite dimension" for a given paracompact space X.

Section 5 contains most proofs, and Section 6 gives a short list of problems—open, to the best of our knowledge.

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2. Extension properties of families of maps

In [7] Dranishnikov and Repovš introduced the concept of $X\tau(L,K)$, where $K \subset L$. It means that any map $f: A \to K$, with A closed in X, extends to $g: X \to L$. Rubin (see [22], Definition 6.1) says in that case that the pair (L,K) is X-connected. In [5] Dranishnikov considers the concept of $X\tau i$, where $i: K \to L$ is a map. This means that for any map $f: A \to K$ defined on the closed subset A of X the composite $if: A \to L$ extends over X. We depict this in the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{f} & K \\ {\scriptstyle \text{incl}}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{F} & L \end{array} \qquad (*)$$

We can also define $X\tau_h i$ by demanding the existence of F so that the diagram $(*)$ commutes up to homotopy. Furthermore, we consider the following notions:

We say that $X\tau_{cm} i$ (respectively, $X\tau_{lc} i$) if for any compact (respectively, locally compact) map f there exists a compact (respectively, locally compact) map F making the diagram $(*)$ commutative. The set A is required to be closed in X.

We generalize the above notions as follows.

Definition 2.1. Suppose $\mathcal{F}$ is a family of maps between CW complexes. Given a paracompact space X we say that $X\tau\mathcal{F}$ or $\mathcal{F} \in AE(X)$ (respectively, $X\tau_h\mathcal{F}$, $X\tau_{cm}\mathcal{F}$, $X\tau_{lc}\mathcal{F}$) if $X\tau i$ (respectively, $X\tau_h i$, $X\tau_{cm} i$, $X\tau_{lc} i$) for all $i \in \mathcal{F}$.

As in the case of absolute extensors, the motivating problem is the relation between the properties $X\tau_h\mathcal{F}$ and $\beta(X)\tau\mathcal{F}$. Here the Čech–Stone compactification of X can be replaced by X if we restrict to compact maps:

Proposition 2.2. *Let X be paracompact and let $\mathcal{F}$ be a family of maps between CW complexes. Then $X\tau_{cm}\mathcal{F}$ if and only if $\beta(X)\tau\mathcal{F}$.*

If $\mathcal{F}$ is a family of maps between finite CW complexes, the following is true.

Proposition 2.3. *Let X be a paracompact space and $\mathcal{F}$ a family of maps between finite (or finitely dominated) CW complexes. Then $X\tau_h\mathcal{F}$ if and only if $\beta(X)\tau\mathcal{F}$.*

Question 2.4. Find necessary and sufficient conditions for $X\tau_h\mathcal{F}$ to imply $\beta(X)\tau\mathcal{F}$ for all $X \in \mathcal{C}$.

We need the notion of *extensionally equivalent* families.

2.1. Extensional equivalence of families

Definition 2.5. Suppose $\mathcal{C}$ is a class of paracompact spaces. Two families $\mathcal{F}$ and $\mathcal{G}$ of maps between CW complexes are *equivalent over* $\mathcal{C}$ (or simply $\mathcal{C}$-equivalent) if $X\tau_h\mathcal{F}$ is equivalent to $X\tau_h\mathcal{G}$ for all $X \in \mathcal{C}$.

More generally, if $X\tau_h\mathcal{F}$ implies $X\tau_h\mathcal{G}$ whenever $X \in \mathcal{C}$, we say that $\mathcal{F}$ *is extensionally smaller than* $\mathcal{G}$ *over* $\mathcal{C}$ and denote it $\mathcal{F} \leqslant \mathcal{G}$ over $\mathcal{C}$ or simply $\mathcal{F} \leqslant_{\mathcal{C}} \mathcal{G}$.

Extensionally, we can always replace a family of maps by a single map.

Theorem 2.6. *Let $\mathcal{F} = \{i_s : K_s \to L_s \mid s \in S\}$ be a family of maps between CW complexes, and form the obvious map $i : \bigvee_s K_s \to \bigvee_s L_s$. Then $\mathcal{F}$ is equivalent to $\{i\}$ over the class of paracompact spaces.*

Remark 2.7. Although it follows from Theorem 2.6 that every family of maps between CW complexes is equivalent over the class of paracompact spaces to a single map between CW complexes, it makes sense to consider families of maps. Namely, in light of Proposition 2.3 we are interested in precisely those maps (or families of maps) which are equivalent to a family of maps between *finite* CW complexes.

The following result shows that extension properties of a family with respect to the class of compact Hausdorff spaces are the same as those with respect to compact metric spaces.

Theorem 2.8. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of maps between CW complexes. If $\mathcal{F} \leqslant \mathcal{G}$ over compact metric spaces, then $\mathcal{F} \leqslant \mathcal{G}$ over compact Hausdorff spaces.*

The class of compact metric spaces is important enough to justify the following definition.

Definition 2.9. Let $\mathcal{C}$ be a class of paracompact spaces. We say that $\mathcal{C}$ is *rich* if it contains all compact metric spaces.

2.2. The notions of quasi-finiteness

For the purpose of answering Question 2.4 we introduce the following notions.

Definition 2.10. Suppose $\mathcal{C}$ is a class of paracompact spaces and $\mathcal{F}$ is a family of maps between CW complexes.

(1) The family $\mathcal{F}$ is *weakly quasi-finite over* $\mathcal{C}$ (or simply weakly $\mathcal{C}$-quasi-finite) if $X\tau_h\mathcal{F}$ implies $\beta(X)\tau\mathcal{F}$ whenever $X \in \mathcal{C}$.
(2) The family $\mathcal{F}$ is *quasi-finite over* $\mathcal{C}$ (or simply $\mathcal{C}$-quasi-finite) if for each member $i : K \to L$ of $\mathcal{F}$ and for each finite subcomplex M of K there is a finite subcomplex $e_i(M)$ of L containing the image $i(M)$ such that all restrictions $j_{i,M} : M \to e_i(M)$ of i satisfy $X\tau j_{i,M}$ if $X \in \mathcal{C}$ satisfies $X\tau_h\mathcal{F}$.
(3) The family $\mathcal{F}$ is *strongly quasi-finite over* $\mathcal{C}$ (or strongly $\mathcal{C}$-quasi-finite) if $\mathcal{F}$ is $\mathcal{C}$-equivalent to a family $\mathcal{G}$ of maps between finite CW complexes.

Definition 2.11. Suppose that $\mathcal{C}$ is a class of paracompact spaces and let K be a CW complex. Then we say that K is *weakly $\mathcal{C}$-quasi-finite* or *$\mathcal{C}$-quasi-finite* or *strongly $\mathcal{C}$-quasi-finite* if the property in question holds for the singleton family $\{\mathrm{id} : K \to K\}$.

It is obvious that the notion of a strongly quasi-finite family $\mathcal{F}$ with respect to an arbitrary class is actually well defined for the equivalence class of $\mathcal{F}$. For the notion of a weakly quasi-finite family the same is true if the reference class of paracompact spaces is rich:

Proposition 2.12. *Let $\mathcal{C}$ be a rich class of paracompact spaces, and suppose that families $\mathcal{F}$ and $\mathcal{G}$ of maps between CW complexes are equivalent over $\mathcal{C}$. If $\mathcal{F}$ is weakly $\mathcal{C}$-quasi-finite, then so is $\mathcal{G}$.*

Proof. By Theorem 2.8, the families $\mathcal{F}$ and $\mathcal{G}$ are equivalent over the class of all compact Hausdorff spaces. Suppose $X \in \mathcal{C}$ and $X\tau_h\mathcal{G}$. Hence $X\tau_h\mathcal{F}$ which implies $\beta(X)\tau\mathcal{F}$. Hence $\beta(X)\tau\mathcal{G}$ and L is weakly quasi-finite with respect to $\mathcal{C}$. □

Definition 2.13. Let $\mathcal{C}$ be a class of paracompact spaces, let $\mathcal{F}$ be a family of maps between CW complexes, and let α be an infinite cardinal number.

- The class $\mathcal{C}$ is *α-saturated* if it is closed under disjoint unions of at most α members. If $\mathcal{C}$ is α-saturated with respect to all α then $\mathcal{C}$ is called *saturated*.
- The family $\mathcal{F}$ is *range-dominated by* α if the ranges of all maps in $\mathcal{F}$ have at most α cells (or are homotopy dominated by such complexes).

Let $\mathcal{C}$ be any class of paracompact spaces. Let $\bar{\mathcal{C}}^\alpha$ denote all possible disjoint unions of at most α members of $\mathcal{C}$, and let $\bar{\mathcal{C}}$ denote all possible disjoint unions of members of $\mathcal{C}$. Then $\bar{\mathcal{C}}^\alpha$ is evidently α-saturated while $\bar{\mathcal{C}}$ is saturated.

We call $\bar{\mathcal{C}}^\alpha$ and $\bar{\mathcal{C}}$ the *α-saturation* and the *saturation* of $\mathcal{C}$, respectively.

We note the evident

Proposition 2.14. *Let $\mathcal{F}$ be quasi-finite (respectively, strongly quasi-finite) with respect to $\mathcal{C}$. Then $\mathcal{F}$ is actually quasi-finite (respectively, strongly quasi-finite) with respect to $\bar{\mathcal{C}}$.*

Theorem 2.15. *Let $\mathcal{C}$ be a class of paracompact spaces and let $\mathcal{F}$ be a family of maps between CW complexes.*

(1) *If $\mathcal{F}$ is $\mathcal{C}$-quasi-finite then $\mathcal{F}$ is weakly $\mathcal{C}$-quasi-finite.*
(2) *If $\mathcal{F}$ is range-dominated by the infinite cardinal number α and $\mathcal{C}$ is α-saturated then $\mathcal{F}$ is $\mathcal{C}$-quasi-finite if and only if $\mathcal{F}$ is weakly $\mathcal{C}$-quasi-finite.*
(3) *If $\mathcal{F}$ is strongly $\mathcal{C}$-quasi-finite and $\mathcal{C}$ is rich then $\mathcal{F}$ is weakly $\mathcal{C}$-quasi-finite.*

Theorem 2.16. *Let $\mathcal{C}$ be a rich class of paracompact spaces. Two strongly $\mathcal{C}$-quasi-finite families are equivalent over $\mathcal{C}$ if and only if they are equivalent over the class of compact metric spaces.*

The following theorem partially answers our motivating problems.

Theorem 2.17. *Let $\mathcal{C}$ be a class of paracompact spaces and suppose that the family $\mathcal{F}$ is $\mathcal{C}$-quasi-finite.*

(1) *If $\mathcal{F}$ is strongly $\mathcal{C}$-quasi-finite and $\mathcal{C}$ is rich then $X\tau_h\mathcal{F}$ is equivalent to $\beta(X)\tau\mathcal{F}$ whenever $X \in \mathcal{C}$.*
(2) *If $X\tau_h\mathcal{F}$ is equivalent to $\beta(X)\tau\mathcal{F}$ whenever $X \in \mathcal{C}$ then $\mathcal{F}$ is strongly $\mathcal{C}$-quasi-finite.*

2.3. Invertible maps

Definition 2.18. Let $\mathcal{C}$ be a class of paracompact spaces and let $\mathcal{F}$ be a family of maps between CW complexes. The map $p: E \to B$ is *$\mathcal{F}$-invertible* with respect to $\mathcal{C}$ if any map $f: X \to B$ with $X\tau_h\mathcal{F}$ and $X \in \mathcal{C}$ has a lift $g: X \to E$.

Remark 2.19. Note that for any class $\mathcal{C}$, a given map $p: E \to B$ is $\mathcal{F}$-invertible with respect to $\mathcal{C}$ if and only if p is $\mathcal{F}$-invertible with respect to the saturation $\bar{\mathcal{C}}$.

Proposition 2.20. *Let $\mathcal{C}$ be a class of paracompact spaces. If there is an $\mathcal{F}$-invertible map $p : E \to I^{\omega}$ with respect to $\mathcal{C}$ where E is a compact Hausdorff space and $E\tau\mathcal{F}$, then $\mathcal{F}$ is weakly $\mathcal{C}$-quasi-finite.*

Recall that the *weight* of a topological space is defined in the following way. Let α be a cardinal number, and let Y be a topological space. We say that the weight of Y is less than α (or is at most α), in symbols $w(Y) \leqslant \alpha$, if the topology on Y admits a basis whose cardinality is at most α.

Theorem 2.21. *Let $\mathcal{C}$ be a class of paracompact spaces and let $\mathcal{F}$ be a family of maps between CW complexes. If $\mathcal{F}$ is $\mathcal{C}$-quasi-finite, then for each infinite cardinal number α there are a compact Hausdorff space E of weight at most α and an $\mathcal{F}$-invertible map $p : E \to I^{\alpha}$ with respect to $\mathcal{C}$ such that $E\tau\mathcal{F}$.*

Corollary 2.22. *(See Karasev–Valov [17], Theorem 2.1.) If K is a quasi-finite CW complex (with respect to paracompact spaces), then for any α there are a compact Hausdorff space E of weight at most α and a K-invertible map $p : E \to I^{\alpha}$ such that $E\tau K$.*

2.4. Geometric realization of families of maps

Definition 2.23. Let $\mathcal{F}$ be a family of maps between CW complexes and let $\mathcal{C}$ be a class of paracompact spaces. A CW complex K is *a geometric $\mathcal{C}$-realization* of $\mathcal{F}$ if $X\tau_h\mathcal{F}$ is equivalent to $X\tau_h K$ whenever $X \in \mathcal{C}$.

Note that a CW complex K is strongly $\mathcal{C}$-quasi-finite if and only if K is a geometric $\mathcal{C}$-realization of a family of maps between finite CW complexes.

Proposition 2.24. *Let $\{\mathcal{F}_s \mid s \in S\}$ be families of maps between CW complexes indexed by the set S. If K_s is a geometric $\mathcal{C}$-realization of $\mathcal{F}_s$ for each s, then $\bigvee_{s\in S} K_s$ is a geometric $\mathcal{C}$-realization of $\bigcup_{s\in S} \mathcal{F}_s$.*

Problem 2.25. Suppose K_s is a geometric realization of $\mathcal{F}_s$ for each $s = 1, 2$. Is $K_1 * K_2$ a geometric realization of $\mathcal{F}_1 * \mathcal{F}_2$?

By $\mathcal{F}_1 * \mathcal{F}_2$ we mean the collection of maps $i_1 * i_2$ for all i_1 and i_2 that belong to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. See the proof of Theorem 3.10 for an explanation of this definition of $\mathcal{F}_1 * \mathcal{F}_2$.

To avoid technical difficulties surrounding the choice of topology on $K_1 * K_2$ (in general, namely, we have to take the compactly generated refinement of the topological join to get the natural CW structure) one can start with the case of K_1 and K_2 countable. In that case the topological join $K_1 * K_2$ is already compactly generated.

Problem 2.26. Is there a family $\mathcal{F} = \{i_s : K_s \to L_s\}_{s\in S}$ such that all K_s, L_s are finite and $\mathcal{F}$ has no geometric realization?

Example 2.27. Let p be a natural number and let $f : K \to K$ be a self-map. If the iterate f^{p+1} is homotopic to f, then f is called *a homotopy p-idempotent* (see [23]). The case of ordinary homotopy idempotents occurs for $p = 1$.

Let Tel_f denote the infinite mapping telescope of f, that is, the quotient space of $K \times N \times [0, 1]$ modulo the relations $(x, n, 1) \simeq (f(x), n + 1, 0)$. If h is a homotopy between the iterates f and f^{p+1}, then a map $u : \mathrm{Tel}_f \to K$ may be defined by

$$[x, n, t] \mapsto f^{n(p-1)}h(x, t).$$

Here $[x, n, t]$ denotes the equivalence class of (x, n, t) in Tel_f. There is an obvious map $d : K \to \mathrm{Tel}_f$ defined by $d(x) = [x, 0, 0]$.

Observe that the composite ud equals f. The p-idempotent f *splits* (see [23]) if the composite du is a homotopy equivalence. In this case $(du)^p$ is homotopic to the identity.

By the cellular approximation theorem we can assume f to be cellular. In that case the infinite mapping telescope Tel_f admits an obvious CW decomposition. The following proposition says that Tel_f is a geometric realization of $\{f\}$ if f is a split p-idempotent.

Proposition 2.28. *Let $f : K \to K$ be a cellular homotopy p-idempotent and let* Tel_f *be the infinite mapping telescope of f. Let X be any space.*

(1) $X\tau_h \mathrm{Tel}_f$ *implies* $X\tau_h f$.
(2) *If f splits, then* $X\tau_h f$ *implies* $X\tau_h \mathrm{Tel}_f$.

Proof. An immediate consequence of Lemma 5.3. □

Corollary 2.29. *Let $f : K \to K$ be a cellular homotopy p-idempotent on the finite-dimensional CW complex K, that is, a cellular map for which f^{p+1}—the $(p+1)$st iterate of f—is homotopic to f. Then the infinite mapping telescope of f is a geometric realization of $\{f : K \to K\}$.*

Proof. Follows from Proposition 2.28 together with Corollary 2.4 of [23]. □

Example 2.30. Let T be a torus with an open disk removed and let ∂T be the boundary of T. If K is a CW complex such that K geometrically realizes the inclusion $i_T : \partial T \to T$, then K is acyclic.

Proof. As in [7] (Theorem 4.8) the n-sphere S^n, $n \geqslant 2$, can be split as $X_1 \cup X_2$ where X_2 is 0-dimensional and X_1 is a countable union of compact metric spaces A_k satisfying $A_k \tau i_T$ for each k. Therefore $X_2 \tau K$ and, by the union theorem for extension theory (see [10]), it follows that $S^n \tau S^0 * K$. This is only possible if $S^0 * K = \Sigma(K)$ is contractible. □

3. Geometric properties of quasi-finite CW complexes

Here is the main property of quasi-finite CW complexes.

Theorem 3.1. *Let K be an infinite CW complex of cardinality α and let $\mathcal{C}$ be a class of paracompact spaces. Consider the following statements.*

(1) *K is strongly $\mathcal{C}$-quasi-finite.*
(2) *K is $\mathcal{C}$-quasi-finite.*
(3) *K is weakly $\mathcal{C}$-quasi-finite.*

The implication (2) ⇒ (1) *always holds while* (1) ⇒ (3) *holds if $\mathcal{C}$ is rich.*

The implication (3) ⇒ (2) *holds if $\mathcal{C}$ is α-saturated.*

In particular, (1), (2), (3) *are equivalent if $\mathcal{C}$ is a rich α-saturated class of paracompact spaces.*

Remark 3.2. For Theorem 7.6 of [22], the ‘cycle’ of implications (d) ⇒ (e) ⇒ (f) ⇒ (c) ⇒ (d) can also be deduced from our results in the following way. (d) ⇒ (f) follows from Theorem 2.21, (f) ⇒ (e) is a tautology, (e) ⇒ (c) follows from Remark 2.19 and Proposition 2.20, and (c) ⇒ (d) follows from Theorem 3.1.

The equivalence (b) ⇔ (c) of [22] shows, in our language, that a CW complex K is weakly quasi-finite with respect to the $\aleph_0$-saturation of compact metric spaces if and only if K is weakly quasi-finite with respect to the (total) saturation of the class of compact metric spaces. We consider that result very interesting.

In light of Proposition 2.14 we immediately deduce

Corollary 3.3. *Let K be an infinite CW complex and $\mathcal{C}$ a rich class of paracompact spaces. Then K is $\mathcal{C}$-quasi-finite if and only if it is strongly $\mathcal{C}$-quasi-finite.*

If K is countable then the notion of quasi-finiteness in our Definition 1.7 is equivalent that of [16].

Corollary 3.4. *If K is a countable CW complex, then the following are equivalent*:

(a) *K is quasi-finite with respect to Polish spaces.*
(b) *K is quasi-finite with respect to paracompact spaces.*

Corollary 3.5. *If K is a countable CW complex, then the following are equivalent*:

(a) *K is quasi-finite with respect to Polish spaces.*
(b) *For each metric compactification $\nu(X)$ of a separable metric space X satisfying $X\tau K$ there is a metric compactification $\gamma(X)$ of X such that $\gamma(X)\tau K$ and $\gamma(X) \geqslant \nu(X)$.*

Theorem 3.6. *If $\{K_s\}_{s\in S}$ is a family of pointed $\mathcal{C}$-quasi-finite* (*respectively, strongly or weakly $\mathcal{C}$-quasi-finite*) *complexes, then their wedge $K = \bigvee_{s\in S} K_s$ is $\mathcal{C}$-quasi-finite* (*respectively, strongly or weakly $\mathcal{C}$-quasi-finite*).

Example 3.7. The wedge $\bigvee_{p \text{ prime}} K(Z_{(p)}, 1)$ is not quasi-finite.

Proof. By the well-known First Theorem of Bockstein the CW complex $\bigvee_{p \text{ prime}} K(Z_{(p)}, 1)$ is equivalent, over the class of compact metric spaces, to S^1. However, Dranishnikov–Repovš–Shchepin [8] proved the existence of a separable metric space X of dimension 2 such that $\dim_{Z_{(p)}}(X) = 1$ for all primes p. Thus, $\bigvee_{p \text{ prime}} K(Z_{(p)}, 1)$ is not equivalent to S^1 over the class of separable metric spaces and is therefore not quasi-finite. □

In [3] Dranishnikov proved that every CW complex is equivalent, over the class of compact Hausdorff spaces, to a wedge of countable CW complexes. Our next result is a variation of that theorem.

Theorem 3.8. *For every $\mathcal{C}$-quasi-finite CW complex K there is a family $\{K_s\}_{s\in S}$ of countable CW complexes so that $\bigvee_{s\in S} K_s$ is $\mathcal{C}$-quasi-finite and $\mathcal{C}$-equivalent to K.*

Theorem 3.9. *Let $j_i : K_i \to L_i$, $i = 1, 2$, be maps between CW complexes. Suppose X is a separable metric space and X_1, X_2 are subsets of X. If $Y_1\tau j_1$ for every subset Y_1 of X_1 and $Y_2\tau j_2$ for every subset Y_2 of X_2, then $X_1 \cup X_2\tau j_1 * j_2$.*

Theorem 3.10. *If K_1 and K_2 are quasi-finite countable complexes, then their join $K_1 * K_2$ is quasi-finite.*

4. Quasi-finite dimension

For each paracompact space X consider all quasi-finite complexes K such that $X\tau_h K$. That class has an initial element K_X (with respect to the relation $K \leqslant L$ over paracompact spaces) described as follows: consider all countable CW complexes K_s, $s \in S_X$, that appear in a decomposition guaranteed by Theorem 3.8 of a quasi-finite CW complex K with $X\tau_h K$. To make sure that S_X is a set, we choose only one representative K_s within its homotopy type. (There is only a set of distinct homotopy types of countable CW complexes.) The wedge K_X of those K_s is quasi-finite. Indeed, K_X is equivalent to the wedge of S_X copies of itself and that wedge is equivalent to a wedge of quasi-finite complexes. By Theorem 3.6 K_X is quasi-finite. Also, $K_X \leqslant K$ for all quasi-finite K satisfying $X\tau_h K$ which means that K_X is indeed an initial element. It is called the *quasi-finite dimension* of X and is denoted by $\dim_{QF}(X)$.

Theorem 4.1. *For every quasi-finite CW complex K there is a compactum X such that* $\dim_{QF}(X) = K$ *and the extension dimension of X equals K.*

Proposition 4.2.

(1) *If X is any paracompact space then* $\dim_{QF}(X) = \dim_{QF}(\beta(X))$.
(2) *If X is a separable metric space then there exists a metric compactification $c(X)$ of X such that* $\dim_{QF}(X) = \dim_{QF}(c(X))$.

(3) *If Y is a separable metric space and $X \subset Y$, then there is a G_δ-subset X' of Y containing X such that* $\dim_{QF}(X) = \dim_{QF}(X')$.

An alternative way to define quasi-finite dimension is to fix a countable family $\{M_i\}_{i=1}^{\infty}$ of finite CW complexes such that any finite CW complex is homotopy equivalent to one of them and assign to each paracompact space X the family of homotopy classes of maps $f : M_i \to M_j$ such that there is a quasi-finite CW complex K_f satisfying $\{\mathrm{id} : K_f \to K_f\} \leqslant f$ over paracompact spaces and $X \tau_h K_f$. The advantage of this approach is that $\dim_{QF}(X)$ would be a countable object. Note that the family of such f would be equivalent to $\{\mathrm{id} : K \to K\}$, where K is the wedge of all K_f and K is quasi-finite. In that sense the two approaches are equivalent.

5. Proofs

Proof of Proposition 1.5. Consider X as a subset of the Hilbert cube I^ω and let $f : \beta(X) \to I^\omega$ be an extension of the inclusion $X \to I^\omega$. By Theorem 1.1 of [20] there is a factorization $f = p \circ g$ through $g : \beta(X) \to Y$ where Y is compact metric and $Y \tau K$. Note that Y is a metric compactification of X. Corollary 3.7 of [15] implies $X \tau K$. □

For convenience we record a few results concerning the relations between $X\tau\mathcal{F}$, $X\tau_h\mathcal{F}$, and $X\tau_{lc}\mathcal{F}$ implied by [9].

Proposition 5.1.

(1) *If f is a locally compact map then $\psi f \varphi$ is locally compact for any maps φ, ψ for which the composition makes sense.*
(2) *If X is first countable or locally compact Hausdorff then any map from X to a CW complex is locally compact. In particular, for such X and i any map between CW complexes, $X\tau i$ if and only if $X \tau_h i$.*
(3) *If X is paracompact and $f : A \to K$ is a locally compact map into the CW complex K, defined on the closed subset A of X, then f extends over X up to homotopy if and only if it extends over X to a locally compact map.*
(4) *Let X be paracompact and let $i : K \to L$ be a map between CW complexes.*
    (a) *$X\tau_h i$ if and only if $X \tau_{lc} i$.*
    (b) *If, in addition, i is locally compact, then $X\tau_h i$ if and only if $X \tau i$ as well.*

Proof. Statement (1) is trivial, while statement (2) follows from Corollary 5.4 of [9]. Statement (3) follows from Corollary 2.13 of [9], while (4) is an immediate consequence of (3). □

Proof of Proposition 2.2. It suffices to consider the case of $\mathcal{F}$ consisting of one map $i : K \to L$. Suppose $X\tau_{cm} i$ and let $f : A \to K$ be a map defined on the closed subset A of $\beta(X)$. The image of f is contained in a finite subcomplex K' of K. By Corollary 2.11 of [9], the map $f : A \to K'$ extends (strictly) over a neighborhood N of A in $\beta(X)$. We may assume that N is closed in $\beta(X)$ and abuse notation to let $f : N \to K'$ denote the extension. Since $f|_{N\cap X} : N \cap X \to K'$ is a compact map, the composite $i \circ f|_{N\cap X}$ admits a compact extension $h : X \to L$, that is, the image of h is contained in a finite subcomplex L' of L. Viewing L' as a subset of the Hilbert cube we may extend h to a map $F : \beta(X) \to L'$. Since $X \cap \mathrm{Int}(N)$ is dense in $\mathrm{Int}(N)$ it follows that $F|_{\mathrm{Int}(N)} = i \circ f|_{\mathrm{Int}(N)}$. In particular, $F|_A = i \circ f$.

Suppose now that $\beta(X)\tau i$ and let $f : A \to K$ be a compact map defined on the closed subset A of X. Certainly f extends over $\beta(A)$ which is closed in $\beta(X)$. Consequently $i \circ f$ extends to a map $\beta(X) \to L$, and the restriction of that extension to X is a compact extension of $i \circ f$. □

The following is a generalization of the Mardešić (or Levin, Rubin, Schapiro, see [20]) factorization theorem.

Theorem 5.2. *Let α be an infinite cardinal. If $f : X \to Y$ is a map of compact Hausdorff spaces with $w(Y) \leqslant \alpha$, then f factors as $f = p \circ g$, where $g : X \to Z$ is surjective, Z is compact, $w(Z) \leqslant \alpha$, and for any family $\mathcal{F}$ of maps between CW complexes $X\tau\mathcal{F}$ implies $Z\tau\mathcal{F}$.*

Proof. Theorem 1.1 of [20] implies that f factors as $f = p \circ g$, where $g : X \to Z$ is surjective, Z is compact metric and for any map $f_0 : A \to M$, with A closed in Z and M a CW complex, f_0 extends over Z if $f_0 \circ g : g^{-1}(A) \to M$ extends over X.

Let $i : K \to L$ be a member of $\mathcal{F}$. Suppose $h : A \to K$ is a map from a closed subset A of Z. Put $f_0 = i \circ h : A \to L$. If $X\tau i$, an extension $F : g^{-1}(A) \to L$ of $f_0 \circ g|_{g^{-1}(A)}$ does exist. Therefore f_0 extends over Z. This shows $Z\tau i$. □

Lemma 5.3. *Let $f : K \to L$ be a map with L and K homotopy dominated by N and M respectively* (*i.e., there are maps $d : N \to L$, $u : L \to N$, $d' : M \to K$, $u' : K \to M$ with du homotopic to* id_L *and $d'u'$ homotopic to* id_K). *For any space X the following are equivalent*:

(i) $X\tau_h f$,
(ii) $X\tau_h(uf)$,
(iii) $X\tau_h(fd')$.

Proof. Note that if $\lambda : L \to L'$ is any map, $X\tau_h f$ implies $X\tau_h(\lambda f)$. Dually, if $\kappa : K' \to K$ is any map, $X\tau_h f$ implies $X\tau_h(f\kappa)$. This establishes implications (i) ⇒ (ii) and (i) ⇒ (iii).

The same argument settles the reverse implications as f is homotopic to $d(uf)$ and $(fd')u'$. □

Lemma 5.4. *Let X be a paracompact space and let $i : K \to L$ be a map between CW complexes. If L is homotopy dominated by a finite CW complex, then $X\tau_h i \Rightarrow \beta(X)\tau i$.*

Consequently if $\mathcal{F}$ is a family of maps with ranges finitely dominated, $X\tau_h\mathcal{F}$ implies $\beta(X)\tau\mathcal{F}$.

Proof. First note that since βX is compact Hausdorff, $\beta X\tau i$ is equivalent to $\beta X\tau_h i$, by (2) of Proposition 5.1. Therefore we may assume that L is in fact finite, by (ii) of Lemma 5.3.

By (4) of Proposition 5.1, $X\tau_{lc} i$ hence $X\tau_{cm} i$ as L is compact. Thus $\beta(X)\tau i$ by Proposition 2.2. □

Lemma 5.5. *Let A be a closed subset in βX. Then $\beta X = \beta(X \cup A)$.*

Proof. Take $g : X \cup A \to I$. We claim that there exists a unique extension to $\beta X \to I$. Certainly there exists an extension $h : \beta X \to I$ of $g|_X : X \to I$. As X is dense in $X \cup A$ and I is Hausdorff, $h|_{X\cup A} = g$. □

Lemma 5.6. *Let X be a paracompact space and let $i : K \to L$ be a map between CW complexes. If K is homotopy dominated by a finite CW complex, then $\beta(X)\tau i$ implies $X\tau_h i$.*

Consequently if $\mathcal{F}$ is a family of maps with finitely dominated domains, $\beta(X)\tau\mathcal{F}$ implies $X\tau_h\mathcal{F}$.

Proof. By (iii) of Lemma 5.3 we may assume that K is in fact finite.

Let $f : A \to K$ be a map defined on the closed subset A of X. As K is a finite CW complex, f extends by [13, Corollary 7.5.39] to a map $F : \beta A \to K$. As X is normal, βA is the closure in βX of A. By assumption on βX, the map iF extends to a map $H : \beta X \to L$. Then $h = H|_X : X \to L$ is the desired extension of if. □

Proof of Proposition 2.3. Follows immediately from Lemmas 5.4 and 5.6. □

Lemma 5.7. *Let $\mathcal{F}$ be a family of maps between CW complexes and let Y be a paracompact space. Suppose that for every finite CW complex M and every map $F : Y \to$ Cone(M) there exists a factorization $F = p \circ g$ through $g : Y \to Z$ where Z is a paracompact space with $Z\tau_{cm}\mathcal{F}$. Then $Y\tau_{cm}\mathcal{F}$. In particular, $Y\tau\mathcal{F}$ if Y is compact.*

Proof. Let $i : K \to L$ be a member of $\mathcal{F}$, and let $f : A \to K$ be a compact map defined on the closed subset A of Y. The image of f is contained in a finite subcomplex M of K. Let $F : Y \to$ Cone(M) denote an extension of the composite $A \xrightarrow{f} M \hookrightarrow$ Cone(M). Let $F = pg$ be the factorization guaranteed by the assumption. Then

$$p|_{p^{-1}(M)} : p^{-1}(M) \to M \hookrightarrow K \xrightarrow{i} L$$

extends to a compact map $G : Z \to L$. The composite Gg is the desired extension of if. □

Proof of Theorem 2.6. Pick $t \in S$ and let $j_t : K_t \to \bigvee_s K_s$ and $q_t : \bigvee_s L_s \to L_t$ be the obvious maps. Evidently $X\tau_h i$ implies $X\tau_h q_t i j_t$. Therefore, $\{i\} \leqslant \mathcal{F}$ over paracompact spaces.

To show $\mathcal{F} \leqslant \{i\}$, let $X\tau_h\mathcal{F}$. In view of (3) and (4) of Proposition 5.1 it suffices to be able to extend any locally compact map $f : A \to \bigvee_s K_s$ defined on an arbitrary closed subset A of X to a locally compact map defined on the whole space X. We can extend the composite $A \xrightarrow{f} \bigvee_s K_s \to \bigvee_s \mathrm{Cone}(K_s)$ to a locally compact map $F : X \to \bigvee_s \mathrm{Cone}(K_s)$ by Corollary 2.10 of [9]. Let $*$ denote the common base point of all the K_s. Set $A' = A \cup F^{-1}(*)$. By Corollary 2.11 of [9] we can extend the restriction $F|_{A'} : A' \to \bigvee_s K_s$ to a locally compact map $f' : N \to \bigvee_s K_s$ for a closed neighborhood N of A'. Define $N_s = f'^{-1}(K_s)$ and $X_s = F^{-1}(\mathrm{Cone}(K_s))$. Note that for distinct t and t', the intersection $X_t \cap X_{t'}$ equals $F^{-1}(*)$ where $*$ is the common base point of all the K_s. By assumption and (4) of Proposition 5.1, the locally compact map

$$N_s \xrightarrow{f'|N_s} K_s \xrightarrow{i_s} L_s$$

extends to a locally compact map $g_s : X_s \cup N_s \to L_s$. By construction, the g_s define a function $g : X \to \bigvee_s L_s$ that extends f'. Let B be a closed neighborhood of A' contained in the interior of N. As $X \setminus B = \bigcup(X_s \setminus B)$ is the disjoint union of open sets, g is continuous. □

Proof of Theorem 2.8. Let X be a compact Hausdorff space with $X\tau\mathcal{F}$. Let M be a finite CW complex and $F : X \to \mathrm{Cone}(M)$ any map. By Theorem 5.2 the map F factors as $F = p \circ g$ through a surjective map $g : X \to Z$ where Z is a compact metric space with $Z\tau\mathcal{F}$. By assumption this implies also $Z\tau\mathcal{G}$. Lemma 5.7 implies $X\tau\mathcal{G}$. □

Proof of Theorem 2.15. (1) Suppose that $\mathcal{F}$ is $\mathcal{C}$-quasi-finite and $X \in \mathcal{C}$. Evidently $X\tau_{cm}\mathcal{F}$ and hence $\beta(X)\tau\mathcal{F}$ by Proposition 2.2.

(2) Suppose that $\mathcal{F}$ is range dominated by α and that $\mathcal{C}$ is α-saturated. By (1) we have to show that if $\mathcal{F}$ is weakly $\mathcal{C}$-quasi-finite then $\mathcal{F}$ is $\mathcal{C}$-quasi-finite. To this end, suppose that for a member $i : K \to L$ of the family $\mathcal{F}$ the function $e = e_i$ of (2) of Definition 2.10 does not exist. Let M_0 be the smallest (finite) subcomplex of L containing $i(M)$ and let $\{M_t \mid t \in T\}$ be all finite subcomplexes of L that contain M_0. For each t there are a space $X_t \in \mathcal{C}$ with $X_t\tau_h\mathcal{F}$, and a map $f_t : A_t \to M$ defined on the closed subset A_t of X_t such that there is no extension of the composite $i \circ f_t$ to a map $X_t \to M_t$. Set $X = \bigoplus_{t\in T} X_t$, $A = \bigoplus_{t\in T} A_t$, and let $f : A \to M$ be induced by $\{f_t\}_{t\in T}$. Obviously, A is closed in X, and, by assumption on $\mathcal{F}$ and $\mathcal{C}$, also $X \in \mathcal{C}$. Evidently $X\tau_h\mathcal{F}$, hence $X\tau_{cm}\mathcal{F}$, by Proposition 2.2. This means that for some t there exists $g : X \to M_t$ with $g|_A = i \circ f$. But then $g_t = g|_{X_t} : X_t \to M_t$ has $g_t|_{A_t} = i \circ f_t$; contradiction.

(3) Suppose $\mathcal{F}$ is strongly $\mathcal{C}$-quasi-finite where $\mathcal{C}$ is rich. Then $\mathcal{F}$ is $\mathcal{C}$-equivalent to a family $\mathcal{G}$ of maps between finite CW complexes. We want to show that $X\tau_h\mathcal{F}$ implies $\beta(X)\tau\mathcal{F}$ to infer that $\mathcal{F}$ is weakly $\mathcal{C}$-quasi-finite. To this end, let $X \in \mathcal{C}$ satisfy $X\tau_h\mathcal{F}$. As $\mathcal{F} \leqslant \mathcal{G}$ over $\mathcal{C}$, this implies $X\tau_h\mathcal{G}$. By Lemma 5.4 it follows that $\beta(X)\tau\mathcal{G}$. Because $\mathcal{C}$ is rich and $\mathcal{G} \leqslant \mathcal{F}$ over $\mathcal{C}$, Theorem 2.8 yields $\beta(X)\tau\mathcal{F}$, as asserted. □

Proof of Theorem 2.16. Suppose $\mathcal{F}$ and $\mathcal{G}$ are two strongly $\mathcal{C}$-quasi-finite families of maps between CW complexes. This means that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{C}$-equivalent to families $\mathcal{F}_1$ and $\mathcal{G}_1$, respectively, of maps between finite CW complexes.

We want to show that if $\mathcal{C}$ is rich then $\mathcal{F} \Leftrightarrow \mathcal{G}$ over compact metric spaces implies $\mathcal{F} \Leftrightarrow \mathcal{G}$ over $\mathcal{C}$. Let $\mathcal{F} \Leftrightarrow \mathcal{G}$ over compact metric spaces. Then also $\mathcal{F}_1 \Leftrightarrow \mathcal{G}_1$ over compact metric spaces. By Theorem 2.8, $\mathcal{F}_1 \Leftrightarrow \mathcal{G}_1$ over the class of all compact Hausdorff spaces.

Pick $X \in \mathcal{C}$ with $X\tau_h\mathcal{F}$. Then $\mathcal{F} \leqslant \mathcal{F}_1$ over $\mathcal{C}$ implies $X\tau_h\mathcal{F}_1$. Proposition 2.3 implies $\beta(X)\tau\mathcal{F}_1$ and since $\mathcal{F}_1 \leqslant \mathcal{G}_1$ over compact Hausdorff spaces, also $\beta(X)\tau\mathcal{G}_1$. Another application of Proposition 2.3 yields $X\tau_h\mathcal{G}_1$. Finally, $\mathcal{G}_1 \leqslant \mathcal{G}$ over $\mathcal{C}$ implies $X\tau_h\mathcal{G}$, as claimed. □

Proof of Theorem 2.17. (1) Assume that $\mathcal{F}$ is strongly $\mathcal{C}$-quasi-finite. Then $\mathcal{F}$ is $\mathcal{C}$-equivalent to a family $\mathcal{G}$ of maps between finite CW complexes. By Theorem 2.8, the richness of $\mathcal{C}$ implies that the families $\mathcal{F}$ and $\mathcal{G}$ are equivalent over compact Hausdorff spaces. The hypothesis is that $\mathcal{F}$ is $\mathcal{C}$-quasi-finite and hence weakly $\mathcal{C}$-quasi-finite by (1) of Theorem 2.15. By definition, this means that $X\tau_h\mathcal{F}$ implies $\beta(X)\tau\mathcal{F}$ whenever $X \in \mathcal{C}$. For the reverse implication, assume that $\beta(X)\tau\mathcal{F}$. Then $\beta(X)\tau\mathcal{G}$ because $\mathcal{F} \leqslant \mathcal{G}$ over compact Hausdorff spaces. This implies $X\tau_h\mathcal{G}$, by Proposition 2.3. Therefore $X\tau_h\mathcal{F}$ since $\mathcal{G} \leqslant \mathcal{F}$ over $\mathcal{C}$.

(2) Assume now that $X\tau_h\mathcal{F}$ is equivalent to $\beta(X)\tau\mathcal{F}$ whenever $X \in \mathcal{C}$. Let $\{e_i \mid i \in \mathcal{F}\}$ be the collection of functions guaranteed by (2) of Definition 2.10. For each $i : K \to L$ that belongs to $\mathcal{F}$ and each finite subcomplex M of K let $j_{i,M} : M \to e_i(M)$ denote the restriction of i to M. Let $\mathcal{G}$ denote the collection of all maps $j_{i,M}$ where the index i

ranges over $\mathcal{F}$ and, for each $i \in \mathcal{F}$, the index M ranges over all finite subcomplexes of the domain of i. We want to show that $\mathcal{F} \Leftrightarrow \mathcal{G}$ over $\mathcal{C}$. The fact that $\mathcal{F} \leqslant \mathcal{G}$ over $\mathcal{C}$ is the content of (2) of Definition 2.10. To show that $\mathcal{G} \leqslant \mathcal{F}$ over $\mathcal{C}$, assume that $X \in \mathcal{C}$ satisfies $X\tau_h\mathcal{G}$. By construction of $\mathcal{G}$ this implies $X\tau_{cm}\mathcal{F}$, and hence $\beta(X)\tau\mathcal{F}$ by Proposition 2.2. By assumption, this is equivalent to $X\tau_h\mathcal{F}$. □

Proof of Proposition 2.20. Let M be a finite CW complex, let $X \in \mathcal{C}$, and assume that $X\tau_h\mathcal{F}$. Furthermore, let $F : X \to \mathrm{Cone}(M)$ be any map. We may assume that $\mathrm{Cone}(M) \subset I^\omega$, and as $X\tau_h\mathcal{F}$, it follows that F admits a lift $G : X \to p^{-1}(\mathrm{Cone}(M))$. As $p^{-1}(\mathrm{Cone}(M))$ is compact, $p^{-1}(\mathrm{Cone}(M))\tau_{cm}\mathcal{F}$, and Lemma 5.7 (together with Proposition 2.2) can be applied to deduce $\beta(X)\tau\mathcal{F}$. □

Proof of Theorem 2.21. Label all maps $Y \to I^\alpha$ where $Y \subset I^\alpha$ is closed and satisfies $Y\tau\mathcal{F}$ as $f_t : Y_t \to I^\alpha$, where t belongs to the indexing set T. Let Y be the Čech–Stone compactification of $Y' = \bigoplus_{t\in T} Y_t$. Let β be the cardinality of T. Since $\mathcal{F}$ is $\mathcal{C}$-quasi-finite, $\mathcal{F}$ is quasi-finite with respect to β-saturation $\bar{\mathcal{C}}^\beta$ of $\mathcal{C}$, by Proposition 2.14. By Theorem 2.15, $\mathcal{F}$ is also weakly $\bar{\mathcal{C}}^\beta$-quasi-finite. Therefore $Y\tau\mathcal{F}$, and Theorem 5.2 implies that the natural map $f : Y \to I^\alpha$ factors as $f = p \circ g$ where $g : Y \to E$ is a surjection onto a compact Hausdorff space E with $E\tau\mathcal{F}$ and $w(E) \leqslant \alpha$. By construction of Y, the map p is $\mathcal{F}$-invertible with respect to $\mathcal{C}$. Indeed, let $X \in \mathcal{C}$ and $X\tau_h\mathcal{F}$. Any $u : X \to I^\alpha$ extends over $\beta(X)$, and factors through some f_t by Theorem 5.2. Since all f_t have lifts to E, so has u. □

Proof of Proposition 2.24. By assumption, $X\tau_h K_s \Leftrightarrow X\tau_h\mathcal{F}_s$ for each $s \in S$ and each $X \in \mathcal{C}$. We are asserting $X\tau_h \bigvee_{s\in S} K_s \Leftrightarrow X\tau_h \bigcup_{s\in S} \mathcal{F}_s$.

If $X\tau_h \bigvee_{s\in S} K_s$ then $X\tau_h K_s$ for all $s \in S$, which means $X\tau_h\mathcal{F}_s$ and of course $X\tau_h \bigcup_{s\in S} \mathcal{F}_s$.

For the reverse implication, $X\tau_h \bigcup_{s\in S} \mathcal{F}_s$ implies $X\tau_h\mathcal{F}_s$ for all $s \in S$ and therefore $X\tau_h\{\mathrm{id} : K_s \to K_s\}$. Therefore $X\tau_h \bigvee_{s\in S} K_s$, by Theorem 2.6. □

Proof of Theorem 3.1. That (3) is equivalent to (2) if $\mathcal{C}$ is α-saturated follows from (2) of Theorem 2.15.

That (1) ⇒ (3) if $\mathcal{C}$ is rich follows from (3) of Theorem 2.15.

We show that (2) ⇒ (1). Let $\mathcal{G}$ be the collection of inclusions $j_M : M \to e(M)$ where M ranges over all finite subcomplexes of K, such that $X\tau_h\mathcal{G}$ if $X \in \mathcal{C}$ satisfies $X\tau_h K$. Suppose that $X \in \mathcal{C}$ satisfies $X\tau\mathcal{G}$. Theorem 2.9 of [9] says that $X\tau_{lc}K$ which, according to Corollary 2.13 of [9], amounts to $X\tau_h K$. Thus K is $\mathcal{C}$-equivalent to $\mathcal{G}$ and K is strongly $\mathcal{C}$-quasi-finite. □

Proof of Corollary 3.4. Only (a) ⇒ (b) is non-trivial. Replace K by a locally finite countable simplicial complex, and let X be a paracompact space. By (4) of Proposition 5.1, $X\tau_h K$ implies $X\tau K$. Given $f : X \to \mathrm{Cone}(M)$ where M is a finite CW complex, we apply Chigogidze's Factorization Theorem [2] to obtain a factorization $F = p \circ G$ through $G : X \to Y$ where Y is a Polish space with $Y\tau K$. Therefore $Y\tau_{cm}K$ and 5.7 says $X\tau_{cm}K$. Thus K is weakly quasi-finite with respect to paracompact spaces and Theorem 3.1 implies that K is quasi-finite with respect to paracompact spaces. □

Proof of Corollary 3.5. (a) ⇒ (b) Let E be a compact metric space with $E\tau K$ and let $p : E \to I^\omega$ be an invertible map with respect to the class of paracompact spaces Y satisfying $Y\tau_h K$ (see 2.21 and 3.4). Given a compactification $\nu(X)$ of a separable metric space X satisfying $X\tau_h K$ let $i : \nu(X) \to I^\omega$ be an embedding. Lift $i|_X : X \to I^\omega$ to a map $j : X \to E$. The closure of $j(X)$ in E is the desired compactification of X.

(b) ⇒ (a) Suppose X is a separable metric space satisfying $X\tau_h K$, and let $f : X \to \mathrm{Cone}(M)$ be a map to the cone over a finite CW complex M. Note that f factors as $X \to \nu(X) \to \mathrm{Cone}(M)$ for some metric compactification $\nu(X)$ of X. Therefore f factors as $X \to \gamma(X) \to \mathrm{Cone}(M)$ for some metric compactification $\gamma(X)$ satisfying $\gamma(X)\tau K$. Hence $X\tau_{cm}K$ by Lemma 5.7, and Theorem 3.1 implies that K is quasi-finite with respect to the class of separable metric spaces. □

Proof of Theorem 3.6. For the case of $\mathcal{C}$-quasi-finiteness suppose that the K_s are $\mathcal{C}$-quasi-finite. Let $\{e_s \mid s \in S\}$ be the set of functions guaranteed by (2) of Definition 2.10. If M is a finite subcomplex of K then M is contained in a finite wedge $\bigvee_{k=1}^n M_{s_k}$ of finite subcomplexes M_{s_k} of K_{s_k}. Let $e(M) = \bigvee_{k=1}^n e_{s_k}(M_{s_k})$. Let $X \in \mathcal{C}$ satisfy $X\tau_h K$. Then also

$X\tau_h K_s$ for all s, and, in particular $X\tau(M_{s_k} \to e_{s_k}(M_{s_k}))$ for all k. By Theorem 2.6, it follows that $X\tau(\bigvee_{k=1}^n M_{s_k} \to e(M))$ and therefore also $X\tau(M \to e(M))$. This shows that e satisfies (2) of Definition 2.10, so K is $\mathcal{C}$-quasi-finite.

For the case of strong $\mathcal{C}$-quasi-finiteness, let the K_s be strongly $\mathcal{C}$-quasi-finite. Then each K_s is equivalent to a family, say $\mathcal{F}_s$, of maps between finite CW complexes. By Proposition 2.24, the wedge $K = \bigvee_s K_s$ is equivalent to the union $\bigcup_s \mathcal{F}_s$. Thus K is also strongly $\mathcal{C}$-quasi-finite.

The remaining case is that of weak $\mathcal{C}$-quasi-finiteness. If the K_s are weakly $\mathcal{C}$-quasi-finite, then note that $\mathcal{F} = \{\mathrm{id}_s : K_s \to K_s\}$ is a weakly $\mathcal{C}$-quasi-finite family. Indeed, if $X \in \mathcal{C}$ then $X\tau_h\mathcal{F}$ is equivalent to $X\tau_h K_s$ for all s, and this in turn implies $\beta(X)\tau K_s$ for all s as the K_s are $\mathcal{C}$-quasi-finite. By Theorem 2.6, the family $\mathcal{F}$ is equivalent to $\{\mathrm{id} : K \to K\}$ over the class of all paracompact spaces and therefore, by Theorem 2.8, also over the class of compact Hausdorff spaces. Thus $\beta(X)\tau\mathcal{F}$ implies $\beta(X)\tau(\mathrm{id} : K \to K)$, as asserted. □

Proof of Theorem 3.8. Let e be the function defined on finite subcomplexes of K that exists according to (2) of Definition 2.10.

By induction, we define a new function E that also meets the requirements of (2) of Definition 2.10 and has the additional property that $E(L) \subset E(L')$ provided $L \subset L'$. If L has only one cell, we put $E(L) = e(L)$. The general induction step is as follows. Add $E(M)$ to $e(L)$ for all proper $M \subset L$ and apply e to that union.

In the next step we construct a family K_M of countable subcomplexes of K indexed by all finite subcomplexes M of K. Put $M_0 = M$ and define M_i inductively by $M_{i+1} = E(M_i)$. Let $K_M = \bigcup_{i=0}^{\infty} M_i$. Note that $M \subset L$ implies $K_M \subset K_L$. Also, by Theorem 2.9 of [9], $X\tau_h K$ implies $X\tau_h K_M$ for all $X \in \mathcal{C}$.

Consider $P = \bigvee_{M\subset K} K_M$. Theorem 2.6 implies that $K \leqslant P$ over $\mathcal{C}$. An application of Theorem 2.9 of [9] yields $P \leqslant K$ over the class of paracompact spaces. Thus P and K are equivalent over $\mathcal{C}$. One can easily check that P is $\mathcal{C}$-quasi-finite using the function e. □

Proof of Theorem 3.9. We may assume that $X_1 \cup X_2 = X$ and that all K_1, K_2, L_1, L_2 are simplicial complexes equipped with the CW topology. For a simplicial complex M let $|M|_w$ and $|M|_m$ denote the underlying topological spaces with the CW ('weak') topology and the metric topology, respectively. Any continuous map $f : Y \to |M|_w$ is locally compact (see [9]). In addition, the following is true. If $f : Y \to |M|_m$ is a continuous map such that every $y \in Y$ has a neighborhood U for which $f(U)$ is contained in a finite subcomplex of $|M|_m$, then f is also continuous as a function from Y to $|M|_w$.

Suppose that C is a closed subset of $X_1 \cup X_2$ and let $f : C \to K_1 * K_2$ be a map. Note that f defines two closed, disjoint subsets $C_1 = f^{-1}(K_1)$, $C_2 = f^{-1}(K_2)$ of C and maps $f_1 : C \setminus C_2 \to K_1$, $f_2 : C \setminus C_1 \to K_2$, $\alpha : C \to [0,1]$ such that:

(1) $\alpha^{-1}(0) = C_1$, $\alpha^{-1}(1) = C_2$,
(2) $f(x) = (1-\alpha(x)) \cdot f_1(x) + \alpha(x) \cdot f_2(x)$ for all $x \in C$.

Indeed, each point x of a simplicial complex M can be uniquely written as $x = \sum_{v\in M^{(0)}} \phi_v(x) \cdot v$ where $M^{(0)}$ is the set of vertices of M and $\{\phi_v(x)\}$ are barycentric coordinates of x. We define $\alpha(x) = \sum_{v\in K_2^{(0)}} \phi_v(f(x))$, and

$$f_1(x) = \frac{1}{1-\alpha(x)} \sum_{v\in K_1^{(0)}} \phi_v\bigl(f(x)\bigr) \cdot v, \qquad f_2(x) = \frac{1}{\alpha(x)} \sum_{v\in K_2^{(0)}} \phi_v\bigl(f(x)\bigr) \cdot v.$$

Since $X_1 \setminus C_2 \tau j_1$, the composite $j_1 \circ f_1$ extends over $(C \cup X_1) \setminus C_2$. Consider a homotopy extension $g_1 : U_1 \to L_1$ of that map over a neighborhood U_1 of $(C \cup X_1) \setminus C_2$ in $X \setminus C_2$. Since $C \setminus C_2$ is closed in U_1, we may assume that g_1 is a genuine extension of $j_1 \circ f_1 : C \setminus C_2 \to L_1$ (see Corollary 2.13 of [9]). Similarly, let $g_2 : U_2 \to L_2$ be an extension of $j_2 \circ f_2$ over a neighborhood U_2 of $(C \cup X_2) \setminus C_1$ in $X \setminus C_1$. Note that $X = U_1 \cup U_2$. Let $\beta : X \to [0,1]$ be an extension of α such that $\beta(X \setminus U_2) \subset \{0\}$ and $\beta(X \setminus U_1) \subset \{1\}$. Define $f' : X \to L_1 * L_2$ by

$$f'(x) = \bigl(1-\beta(x)\bigr) \cdot g_1(x) + \beta(x) \cdot g_2(x) \quad \text{for all } x \in U_1 \cap U_2,$$
$$f'(x) = g_1(x) \quad \text{for all } x \in U_1 \setminus U_2,$$

and

$$f'(x) = g_2(x) \quad \text{for all } x \in U_2 \setminus U_1.$$

Note that f' is a pointwise extension of $(j_1 * j_2) \circ f$. To finish the proof it suffices to show that $f' : X \to |L_1 * L_2|_m$ is continuous. Assuming this, the composite of f' and a homotopy inverse to the identity map $|L_1 * L_2|_w \to |L_1 * L_2|_m$ is an extension of $(j_1 * j_2) \circ f : C \to |L_1 * L_2|_w$ up to homotopy. This implies the existence of a genuine extension by Corollary 2.13 of [9].

To prove continuity of $f' : X \to |L_1 * L_2|_m$ we need to show that the composites $\phi_v \circ f'$ are continuous for all vertices v of $L_1 * L_2$ (see Theorem 8 on p. 301 in [21]). Without loss of generality, we may assume that $v \in L_1$. Then,

$$\phi_v\big(f'(x)\big) = \big(1 - \beta(x)\big) \cdot \phi_v\big(g_1(x)\big) \quad \text{for all } x \in U_1$$

and

$$\phi_v\big(f'(x)\big) = 0 \quad \text{for all } x \in U_2 \setminus U_1.$$

Clearly, the restriction $\phi_v \circ f'|_{U_1}$ is continuous. If $x_0 \in U_2 \setminus U_1$ is the limit of a sequence $\{x_n | n\}$ contained in U_1 then the sequence $\{\beta(x_n)\}$ converges to 1 and $0 \leqslant \phi_v(g_1(x_n)) \leqslant 1$ for all n. Consequently, the sequence $\{\phi_v(f'(x_n))\}$ converges to $0 = \phi_v(f'(x_0))$ which finishes the proof. □

Proof of Theorem 3.10. Let e_K be a function defined on the family of all finite subcomplexes of K such that $M \subset e_K(M)$ for all M and $X \tau i_M$ for all Polish spaces X satisfying $X \tau K$, where i_M is the inclusion $M \to e_K(M)$. Let e_L be the analogous function for L. Define e for $K * L$ as follows: given a finite subcomplex M of $K * L$ find finite subcomplexes K_0 of K and L_0 of L so that $M \subset K_0 * L_0$. Define $e(M)$ as $e_K(K_0) * e_L(L_0)$.

Suppose X is a Polish space and $X \tau K * L$. Express X as $X_1 \cup X_2$ so that $X_1 \tau K$ and $X_2 \tau L$ (see the main result of [6]). Notice $Y \tau K$ for every subset Y of X_1 and $Y \tau L$ for every subset Y of X_2. Therefore (see 3.9) $X \tau i_M * i_P$ for every finite subcomplex M of K and every finite subcomplex P of L. That readily implies $X \tau i_M$ for every finite subcomplex M of $K * L$, i.e. $K * L$ is quasi-finite. □

Proof of Theorem 4.1. Consider a K-invertible map $p : X \to I^\omega$ to the Hilbert cube I^ω such that $X \tau K$ and X is a compact metric space (such a map exists by Corollary 2.22).

Suppose $X \tau L$ for some CW complex L. We need to show that $K \leqslant L$ over the class of compact metric spaces (see Theorem 2.16). Suppose that $Y \tau K$ where Y is a compact metric space. We may assume that $Y \subset I^\omega$ in which case there is a lift $g : Y \to X$ of the inclusion $Y \to I^\omega$. Thus Y is homeomorphic to $g(Y)$, and hence $Y \tau L$. □

Proof of Proposition 4.2. (1) is obvious as $X \tau_h K$ is equivalent to $\beta(X) \tau K$ for all quasi-finite CW complexes by Theorems 2.17 and 3.1.

(2) Pick a quasi-finite complex K so that $K = \dim_{QF}(X)$. By Corollary 2.22 there exists a K-invertible map $p : E \to I^\omega$ from a compact metric space E with $E \tau K$ to the Hilbert cube. As in the proof of Theorem 4.1, the equality $\dim_{QF}(E) = K$ holds. Since X embeds in I^ω, a lift of that embedding is an embedding of X in E. Let $c(X)$ be the closure of X in E. Since $\dim_{QF}(X) \leqslant \dim_{QF}(c(X)) \leqslant \dim_{QF}(E) = K = \dim_{QF}(X)$, we infer $\dim_{QF}(X) = \dim_{QF}(c(X))$.

(3) By (2) there exists a metric compactification $c(X)$ of X of the same quasi-finite dimension as X. By Lusin's Theorem there is a G_δ-subset X' of Y containing X so that X' is homeomorphic to a subset of $c(X)$. Thus $\dim_{QF}(X) \leqslant \dim_{QF}(X') \leqslant \dim_{QF}(c(X))$. □

6. Open problems

Problem 6.1. Let X be a separable metric space and let $\mathcal{F}$ be a family of maps between CW complexes. Find characterizations of families $\mathcal{F}$ for which $\beta(X) \tau \mathcal{F}$ implies $X \tau \mathcal{F}$.

Corollary 3.4 raises the following issue:

Problem 6.2. Suppose K is a CW complex that is quasi-finite with respect to separable metric spaces. Is K quasi-finite (with respect to paracompact spaces)?

Chigogidze [2] proved the following

Theorem 6.3. *Suppose K is a countable CW complex. If $f : X \to Y$ is a map of paracompact spaces where Y is a Polish space and X satisfies $X\tau K$, then f factors as $f = p \circ g$ where $g : X \to Z$ is surjective, Z is a Polish space, and $Z\tau K$.*

In view of Theorem 6.3 the following problem is natural:

Problem 6.4. Suppose $\mathcal{F}$ is a countable family of maps between countable CW complexes. If $f : X \to Y$ is a map of paracompact spaces such that Y is Polish and $X\tau_h\mathcal{F}$, does f factor as $f = p \circ g$, where $g : X \to Z$ is surjective, Z is Polish, and $Z\tau\mathcal{F}$?

A positive answer to 6.4 would imply a positive answer to 6.2.

Problem 6.5. Is there a countable quasi-finite CW complex K that is not equivalent (over the class of paracompact spaces) to a wedge of finitely dominated CW complexes?

Problem 6.6. Is there a quasi-finite (with respect to the class of paracompact spaces) CW complex K that is not equivalent (over the class of paracompact spaces) to a countable quasi-finite CW complex?

Problem 6.7. Is there a quasi-finite (with respect to the class of separable metric spaces) CW complex K that is not quasi-finite with respect to the class of paracompact spaces?

Problem 6.8. If K_1 and K_2 are not necessarily countable quasi-finite CW complexes, is their join $K_1 * K_2$ (with the CW topology) quasi-finite?

Problem 6.9. Suppose $K(G, 1)$ is quasi-finite. Is it equivalent over compact metric spaces to S^1? Is it equivalent over separable metric spaces to S^1?

Problem 6.10. Suppose K is a countable CW complex whose infinite symmetric product $SP(K)$ is quasi-finite. Is $SP(K)$ equivalent over compact metric spaces to S^1? Is it equivalent over separable metric spaces to S^1?

Problem 6.11. Is there a family $\mathcal{F}$ of maps between finite CW complexes without a geometric realization?

Problem 6.12. Characterize those maps $f : K \to K$ whose infinite mapping telescope of $K \to K \to K \to \cdots$ is quasi-finite.

Problem 6.13. Characterize those maps $f : K \to K$ whose infinite mapping telescope of $K \to K \to K \to \cdots$ is a geometric realization of f over paracompact spaces.

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