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The spectra of the adjacency matrix and Laplacian matrix for some balanced trees ☆

Oscar Rojo *, Ricardo Soto [1]

Departamento de Matemáticas, Universidad Católica del Norte, Casilla 1280, Antofagasta, Chile

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Abstract

Let $\mathscr{T}$ be an unweighted rooted tree of k levels such that in each level the vertices have equal degree. Let d_{k-j+1} denotes the degree of the vertices in the level j. We find the eigenvalues of the adjacency matrix and of the Laplacian matrix of $\mathscr{T}$. They are the eigenvalues of principal submatrices of two nonnegative symmetric tridiagonal matrices of order $k \times k$. The codiagonal entries for both matrices are $\sqrt{d_j - 1},\ 2 \leqslant j \leqslant k-1$, and $\sqrt{d_k}$, while the diagonal entries are zeros, in the case of the adjacency matrix, and $d_j,\ 1 \leqslant j \leqslant k$, in the case of the Laplacian matrix. Moreover, we give some results concerning to the multiplicity of the above mentioned eigenvalues.

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* Corresponding author. Tel.: +56 55 355593; fax: +56 55 355599.
E-mail addresses: orojo@socompa.ucn.cl, orojo@ucn.cl (O. Rojo), rsoto@ucn.cl (R. Soto).
[1] Part of this research was conducted while the authors were visitors at the Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, Brasil.

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1. Notations and preliminaries

Let $\mathscr{G}$ be a simple graph. Let $A(\mathscr{G})$ be the adjacency matrix of $\mathscr{G}$ and let $D(\mathscr{G})$ be the diagonal matrix of vertex degrees. The Laplacian matrix of $\mathscr{G}$ is $L(\mathscr{G}) = D(\mathscr{G}) - A(\mathscr{G})$. Clearly, $L(\mathscr{G})$ is a real symmetric matrix. From this fact and Geršgorin's theorem, it follows that its eigenvalues are nonnegative real numbers. Moreover, since its rows sum to 0, 0 is the smallest eigenvalue of $L(\mathscr{G})$. In [4], some of the many results known for Laplacian matrices are given. Fiedler [2] proved that $\mathscr{G}$ is a connected graph if and only if the second smallest eigenvalue of $L(\mathscr{G})$ is positive. This eigenvalue is called the algebraic connectivity of $\mathscr{G}$.

We recall that a tree is a connected acyclic graph. Here we consider an unweighted rooted tree $\mathscr{T}$ such that in each level the vertices have equal degree. We agree that the root vertex is at level 1 and that $\mathscr{T}$ has k levels. Thus the vertices in the level k have degree 1.

For $j = 1, 2, 3, \ldots, k$, the numbers d_{k-j+1} and n_{k-j+1} denote the degree of the vertices and the number of vertices in the level j, respectively. Then, for $j = 2, 3, \ldots k-1$,

$$n_{k-j} = (d_{k-j+1} - 1)n_{k-j+1}. \tag{1}$$

Observe that d_k is the degree of the root vertex, $d_1 = 1$ is the degree of the vertices in the level k, $n_k = 1$, $n_{k-1} = d_k$, n_{j+1} divides n_j for all $j = 1, \ldots, k-1$ and that the total number of vertices in the tree is

$$n = \sum_{j=1}^{k-1} n_j + 1.$$

We introduce the following notations:

If all the eigenvalues of an $n \times n$ matrix A are real numbers, we write

$$\lambda_n(A) \leqslant \lambda_{n-1}(A) \leqslant \cdots \leqslant \lambda_2(A) \leqslant \lambda_1(A).$$

0 is the all zeros matrix.

The order of 0 will be clear from the context in which it is used.

I_m is the identity matrix of order $m \times m$.

$\mathbf{e}_m$ is the all ones column vector of dimension m.

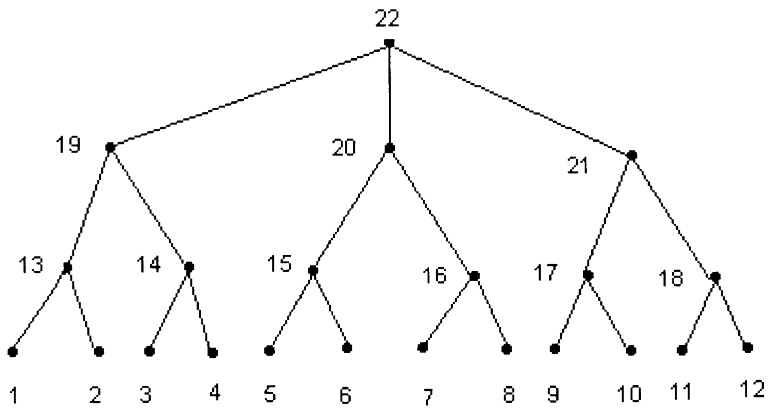
For $j = 1, 2, \ldots, k-1$, C_j is the block diagonal matrix defined by

$$C_j = \begin{bmatrix} \mathbf{e}_{\frac{n_j}{n_{j+1}}} & 0 & \cdots & 0 \\ 0 & \mathbf{e}_{\frac{n_j}{n_{j+1}}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{e}_{\frac{n_j}{n_{j+1}}} \end{bmatrix}, \tag{2}$$

with n_{j+1} diagonal blocks. Thus, the order of C_j is $n_j \times n_{j+1}$. Observe that $C_{k-1} = \mathbf{e}_{n_{k-1}}$.

Let us illustrate the notations above introduced and our labeling for $\mathscr{T}$ with the following example.

Example 1. Let $\mathscr{T}$ be the tree



We see that this tree has 4 levels, $n_1 = 12, n_2 = 6, n_3 = 3, n_4 = 1$ and the vertex degrees are $d_1 = 1, d_2 = 3, d_3 = 3, d_4 = 3$. Then, $\frac{n_1}{n_2} = 2$, $\frac{n_2}{n_3} = 2$ and $\frac{n_3}{n_4} = 3$. The matrices defined in (2) are

$$C_1 = \begin{bmatrix} 1 & & & & & \\ 1 & & & & & \\ & 1 & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & 1 & \\ & & & & & 1 \\ & & & & & 1 \end{bmatrix} = \operatorname{diag}\{\mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_2\},$$

$$C_2 = \begin{bmatrix} 1 & & \\ 1 & & \\ & 1 & \\ & 1 & \\ & & 1 \\ & & 1 \end{bmatrix} = \operatorname{diag}\{\mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_2\},\ C_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{e}_3.$$

In general, using the labels $1, 2, 3, \ldots, n$, in this order, our labeling for the vertices of $\mathscr{T}$ is: *Label the vertices from the bottom to the root vertex and, in each level, from the left to the right*.

For this labeling the adjacency matrix $A(\mathscr{T})$ and Laplacian matrix $L(\mathscr{T})$ of the tree in Example 1 become

$$A(\mathscr{T}) = \begin{bmatrix} 0 & C_1 & 0 & 0 \\ C_1^{\mathrm{T}} & 0 & C_2 & 0 \\ 0 & C_2^{\mathrm{T}} & 0 & C_3 \\ 0 & 0 & C_3^{\mathrm{T}} & 0 \end{bmatrix}$$

and

$$L(\mathscr{T}) = \begin{bmatrix} I_{12} & -C_1 & 0 & 0 \\ -C_1^{\mathrm{T}} & 3I_6 & -C_2 & 0 \\ 0 & -C_2^{\mathrm{T}} & 3I_2 & -C_3 \\ 0 & 0 & -C_3^{\mathrm{T}} & 3 \end{bmatrix}.$$

with C_1, C_2 and C_3 as in Example 1.

In general, our labeling yields to

$$A(\mathscr{T}) = \begin{bmatrix} 0 & C_1 & 0 & \cdots & \cdots & 0 \\ C_1^{\mathrm{T}} & 0 & C_2 & \ddots & & \vdots \\ 0 & C_2^{\mathrm{T}} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & C_{k-2} & 0 \\ \vdots & & \ddots & C_{k-2}^{\mathrm{T}} & 0 & C_{k-1} \\ 0 & \cdots & \cdots & 0 & C_{k-1}^{\mathrm{T}} & 0 \end{bmatrix} \tag{3}$$

and

$$L(\mathscr{T}) = \begin{bmatrix} I_{n_1} & -C_1 & 0 & \cdots & \cdots & 0 \\ -C_1^{\mathrm{T}} & d_2 I_{n_2} & C_2 & \ddots & & \vdots \\ 0 & -C_2^{\mathrm{T}} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_{k-2} I_{n_{k-2}} & -C_{k-2} & 0 \\ \vdots & & \ddots & -C_{k-2}^{\mathrm{T}} & d_{k-1} I_{n_{k-1}} & -C_{k-1} \\ 0 & \cdots & \cdots & 0 & -C_{k-1}^{\mathrm{T}} & d_k \end{bmatrix}. \tag{4}$$

The following lemma plays a fundamental role in this paper.

Lemma 1. *Let*

$$M = \begin{bmatrix} \alpha_1 I_{n_1} & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ C_1^{\mathrm{T}} & \alpha_2 I_{n_2} & C_2 & \ddots & & & \\ 0 & C_2^{\mathrm{T}} & & & \ddots & & \\ \vdots & \ddots & & & & \ddots & \\ \vdots & & \ddots & & \alpha_{k-2} I_{n_{k-2}} & C_{k-2} & 0 \\ \vdots & & & \ddots & C_{k-2}^{\mathrm{T}} & \alpha_{k-1} I_{n_{k-1}} & C_{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & C_{k-1}^{\mathrm{T}} & \alpha_k \end{bmatrix}.$$

Let

$$\beta_1 = \alpha_1$$

and

$$\beta_j = \alpha_j - \frac{n_{j-1}}{n_j} \frac{1}{\beta_{j-1}}, \quad j = 2, 3, \ldots, k, \; \beta_{j-1} \neq 0.$$

If $\beta_j \neq 0$ *for all* $j = 1, 2, \ldots, k-1$,

$$\det M = \beta_1^{n_1} \beta_2^{n_2} \ldots \beta_{k-2}^{n_{k-2}} \beta_{k-1}^{n_{k-1}} \beta_k. \tag{5}$$

Proof. Suppose $\beta_j \neq 0$ for all $j = 1, 2, \ldots, k-1$. We apply the Gaussian elimination procedure, without row interchanges, to reduce the matrix M to an upper triangular matrix. Just before the last step, we have the matrix

$$\begin{bmatrix} \beta_1 I_{n_1} & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \beta_2 I_{n_2} & C_2 & & & & \vdots \\ 0 & 0 & \beta_3 I_{n_3} & C_3 & & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & C_{k-2} & 0 \\ \vdots & & & & 0 & \beta_{k-1} I_{n_{k-1}} & C_{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & C_{k-1}^{\mathrm{T}} & \alpha_k \end{bmatrix}.$$

Finally, the Gaussian elimination gives

$$\begin{bmatrix} \beta_1 I_{n_1} & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \beta_2 I_{n_2} & C_2 & & & & \vdots \\ 0 & 0 & \beta_3 I_{n_3} & C_3 & & & \vdots \\ \vdots & & 0 & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & C_{k-2} & 0 \\ \vdots & & & & 0 & \beta_{k-1} I_{n_{k-1}} & C_{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \alpha_k - n_{k-1}\frac{1}{\beta_{k-1}} \end{bmatrix}. \quad (6)$$

Thus, (5) is proved. □

2. The spectrum of the Laplacian matrix of $\mathscr{T}$

Let

$$\Phi = \{1, 2, 3, \ldots, k-1\}.$$

We consider the following subset of Φ,

$$\Omega = \{j \in \Phi : n_j > n_{j+1}\}.$$

Since $n_{k-1} > n_k = 1$, the index $k-1 \in \Omega$. Observe that if $i \in \Phi - \Omega$ then $n_i = n_{i+1}$ and thus, from (2), $C_i = I_{n_i}$.

Theorem 2. *Let*

$$P_0(\lambda) = 1, \quad P_1(\lambda) = \lambda - 1$$

and

$$P_j(\lambda) = (\lambda - d_j)P_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} P_{j-2}(\lambda) \quad \text{for } j = 2, 3, \ldots, k. \quad (7)$$

Hence

(a) *If $P_j(\lambda) \neq 0$, for all $j = 1, 2, \ldots, k-1$, then*

$$\det(\lambda I - L(\mathscr{T})) = P_k(\lambda) \prod_{j \in \Omega} P_j^{n_j - n_{j+1}}(\lambda). \quad (8)$$

(b)

$$\sigma(L(\mathscr{T})) = (\cup_{j\in\Omega}\{\lambda \in \mathbb{R} : P_j(\lambda) = 0\}) \cup \{\lambda \in \mathbb{R} : P_k(\lambda) = 0\}. \quad (9)$$

Proof. (a) We apply Lemma 1 to the matrix $M = \lambda I - L(\mathscr{T})$. For this matrix $\alpha_1 = \lambda - 1$ and $\alpha_j = \lambda - d_j$ for $j = 2, 3, \ldots, k$. Let $\beta_1, \beta_2, \ldots, \beta_k$ be as in Lemma 1. Suppose that $\lambda \in \mathbb{R}$ is such that $P_j(\lambda) \neq 0$ for all $j = 1, 2, \ldots, k-1$. We have

$$\beta_1 = \lambda - 1 = \frac{P_1(\lambda)}{P_0(\lambda)} \neq 0,$$

$$\beta_2 = (\lambda - d_2) - \frac{n_1}{n_2}\frac{1}{\beta_1} = (\lambda - d_2) - \frac{n_1}{n_2}\frac{P_0(\lambda)}{P_1(\lambda)}$$

$$= \frac{(\lambda - d_2)P_1(\lambda) - \frac{n_1}{n_2}P_0(\lambda)}{P_1(\lambda)} = \frac{P_2(\lambda)}{P_1(\lambda)} \neq 0,$$

$$\beta_3 = (\lambda - d_3) - \frac{n_2}{n_3}\frac{1}{\beta_2} = (\lambda - d_3) - \frac{n_2}{n_3}\frac{P_1(\lambda)}{P_2(\lambda)}$$

$$= \frac{(\lambda - d_3)P_2(\lambda) - \frac{n_2}{n_3}P_1(\lambda)}{P_2(\lambda)} = \frac{P_3(\lambda)}{P_2(\lambda)} \neq 0,$$

$$\vdots$$

$$\beta_{k-1} = (\lambda - d_{k-1}) - \frac{n_{k-2}}{n_{k-1}}\frac{1}{\beta_{k-2}} = (\lambda - d_{k-1}) - \frac{n_{k-2}}{n_{k-1}}\frac{P_{k-3}(\lambda)}{P_{k-2}(\lambda)}$$

$$= \frac{(\lambda - d_{k-1})P_{k-2}(\lambda) - \frac{n_{k-2}}{n_{k-1}}P_{k-3}(\lambda)}{P_{k-2}(\lambda)} = \frac{P_{k-1}(\lambda)}{P_{k-2}(\lambda)} \neq 0,$$

$$\beta_k = (\lambda - d_k) - \frac{n_{k-1}}{n_k}\frac{1}{\beta_{k-1}} = (\lambda - d_k) - \frac{n_{k-1}}{n_k}\frac{P_{k-2}(\lambda)}{P_{k-1}(\lambda)}$$

$$= \frac{(\lambda - d_k)P_{k-1}(\lambda) - \frac{n_{k-1}}{n_k}P_{k-2}(\lambda)}{P_{k-1}(\lambda)} = \frac{P_k(\lambda)}{P_{k-1}(\lambda)}.$$

From (5)

$$\det(\lambda I - L(\mathscr{T})) = \frac{P_1^{n_1}(\lambda)}{P_0^{n_1}(\lambda)}\frac{P_2^{n_2}(\lambda)}{P_1^{n_2}(\lambda)}\frac{P_3^{n_3}(\lambda)}{P_2^{n_3}(\lambda)}\cdots\frac{P_{k-2}^{n_{k-2}}(\lambda)}{P_{k-3}^{n_{k-2}}(\lambda)}\frac{P_{k-1}^{n_{k-1}}(\lambda)}{P_{k-2}^{n_{k-1}}(\lambda)}\frac{P_k(\lambda)}{P_{k-1}(\lambda)}$$

$$= P_1^{n_1-n_2}(\lambda)P_2^{n_2-n_3}(\lambda)P_3^{n_3-n_4}(\lambda)\ldots P_{k-1}^{n_{k-1}-1}(\lambda)P_k(\lambda)$$

$$= P_k(\lambda)\prod_{j\in\Omega} P_j^{n_j-n_{j+1}}(\lambda).$$

Thus, (8) is proved.

(b) From (8), if $\lambda \in \mathbb{R}$ is such that $P_j(\lambda) \neq 0$, for all $j = 1, 2, \ldots, k-1, k$, then $\det(\lambda I - L(\mathscr{T})) \neq 0$. That is

$$\cap_{j=1}^{k}\{\lambda \in \mathbb{R} : P_j(\lambda) \neq 0\} \subseteq (\sigma(L(\mathscr{T})))^c.$$

That is

$$\sigma(L(\mathscr{T})) \subseteq \left(\cup_{j=1}^{k-1}\{\lambda \in \mathbb{R} : P_j(\lambda) = 0\}\right) \cup \{\lambda \in \mathbb{R} : P_k(\lambda) = 0\}. \tag{10}$$

We claim that

$$\sigma(L(\mathscr{T})) \subseteq (\cup_{j\in\Omega}\{\lambda \in \mathbb{R} : P_j(\lambda) = 0\}) \cup \{\lambda \in \mathbb{R} : P_k(\lambda) = 0\}. \tag{11}$$

If $\Omega = \Phi = \{1, 2, \ldots, k-1\}$ then (11) is (10) and there is nothing to prove. Suppose that Ω is a proper subset of Φ. Clearly, (11) is equivalent to

$$\cap_{j\in\Omega}\{\lambda \in \mathbb{R} : P_j(\lambda) \neq 0\} \cap \{\lambda \in \mathbb{R} : P_k(\lambda) \neq 0\} \subseteq (\sigma(L(\mathscr{T})))^c.$$

Suppose that $\lambda \in \mathbb{R}$ is such that $P_j(\lambda) \neq 0$ for all $j \in \Omega$ and $P_k(\lambda) \neq 0$. Since $k-1 \in \Omega$, $P_{k-1}(\lambda) \neq 0$. If in addition $P_j(\lambda) \neq 0$ for all $j \in \Phi - \Omega$ then (8) holds and consequently $\det(\lambda I - L(\mathscr{T})) \neq 0$. That is, $\lambda \in (\sigma(L(\mathscr{T})))^c$. If $P_i(\lambda) = 0$ for some $i \in \Phi - \Omega$, let l be the first index in $\Phi - \Omega$ such that $P_l(\lambda) = 0$. Then, $\beta_j \neq 0$ for all $j = 1, 2, \ldots, l-1$, $\beta_l = 0$ and

$$P_{l+2}(\lambda) = (\lambda - d_{l+2})P_{l+1}(\lambda).$$

We observe that $P_{l+1}(\lambda) \neq 0$. Otherwise, a back sustitution in (7) gives $P_0(\lambda) = 0$. Therefore, $\beta_{l+2} = \frac{P_{l+2}(\lambda)}{P_{l+1}(\lambda)} = \lambda - d_{l+2}$. Since $l \in \Phi - \Omega$, then $n_l = n_{l+1}$, $C_l = I_{n_l}$ and the Gaussian elimination procedure applied to $M = \lambda I - L(\mathscr{T})$ yields to the intermediate matrix

$$\begin{bmatrix} \beta_1 I_{n_1} & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & 0 & 0 & I_{n_l} & 0 & & \vdots \\ \vdots & \ddots & I_{n_l} & (\lambda - d_{l+1})I_{n_{l+1}} & C_{l+1} & \ddots & \vdots \\ \vdots & & \ddots & C_{l+1}^{\mathrm{T}} & (\lambda - d_{l+2})I_{n_{l+2}} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & C_{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & C_{k-1} & \lambda - d_k \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1 I_{n_1} & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & 0 & 0 & I_{n_l} & 0 & & \vdots \\ \vdots & \ddots & I_{n_l} & (\lambda - d_{l+1})I_{n_{l+1}} & C_{l+1} & \ddots & \vdots \\ \vdots & & \ddots & C_{l+1}^{\mathrm{T}} & \beta_{l+2} I_{n_{l+2}} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & C_{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & C_{k-1} & \lambda - d_k \end{bmatrix}.$$

Next, a number of n_l row interchanges gives the matrix

$$\begin{bmatrix} \beta_1 I_{n_1} & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & 0 & I_{n_l} & (\lambda - d_{l+1}) I_{n_{l+1}} & C_{l+1} & & \vdots \\ \vdots & \ddots & 0 & I_{n_l} & 0 & \ddots & \vdots \\ \vdots & & \ddots & C_{l+1}^{\mathrm{T}} & \beta_{l+2} I_{n_{l+2}} & \ddots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & C_{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & C_{k-1}^{\mathrm{T}} & \lambda - d_k \end{bmatrix}.$$

Therefore

$$\det(\lambda I - L(\mathscr{T})) = (-1)^{n_l} \beta_1^{n_1} \beta_2^{n_2} \ldots \beta_{l-1}^{n_{l-1}} \det \begin{bmatrix} \beta_{l+2} I_{n_{l+2}} & \ddots & 0 \\ \ddots & \ddots & C_{k-1} \\ 0 & C_{k-1}^{\mathrm{T}} & \lambda - d_k \end{bmatrix}.$$

Now, if there exists $j \in \Phi - \Omega$, $l + 2 \leqslant j \leqslant k - 2$, such that $P_j(\lambda) = 0$, we apply the above procedure to the matrix

$$\begin{bmatrix} \beta_{l+2} I_{\frac{n_{l+2}}{2}} & \ddots & 0 \\ \ddots & \ddots & C_{k-1} \\ 0 & C_{k-1}^{\mathrm{T}} & \lambda - d_k \end{bmatrix}.$$

Finally, we obtain

$$\det(\lambda I - L(\mathscr{T})) = \gamma \beta_k = \gamma \frac{P_k(\lambda)}{P_{k-1}(\lambda)}, \tag{12}$$

where γ is a factor different from 0. By hypothesis, $P_{k-1}(\lambda) \neq 0$ and $P_k(\lambda) \neq 0$. Therefore, $\det(\lambda I - L(\mathscr{T})) \neq 0$ and thus $\lambda \notin \sigma(L(\mathscr{T}))$. Hence, (11) is proved. Now, we claim that

$$(\cup_{j\in\Omega}\{\lambda \in \mathbb{R} : P_j(\lambda) = 0\}) \cup \{\lambda \in \mathbb{R} : P_k(\lambda) = 0\} \subseteq \sigma(L(\mathscr{T})).$$

Let $\lambda \in \cup_{j\in\Omega}\{\lambda \in \mathbb{R} : P_j(\lambda) = 0\}$. Let l be the first index in Ω such that $P_l(\lambda) = 0$. Then, $\beta_l = \frac{P_l(\lambda)}{P_{l-1}(\lambda)} = 0$. The corresponding intermediate matrix in the Gaussian elimination procedure applied to the matrix $M = \lambda I - L(\mathscr{T})$ is

$$\begin{bmatrix} \beta_1 I_{n_1} & C_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ 0 & \ddots & 0 & C_l & & \vdots \\ \vdots & \ddots & C_l^{\mathrm{T}} & (\lambda - d_{l+1}) I_{n_{l+1}} & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & C_{k-1} \\ 0 & \cdots & \cdots & 0 & C_{k-1}^{\mathrm{T}} & \lambda - d_k \end{bmatrix}. \tag{13}$$

Since $l \in \Omega$, $n_l > n_{l+1}$ and C_l is a matrix with more rows than columns. Therefore, the matrix in (13) has at least two equal rows. Thus, $\det(\lambda I - L(\mathscr{T})) = 0$. That is, $\lambda \in (L(\mathscr{T}))$. Hence

$$\cup_{j \in \Omega} \{\lambda \in \mathbb{R} : P_j(\lambda) = 0\} \subseteq \sigma(L(\mathscr{T})). \tag{14}$$

Now let $\lambda \in \{\lambda \in \mathbb{R} : P_k(\lambda) = 0\}$. Observe that $P_{k-1}(\lambda) \neq 0$. Otherwise, a back substitution in (7) yields to $P_0(\lambda) = 0$. If $P_j(\lambda) = 0$ for some $j \in \Omega$ then the use of (14) gives $\lambda \in \sigma(L(\mathscr{T}))$. Hence, we may suppose that $P_j(\lambda) \neq 0$ for all $j \in \Omega$. If in addition $P_j(\lambda) \neq 0$ for all $j \in \Phi - \Omega$ then (8) holds and thus $\det(\lambda I - L(\mathscr{T})) = 0$ because $P_k(\lambda) = 0$. If $P_i(\lambda) = 0$ for some $i \in \Phi - \Omega$ then we have the assumptions under which (12) was obtained. Therefore

$$\det(\lambda I - L(\mathscr{T})) = \gamma \beta_k = \gamma \frac{P_k(\lambda)}{P_{k-1}(\lambda)} = 0.$$

Thus, we have proved that

$$\{\lambda \in \mathbb{R} : P_k(\lambda) = 0\} \subseteq \sigma(L(\mathscr{T})). \tag{15}$$

From (14) and (15),

$$(\cup_{j \in \Omega} \{\lambda \in \mathbb{R} : P_j(\lambda) = 0\}) \cup \{\lambda \in \mathbb{R} : P_k(\lambda) = 0\} \subseteq \sigma(L(\mathscr{T})). \tag{16}$$

Finally, (11) and (16) imply (9). □

Lemma 3. *For $j = 1, 2, 3, \ldots, k-1$, let T_j be the $j \times j$ principal submatrix of the $k \times k$ symmetric tridiagonal matrix*

$$T_k = \begin{bmatrix} 1 & \sqrt{d_2 - 1} & 0 & \cdots & \cdots & 0 \\ \sqrt{d_2 - 1} & d_2 & \sqrt{d_3 - 1} & \ddots & & \vdots \\ 0 & \sqrt{d_3 - 1} & d_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \sqrt{d_{k-1} - 1} & 0 \\ \vdots & & \ddots & \sqrt{d_{k-1} - 1} & d_{k-1} & \sqrt{d_k} \\ 0 & \cdots & \cdots & 0 & \sqrt{d_k} & d_k \end{bmatrix}.$$

Then

$$\det(\lambda I - T_j) = P_j(\lambda), \quad j = 1, 2, \ldots, k.$$

Proof. It is well known (see for instance [1, p.229]) that the characteristic polynomials, Q_j, of the $j \times j$ principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$\begin{bmatrix} a_1 & b_1 & 0 & \cdots & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & & \vdots \\ 0 & b_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & a_{k-1} & b_{k-1} \\ 0 & \cdots & \cdots & 0 & b_{k-1} & a_k \end{bmatrix},$$

satisfy the three-term recursion formula

$$Q_j(\lambda) = (\lambda - a_j)Q_{j-1}(\lambda) - b_{j-1}^2 Q_{j-2}(\lambda),$$

with

$$Q_0(\lambda) = 1 \quad \text{and} \quad Q_1(\lambda) = \lambda - a_1.$$

In our case, $a_1 = 1, a_j = d_j$ for $j = 2, 3, , \ldots, k$ and $b_j = \sqrt{\frac{n_j}{n_{j+1}}}$ for $j = 1, 2, \ldots,$ $k - 1$. For these values, the above recursion formula gives the polynomials P_j, $j = 0, 1, 2, \ldots, k$. Now, we use (1), to see that $\sqrt{\frac{n_j}{n_{j+1}}} = \sqrt{d_j - 1}$ for $j = 1, 2, \ldots, k - 2$ and $\sqrt{\frac{n_{k-1}}{n_k}} = \sqrt{n_{k-1}} = \sqrt{d_k}$. □

Theorem 4. *Let T_j, $j = 1, 2, \ldots, k - 1$ and T_k be the symmetric tridiagonal matrices defined in Lemma 3. Then*

(a)

$$\sigma(L(\mathscr{T})) = (\cup_{j\in\Omega}\sigma(T_j)) \cup \sigma(T_k).$$

(b) *The multiplicity of each eigenvalue of the matrix T_j, as an eigenvalue of $L(\mathscr{T})$, is at least $(n_j - n_{j+1})$ for $j \in \Omega$ and 1 for $j = k$.*

Proof. We recall that the eigenvalues of any symmetric tridiagonal matrix with nonzero codiagonal entries are simple. Then, (a) and (b) are immediate consecuences of this fact, Theorem 2 and Lemma 3. □

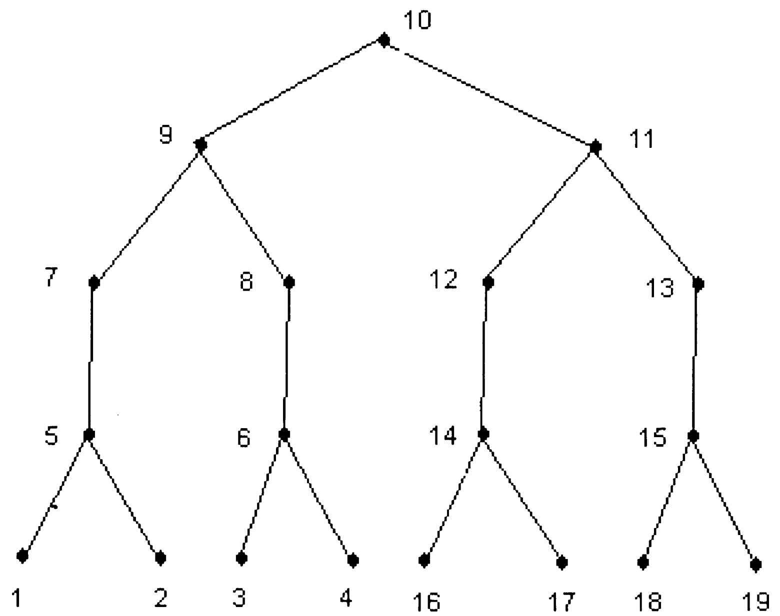
Example 2. Let $\mathscr{T}$ be the tree in Example 1. For this tree, $k = 4$, $d_1 = 1$, $d_2 = 3$, $d_3 = 3$, $d_4 = 3$, $n_1 = 12$, $n_2 = 6$ and $n_3 = 3$. Hence

$$T_4 = \begin{bmatrix} 1 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 3 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 3 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 3 \end{bmatrix}$$

and $\Omega = \{1, 2, 3\}$. The eigenvalues of $L(\mathscr{T})$ are the eigenvalues of T_1, T_2, T_3 and T_4. To four decimal places these eigenvalues are:

T_1 :	1			
T_2 :	0.2679	3.7321		
T_3 :	0.0968	2.1939	4.709	
T_4 :	0	1.1864	3.4707	5.3429

Example 3. Let $\mathscr{T}$ be the tree



For this tree, $k = 5$, $n_1 = 8$, $n_2 = n_3 = 4$, $n_4 = 2$, $n_5 = 1$, $d_1 = 1$, $d_2 = 3$, $d_3 = 2$, $d_4 = 3$ and $d_5 = 2$. Hence the matrix T_5 is

$$T_5 = \begin{bmatrix} 1 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 3 & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} & 2 \end{bmatrix}$$

and $\Omega = \{1, 3, 4\}$. Thus the spectrum of $L(\mathscr{T})$ is the union of the spectra of T_1, T_3, T_4 and T_5:

T_1 :	1				
T_3 :	0.1392	1.7459	4.1149		
T_4 :	0.0646	1	3.4626	4.4728	
T_5 :	0	0.5617	1.8614	3.8202	4.7566

We recall the following interlacing property [3]:

Let T be a symmetric tridiagonal matrix with nonzero codiagonal entries and $\lambda_i^{(j)}$ be the ith smallest eigenvalue of its $j \times j$ principal submatrix. Then,

$$\lambda_{j+1}^{(j+1)} < \lambda_j^{(j)} < \lambda_j^{(j+1)} < \cdots < \lambda_{i+1}^{(j+1)} < \lambda_i^{(j)} < \lambda_i^{(j+1)} < \cdots < \lambda_2^{(j+1)} < \lambda_1^{(j)} < \lambda_1^{(j+1)}.$$

Theorem 5. *Let $L(\mathscr{T})$ be the Laplacian matrix of $\mathscr{T}$. Then*

(a) $\sigma(T_{j-1}) \cap \sigma(T_j) = \phi$ *for* $j = 2, 3, \ldots, k$.
(b) *The largest eigenvalue of T_k is the largest eigenvalue of $L(\mathscr{T})$.*
(c) *The smallest eigenvalue of T_{k-1} is the algebraic connectivity of $\mathscr{T}$.*
(d) *The largest eigenvalue of T_{k-1} is the second largest eigenvalue of $L(\mathscr{T})$.*
(e) $\det T_j = 1$ *for* $j = 1, 2, \ldots, k-1$.
(f) *If λ is an integer eigenvalue of $L(\mathscr{T})$ and $\lambda > 1$ then $\lambda \in \sigma(T_k)$.*

Proof. First we observe that $k - 1 \in \Omega$. Thus the eigenvalues of T_{k-1} are always eigenvalues of $L(\mathscr{T})$. Now (a), (b), (c) and (d) follow from the interlacing property and Theorem 4. Clearly, $\det T_1 = 1$. Let $2 \leqslant j \leqslant k-1$. We apply the Gaussian elimination procedure, without row interchanges, to reduce the matrix T_j to the upper triangular matrix

$$\begin{bmatrix} 1 & \sqrt{d_2 - 1} & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \sqrt{d_3 - 1} & & & \vdots \\ 0 & 0 & 1 & \sqrt{d_4 - 1} & & \vdots \\ \vdots & \ddots & & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & 1 & \sqrt{d_j - 1} \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

Thus, (e) is proved. Since $P_0(\lambda) = 1$ and $P_1(\lambda) = \lambda - 1$, it follows from the recursion formula $P_j(\lambda) = (\lambda - d_j)P_{j-1} - \frac{n_{j-1}}{n_j} P_{j-2}(\lambda)$ that $P_j(\lambda)$ is a polynomial with integer coefficients. Therefore, if λ is an eigenvalue of T_j then λ exactly divides $P_j(0)$. Moreover, $P_j(0) = (-1)^j \det T_j = (-1)^j$. Consequently, no integer greater than 1 is an eigenvalue of T_j. $\square$

3. The spectrum of the adjacency matrix of $\mathscr{T}$

Let

$$D = \begin{bmatrix} -I_{n_1} & 0 & 0 & \cdots & \cdots & 0 \\ 0 & I_{n_2} & 0 & \ddots & & \vdots \\ 0 & 0 & -I_{n_3} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & (-1)^{k-1} I_{n_{k-1}} & 0 \\ 0 & \cdots & \cdots & 0 & 0 & (-1)^k \end{bmatrix}.$$

From (3),

$$A(\mathscr{T}) = \begin{bmatrix} 0 & C_1 & 0 & \cdots & \cdots & 0 \\ C_1^{\mathrm{T}} & 0 & C_2 & \ddots & & \vdots \\ 0 & C_2^{\mathrm{T}} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & C_{k-2} & 0 \\ \vdots & & \ddots & C_{k-2}^{\mathrm{T}} & 0 & C_{k-1} \\ 0 & \cdots & \cdots & 0 & C_{k-1}^{\mathrm{T}} & 0 \end{bmatrix}.$$

It is easily see that

$$D(\lambda I + A(\mathscr{T}))D^{-1} = \lambda I - A(\mathscr{T}).$$

This fact will be used in the proof of the following theorem.

Theorem 6. *Let*

$$S_0(\lambda) = 1, \quad S_1(\lambda) = \lambda$$

and

$$S_j(\lambda) = \lambda S_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} S_{j-2}(\lambda) \quad \textit{for } j = 2, 3, \ldots, k.$$

Then

(a) *If* $S_j(\lambda) \neq 0$, *for all* $j = 1, 2, \ldots, k-1$, *then*

$$\det(\lambda I - A(\mathscr{T})) = S_k(\lambda) \prod_{j \in \Omega} S_j^{n_j - n_{j+1}}(\lambda).$$

(b)

$$\sigma(A(\mathscr{T})) = (\cup_{j \in \Omega}\{\lambda \in \mathbb{R} : S_j(\lambda) = 0\}) \cup \{\lambda \in \mathbb{R} : S_k(\lambda) = 0\}.$$

Proof. Similar to the proof of Theorem 2. Apply Lemma 1 to the matrix $M = \lambda I + A(\mathscr{T})$. For this matrix $\alpha_j = \lambda$ for $j = 1, 2, \ldots, k$. Finally, use the fact that $\det(\lambda I - A(\mathscr{T})) = \det(\lambda I + A(\mathscr{T}))$. □

Lemma 7. *For $j = 1, 2, 3, \ldots, k-1$, let R_j be the $j \times j$ principal submatrix of the $k \times k$ tridiagonal matrix*

$$R_k = \begin{bmatrix} 0 & \sqrt{d_2 - 1} & 0 & \cdots & \cdots & 0 \\ \sqrt{d_2 - 1} & 0 & \sqrt{d_3 - 1} & \ddots & & \vdots \\ 0 & \sqrt{d_3 - 1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \sqrt{d_{k-2} - 1} & 0 \\ \vdots & & \ddots & \sqrt{d_{k-2} - 1} & 0 & \sqrt{d_k} \\ 0 & \cdots & \cdots & 0 & \sqrt{d_k} & 0 \end{bmatrix}.$$

Then

$$\det(\lambda I - R_j) = S_j(\lambda), \quad j = 1, 2, \ldots, k.$$

Proof. Similar to the proof of Lemma 3. □

Theorem 8. *Let R_j, $j = 1, 2, \ldots, k-1$ and R_k be the symmetric tridiagonal matrices defined in Lemma 7.*

(a)

$$\sigma(A(\mathscr{T})) = (\cup_{j \in \Omega} \sigma(R_j)) \cup \sigma(R_k).$$

(b) *The multiplicity of each eigenvalue of the matrix R_j, as an eigenvalue of $A(\mathscr{T})$, is at least $(n_j - n_{j+1})$ for $j \in \Omega$ and 1 for $j = k$.*

Proof. (a) and (b) are immediate consequences of Theorem 6, Lemma 7 and the fact that the eigenvalues of any symmetric tridiagonal matrix with nonzero codiagonal entries are simple. □

Example 4. Let $\mathscr{T}$ be the tree in Example 1. Then, $k = 4$, $d_1 = 1$, $d_2 = 3$, $d_3 = 3$ and $d_4 = 3$. Hence

$$R_4 = \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}.$$

and $\Omega = \{1, 2, 3\}$. To four decimal places these eigenvalues are

R_1 :	0			
R_2 :	−1.4142	1.4142		
R_3 :	−2	0	2	
R_4 :	−2.4495	−1	1	2.4495

4. Applications to some trees

In this section, we apply the results of the previous sections to some specific trees.

4.1. Balanced binary tree

In a balanced binary tree $\mathscr{B}_k$ of k levels, we have $d_k = 2$ for the root vertex degree and $d_{k-j+1} = 3$ for $j = 2, 3, \ldots, k-1$. Clearly, $\Omega = \{1, 2, \ldots, k-1\}$. Then

$$\sigma(L(\mathscr{B}_k)) = \cup_{j=1}^{k} \sigma(T_j)$$

where for $j = 1, \ldots., k-1$, T_j is the $j \times j$ principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$T_k = \begin{bmatrix} 1 & \sqrt{2} & 0 & \cdots & \cdots & 0 \\ \sqrt{2} & 3 & \ddots & & & \vdots \\ 0 & \ddots & & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ \vdots & & & \ddots & 3 & \sqrt{2} \\ 0 & \cdots & \cdots & 0 & \sqrt{2} & 2 \end{bmatrix}.$$

This is the main result in [6]. In [5] quite tight lower and upper bounds for the algebraic connectivity of $\mathscr{B}_k$ are given and in [7] the integer eigenvalues of $L(\mathscr{B}_k)$ are found.

For the adjacency matrix of $\mathscr{B}_k$ we have

$$\sigma(A(\mathscr{B}_k)) = \cup_{j=1}^{k} \sigma(R_j),$$

where, for $j = 1, \ldots, k-1$, R_j is the $j \times j$ principal submatrix of

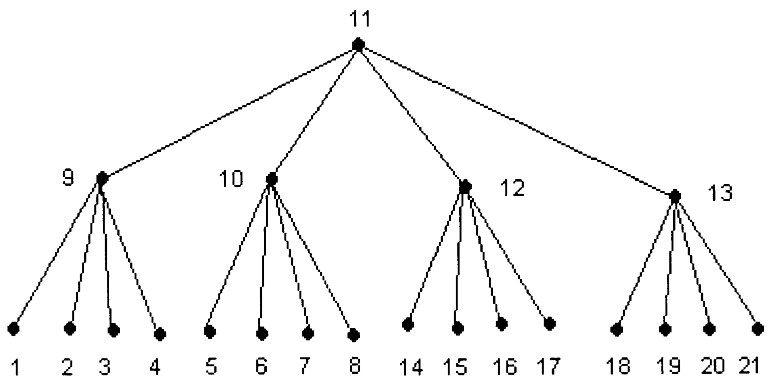
$$R_k = \begin{bmatrix} 0 & \sqrt{2} & 0 & \cdots & \cdots & 0 \\ \sqrt{2} & 0 & \ddots & & & \vdots \\ 0 & \ddots & & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ \vdots & & & \ddots & 0 & \sqrt{2} \\ 0 & \cdots & \cdots & 0 & \sqrt{2} & 0 \end{bmatrix}$$

of order $k \times k$.

4.2. Balanced 2^p-ary tree

In the tree $\mathscr{B}_k$, from the root vertex until the vertices in the level $(k-1)$, each vertex originates two more new vertices. Let us consider a tree of k levels in which from the root vertex until the vertices in the level $(k-1)$, each vertex originates 2^p more new vertices. We call this tree a balanced 2^p–ary tree and we denote it by $\mathscr{B}_k^p$.

Example 5. The tree $\mathscr{B}_3^2$ is



The total number of vertices in $\mathscr{B}_k^p$ is

$$n = 1 + 2^p + \cdots + 2^{(k-1)p} = \frac{2^{kp}-1}{2^p-1}.$$

Now $d_k = 2^p$ for the root vertex degree, $d_{k-j+1} = 2^p + 1$ and $\frac{n_{k-j}}{n_{k-j+1}} = 2^p$ for $j = 2, 3, \ldots, k-1$, and $n_{k-1} = 2^p$. Then

$$\sigma(L(\mathscr{B}_k^p)) = \cup_{j=1}^{k} \sigma(T_j),$$

where $T_j,\ j = 1, \ldots, k-1$, is the $j \times j$ principal submatrix of $k \times k$ symmetric tridiagonal matrix

$$T_k = \begin{bmatrix} 1 & \sqrt{2^p} & 0 & \cdots & \cdots & 0 \\ \sqrt{2^p} & 2^p+1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ \vdots & & & \ddots & 2^p+1 & \sqrt{2^p} \\ 0 & \cdots & \cdots & 0 & \sqrt{2^p} & 2^p \end{bmatrix}.$$

For the adjacency matrix of $\mathscr{B}_k^p$ we have

$$\sigma(A(\mathscr{B}_k^p)) = \cup_{j=1}^k \sigma(R_j),$$

where, for $j = 1, \ldots, k-1$, the matrix R_j is the $j \times j$ principal submatrix of the $k \times k$ symmetric tridiagonal matrix

$$R_k = \begin{bmatrix} 0 & \sqrt{2p} & 0 & \cdots & \cdots & 0 \\ \sqrt{2p} & 0 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ \vdots & & & \ddots & 0 & \sqrt{2p} \\ 0 & \cdots & \cdots & 0 & \sqrt{2p} & 0 \end{bmatrix}.$$

Example 6. The eigenvalues of the Laplacian matrix and adjacency matrix of $\mathscr{B}_3^2$ are the eigenvalues of the principal submatrices of the matrices

$$T_3 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

and

$$R_3 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix},$$

respectively. These eigenvalues are

T_1 :	1		
T_2 :	0.1716	5.8284	
T_3 :	0	3	7

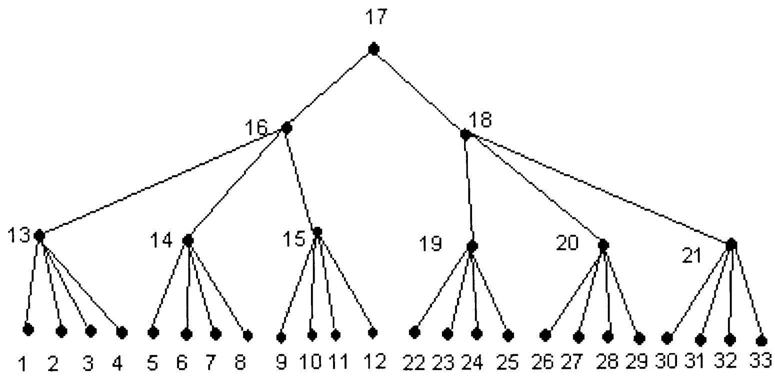
for the Laplacian matrix and

R_1 :	0		
R_2 :	−2	2	
R_3 :	−2.8284	0	2.8284

for the adjacency matrix.

4.3. Balanced factorial tree

We introduce a balanced tree of k levels in which, from the root vertex until the vertices in the level $(k-1)$, each vertex in the level j originates $(j+1)$ new vertices. Let us denote this tree by $\mathscr{F}_k$. For example, the tree $\mathscr{F}_4$ is



The degree of the vertices in each level of $\mathscr{F}_k$ is as follows:

$$\begin{array}{cc} j-\text{level} & \\ j=1 & d_k=2 \\ j=2 & d_{k-1}=4 \\ j=3 & d_{k-2}=5 \\ j=4 & d_{k-3}=6 \\ \vdots & \\ j=k-1 & d_2=k+1 \\ j=k & d_1=1. \end{array}$$

Then

$$\sigma(L(\mathscr{F}_k)) = \cup_{j=1}^{k}\sigma(T_j),$$

where, for $j = 1, \ldots, k-1$, T_j is the $j \times j$ principal submatrix of the $k \times k$ tridiagonal matrix

$$T_k = \begin{bmatrix} 1 & \sqrt{k} & 0 & \cdots & \cdots & 0 \\ \sqrt{k} & k+1 & \sqrt{k-1} & & & \vdots \\ 0 & \sqrt{k-1} & k & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & 0 \\ \vdots & & & \ddots & 4 & \sqrt{2} \\ 0 & \cdots & \cdots & 0 & \sqrt{2} & 2 \end{bmatrix}$$

and

$$\sigma(A(\mathscr{F}_k)) = \cup_{j=1}^{k}\sigma(R_j),$$

where, for $j = 1, \ldots, k-1$, R_j is the $j \times j$ principal submatrix of the $k \times k$ tridiagonal matrix

$$R_k = \begin{bmatrix} 0 & \sqrt{k} & 0 & \cdots & \cdots & 0 \\ \sqrt{k} & 0 & \sqrt{k-1} & & & \vdots \\ 0 & \sqrt{k-1} & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 0 & \sqrt{2} \\ 0 & \cdots & \cdots & 0 & \sqrt{2} & 0 \end{bmatrix}.$$

Example 7. For the tree $\mathscr{F}_4$ we have

$$T_4 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 5 & \sqrt{3} & 0 \\ 0 & \sqrt{3} & 4 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 2 \end{bmatrix}$$

and the eigenvalues of $L(\mathscr{F}_4)$ to four decimal places are

T_1 :	1			
T_2 :	0.1716	5.8284		
T_3 :	0.0464	3.1794	6.7742	
T_4 :	0	1.2363	3.8748	6.8890

The eigenvalues of the adjacency matrix $A(\mathscr{F}_4)$ are the eigenvalues of the principal submatrices of

$$R_4 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & \sqrt{3} & 0 \\ 0 & \sqrt{3} & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}$$

and they are

R_1 :	0			
R_2 :	−2	2		
R_3 :	−2.6458	0	2.6458	
R_4 :	−2.8284	−1	1	2.8284

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