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Sequences in a Fuzzy Metric Space

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Abstract—In this paper, we introduce the concept of bounded sequences in a fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$ and show that every convergent sequence in X is bounded. We also discuss the space of convergent sequences in a complete fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$.

Keywords—Sequences in a fuzzy metric space, Convergent and bounded sequences in a fuzzy metric space.

1. INTRODUCTION

Nanda [1] proved that the spaces of bounded and convergent sequences of fuzzy numbers are complete metric spaces. In this paper, we introduce the concept of bounded sequences in a fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$ and show that every convergent sequence in X is bounded. We also discuss the space of convergent sequences in a complete fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$ by partitioning the set of all convergent sequences in X and proving that the collection of these equivalence classes is complete. Our results are motivated by the results obtained by Nanda [1], and Kaleva [2].

2. PRELIMINARIES

In this section, we provide some relevant backgrounds which will be used later.

Let R^1 denote the set of all real numbers. A fuzzy number is a fuzzy set $u : R^1 \to [0, 1]$ with the following properties:

(1) u is upper semicontinuous,
(2) u is fuzzy convex,
(3) u is normal, i.e., there exists a $t_0 \in R^1$ such that $u(t_0) = 1$.

The α-level set of a fuzzy number u, $0 < \alpha \le 1$, denoted by $[u]_\alpha$, is defined as

$$[u]_\alpha = \left\{ t \in R^1 \mid u(t) \ge \alpha \right\}.$$

It can be seen easily that the α-level set of a fuzzy number is a closed interval $[a^\alpha, b^\alpha]$, where $a^\alpha = -\infty$ and $b^\alpha = \infty$ are admissible.

Let $\mathcal{F}$ denote the set of all fuzzy numbers. Since each $r \in R^1$ can be considered as a fuzzy number $\tilde{r}$ defined by

$$\tilde{r}(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \ne r, \end{cases}$$

R^1 can be embedded in $\mathcal{F}$.

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A fuzzy number u is called nonnegative if $u(t) = 0$ for all $t < 0$. We will denote by $\mathcal{F}^*$ the set of all nonnegative fuzzy numbers of $\mathcal{F}$.

The arithmetic operation $+$ on $\mathcal{F} \times \mathcal{F}$ can be defined in [3] as:

$$(x+y)(t) = \sup_{s \in R^1} \mathrm{Min}(x(s), y(t-s)), \qquad t \in R^1.$$

If $x, y \in \mathcal{F}$ and $[x]_\alpha = [a_1^\alpha, b_1^\alpha]$, $[y]_\alpha = [a_2^\alpha, b_2^\alpha]$, then a straightforward calculation yields

$$[x+y]_\alpha = [a_1^\alpha + a_2^\alpha, b_1^\alpha + b_2^\alpha]. \tag{2.1}$$

A sequence $\{u_n\}$ in $\mathcal{F}$ converges to u in $\mathcal{F}$, denoted by

$$u = \lim_{n \to \infty} u_n,$$

if and only if

$$\lim_{n \to \infty} a_n^\alpha = a^\alpha \quad \text{and} \quad \lim_{n \to \infty} b_n^\alpha = b^\alpha,$$

for $\alpha \in (0,1]$, where $[u_n]_\alpha = [a_n^\alpha, b_n^\alpha]$ and $[u]_\alpha = [a^\alpha, b^\alpha]$.

Let X be a nonempty set, $d : X \times X \to \mathcal{F}^*$, and let the mappings L, $R : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$. Write

$$[d(x,y)]_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)],$$

for all x, $y \in X$, $0 < \alpha \le 1$. The quadruple (X, d, L, R) is called a fuzzy metric space, if the following conditions are fulfilled:

(1) $d(x,y) = \tilde{0}$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$ for all x, $y \in X$,
(3) for all x, y, $z \in X$,
  (i) $d(x,y)(s+t) \ge L(d(x,z)(s), d(z,y)(t))$ whenever $s \le \lambda_1(x,z), t \le \lambda_1(z,y)$ and $s+t \le \lambda_1(x,y)$, and
  (ii) $d(x,y)(s+t) \le R(d(x,z)(s), d(z,y)(t))$ whenever $s \ge \lambda_1(x,z), t \ge \lambda_1(z,y)$ and $s+t \ge \lambda_1(x,y)$.

The last condition is the triangle inequality.

It is known (see [3]) that the triangle inequality (3)(ii) with $R = \mathrm{Max}$ is equivalent to the triangle inequality

$$\rho_\alpha(x,y) \le \rho_\alpha(x,z) + \rho_\alpha(z,y), \tag{2.2}$$

for all $\alpha \in (0,1]$ and x, y, $z \in X$, and that the triangle inequality (3)(i) with $L = \mathrm{Min}$ is equivalent to the triangle inequality

$$\lambda_\alpha(x,y) \le \lambda_\alpha(x,z) + \lambda_\alpha(z,y), \tag{2.3}$$

for all $\alpha \in (0,1]$ and x, y, $z \in X$.

Kaleva and Seikkala [3] defined a partial ordering $\preceq$ in $\mathcal{F}$ by

$$u \preceq v, \qquad \text{if and only if } a_1^\alpha \le b_1^\alpha \text{ and } a_2^\alpha \le b_2^\alpha,$$

for $\alpha \in (0,1]$, where $[u]_\alpha = [a_1^\alpha, b_1^\alpha]$ and $[v]_\alpha = [a_2^\alpha, b_2^\alpha]$.

They also proved the following result.

LEMMA 2.1. *In a fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$ the triangle inequality (3) is equivalent to*

$$d(x,y) \preceq d(x,z) + d(z,y). \tag{2.4}$$

The fuzzy metric spaces (X, d, L, R) and (X', d', L', R') are isometric if there exists a one-to-one mapping φ from X onto X' such that

$$d(x, y) = d'(\varphi(x), \varphi(y)), \qquad \text{for all } x, y \in X.$$

Let (X, d, L, R) be a fuzzy metric space. A sequence $\{x_n\} \subset X$ is said to converge to $x \in X$, denoted by

$$x = \lim_{n\to\infty} x_n,$$

if and only if

$$\lim_{n\to\infty} d(x_n, x) = \tilde{0},$$

i.e.,

$$\lim_{n\to\infty} \lambda_\alpha (x_n, x) = \lim_{n\to\infty} \rho_\alpha (x_n, x) = 0,$$

for $\alpha \in (0, 1]$.

It is known (see [3]) that in a fuzzy metric space (X, d, L, R) with $\lim_{a\to 0^+} R(a, a) = 0$, the limit of a convergent sequence is uniquely determined and all subsequences of a convergent sequence converge to the limit of the convergent sequence.

A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if

$$\lim_{m,n\to\infty} d(x_m, x_n) = \tilde{0},$$

i.e.,

$$\lim_{m,n\to\infty} \rho_\alpha (x_n, x_m) = 0,$$

for $\alpha \in (0, 1]$.

From (2.2) it follows that in a fuzzy metric space (X, d, L, Max) every convergent sequence is also a Cauchy sequence.

A fuzzy metric space (X, d, L, R) with $\lim_{a\to 0^+} R(a, a) = 0$ is said to be complete if every Cauchy sequence in X converges.

3. THE RESULTS

Let S be a subset in a fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$, and choose $x_0 \in X$. S is said to be bounded if there exist fuzzy numbers u and v such that

$$u \preceq d(x, x_0) \preceq v, \qquad \text{for all } x \in X.$$

By Lemma 2.1, we see that the definition is independent of the choice of x_0.

Now, we introduce the concept of bounded sequences in a fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$.

DEFINITION 3.1. *A sequence $\{x_n\}$ in a fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$ is said to be bounded if $\{x_n\}$ is a bounded subset in X.*

From the inequalities (2.1)–(2.3), we have the following theorem.

THEOREM 3.1. *In a fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$, every convergent sequence is bounded.*

We now discuss the space of convergent sequences in a fuzzy metric space $(X, d, \mathrm{Min}, \mathrm{Max})$ by using the technique of Kaleva [2].

Two convergent sequences $\{x_n\}$, $\{y_n\}$ are called equivalent, denoted by $\{x_n\} \sim \{y_n\}$, if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Clearly $\sim$ is an equivalence relation.

Let c denote the set of all convergent sequences in X, and c^* the collection of equivalence classes of c determined by the equivalence relation $\sim$.

Let $x^*, y^* \in c^*$, $\{x_n\} \in x^*$, $\{y_n\} \in y^*$. Let $\{x_n\}$ and $\{y_n\}$ converge to x and $y \in X$, respectively. The equation

$$d^*(x^*, y^*) = d(x, y),$$

where $\{x_n\} \in x^*$, $\{y_n\} \in y^*$, $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} y_n = y$, defines a fuzzy metric $d^* : c^* \times c^* \to \mathcal{F}^*$. So we have the following lemma.

LEMMA 3.1. *$(c^*, d^*, \mathrm{Min}, \mathrm{Max})$ is a fuzzy metric space with the metric d^* defined by*

$$d^*(x^*, y^*) = d(x, y),$$

where $\{x_n\} \in x^*$, $\{y_n\} \in y^*$, $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} y_n = y$.

THEOREM 3.2. *Let $(X, d, \mathrm{Min}, \mathrm{Max})$ be a complete fuzzy metric space. Then $(c^*, d^*, \mathrm{Min}, \mathrm{Max})$ is complete.*

PROOF. Define $\varphi : X \to c^*$ by setting $\varphi(x)$ the equivalence class of the sequence $\{x, x, \dots\}$. Then φ is a one-to-one mapping from X onto c^* and by

$$d^*(x^*, y^*) = d^*(\varphi(x), \varphi(y)) = d(x, y).$$

Thus, φ is also an isometry, since $(X, d, \mathrm{Min}, \mathrm{Max})$ is complete and consequently $(c^*, d^*, \mathrm{Min}, \mathrm{Max})$ is complete. This completes the proof.

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